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Fredholmness and finite section method for Toeplitz operators in $l^p(\mathbf{Z}_+ \times \mathbf{Z}_+)$ with piecewise continuous symbols I[1])

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Wir betrachten diskrete Toeplitzoperatoren im Raum l^p über der Viertelebene für eine Klasse von stückweise stetigen Symbolen. Diese Klasse wird gewöhnlich mit $PC_p(\mathbf{T}^2)$ bezeichnet und enthält insbesondere alle endlichen Summen der Form $\sum a_i(\xi)\, b_i(\eta)$, $(\xi, \eta) \in \mathbf{T}^2$, wobei a_i und b_i von beschränkter Variation sind. Wir geben notwendige und hinreichende Bedingungen dafür an, daß ein solcher Operator noethérisch ist und ebenso dafür, daß auf ihn das Reduktionsverfahren anwendbar ist. Der vorliegende Teil I enthält die nötigen Definitionen, die Formulierung der Hauptresultate und die Beweise der Notwendigkeit der angegebenen Bedingungen. Teil II dieser Arbeit ist den Beweisen der Hinlänglichkeit dieser Bedingungen gewidmet.

We consider discrete Toeplitz operators on the space l^p over the quarter-plane for a class of piecewise continuous symbols. This class of symbols is usually denoted by $PC_p(\mathbf{T}^2)$ and it contains, in particular, all finite sums of the form $\sum a_i(\xi)\, b_i(\eta)$, $(\xi, \eta) \in \mathbf{T}^2$, where a_i and b_i are of bounded variation. Necessary and sufficient conditions for Fredholmness of such operators and for the applicability of the finite section method to them are obtained. The present part I contains the necessary definitions, the formulation of the main results, and the proofs of the necessity of the given conditions. Their sufficiency will be proved in part II of this work.

Рассматриваются дискретные операторы теплица в пространстве l^p на квадранте для одного класса кусочно-непрерывных символов, обозначаемого обычно через $PC_p(\mathbf{T}^2)$ и содержащего, в частности, все конечные суммы вида $\sum a_i(\xi)\, b_i(\eta)$, $(\xi, \eta) \in \mathbf{T}^2$, где a_i и b_i — функции ограниченной вариации. Получены критерии нетеровости и применимости метода редукции для таких операторов. Часть I содержит необходимые определения, формулировку главных результатов и доказательства необходимости указанных условий. Часть II этой работы посвящена доказательствам достаточности этих условий.

§ 1. Introduction

With the one-dimensional Toeplitz operator $T(a)$ defined by

$$\left(T(a)\,\varphi\right)_i = \sum_{j=0}^{\infty} a_{i-j}\varphi_j \qquad (i \geqq 0)$$

on the space $l^p (1 < p < \infty)$ we associate the function $a(t) = \sum_{j=0}^{\infty} a_j t^j$ $(|t| = 1)$ and refer to a as the symbol of $T(a)$. Besides the question of Fredholmness and invertibility, the finite section method, as a very natural procedure for the approximate solution of the equation $T(a)\,\varphi = f$, has been the subject of numerous investigations since the earliest studies of Toeplitz operators. We say that the finite section method is *applicable* to $T(a)$ in l^p if for every $f \in l^p$ the equation

$$\sum_{j=0}^{n} a_{i-j}\varphi_j^{(n)} = f_i \qquad (i = 0, 1, \ldots, n) \tag{1}$$

[1]) Der abschließende Teil II wird in Kürze ebenfalls in dieser Zeitschrift erscheinen.

has a unique solution for all sufficiently large n and if $\varphi^{(n)} = \{\varphi_j^{(n)}\}_{j=0}^{n}$ converges in the norm of l^p to a solution φ of the equation $T(a)\,\varphi = f$. In this case we write $T(a) \in \Pi_p\{P_n\}$.

After the fundamental paper [19] of M. G. KREIN a first systematical treatment of Fredholmness, invertibility, and finite section method for Toeplitz operators was given by I. C. GOHBERG and I. A. FELDMAN in the book [15].

It turns out that a one-dimensional Toeplitz operator is invertible on l^p if and only if it is Fredholm and has index zero. For a Toeplitz operator $T(a)$ with continuous symbol a to be Fredholm on l^p it is necessary and sufficient that $a(t) \neq 0$, $t \in \mathbf{T}$. Then Ind $T(a) = -$ind a, where ind a is the winding number of the range of a with respect to the origin. Finally, for continuous symbols we have $T(a) \in \Pi_p\{P_n\}$ if and only if $T(a)$ is invertible on l^p. These results may be found in [15, 4, 23, 24, 14, 21, 32].

Having solved the fundamental problems for Toeplitz operators with continuous symbols, the interest in discontinuous, first of all in piecewise continuous, symbols arose.

The first result in this direction was concerned with the space l^2: a Toeplitz operator $T(a)$ with piecewise continuous symbol a is Fredholm on l^2 if and only if the origin does not ly on the continuous closed curve a_2 obtained from the range of a by filling in the *straight line segments* joining $a(t_j - 0)$ to $a(t_j + 0)$ for each discontinuity (cf. [31, 13–15, 21]).

While for continuous symbols the results concerning Fredholmness did, roughly speaking, not differ in the cases $p = 2$ and $p \neq 2$, the difference between these two cases became appearent when succeeding for piecewise continuous symbols: the Toeplitz operator $T(a)$ is Fredholm on l^p if and only if the origin does not lie on the continuous closed curve a_p obtained from the range of a by filling in certain *circular arcs* joining $a(t_j - 0)$ to $a(t_j + 0)$ for each discontinuity (cf. [14, 8]).

In [15] the problem of the applicability of the finite section method to Toeplitz operators with piecewise continuous symbols having only a finite number of discontinuities was solved for the case $p = 2$: $T(a) \in \Pi_2\{P_n\}$ if and only if $T(a)$ is invertible (which is equivalent to $0 \notin a_2$ and ind $a_2 = 0$). But for $p \neq 2$ or for piecewise continuous symbols with countably many discontinuities the problem has been open for a long time. In [30], I. E. VERBICKIĬ and N. YA. KRUPNIK obtained a necessary and sufficient condition for $T(a) \in \Pi_p\{P_n\}$ if a has only one discontinuity. This result was generalized in [2] to the case of a finite number of discontinuities by an argument which we could call "separation of singularities". But the final solution of the problem was given by B. SILBERMANN in [26] only recently. He succeeded by developing a method which allows to carry over the local principle of I. C. GOHBERG and N. YA. KRUPNIK [14] (which has been so useful for Fredholmness) to the investigation of the finite section method. In this way he reduced the problem to the case of only one discontinuity, which had, fortunately, already been considered by I. E. VERBICKIĬ and N. YA. KRUPNIK.

