Zeitschrift für Analysis
und ihre Anwendungen
Bd. 3 (1) (1984, S. 87–95

Estimates by Lozinsky's functional improved in the linear autonomous case

B. M. GARAY and V. KERTÉSZ

Mit Hilfe eines von LOZINSKIJ eingeführten und auf Matrizen definierten reellen Funktionals μ kann eine untere und obere Abschätzung für die Norm der Lösungen von linearen Differentialgleichungen erhalten werden. Dieses Funktional hängt auch von der angewandten Norm ab. Es wird gezeigt, daß man durch eine entsprechende lineare Transformation der Differentialgleichung die bestmögliche Abschätzung erhalten kann. Im reellen, autonomen Fall kann eine reelle, von der Zeit unabhängige, lineare Transformation verwendet werden, wenn für die Definition von μ die euklidische Norm angenommen wird. Es gibt einen engen Zusammenhang zwischen dem so definierten Funktional μ und den quadratischen Ljapunov-Funktionen. Die Abschätzungen können im Fall anderer Normen allgemein nicht durch eine von der Zeit unabhängige lineare Transformation zur schärfsten verbessert werden.

Вещественнозначный, определенный на матрицах функционал Лозинского μ дает нижние и верхние оценки решений дифференциальных уравнений. Функционал зависит и от выбора нормы. В статье показывается, как можно сделать оценки точными с помощью подходящего преобразования переменных. В вещественном автономном случае преобразование вещественное, линейное, не зависящее от времени, если μ определяется евклидовой нормой. Показана тесная связь между так определенным μ и квадратичными функциями Ляпунова. Доказывается, что в общем случае, для любой нормы, оценки не могут быть сделаны точными линейным преобразованием, не зависящим от времени.

Using a real-valued functional (denoted by μ) introduced by LOZINSKY and defined on matrices, lower and upper bounds can be given for the norm of solutions of linear differential equations. This functional also depends on the norm applied. It is pointed out that the best possible bounds can be obtained applying an appropriate linear transformation of the differential equation. In the real and autonomous case this appropriate linear transformation is real and does not depend on the time if μ is induced by the Euclidean norm. Moreover, close correspondence between μ and quadratic Liapunov functions is shown. It is proved that in the general case (not Euclidean norms) the best possible bounds can, generally, not be obtained if the linear transformation does not depend on the time.

Introduction

Let $|\cdot|$ be a norm in the n-dimensional (real or complex) vector space $\mathbf{K}^n$ ($\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$). The matrix norm induced by $|\cdot|$ is defined as $|A| = \sup_{|x|=1} |Ax|$. Even though single bars are used for norms in different spaces, no confusion should arise.

The following definition makes sense [6]

$$\mu(A) = \lim_{h \to +0} \frac{|I + hA| - 1}{h},$$

where I is the unit matrix. μ is a special real-valued function on matrices induced by the norm $|\cdot|$. In some cases, to emphasize the dependence of $\mu(A)$ upon $|\cdot|$, we write $\mu(A) = \mu_{|\cdot|}(A)$.

We recall some basic facts about μ [3, 4, 6]:

$$\mu(A + B) \leqq \mu(A) + \mu(B), \tag{1}$$

$$\mu(cA) = c\mu(A), \quad c \geqq 0, \tag{2}$$

$$\mu(I) = 1, \tag{3}$$

$$|\mu(A)| \leqq |A|. \tag{4}$$

The eigenvalues of A are denoted by $\lambda_1, \ldots, \lambda_n$. Let $\lambda(A)$ and $\nu(A)$ denote $\max\limits_{1\leqq i\leqq n} \operatorname{Re} \lambda_i$ and $\min\limits_{1\leqq i\leqq n} \operatorname{Re} \lambda_i$, respectively. Then

$$\lambda(A) \leqq \mu(A). \tag{5}$$

Since $\lambda(-A) = -\nu(A)$, (5) implies

$$-\mu(-A) \leqq \nu(A). \tag{6}$$

Further, if the norm $|\cdot| = |\cdot|_V$ is defined by a positive definite Hermitian matrix V, i.e. $|x|_V = (x^* Vx)^{1/2}$, then for $\mu_{|\cdot|_V}$ (or briefly μ_V) we have

$$\mu_V(A) = \sup_{|x|=1} \operatorname{Re} x^* VAx = \lambda\left(\frac{A^*V + VA}{2}\right), \tag{7}$$

where * denotes the conjugate transpose. Obviously, $|\cdot|_I$ is the usual Euclidean norm.