The result obtained in [26] reads: for a piecewise continuous symbol a (with, possibly, a countable number of discontinuities) we have $T(a) \in \Pi_p\{P_n\}$ if and only if $T(a)$ is invertible on both l^p and $l^q (1/p + 1/q = 1)$. Geometrically speaking, this means that the origin must not lie in the region obtained from the range of a by adding certain *lentiform domains* joining $a(t_j - 0)$ to $a(t_j + 0)$ for each discontinuity and that the curve a_2, completely contained in this region, has index zero.

All the problems considered above are emerging for higher-dimensional Toeplitz operators as well. The two-dimensional Toeplitz operator $W(a)$ is defined by

$$\big(W(a)\,\varphi\big)_{i,j} = \sum_{k,l=0}^{\infty} a_{i-k,j-l}\varphi_{kl} \qquad (i, j \geqq 0)$$

on the space $l^p \otimes l^p \cong l^p(\mathbf{Z}_+ \times \mathbf{Z}_+)$ $(1 < p < \infty)$. Now the function $a(\xi, \eta) = \sum_{i,j=-\infty}^{\infty} a_{ij}\xi^i\eta^j$ given on the torus $\mathbf{T}^2$ is referred to as the symbol of $W(a)$. Applicability of the finite section method is defined in a similar way as for the one-dimensional case, only with (1) replaced by

$$\sum_{k,l=0}^{n} a_{i-k,j-l}\varphi_{kl}^{(n)} = f_{ij} \qquad (0 \leq i, j \leq n).$$

Instead of $T(a) \in \Pi_p\{P_n\}$ we now write $W(a) \in \Pi_p\{P_n \otimes P_n\}$.

The first deeper results on multidimensional Toeplitz operators were obtained by I. B. SIMONENKO by means of his local principle: if $a(\xi, \eta)$ is continuous, then $W(a)$ is Fredholm on $l^p \otimes l^p$ if and only if $a(\xi, \eta) \neq 0$, $(\xi, \eta) \in \mathbf{T}^2$, and $\operatorname{ind}_\xi a = \operatorname{ind}_\eta a = 0$ (cf. [28]; see also [5] for $p = 2$).

A. V. KOZAK was the first who realized that local principles can be applied to the investigation of the finite section method and his approach, based upon an essential generalization of the local principle of I. B. SIMONENKO, led to a series of remarkable results on multidimensional Toeplitz operators with continuous symbols (see [16—18], but also [12]). Finally, in [22], V. S. PILIDI developed a local method which can advantageously be used to derive results on higher-dimensional operators from the one-dimensional situation.

Two-dimensional Toeplitz operators with piecewise continuous symbols (though of a special kind, namely, from appropriate tensor products) have been considered as well. In [10], R. V. DUDUČAVA solved the problem of Fredholmness for two-dimensional Toeplitz operators with piecewise continuous symbols in $l^2 \otimes l^2$. In [3] a criterion for the applicability of the finite section method to such operators was obtained, again for the case $p = 2$.

But at present we do not known anything about Fredholmness or the finite section method for two-dimensional Toeplitz operators with piecewise continuous symbols in the case $p \neq 2$. It is the aim of this paper to fill out this gap.

It should be noted that the problem of invertibility for higher-dimensional Toeplitz operators is extremely difficult and that its solution is hardly to be expected at the given moment (even for continuous symbols). But in general, the problem of the applicability of the finite section method is considered as solved if it has been reduced to that of invertibility.

Furthermore, note that all the problems touched upon here for (discrete) Toeplitz operators arise for their continuous analogue, the Wiener-Hopf integral operators, too. See [19, 15, 6, 7, 11, 23, 24, 32, 1] for the one-dimensional case and in the higher-dimensional case we refer to [27, 17] for continuous and to [9] for piecewise continuous symbols.

In particular, in [9] a criterion for a two-dimensional Wiener-Hopf integral operator with piecewise continuous symbol to be Fredholm on $L^p(\mathbf{R}_+ \times \mathbf{R}_+)$ was established. This problem was solved by applying the local principle of I. C. GOHBERG and N. YA. KRUPNIK [14] with the strategy of V. S. PILIDI [22], but making use of some features of integral operators. Due to the latter fact, the method of [9] cannot be carried over to the discrete case.

In this connection B. Silbermann drew my attention to N. YA. KRUPNIK's paper [20], where the local principle of R. G. DOUGLAS [4] for C^*-algebras was generalized to Banach algebras. His intuition was right — as we shall demonstrate in the given paper, the local principle of [20] applied with the method of V. S. PILIDI [22] is indeed powerful enough to solve the problems considered here.

The present paper is very voluminous. This is due to the fact that it are not the results (formulated in the Theorems 1 and 2) which are of our primary interest. These results are easy to guess if one puzzles together former results from [26, 9, 10] and [3]. It is rather a problem of *how* to prove these results which seems to be of interest. The main dilemma is that C^*-algebra techniques, playing an important part in the case $p = 2$ (see [10, 3]), fail for $p \neq 2$. Moreover, everyone who has already been concerned with similar problems will know how carefully one has to work in order to pass over from the case $p = 2$ to the case $p \neq 2$. Therefore we were constrained to give all essential details of the proofs, which, consequently, led to a considerable enlargement of the volume of this paper.

I express my sincere thanks B. Silbermann, without whose helpful suggestions and continuous interest in the subject considered here I would not have been able to accomplish this paper. I am grateful to V. B. Dybin, R. V. Dudučava, and A. V. Kozak for stimulating discussions.

§ 2. One-dimensional Toeplitz operators in l^p with piecewise continuous symbols

For $1 < p < \infty$ let l^p be the customary Banach space of all sequences $\varphi = \{\varphi_n\}_{n=0}^{\infty}$ satisfying $\|\varphi\|_p = (\sum |\varphi_n|^p)^{1/p} < \infty$. By $\mathfrak{L}(l^p)$, $\mathfrak{K}_p = \mathfrak{K}(l^p)$ and $\Phi(l^p)$ we denote the Banach algebra of all bounded (linear) operators on l^p, the ideal of compact operators in $\mathfrak{L}(l^p)$ and the collection of all Fredholm (Noetherian) operators in $\mathfrak{L}(l^p)$, respectively. Given a Banach algebra $\mathfrak{A}$ with unit, we will denote by $G\mathfrak{A}$ the group of invertible elements of $\mathfrak{A}$ throughout the paper.