Let t_0 be a fixed real number and consider the linear differential equation

$$\dot{x}(t) = A(t)\, x(t), \quad t \geqq t_0, \tag{8}$$

where $A(t)$ is a continuous matrix function defined for $t \geqq t_0$. Then, for any solution $x(t)$ of (8), the following inequality holds:

$$|x(t_0)| \exp\left(-\int_{t_0}^{t} \mu\big(-A(\tau)\big)\, d\tau\right) \leqq |x(t)| \leqq |x(t_0)| \exp\left(\int_{t_0}^{t} \mu\big(A(\tau)\big)\, d\tau\right), \tag{*}$$

$$t \geqq t_0.$$

The aim of this paper is to investigate the connection between (*) and transformations of variables in the autonomous case. By transforming the variable x, inequality (*) can be improved. Moreover, in case of $|\cdot| = |\cdot|_V$, one can obtain the best possible upper and lower bounds for $|x(t)|$, applying special transformations. A close correspondence between μ and quadratic Liapunov functions is shown as well. On the other side, we show by examples that, in general, although we allow transformations of the variable x, inequality (*) is too weak to give strong estimates for the norm of solutions of (8).

Remark 1: By (4), (*) is an improvement of the well-known inequality

$$|x(t_0)| \exp\left(-\int_{t_0}^{t} |A(\tau)|\, d\tau\right) \leqq |x(t)| \leqq |x(t_0)| \exp\left(\int_{t_0}^{t} |A(\tau)|\, d\tau\right),$$

$$t \geqq t_0.$$

For example, if $A(\tau)$ is a nonzero skew-symmetric matrix function, then, applying (7), $\mu_I\big(A(\tau)\big) = \mu_I\big(-A(\tau)\big) = 0$, but $|A(\tau)|_I > 0$. Similarly, if A is a real asymptotically stable diagonal matrix, then $\mu_I(A) = \lambda(A) < 0$ but $|A|_I = |\nu(A)| > 0$.

Remark 2: It follows from (*) that the asymptotic stability of the zero solution of (8) is implied by the condition: $\mu(A(\tau)) < \delta < 0$ for $\tau \geqq t_0$. However, the converse of this statement is not true even for the autonomous case. This is shown by the following

Example 1: Let A be defined as follows:

$$A = \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix}, \qquad \text{where } a < 0,\ c < 0,\ b^2 > ac.$$

It is easy to see that A is asymptotically stable but $\mu_I(A) > 0$. But, as we shall see in the autonomous case, the converse of our statement is true if the variable x is appropriately transformed. It is valid that if A is asymptotically stable then $\mu(C) < 0$, where $\dot{y} = Cy$ is the transformed equation.

Substituting $y = Q^{-1}(t)\,x$, where $Q(t)$ is a continuously differentiable nonsingular matrix function defined for $t \geqq t_0$, (8) goes over into

$$\dot{y} = \left(\dot{Q}^{-1}(t)\,Q(t) + Q^{-1}(t)\,A(t)\,Q(t)\right) y = C(t)\,y. \tag{9}$$

Applying inequality (*) for (9), we obtain

$$|x(t_0)|\,|Q(t_0)|^{-1}\,|Q^{-1}(t)|^{-1} \exp\left(-\int_{t_0}^{t} \mu(-C(\tau))\,d\tau\right) \leqq |x(t)|$$

$$\leqq |x(t_0)|\,|Q^{-1}(t_0)|\,|Q(t)| \exp\left(\int_{t_0}^{t} \mu(C(\tau))\,d\tau\right), \qquad t \geqq t_0. \tag{10}$$