Let $\mathbf{T}$ be the complex unit circle and $a \in L^\infty(\mathbf{T})$ a function with bounded variation on $\mathbf{T}$. Then the Toeplitz operator $T(a)$, induced by the semi-infinite matrix

$$T(a) = \{a_{j-k}\}_{j,k=0}^{\infty}, \qquad a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\psi})\, e^{-in\psi}\, d\psi,$$

is bounded on l^p, $1 < p < \infty$ [29: Lemma 10]. The function a is referred to as the symbol of $T(a)$. By PC_0 we denote the collection of all piecewise constant functions on $\mathbf{T}$ having only finitely many discontinuities. Thus for $a \in PC_0$ we have $T(a) \in \mathfrak{L}(l^p)$ for every p, $1 < p < \infty$. Let $PC_p(\mathbf{T})$ be the closure of PC_0 with respect to the norm $\|a\|_p = \|T(a)\|_{\mathfrak{L}(l^p)}$. Note that $PC_2(\mathbf{T})$ consists of all functions a that are continuous with exception of at most countably many points, where, however, the limits $a(t \pm 0)$ exist and are finite. Furthermore, we have

$$PC_p(\mathbf{T}) = PC_q(\mathbf{T}) \subset PC_r(\mathbf{T}) = PC_s(\mathbf{T}) \subset PC_2(\mathbf{T})$$

for $1 < p < r < 2$, $1/p + 1/q = 1$, $1/r + 1/s = 1$, and all piecewise continuous functions with bounded variation belong to $PC_p(\mathbf{T})$, $1 < p < \infty$ (cf. [8, 29]).

By $\mathfrak{B}_p$ we denote the closure in $\mathfrak{L}(l^p)$ of the collection of all operators of the form

$$A = \sum_{j=1}^{r} \prod_{k=1}^{s} T(a_{jk}),$$

$r, s \in \mathbf{Z}_+$, $a_{jk} \in PC_0$. We list some properties of the Banach algebra $\mathfrak{B}_p$ (cf. [8, 11, 13, 14]): we have $\mathfrak{K}_p \subset \mathfrak{B}_p$ and $\mathfrak{K}_p$ forms a closed two-sided ideal in $\mathfrak{B}_p$; the quotient space $\mathfrak{B}_p/\mathfrak{K}_p$ is a commutative Banach algebra with unit. Let σ_p denote the canonical projection of $\mathfrak{B}_p$ onto $\mathfrak{B}_p/\mathfrak{K}_p$, $\mathfrak{M}_p$ the maximal ideal space of $\mathfrak{B}_p/\mathfrak{K}_p$ and $\Gamma_{\mathfrak{M}_p}$ the Gelfand map of $\mathfrak{B}_p/\mathfrak{K}_p$ into $C(\mathfrak{M}_p)$. Note that $\mathfrak{M}_p$ is homeomorphic to the cylinder

$\mathbf{T} \times [0, 1]$, equipped with an exotic topology [13]. In what follows we shall often identify $\mathfrak{N}_p$ with $\mathbf{T} \times [0, 1]$.

Let $a \in PC_p(\mathbf{T})$ and $N = (\zeta, \mu) \in \mathfrak{N}_p$. Then

$$\begin{aligned}(\Gamma_{\mathfrak{N}_p} \sigma_p T(a))(N) &= (\Gamma_{\mathfrak{N}_p} \sigma_p T(a))(\zeta, \mu) \\ &= (1 - s_p(\mu))\, a(\zeta - 0) + s_p(\mu)\, a(\zeta + 0).\end{aligned} \tag{1}$$

Here $s_p(\mu)$ is defined for $\mu \in [0, 1]$ by

$$\begin{aligned}s_2(\mu) &= \mu, \\ s_p(\mu) &= \frac{\sin \vartheta\mu \cdot \exp(i\vartheta\mu)}{\sin \vartheta \cdot \exp(i\vartheta)}, \quad \vartheta = \pi\left(1 - \frac{2}{p}\right), \quad p \neq 2.\end{aligned} \tag{2}$$

If μ runs from 0 to 1, then $s_p(\mu)$ runs in $\mathbf{C}$ for $1 < p < 2$ (resp. $2 < p < \infty$) along a circular arc joining 0 to 1 and lying on the left (resp. on the right) of the line segment $[0, 1]$. Thus for $a \in PC_p(\mathbf{T})$ by

$$a_p(\zeta, \mu) = (1 - s_p(\mu))\, a(\zeta - 0) + s_p(\mu)\, a(\zeta + 0) \tag{3}$$

a continuous, closed, oriented curve is given in $\mathbf{C}$ if (ζ, μ) runs through $\mathbf{T} \times [0, 1]$.

The fundamental results about one-dimensional Toeplitz operators with piecewise continuous symbols are summarized in the following two theorems.

Theorem Φ [8, 14]: *Let $a \in PC_p(\mathbf{T})$. Then $T(a) \in \Phi(l^p)$ if and only if $a_p(\zeta, \mu) \neq 0$ for $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$. In this case* $\operatorname{Ind} T(a) = -\operatorname{ind} a_p(\zeta, \mu)$.

Theorem G [8, 14]: *Let $a \in PC_p(\mathbf{T})$. Then $T(a) \in G\mathfrak{L}(l^p)$ if and only if $T(a) \in \Phi(l^p)$ and* $\operatorname{Ind} T(a) = 0$.

Here $\operatorname{Ind} T(a) = \dim \operatorname{Ker} T(a) - \dim \operatorname{Coker} T(a)$ and $\operatorname{ind} a_p(\zeta, \mu)$ denotes the winding number of the curve $a_p(\zeta, \mu)$ with respect to the origin.

We define the projection P_n in l^p by

$$P_n : \{\varphi_0, \varphi_1, \varphi_2, \ldots\} \mapsto \{\varphi_0, \varphi_1, \ldots, \varphi_n, 0, 0, \ldots\}$$

and set $Q_n = I - P_n$. The operator W_n is defined in l^p by

$$W_n : \{\varphi_0, \varphi_1, \varphi_2, \ldots\} \mapsto \{\varphi_n, \varphi_{n-1}, \ldots, \varphi_0, 0, 0, \ldots\};$$

it will play an important role later on.

For $a \in PC_p(\mathbf{T})$ we set

$$T_n(a) = P_n T(a) P_n \mid \operatorname{Im} P_n = \{a_{j-k}\}_{j,k=0}^{n}.$$

We say that the finite section method is *applicable* to $T(a)$ in l^p (and write $T(a) \in \Pi_p\{P_n\}$ in this case) if the operators $T_n(a)\colon \operatorname{Im} P_n \to \operatorname{Im} P_n$ are invertible for all sufficiently large n (say $n \geq n_0$) and if $\sup_{n \geq n_0} \|T_n^{-1}(a) P_n\|_{\mathfrak{L}(l^p)} < \infty$. As a consequence of $T(a) \in \Pi_p\{P_n\}$ we have $T(a) \in G\mathfrak{L}(l^p)$ [15: p. 111] and $T_n^{-1}(a) P_n \to T^{-1}(a)$, strongly. The following theorem was proved in [26].