In case of $Q(t)$ being constant, we have $C(t) = Q^{-1}A(t)\,Q$, and inequality (10) turns into

$$|x(t_0)|\,|Q|^{-1}\,|Q^{-1}|^{-1} \exp\left(-\int_{t_0}^{t} \left(-Q^{-1}A(\tau)\,Q\right) d\tau\right) \leqq |x(t)|$$

$$\leqq |x(t_0)|\,|Q^{-1}|\,|Q| \exp\left(\int_{t_0}^{t} \mu\left(Q^{-1}A(\tau)\,Q\right) d\tau\right), \qquad t \geqq t_0. \tag{11}$$

On matrix theory used in this paper, see [1]. Through this paper, excepting Remark 5, (8) will be autonomous. The nonautonomous case as well as the case of nonlinear perturbations are treated in [5].

Estimates for some special norms

Proposition 1: *Let* $\mathrm{K} = \mathrm{C}$. *Let A be an $n \times n$ matrix. Then*

$$\inf\{\mu_I(Q^{-1}AQ) \mid Q \text{ complex, nonsingular}\} = \lambda(A),$$

$$\sup\{-\mu_I(-Q^{-1}AQ) \mid Q \text{ complex, nonsingular}\} = \nu(A).$$

Further the infimum is attained if and only if for any λ_i such that $\operatorname{Re}\lambda_i = \lambda(A)$ *the corresponding Jordan blocks are diagonal.*

The supremum is attained if and only if for any λ_i such that $\operatorname{Re}\lambda_i = \nu(A)$ *the corresponding Jordan blocks are diagonal.*

Proof: We restrict ourselves to the proof of the statements concerning infimum. By $\lambda(-A) = -\nu(A)$, the other part of the proposition follows immediately.

By (5), it is clear that $\mu_I(Q^{-1}AQ) \geqq \lambda(A)$ for any Q. If (8) has a solution x_0 for which $|x_0(t)| = t\, e^{\lambda(A)t}$, it follows from (11) that the infimum cannot be attained.

We put A into its Jordan canonical form $J: T^{-1}AT = J$. J is a blocked diagonal matrix whose diagonal entries are of the form

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \qquad (k = 1, \ldots, N(J) \leqq n).$$

J_k is an $r_k \times r_k$ matrix, $\sum_{k=1}^{N(3)} r_k = n$, λ_k is an eigenvalue of A. For any real $\varepsilon \neq 0$ the diagonal matrix J_k is similar to $\tilde{J}_k(\varepsilon)$, where

$$\tilde{J}_k(\varepsilon) = \begin{pmatrix} \lambda_k & \varepsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda_k \end{pmatrix}.$$

In fact, $S_k^{-1}(\varepsilon)\, J_k S_k(\varepsilon) = \tilde{J}_k(\varepsilon)$, where

$$S_k(\varepsilon) = \begin{pmatrix} 1 & & & & \\ & \varepsilon & & & \\ & & \varepsilon^2 & & \\ & & & \ddots & \\ & & & & \varepsilon^{r_k - 1} \end{pmatrix}.$$

Thus, $S^{-1}(\varepsilon)\, JS(\varepsilon) = \tilde{J}(\varepsilon)$, where $S(\varepsilon)$ is a blocked diagonal matrix whose diagonal entries are $S_k(\varepsilon)$. It is clear that $\lambda\left(\frac{\tilde{J}(\varepsilon) + \tilde{J}^*(\varepsilon)}{2}\right) \to \lambda(A)$ as $\varepsilon \to 0$. Thus, for $Q(\varepsilon)$ defined as $TS(\varepsilon)$, $\mu_I(Q^{-1}(\varepsilon)\, AQ(\varepsilon)) \to \lambda(A)$ as $\varepsilon \to 0$. If for any λ_i for which $\operatorname{Re} \lambda_i = \lambda(A)$ the corresponding Jordan blocks are diagonal, it is clear that there is an $\varepsilon_0 > 0$ such that for ε $(0 < |\varepsilon| < \varepsilon_0)$

$$\mu(Q^{-1}(\varepsilon)\, AQ(\varepsilon)) = \lambda(A) \;\blacksquare$$

Proposition 2: *Let* $\mathbf{K} = \mathbf{R}$. *Let A be an $n \times n$ real matrix. Then the previous proposition remains true if Q is restricted to be taken real.*

Proof: Suppose that λ_l is a non-real eigenvalue of A and J_l is a Jordan block belonging to it. Then J_l^*, the conjugate transpose of J_l, is a Jordan block as well, belonging to the eigenvalue $\bar{\lambda}_l$. $\tilde{J}_l(\varepsilon)$ is defined as in the proof of Proposition 1. Rearranging the blocks of $\tilde{J}(\varepsilon)$ if necessary, we join $\tilde{J}_l(\varepsilon)$ and $\tilde{J}_l^*(\varepsilon)$ together as $H_l(\varepsilon)$, where

$$H_l(\varepsilon) = \left(\begin{array}{cccc|cccc} \lambda_l & \varepsilon & & & & & & \\ & \ddots & \ddots & & & & 0 & \\ & & \ddots & \varepsilon & & & & \\ & & & \lambda_l & & & & \\ \hline & & & & \bar{\lambda}_l & \varepsilon & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & \ddots & \varepsilon \\ & & & & & & & \lambda_l \end{array}\right).$$

$H_l(\varepsilon)$ is similar to $\tilde{\tilde{J}}_l(\varepsilon)$, where $\alpha_l = \text{Re}\,\lambda_l$, $\beta_l = \text{Im}\,\lambda_l$ and

$$\tilde{\tilde{J}}_l(\varepsilon) = \left(\begin{array}{ccc|ccc} \alpha_l & \varepsilon & & -\beta_l & & \\ & \ddots & \varepsilon & & \ddots & \\ & & \alpha_l & & & -\beta_l \\ \hline \beta_l & & & \alpha_l & \varepsilon & \\ & \ddots & & & \ddots & \varepsilon \\ & & \beta_l & & & \alpha_l \end{array}\right).$$

In fact, $\tilde{\tilde{J}}_l(\varepsilon) = Z_l^{-1} H_l(\varepsilon)\, Z_l$, where

$$Z_l = \frac{1}{2}\left(\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \\ \hline 1 & & & -i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & -i \end{array}\right).$$

Z_l is a $2r_l \times 2r_l$ matrix which is partitioned into four $r_l \times r_l$ blocks. If λ_l is real, let Z_l be the $r_l \times r_l$ unit matrix. Thus, $Z^{-1}\tilde{J}(\varepsilon)\, Z = \tilde{\tilde{J}}(\varepsilon)$.

It is clear that $\lambda\left(\dfrac{\tilde{\tilde{J}}(\varepsilon) + \tilde{\tilde{J}}^*(\varepsilon)}{2}\right) \to \lambda(A)$ as $\varepsilon \to 0$. Thus, for $Q(\varepsilon)$ defined as $TS(\varepsilon)Z$, $\mu_I\big(Q^{-1}(\varepsilon)\, AQ(\varepsilon)\big) \to \lambda(A)$ as $\varepsilon \to 0$. It is a lengthy but straightforward task to prove that $Q(\varepsilon)$ is real. The remaining part of the proof is similar to the one of Proposition 1 ∎

Proposition 3: *Let* $\mathbf{K} = \mathbf{C}$. *Proposition* 1 *remains true if* μ *is induced by the norms* $|\cdot|_1$ *or* $|\cdot|_2$, *where for* $x \in \mathbf{K}^n$, $x = (x_1, \ldots, x_n)$, $|x|_1$ *and* $|x|_2$ *are defined as* $|x|_1 = \max_i |x_i|$ *and* $|x|_2 = \sum_i |x_i|$, *respectively.*