Theorem Π [26]: *Let $a \in PC_p(\mathbf{T})$. Then $T(a) \in \Pi_p\{P_n\}$ if and only if $T(a) \in G\mathfrak{L}(l^p)$ and $T(\tilde{a}) \in G\mathfrak{L}(l^p)$, where $\tilde{a}$ is defined by $\tilde{a}(t) = a(1/t)$, $t \in \mathbf{T}$.*

§3. Two-dimensional Toeplitz operators in $l^p \otimes l^p$ with piecewise continuous symbols

By $l^p \otimes l^p$ we denote the projective tensor product of the space l^p with itself. Obviously,

$$l^p \otimes l^p \cong l^p(\mathbf{Z}_+ \times \mathbf{Z}_+) = \left\{\varphi = \{\varphi_{ij}\}_{i,j=0}^{\infty} : \|\varphi\|_p = \left(\sum_{i,j=0}^{\infty} |\varphi_{ij}|^p\right)^{1/p} < \infty\right\}.$$

Let $\mathfrak{L}(l^p \otimes l^p)$, $\mathfrak{K}(l^p \otimes l^p)$, $\Phi(l^p \otimes l^p)$ denote the Banach algebra of all bounded, the ideal in $\mathfrak{L}(l^p \otimes l^p)$ of all compact, the collection of all Fredholm (Noetherian) operators on $l^p \otimes l^p$, respectively. Note that $\mathfrak{K}(l^p \otimes l^p)$ coincides with the projective tensor product $\mathfrak{K}(l^p) \otimes \mathfrak{K}(l^p)$.

Suppose $b_j, c_j \in PC_0$ $(j = 1, \ldots, n)$ and let

$$a(\xi, \eta) = \sum_{j=1}^{n} b_j(\xi)\, c_j(\eta), \qquad (\xi, \eta) \in \mathbf{T}^2. \tag{1}$$

Then the operator $W(a)$ defined by $W(a) = \sum_{j=1}^{n} T(b_j) \otimes T(c_j)$ is bounded on $l^p \otimes l^p$. We define $PC_p(\mathbf{T}^2)$ to be the closure of the collection of all functions of the form (1) with respect to the norm $\|a\|_p = \|W(a)\|_{\mathfrak{L}(l^p \otimes l^p)}$. Then $PC_2(\mathbf{T}^2)$ consists of all functions $a(\xi, \eta) \in L^\infty(\mathbf{T}^2)$ which have finite limit values $a(\xi_0 \pm 0, \eta_0)$ and $a(\xi_0, \eta_0 \pm 0)$ in the uniform norm at each point $(\xi_0, \eta_0) \in \mathbf{T}^2$. Furthermore,

$$PC_p(\mathbf{T}^2) = PC_q(\mathbf{T}^2) \subset PC_r(\mathbf{T}^2) = PC_s(\mathbf{T}^2) \subset PC_2(\mathbf{T}^2)$$

for $1 < p < r < 2$, $1/p + 1/q = 1$, $1/r + 1/s = 1$, and each function of the form $\sum_{i=1}^{n} b_i(\xi)\, c_i(\eta)$, $(\xi, \eta) \in \mathbf{T}^2$, with $b_i, c_i \in PC_p(\mathbf{T}^2)$ $(i = 1, \ldots, n)$ belongs to $PC_p(\mathbf{T}^2)$. All the facts stated here one can find in [9, 10].

Let $a \in PC_p(\mathbf{T}^2)$ and

$$a_{nm} = \frac{1}{(2\pi)^2} \int_0^{2\pi}\!\!\int_0^{2\pi} a(e^{i\varphi}, e^{i\vartheta})\, e^{-in\varphi}\, e^{-im\vartheta}\, d\varphi\, d\vartheta$$

$(n, m \in \mathbf{Z})$ be its Fourier coefficients. Then the operator $W(a)$ defined by

$$\big(W(a)\,\varphi\big)_{i,j} = \sum_{k,l=0}^{\infty} a_{i-k,j-l}\varphi_{kl} \qquad (i, j \geqq 0)$$

is bounded on $l^p \otimes l^p$. Moreover, $W(a)$ belongs to the projective tensor product $\mathfrak{B}_p \otimes \mathfrak{B}_p$. The operator $W(a)$ is called *two-dimensional Toeplitz operator* and a is referred to as the symbol of $W(a)$.

For $a \in PC_p(\mathbf{T}^2)$ and $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$ we define two functions $a^1_{\zeta,\mu}$ and $a^2_{\zeta,\mu}$ from $PC_p(\mathbf{T})$ by (recall the notation (2.2))

$$\begin{aligned} a^1_{\zeta,\mu}(t) &= \big(1 - s_p(\mu)\big)\, a(t, \zeta - 0) + s_p(\mu)\, a(t, \zeta + 0), \\ a^2_{\zeta,\mu}(t) &= \big(1 - s_p(\mu)\big)\, a(\zeta - 0, t) + s_p(\mu)\, a(\zeta + 0, t) \end{aligned} \tag{2}$$

$(t \in \mathbf{T})$. Several times, in order to emphasize the dependence on p, we shall write $a^1_{p,\zeta,\mu}$ and $a^2_{p,\zeta,\mu}$.

One main result of the present paper is the following theorem.

Theorem 1: *Let* $a \in PC_p(\mathbf{T}^2)$. *Then* $W(a) \in \Phi(l^p \otimes l^p)$ *if and only if*

$$T(a^1_{\zeta,\mu}) \in G\mathfrak{L}(l^p) \quad \text{and} \quad T(a^2_{\zeta,\mu}) \in G\mathfrak{L}(l^p) \tag{3}$$

for each $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$. *In this case* $\operatorname{Ind} W(a) = 0$.

This theorem was proved for the case $p = 2$ in [10] and its integral analogue was proved in [9] for general p.

Thus, recalling Theorem G of section 2, we find that the operator $W(a)$ is Fredholm on $l^p \otimes l^p$ if and only if for each fixed $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$ the origin belongs neither to the curve $(a^1_{\zeta,\mu})_p(\vartheta, \lambda)$ nor to the curve $(a^2_{\zeta,\mu})_p(\vartheta, \lambda)$ ((ϑ, λ) running through $\mathbf{T} \times [0, 1]$) and if both these curves have the winding number zero. Note that

$$\begin{aligned}(a^1_{\zeta,\mu})_p(\vartheta, \lambda) = {} & \big(1 - s_p(\lambda)\big)\big(1 - s_p(\mu)\big)\, a(\vartheta - 0, \zeta - 0) \\ & + \big(1 - s_p(\lambda)\big) s_p(\mu)\, a(\vartheta - 0, \zeta + 0) \\ & + s_p(\lambda)\big(1 - s_p(\mu)\big)\, a(\vartheta + 0, \zeta - 0) \\ & + s_p(\lambda)\, s_p(\mu)\, a(\vartheta + 0, \zeta + 0)\end{aligned} \tag{4}$$

and that we might write down a similar expression for $(a^2_{\zeta,\mu})_p(\vartheta, \lambda)$.

The equality $\operatorname{Ind} W(a) = 0$ is trivial; it follows from the conditions (3) by a simple homotopy argument.

Let now P_n be the projection defined in section 2. Then $P_n \otimes P_n$ acts in $l^p \otimes l^p$ by the rule

$$[(P_n \otimes P_n)\, \varphi]_{i,j} = \begin{cases} \varphi_{ij}, & 0 \leqq i, j \leqq n \\ 0, & \text{otherwise}. \end{cases}$$

For $a \in PC_p(\mathbf{T}^2)$ we set

$$W_n(a) = (P_n \otimes P_n)\, W(a)\, (P_n \otimes P_n) \mid \operatorname{Im}(P_n \otimes P_n). \tag{5}$$

We say that to $W(a)$ the finite section method is applicable in $l^p \otimes l^p$ if the operators $W_n(a)\colon \operatorname{Im}(P_n \otimes P_n) \to \operatorname{Im}(P_n \otimes P_n)$ are invertible for n large enough (say $n \geqq n_0$) and if $\sup_{n=n_0} \|W_n^{-1}(a)\,(P_n \otimes P_n)\|_{\mathfrak{L}(l^p \otimes l^p)} < \infty$. In this case we write $W(a) \in \Pi_p\{P_n \otimes P_n\}$. From $W(a) \in \Pi_p\{P_n \otimes P_n\}$ follows the invertibility of $W(a)$ (cf. [15: p. 111]) and that $W_n^{-1}(a)\,(P_n \otimes P_n) \to W^{-1}(a)$, strongly.

For $a \in PC_p(\mathbf{T}^2)$ we define $a_1, a_2, a_{12} \in PC_p(\mathbf{T}^2)$ by

$$\begin{aligned} &a_1(\xi, \eta) = a(1/\xi, \eta), \qquad a_2(\xi, \eta) = a(\xi, 1/\eta), \\ &a_{12}(\xi, \eta) = a(1/\xi, 1/\eta), \qquad (\xi, \eta) \in \mathbf{T}^2. \end{aligned}$$

The following theorem is the second main result of the paper.

Theorem 2: *Let* $a \in PC_p(\mathbf{T}^2)$. *Then* $W(a) \in \Pi_p\{P_n \otimes P_n\}$ *if and only if the four operators* $W(a)$, $W(a_1)$, $W(a_2)$ *and* $W(a_{12})$ *are invertible in* $\mathfrak{L}(l^p \otimes l^p)$.

This theorem was proved for continuous symbols in [16, 18, 12] and for piecewise continuous symbols in the case $p = 2$ in [3].

The necessity of the conditions given in Theorem 2 is trivial. Indeed, with the operator W_n defined in section 2 we have

$$\begin{aligned} W_n(a_1) &= (W_n \otimes P_n)\, W_n(a)\, (W_n \otimes P_n), \\ W_n(a_2) &= (P_n \otimes W_n)\, W_n(a)\, (P_n \otimes W_n), \\ W_n(a_{12}) &= (W_n \otimes W_n)\, W_n(a)\, (W_n \otimes W_n); \end{aligned}$$

and if $W(a) \in \prod_p \{P_n \otimes P_n\}$ then the invertibility of $W_n(a)$ implies that of $W_n(a_1)$ and from

$$\|W_n^{-1}(a_1)\| = \|(W_n \otimes P_n)\, W_n^{-1}(a)\, (W_n \otimes P_n)\| = \|W_n^{-1}(a)\|$$

we get $\sup \|W_n^{-1}(a_1)\| < \infty$, i.e. $W(a_1) \in \prod_p\{P_n \otimes P_n\}$. Thus $W(a_1) \in G\mathfrak{L}(l^p \otimes l^p)$. Analogously can be shown that $W(a_2)$, $W(a_{12}) \in G\mathfrak{L}(l^p \otimes l^p)$.

§ 4. Necessity of the conditions in Theorem 1

Under the assumption that we have already proved the sufficiency part of Theorem 1, we are going to prove the necessity of the conditions. The sufficiency will be shown in Section 8 contained in part II of this paper.

In what follows $\otimes$ always denotes the projective tensor product.

Let the maps ϑ, ϑ_1, ϑ_2 be defined by

$$\vartheta\colon \mathfrak{B}_p \bar{\otimes} \mathfrak{B}_p \to \mathfrak{B}_p/\mathfrak{K}_p \otimes \mathfrak{B}_p/\mathfrak{K}_p, \qquad \sum A_i \otimes B_i \mapsto \sum \sigma_p A_i \otimes \sigma_p B_i,$$

$$\vartheta_1\colon \mathfrak{B}_p \otimes \mathfrak{B}_p \to \mathfrak{B}_p/\mathfrak{K}_p \otimes \mathfrak{B}_p, \qquad \sum A_i \otimes B_i \mapsto \sum \sigma_p A_i \otimes B_i,$$

$$\vartheta_2\colon \mathfrak{B}_p \otimes \mathfrak{B}_p \to \mathfrak{B}_p \otimes \mathfrak{B}_p/\mathfrak{K}_p, \qquad \sum A_i \otimes B_i \mapsto \sum A_i \otimes \sigma_p B_i.$$

Lemma 1: *ϑ, ϑ_1, ϑ_2 are continuous algebraic homomorphisms.*

Proof: For finite sums we have

$$\|\vartheta(\sum A_i \otimes B_i)\| = \|\sum \sigma_p A_i \otimes \sigma_p B_i\|$$
$$\underset{\text{def}}{=} \inf\{\sum \|\sigma_p C_i\|\, \|\sigma_p D_i\|\colon \sum \sigma_p A_i \otimes \sigma_p B_i = \sum \sigma_p C_i \otimes \sigma_p D_i\}$$
$$\leqq \inf\{\sum \|C_i\|\, \|D_i\|\colon \sum A_i \otimes B_i = \sum C_i \otimes D_i\} \underset{\text{def}}{=} \|\sum A_i \otimes B_i\|$$

and now it is clear that $\|\vartheta C\| \leqq \|C\|$ for every $C \in \mathfrak{B}_p \otimes \mathfrak{B}_p$. Analogously we may prove the assertion for ϑ_1 and ϑ_2 ∎

Lemma 2: *Let $\mathfrak{A}$ be a Banach space and φ be a linear functional on $\mathfrak{A}$ with $\|\varphi\| \leqq 1$. Suppose that for*

$$\sum_{i=1}^{n} B_i \otimes C_i = \sum_{j=1}^{m} F_j \otimes G_j \in \mathfrak{A} \otimes \mathfrak{A}$$

always

$$\sum_{i=1}^{n} \varphi(C_i)\, B_i = \sum_{j=1}^{m} \varphi(G_j)\, F_j \in \mathfrak{A}$$

holds. Then

$$\left\| \sum_{i=1}^{n} \varphi(C_i)\, B_i \right\|_{\mathfrak{A}} \leqq \left\| \sum_{i=1}^{n} B_i \otimes C_i \right\|_{\mathfrak{A} \otimes \mathfrak{A}}$$

for every $\sum_{i=1}^{n} B_i \otimes C_i \in \mathfrak{A} \otimes \mathfrak{A}$.