Proof: It is known [3] that

$$\mu_1(A) = \max_i \Big(\text{Re}\, a_{ii} + \sum_{k,k\neq i} |a_{ik}|\Big), \qquad \mu_2(A) = \max_k \Big(\text{Re}\, a_{kk} + \sum_{i,i\neq k} |a_{ik}|\Big).$$

Without any modification, the proof of Proposition 1 can be repeated ∎

Example 2: Let A be the following 2×2 real matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then, for $i = 1, 2$

$$\inf \{\mu_i(Q^{-1}AQ) \mid Q \text{ complex, nonsingular}\} = 1,$$

but

$$\inf \{\mu_i(Q^{-1}AQ) \mid Q \text{ real, nonsingular}\} = 2.$$

Since $\lambda(A) = 1$, $\inf \{\mu_i(Q^{-1}AQ) \mid Q \text{ complex, nonsingular}\} = 1$ by Proposition 3. The infimum is attained if $Q = T$.

If C is a real matrix similar to A, then C has the form

$$C = \begin{bmatrix} 1 + a & b \\ c & 1 - a \end{bmatrix}, \quad \text{where } 1 - a^2 - bc = 2.$$

Then

$$\mu_1(C) = \max(1 + a + |b|, 1 - a + |c|) \geqq (1 + a + |b| + 1 - a + |c|)/2$$
$$= 1 + \frac{|b| + |c|}{2} \geqq 1 + \sqrt{|bc|} = 1 + \sqrt{a^2 + 1} \geqq 2.$$

Similarly, $\mu_2(C) \geqq 2$. For $C = A$ we have $\mu_1(C) = \mu_2(C) = 2$.

Proposition 4: *Substituting μ_I by μ_V, Proposition 1 and Proposition 2 remain true.*

Proof: By (7),

$$\mu_V(A) = \sup_{x \neq 0} \frac{x^*(A^*V + VA)}{2x^*Vx}.$$

Let W be a matrix for which $V = W^*W$. W is nonsingular and can be taken real if V is real. By elementary operations and (7)

$$\mu_V(A) = \sup_{Wx \neq 0} \frac{x^*W^*[(WAW^{-1})^* + WAW^{-1}]\,Wx}{2x^*W^*Wx} = \mu_I(WAW^{-1}).$$

Since $\lambda(WAW^{-1}) = \lambda(A)$ and $\nu(WAW^{-1}) = \nu(A)$, Proposition 1 and Proposition 2 can be applied directly. We have to use the fact that $\{Q^{-1}WAW^{-1}Q \mid Q$ nonsingular$\}$ $= \{Q^{-1}AQ \mid Q$ nonsingular$\}$, too ∎

Corollary: *Recall that for any V the expression $A^*V + VA$ is the derivative of the quadratic form $v(x) = x^*Vx$ with respect to the differential equation* (8). *Let V be positive definite. It is immediately seen from* (7) *that V is a quadratic Liapunov function with respect to* (8) *— i.e. the derivative of $V(x)$ with respect to* (8) *is negative definite — if and only if $\mu_V(A) < 0$. For any A such that $\lambda(A) < 0$ there exist quadratic Liapunov functions, since* $\inf\{\mu_V(A) \mid V$ *positive definite*$\} = \lambda(A)$. *For example, W^*W is a quadratic Liapunov function if $|\varepsilon|$ is sufficiently small and $W^{-1} = TS(\varepsilon)\,Z$.*

Remark 3: The Corollary above is implicitely contained e.g. in [7].

Estimates for arbitrary norms

Lemma: *Let $\mathbf{K} = \mathbf{R}$ and let $|\cdot|$ be an arbitrary norm on $\mathbf{R}^2$. Let P be a projection of $\mathbf{R}^2$ onto a one-dimensional subspace. Assume that $|P| \geqq 1 + \alpha$, where $\alpha > 0$. Then*

$$\mu(P) \geqq 1 + \frac{\alpha}{\alpha + 1}\,\varrho(\alpha), \qquad \textit{where} \qquad \varrho(\alpha) = \sup_{\lambda \in (0,1)} \frac{1 - \lambda}{\ln\left(1 + \frac{2}{\alpha}\right) - \ln \lambda}.$$

Remark 4: A straightforward but somewhat lengthy calculation shows that $\varrho(\alpha)$ is the unique solution of the equation

$$\ln \varrho + \frac{1}{\varrho} = 1 + \ln\left(1 + \frac{2}{\alpha}\right)$$

in the (0, 1) interval. It is easy to see that $\lim_{\alpha \to 0} \varrho(\alpha) \dfrac{1}{\ln \frac{1}{\alpha}} = 1$.

Proof of the Lemma: Assume that our two-dimensional real normed space is represented on the usual Euclidean plane, i.e. vectors of our two-dimensional real normed space are identified with points of the usual Euclidean plane. The unit sphere $\{x \in \mathbf{R}^2 \mid |x| = 1\}$ is represented as a convex closed curve Γ symmetric to the origin.

Pick a point M on Γ such that $|P(M)| = 1 + \alpha$. Without loss of generality we can assume that the coordinates of $P(M)$ are $(A(1 + \alpha), 0)$, $A > 0$, and that those of M are $(A(1 + \alpha) + v_1, v_2)$, $v_2 > 0$ (see Fig. 1). The intersection point of Γ and the

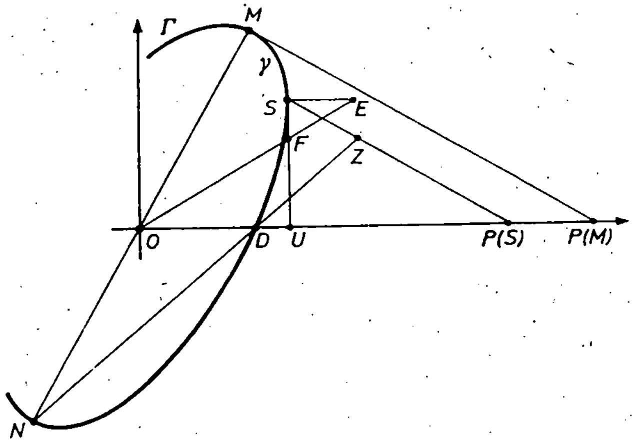

Fig. 1

ray $OP(M)$ is denoted by D, the shorter arc on Γ between D and M is denoted by γ. γ can be parametrized as

$$\{(t + s(t)\, v_1, s(t)\, v_2) \mid A \leqq t \leqq (1 + \alpha)\, A\},$$

where s is a continuous, monotonically increasing convex function, $s(A) = 0$, $s(A(1 + \alpha)) = 1$ and — if M is properly chosen —, $\dot{s}$, the left-hand derivative of s, satisfies $0 < \dot{s}(t) < \infty$, $t \in [A, (1 + \alpha)\, A]$.

Pick a point S on γ. The coordinates of S are $(t + s(t)\, v_1, s(t)\, v_2)$, and those of $P(S)$ are $(t, 0)$. The opposite of M is denoted by N. By convexity, the line passing through the points N and D intersects the segment $SP(S)$ at a point Z. The first coordinate of Z is $t + v_1 \dfrac{t - A}{(2 + \alpha)\, A}$, therefore

$$s(t) \geqq (t - A)\, \frac{1}{(2 + \alpha)\, A}. \tag{12}$$

Let $h > 0$. For brevity, $S + hP(S)$ is denoted by E. The left-hand tangent of the arc γ at S cuts the ray OE in a point F and the segment $DP(M)$ in a point U. The coordinates of U are $(t - s(t)/\dot{s}(t), 0)$. Consequently,