Proof: In accordance with the definition of the norm in the projective tensor product we have

$$\|\sum B_i \otimes C_i\| = \inf\{\sum \|F_j\|\, \|G_j\|\colon \sum F_j \otimes G_j = \sum B_i \otimes C_i\}.$$

Thus, given an arbitrary $\varepsilon > 0$ we can choose $B_j', C_j' \in \mathfrak{A}$ such that on the one hand

$$\sum B_j' \otimes C_j' = \sum B_i \otimes C_i$$

and on the other hand

$$\sum \|B_j'\| \, \|C_j'\| \leqq (1 + \varepsilon) \left\|\sum B_i \otimes C_i\right\|.$$

Hence

$$\left\|\sum \varphi(C_i) B_i\right\| = \left\|\sum \varphi(C_j') B_j'\right\| \leqq \sum |\varphi(C_j')| \cdot \|B_j'\|$$
$$\leqq \sum \|C_j'\| \cdot \|B_j'\| \leqq (1 + \varepsilon) \left\|\sum B_i \otimes C_i\right\| \blacksquare$$

Lemma 3: *Suppose $a \in PC_p(\mathbf{T}^2)$ and $N = (\zeta, \mu) \in \mathfrak{N}_p$ ($\zeta \in \mathbf{T}$, $\mu \in [0, 1]$). Then $W(a) - T(a^1_{\zeta,\mu}) \otimes I$ can be approximated in the norm of $\mathfrak{L}(l^p \otimes l^p)$ as closely as desired by a finite sum of the form $\sum D_i \otimes Z_i$, where $D_i \in \mathfrak{B}_p$, $Z_i \in \mathfrak{B}_p$ and $\sigma_p Z_i \in N$.*

Proof: Suppose for a moment that a is a finite sum of the form

$$a(\xi, \eta) = \sum b_i(\xi) \, c_i(\eta), \qquad (\xi, \eta) \in \mathbf{T}^2, \tag{1}$$

where $b_i, c_i \in PC_p(\mathbf{T})$. By (3.2) and (2.1) we have

$$\begin{aligned} a^1_{\zeta,\mu}(t) &= \big(1 - s_p(\mu)\big) \, a(t, \zeta - 0) + s_p(\mu) \, a(t, \zeta + 0) \\ &= \sum \left[\big(1 - s_p(\mu)\big) \, c_i(\zeta - 0) + s_p(\mu) \, c_i(\zeta + 0)\right] b_i(t) \\ &= \sum \big(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\big) (\zeta, \mu) \, b_i(t). \end{aligned} \tag{2}$$

Hence

$$\begin{aligned} &W(a) - T(a^1_{\zeta,\mu}) \otimes I \\ &= \sum T(b_i) \otimes T(c_i) - \sum \big(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\big) (\zeta, \mu) \, T(b_i) \otimes I \\ &= \sum T(b_i) \otimes \left[T(c_i) - \big(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\big) (\zeta, \mu) \, I\right] \end{aligned}$$

and because

$$\begin{aligned} &\big(\Gamma_{\mathfrak{N}_p} \sigma_p \left[T(c_i) - (\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)) (\zeta, \mu) \, I\right] (\zeta, \mu) \\ &= \big(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\big) (\zeta, \mu) - \big(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\big) (\zeta, \mu) \cdot (\Gamma_{\mathfrak{N}_p} \sigma_p I) (\zeta, \mu) = 0 \end{aligned}$$

we obtain that $W(a) - T(a^1_{\zeta,\mu}) \otimes I$ is a finite sum of the form $\sum D_i \otimes Z_i$, where $D_i, Z_i \in \mathfrak{B}_p$ and $\sigma_p Z_i \in N$. Thus the asertion is true for functions of the form (1). For an arbitrary function $a \in PC_p(\mathbf{T}^2)$ we can choose functions $a^{(j)}(\xi, \eta) = \sum b_i^{(j)}(\xi) \, c_i^{(j)}(\eta)$ of the form (1) such that

$$\|W(a) - W(a^{(j)})\|_{\mathfrak{L}(l^p \otimes l^p)} \to 0 \qquad (j \to \infty).$$

In view of

$$\begin{aligned} &\|W(a) - T(a^1_{\zeta,\mu}) \otimes I\| \\ &\leqq \|W(a) - W(a^{(j)})\| + \|W(a^{(j)}) - T[(a^{(j)})^1_{\zeta,\mu}] \otimes I\| + \|T(a^1_{\zeta,\mu}) - T[(a^{(j)})^1_{\zeta,\mu}]\| \end{aligned}$$

the assertion will follow if we only prove that

$$\|T(a^1_{\zeta,\mu}) - T[(a^{(j)})^1_{\zeta,\mu}]\| \to 0 \qquad (j \to \infty). \tag{3}$$

But from the expression (2) for $(a^{(j)})^1_{\zeta,\mu}$ and Lemma 2 we can easily conclude that $\{T[(a^{(j)})^1_{\zeta,\mu}]\}_{j=1}^{\infty}$ forms a Cauchy sequence in $\mathfrak{B}_p$; then standard arguments give (3) $\blacksquare$

Since C^*-algebra techniques fail in the situation considered here and a theorem like [25: 10.18] seems not to be applicable, we shall make use of arguments having to do with joint topological divisors of zero. The following theorem we have found in [33].

Theorem Z [33: 15.12]: *Let $\mathfrak{A}$ be a commutative Banach algebra with unit, $\mathfrak{X}$ be its maximal ideal space, and let N be a maximal ideal belonging to the Shilov boundary $\partial_S\mathfrak{X}$. Then*

$$\inf\left\{\sum_{i=1}^{m}\|Z_iU\|: U \in \mathfrak{A}, \|U\| = 1\right\} = 0$$

for every finite subset $\{Z_1, \ldots, Z_m\} \subset N$.

It is not hard to show that the Shilov boundary of the maximal ideal space $\mathfrak{N}_p$ of $\mathfrak{B}_p/\mathfrak{K}_p$ coincides with the whole space $\mathfrak{N}_p$, i.e. with the whole cylinder $\mathbf{T} \times [0, 1]$. For $p = 2$ this follows immediately from $\operatorname{Im} \Gamma_{\mathfrak{N}_2} = C(\mathfrak{N}_2)$ (cf. [13]). Thus let $p \neq 2$. By [33: 15.3] it suffices to show that for each point $(\zeta_0, \mu_0) \in \mathbf{T} \times [0, 1]$ and for each neighborhood U of (ζ_0, μ_0) (with respect to the topology [13] of $\mathbf{T} \times [0, 1]$) there exists a $\sigma_pA \in \mathfrak{B}_p/\mathfrak{K}_p$ such that

$$\sup_{\mathfrak{B}_p\setminus U} |\Gamma_{\mathfrak{N}_p}\sigma_pA| < \sup_U |\Gamma_{\mathfrak{N}_p}\sigma_pA|.$$

A little thought shows that such an A may in fact be found among the collection of Toeplitz operators $T(a)$ with $a \in PC_0$.