$$A \leqq t - \frac{s(t)}{\dot{s}(t)} \leqq A(1 + \alpha). \tag{13}$$

By convexity of Γ, $F \geqq 1$. As $|E|/|F|$ is the quotient of the first coordinates of E and of F, a direct computation shows that

$$\frac{|E|}{|F|} = 1 + h\, \frac{t}{t - s(t)/\dot{s}(t)}.$$

Therefore, using (13) as well,

$$\frac{|S + hP(S)| - 1}{h} = \frac{|E| - 1}{h} \geqq \frac{|E|/|F| - 1}{h} = \frac{t}{t - s(t)/\dot{s}(t)}$$
$$= 1 + \frac{s(t)/\dot{s}(t)}{t - s(t)/\dot{s}(t)} \geqq 1 + \frac{s(t)/\dot{s}(t)}{A(1 + \alpha)}.$$

Therefore $\mu(P) \geqq 1 + \frac{1}{A(1 + \alpha)} \beta$, where

$$\beta = \sup \{s(t)/\dot{s}(t) \mid A \leqq t \leqq A(1 + \alpha)\}.$$

As a corollary of (13), $0 < \beta < \infty$.

By the definition of β,

$$\frac{\dot{s}(t)}{s(t)} \geqq \frac{1}{\beta}, \qquad t \in [A, A(1 + \alpha)].$$

Let λ be a fixed number satisfying $0 < \lambda < 1$. Integrating from $A(1 + \alpha\lambda)$ to $A(1 + \alpha)$, we obtain

$$\ln s\big(A(1 + \alpha)\big) - \ln s\big(A(1 + \alpha\lambda)\big) \geqq A\alpha(1 - \lambda)\beta.$$

By (12),

$$-\ln \frac{\alpha\lambda}{2 + \alpha} \geqq A\alpha(1 - \lambda)\beta, \qquad \beta \geqq A\alpha(1 - \lambda)/\ln \frac{2 + \alpha}{\alpha\lambda}.$$

Therefore

$$\mu(P) \geqq 1 + \alpha \frac{1}{1 + \alpha} \sup_{\lambda \in (0,1)} \frac{1 - \lambda}{\ln\left(1 + \frac{2}{\alpha}\right) - \ln \lambda},$$

which was to be proved ∎

Proposition 5: *Let $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$. Let $|\cdot|$ be a norm on $\mathbf{K}^3$. The set*

$$\{P: \mathbf{K}^3 \to \mathbf{K}^3 \mid P \text{ is a projection, rank } (P) = 2\}$$

is denoted by $\mathcal{P}$. Then the following conditions on the norm $|\cdot|$ are equivalent:

(i) *For any $P \in \mathcal{P}$, $|P| > 1$.*

(ii) *There is a $P_0 \in \mathcal{P}$ such that* $\inf \{\mu(Q^{-1}P_0Q) \mid Q$ *nonsingular*$\} > 1$.

Proof: It is obvious that $\{Q^{-1}P_0Q \mid Q \text{ nonsingular}\} = \mathcal{P}$ for any $P_0 \in \mathcal{P}$, therefore $\inf \{\mu(Q^{-1}P_0Q) \mid Q \text{ nonsingular}\} = \inf \{\mu(P) \mid P \in \mathcal{P}\}$. We make use of this fact in both parts of proving our proposition.

(i) → (ii): Suppose that Condition (i) holds. Then, by a compactness argument, there exists an $\alpha > 0$ such that $|P| \geqq 1 + \alpha$ for any $P \in \mathcal{P}$. Let $P \in \mathcal{P}$ be arbitrarily chosen and let $x_0 \in \mathbf{K}^3$ be such that $|Px_0| \geqq (1 + \alpha)|x_0|$. Consider the set $X = \{\lambda x_0 + \mu P x_0 \mid \lambda, \mu \text{ real}\}$. Let P' be the restriction of P on X and $|\cdot|'$ be the restriction of $|\cdot|$ on X. It follows directly from definition that X is a two-dimensional real vector space and $|\cdot|'$ is a norm on X. Further, P' is a projection of X onto a one-dimensional subspace, $|P'|' \geqq 1 + \alpha$ and $\mu_{|\cdot|}(P) \geqq \mu_{|\cdot|'}(P')$, where, using the lemma, $\mu_{|\cdot|'}(P') \geqq 1 + \alpha\varrho(\alpha) > 1$, which was to be proven.

(ii) → (i): This is evident by inequality (4) ∎

Thus we have proven that — in general — inequality (*) cannot be essentially improved by transforming variables. In fact, in Example 2 and Proposition 5 we have shown the existence of norms $|\cdot|$ and of matrices A such that $\lambda(A) < \inf \{\mu_{|\cdot|}(Q^{-1}AQ) \mid Q \text{ nonsingular}\}$. On construction and some properties of norms satisfying Condition (i) see [2].

Remark 5: If the transition matrix Q is allowed to be dependent on t, then — transforming the variable x — the best estimation can be achieved by (*). This remark holds as well in case of (8) being nonautonomous. In fact, let $Q(t) = X(t)$, where $X(t)$ is the fundamental matrix. Then (8) goes over into $\dot{y} = 0$; $y = 0$. Since $\mu(0) = 0$, (10) implies

$$|X(t_0)|^{-1}\,|X^{-1}(t)|^{-1}\,|x(t_0)| \leqq |x(t)| \leqq |X(t)|\,|X^{-1}(t_0)|\,|x(t_0)|, \qquad t \geqq t_0. \tag{14}$$

In case of (8) being autonomous, the transition matrix function can as well be chosen in the following manner:

For $\varepsilon \neq 0$, let $Q_\varepsilon(t) = \tilde{J}(\varepsilon)\,U(t)$, where $U(t)$ is a blocked diagonal matrix function with diagonal entries $U_k(t) = \exp \lambda_k(t - t_0) \cdot I_k$, $k = 1, 2, \ldots, N(A)$; I_k being the $r_k \times r_k$ unit matrix, $\tilde{J}(\varepsilon)$ being the same as in the proof of Proposition 1. Since

$$\dot{Q}_\varepsilon^{-1}(t)\,Q_\varepsilon(t) + Q_\varepsilon^{-1}(t)\,AQ_\varepsilon(t) = \varepsilon E, \tag{15}$$

where E is a blocked diagonal matrix with diagonal entries $E_k = J_k - \lambda_k I_k$, it is easy to show that the best lower and upper exponential bounds on norms of solutions of (8) are assured by inequality (10) as $\varepsilon \to 0$.

REFERENCES

[1] Bellmann, R.: Introduction to Matrix Analysis. McGraw-Hill: New York—Toronto—London 1960.

[2] Bosznay, A. P., and B. M. Garay: On norms of projections (to appear).

[3] Coppel, W. A.: Stability and asymptotic behaviour of differential equations. D. C. Heath: Boston 1965.

[4] Dahlquist, G.: Stability and error bounds in the numerical integration of ordinary differential equations. Kungl. Tekn. Högsk. Handl. Stockholm No 130 (1959), 87 pp.

[5] Kertész, V.: Application of positive definite Ljapunov functions for stability investigations (to appear).

[6] Lozinsky, S. M.: Error estimates for the numerical integration of ordinary differential equation (in Russian). Izv. Vyss. Ucebn. Zaved. Matematika 5 (6) (1958), 52—90.

[7] Petrovsky, I. G.: Lectures on ordinary differential equations (in Russian). Izd. Nauka: Moscow 1970.

Manuskripteingang: 10. 07. 1982

VERFASSER:

Dr. Barnabas Garay and Doz. Dr. Viktor Kertész
Department of Mathematics, Faculty of Mechanical Engineering
University of Technology
H-1521 Budapest, Muegyetem rkp. 3