Now, in Proposition 1, we shall prove that $W(a) \in \Phi(l^p \otimes l^p)$ implies the Fredholmness of $T(a^1_{\zeta,\mu})$ for every $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$ and then, in Proposition 2, it will be shown that $T(a^1_{\zeta,\mu})$ is even invertible in $\mathfrak{L}(l^p)$. Since the same can be done for $T(a^2_{\zeta,\mu})$, the necessity part of Theorem 1 follows.

Proposition 1: *Let $a \in PC_p(\mathbf{T}^2)$ and $W(a) \in \Phi(l^p \otimes l^p)$. Then $T(a^1_{\zeta,\mu}) \in \Phi(l^p)$ for every $(\zeta, \mu) \in T \times [0, 1]$.*

Proof: Suppose that there is a $(\zeta_0, \mu_0) \in \mathbf{T} \times [0, 1]$ such that $T(a^1_{\zeta_0,\mu_0})$ is not Fredholm. This implies the existence of an $M \in \mathfrak{N}_p$ such that $\sigma_pT(a^1_{\zeta_0,\mu_0}) \in M$. From $\partial_S\mathfrak{N}_p = \mathfrak{N}_p$ and Theorem Z we obtain

$$\inf\{\|\sigma_pT(a^1_{\zeta_0,\mu_0}) \cdot \sigma_pB\|: B \in \mathfrak{B}_p, \|\sigma_pB\| = 1\} = 0. \tag{4}$$

Due to Lemma 3 there exist two finite sequences $D_1, \ldots, D_m$ and $Z_1, \ldots, Z_m$ $(D_i, Z_i \in \mathfrak{B}_p)$ such that $(\Gamma_{\mathfrak{N}_p}\sigma_pZ_i)(\zeta_0, \mu_0) = 0$ and

$$A := T(a^1_{\zeta_0,\mu_0}) \otimes I + \sum_{i=1}^{m} D_i \otimes Z_i \in \Phi(l^p \otimes l^p) \tag{5}$$

(note that $\Phi(l^p \otimes l^p)$ forms an open subset in $\mathfrak{L}(l^p \otimes l^p)$). Again by Theorem Z

$$\inf\left\{\sum_{i=1}^{m}\|\sigma_pZ_i \cdot \sigma_pU\|: U \in \mathfrak{B}_p, \|\sigma_pU\| = 1\right\} = 0. \tag{6}$$

Because of (4) there are $B_j \in \mathfrak{B}_p$, $\|\sigma_pB_j\| = 1$, and $K_j \in \mathfrak{K}_p$ $(j = 1, 2, 3, \ldots)$ such that

$$T(a^1_{\zeta_0,\mu_0})\, B_j - K_j = C_j', \qquad \|C_j'\|_{\mathfrak{L}(l^p)} \to 0 \quad (j \to \infty)$$

and (6) yields the existence of $U_j \in \mathfrak{B}_p$, $\|\sigma_pU_j\| = 1$, and $K_{ij} \in \mathfrak{K}_p$ $(i = 1, \ldots, m;$ $j = 1, 2, 3, \ldots)$ such that

$$Z_iU_j - K_{ij} = C''_{ij}, \qquad \|C''_{ij}\|_{\mathfrak{L}(l^p)} \to 0 \qquad (j \to \infty, \ \forall i).$$

Let P_n be the projection introduced in Section 2 and put $Q_n = I - P_n$. Obviously, $Q_n \to 0$, strongly. Now, given an arbitrary $R \in \mathfrak{B}_p$, $\|\sigma_p R\| = 1$, there exists an $n_0 = n_0(R)$ such that $\|RQ_n\| \leqq 3$ for all $n \geqq n_0$. Indeed, because of $\|\sigma_p R\| = 1$ there is a $K \in \mathfrak{K}_p$ with $\|R + K\| \leqq 2$ and, consequently,

$$\|RQ_n\| = \|(R + K)\, Q_n - KQ_n\| \leqq \|(R + K)\, Q_n\| + \|KQ_n\| \\ \leqq \|R + K\|\, \|Q_n\| + \|KQ_n\| \leqq 2 \cdot 1 + 1 = 3, \tag{7}$$

since $\|KQ_n\| \to 0$ $(n \to \infty)$. Now, we have (with A defined by (5))

$$A(B_jQ_n \otimes U_jQ_n) \\ = K_jQ_n \otimes U_jQ_n + C_j'Q_n \otimes U_jQ_n \\ + \sum_{i=1}^{m} D_iB_jQ_n \otimes K_{ij}Q_n + \sum_{i=1}^{m} D_iB_jQ_n \otimes C_{ij}''Q_n. \tag{8}$$

On account of (5) there is an $R \in \mathfrak{L}(l^p \otimes l^p)$ such that $RA - I \otimes I \in \mathfrak{K}(l^p \otimes l^p) = \mathfrak{K}_p \otimes \mathfrak{K}_p$. Thus $RA \in \mathfrak{B}_p \otimes \mathfrak{B}_p$. From Lemma 1 we get $\vartheta(RA) = \sigma_p I \otimes \sigma_p I$. $P_n \in \mathfrak{K}_p$ gives $Q_n = I - P_n \in \mathfrak{B}_p$, consequently, $B_jQ_n \otimes U_jQ_n \in \mathfrak{B}_p \otimes \mathfrak{B}_p$ and, again by Lemma 1,

$$\vartheta(RA)\, \vartheta(B_jQ_n \otimes U_jQ_n) = \vartheta\big(RA(B_jQ_n \otimes U_jQ_n)\big).$$

Thus

$$\|\sigma_pB_jQ_n \otimes \sigma_pU_jQ_n\| \\ = \|(\sigma_pI \otimes \sigma_pI)\, (\sigma_pB_jQ_n \otimes \sigma_pU_jQ_n)\| \\ = \|\vartheta(RA)\, \vartheta(B_jQ_n \otimes U_jQ_n)\| = \big\|\vartheta\big(RA(B_jQ_n \otimes U_jQ_n)\big)\big\| \\ \leqq \|RA(B_jQ_n \otimes U_jQ_n)\| \quad \text{(Lemma 1)} \\ \leqq \|R\|\, \|A(B_jQ_n \otimes U_jQ_n)\|. \tag{9}$$

From $B_jP_n \in \mathfrak{K}_p$ we obtain

$$\|\sigma_pB_jQ_n\| = \|\sigma_pB_j - \sigma_pB_jP_n\| = \|\sigma_pB_j\| = 1$$

and analogously we can derive $\|\sigma_pU_jQ_n\| = 1$. Hence

$$1 = \|\sigma_pB_jQ_n \otimes \sigma_pU_jQ_n\| \tag{10}$$

for every $j, n > 0$. Now we are going to prove that for a suitable choice of $j = j_0$ and $n = n_0$

$$\|A(B_{j_0}Q_{n_0} \otimes U_{j_0}Q_{n_0})\| < \varepsilon = 1/\|R\|.$$

Then (9) and (10) are contradictory and our assertion will therefore be proved.
First choose j_0 large enough, such that

$$\|C_{j_0}'\| < \varepsilon/12, \qquad \|D_i\|\, \|C_{ij_0}''\| < \varepsilon/12m \qquad (i = 1, \ldots, m)$$

and then choose n_0 such that

$$\|K_{j_0}Q_{n_0}\| < \varepsilon/12, \qquad \|U_{j_0}Q_{n_0}\| \leqq 3, \qquad \|B_{j_0}Q_{n_0}\| \leqq 3, \\ \|D_i\|\, \|K_{ij_0}Q_{n_0}\| < \varepsilon/12m \qquad (i = 1, \ldots, m)$$

(cf. (7)). Consequently,

$$\|K_{j_0}Q_{n_0} \otimes U_{j_0}Q_{n_0}\| = \|K_{j_0}Q_{n_0}\| \, \|U_{j_0}Q_{n_0}\| < \varepsilon/12 \cdot 3 = \varepsilon/4,$$

$$\|C'_{j_0}Q_{n_0} \otimes U_{j_0}Q_{n_0}\| \leqq \|C'_{j_0}\| \, \|U_{j_0}Q_{n_0}\| < \varepsilon/12 \cdot 3 = \varepsilon/4,$$

$$\Big\|\sum_i D_i B_{j_0} Q_{n_0} \otimes K_{ij_0} Q_{n_0}\Big\|$$

$$\leqq \sum_i \|D_i\| \, \|B_{j_0}Q_{n_0}\| \, \|K_{ij_0}Q_{n_0}\| < \sum_i \varepsilon/12m \cdot 3 = \varepsilon/4,$$

$$\Big\|\sum_i D_i B_{j_0} Q_{n_0} \otimes C''_{ij_0} Q_{n_0}\Big\|$$

$$\leqq \sum_i \|D_i\| \, \|B_{j_0}Q_{n_0}\| \, \|C''_{ij_0}\| < \sum_i \varepsilon/12m \cdot 3 = \varepsilon/4.$$

Taking into account (8) we arrive at $\|A(B_{j_0}Q_{n_0} \otimes U_{j_0}Q_{n_0})\| < \varepsilon$ ∎

Proposition 2: *If $a \in PC_p(\mathbf{T}^2)$ and $W(a) \in \Phi(l^p \otimes l^p)$ then $T(a^1_{\zeta,\mu}) \in G\mathfrak{L}(l^p)$ and $T(a^2_{\zeta,\mu}) \in G\mathfrak{L}(l^p)$ for every $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$.*

Proof: By Proposition 1 we have $T(a^1_{\zeta,\mu}) \in \Phi(l^p)$ for every $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$. It follows that $T(a^1_{\zeta,\mu})$ is homotopic through Fredholm operators to both $T(a^1_{\zeta,0})$ and $T(a^1_{\zeta,1})$. Thus, in particular,

$$\operatorname*{ind}_{t\in\mathbf{T}} a(t, \zeta - 0) = -\operatorname{Ind} T(a^1_{\zeta,0}) = -\operatorname{Ind} T(a^1_{\zeta,1}) = \operatorname*{ind}_{t\in\mathbf{T}} a(t, \zeta + 0)$$

(cf. Theorem Φ in Section 2) and it results that

$$\operatorname*{ind}_{t\in\mathbf{T}} a(t, \zeta \pm 0) = \varkappa = \text{const} \qquad (\zeta \in \mathbf{T}).$$

Equally

$$\operatorname*{ind}_{t\in\mathbf{T}} a(\zeta \pm 0, t) = \lambda = \text{const} \qquad (\zeta \in \mathbf{T}).$$

Thus we have $a(\xi, \eta) = \xi^\varkappa \eta^\lambda a_0(\xi, \eta)$, $(\xi, \eta) \in \mathbf{T}^2$, where a_0 satisfies the conditions (3) of Theorem 1. In case $\varkappa \geqq 0$, $\lambda \geqq 0$ we get $W(a) = W(a_0) \, W(\xi^\varkappa \eta^\lambda)$. For supposing that the sufficiency part of Theorem 1 is already proved, it follows that $W(a_0) \in \Phi(l^p \otimes l^p)$. Then $W(a) \in \Phi(l^p \otimes l^p)$ gives $W(\xi^\varkappa \eta^\lambda) \in \Phi(l^p \otimes l^p)$, but since, obviously, $\dim \operatorname{Coker} W(\xi^\varkappa \eta^\lambda) = \infty$ if $\varkappa > 0$ or $\lambda > 0$, we deduce that $\varkappa \leqq 0$ and $\lambda \leqq 0$.

Assume at least one of the integers $\varkappa$ and λ is negative. Let $\varkappa < 0$, so that

$$T(a^1_{\zeta,\mu}) = T(\xi^{-|\varkappa|}) \, T(f^1_{\zeta,\mu}),$$

with $f^1_{\zeta,\mu} \in PC_p(\mathbf{T})$ and $T(f^1_{\zeta,\mu}) \in G\mathfrak{L}(l^p)$. Obviously, $\dim \operatorname{Ker} T(\xi^{-|\varkappa|}) \, T(f^1_{\zeta,\mu}) = |\varkappa| > 0$. Take $\varphi_0 \in \operatorname{Ker} T(\xi^{-|\varkappa|}) \, T(f^1_{\zeta,\mu})$ and $F \in (l^p)^*$, $\|F\| = 1$, and define $H \in \mathfrak{L}(l^p)$ by $H\psi = (F\psi) \, \varphi_0$, $\psi \in l^p$. So

$$H \in \mathfrak{K}_p \subset \mathfrak{B}_p, \qquad \|H\| = 1, \qquad T(\xi^{-|\varkappa|}) \, T(f^1_{\zeta,\mu}) \, H = 0.$$

With the operator A defined by (5) and the operators D_i, Z_i, U_i, K_{ij}, C''_{ij} introduced in the proof of Proposition 1 we obtain

$$A(H \otimes U_j Q_n) = \sum_{i=1}^m D_i H \otimes K_{ij} Q_n + \sum_{i=1}^m D_i H \otimes C''_{ij} Q_n.$$

Now, analogously as in the proof of Proposition 1 (with ϑ replaced by ϑ_2) we can derive

$$\|H \otimes \sigma_p U_j Q_n\| \leqq \|R\| \, \|A(H \otimes U_j Q_n)\|,$$

what as above leads to a contradiction. Thus $\varkappa_l = \lambda = 0$ and the assertion follows from Theorem G in Section 2 ∎

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