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Maximal inequalities and Fourier multipliers for spaces with mixed quasinorms. Applications

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Es werden Maximalungleichungen und Fouriersche Multiplikatoren für Systeme ganzer analytischer Funktionen exponentiellen Typs betrachtet, die zu Räumen mit gemischten Quasinormen gehören. Als Anwendung werden 4 Typen von Räumen von Funktionen mit dominierenden gemischten Glattheitseigenschaften eingeführt. Beziehungen zu klassischen Räumen von diesem Typ (Räume von S. M. Nikol'skij, T. I. Amanov, P. I. Lizorkin) werden behandelt. Als Spezialfall sind die Sobolev-Räume mit einer dominierenden gemischten Ableitung enthalten.

Рассматриваются максимальные неравенства и мультипликаторы Фурье для систем функций экспоненциального типа, принадлежащихся пространствам со смешанными квасинормами. Как приложения определяются 4 типы функциональных пространств с доминирующей смешанной производной. Исследуются соотношения к классическим пространствам такого типа (С. М. Никольского, Т. И. Аманова, П. И. Лизоркина). В рассматриваемых классах содержатся, в частности, пространства Соболева с доминирующей смешанной производной.

The paper concerns with maximal inequalities and Fourier multipliers for systems of entire analytic functions of exponential type belonging to mixed quasinormed spaces. As an application 4 types of spaces of functions with dominating mixed smoothness properties are introduced. Relations to classical spaces of that type (spaces of S. M. NIKOL'SKIJ, T. I. AMANOV, P. I. LIZORKIN) are treated. As a special case the Sobolev spaces with a dominating mixed derivative are contained.

1. Introduction

In connection with the study of eigenvalues of integral operators, A. PIETSCH [12, 13] introduced the spaces $[B^s_{p,q}, B^t_{u,v}]$ of functions $f(x, y)$ of two variables to describe smoothness properties of the kernels of the operators which ensure mappings between Besov spaces. This idea was taken up by H. TRIEBEL [26, 27], who described such spaces within the framework of Fourier analysis. It was more or less obvious that the spaces considered are related to those of S. M. NIKOL'SKIJ [10], T. I. AMANOV [2, 3], and H. TRIEBEL [21–23], which are characterized by dominating mixed derivatives. Triebel's approach was in keeping with the recent developments in the theory of function spaces. However, a systematic investigation by means of Fourier analysis in the sense of the modern theory of the Besov and Triebel-Lizorkin spaces (cf. H. TRIEBEL [24, 25]) remained open. The first step in this direction was done in the paper [14].

The aim of the paper presented here is twofold. First we want to establish maximal inequalities and Fourier multiplier theorems for various types of spaces with mixed quasinorms. These spaces are found to be the basis for the treatment of the indicated unction spaces. Second we want to clarify the relations of these function spaces to the classical Sobolev-Lebesgue spaces with dominating mixed derivatives in the sense of P. I. LIZORKIN and S. M. NIKOL'SKIJ [9].

The paper is organized as follows. Section 2 contains all definitions needed and some preliminaries connected with classical Hardy-Littlewood maximal functions and mixed quasinormed spaces. In section 3 we state and prove our main theorems. Section 4 deals with possible applications to the function spaces of Besov-Triebel-Lizorkin type, the comparison with the classical spaces, and remarks concerning integral operators.

Our treatment follows H.-J. SCHMEISSER [17] where the essential results can also be found.

2. Preliminaries

2.1. Spaces with mixed quasinorms

Let (X_i, S_i, μ_i) $(i = 1, \ldots, n)$ be n given, totally σ-finite measure spaces and $\bar{p} = (p_1, \ldots, p_n)$ a given n-tuple with $0 < p_i \leqq \infty$. It is supposed that none of the spaces (X_i, S_i, μ_i) admits only the constant functions. A function $f(x_1, \ldots, x_n)$ measurable in the product space

$$(X, S, \mu) = \left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} S_i, \prod_{i=1}^{n} \mu_i\right)$$

is said to belong to $L_{\bar{p}}(X)$ if the number obtained after taking successively the p_1-quasinorm in x_1, the p_2-quasinorm in $x_2, \ldots$, the p_n-quasinorm in x_n, and in that order, is finite. The number obtained in such a way will be denoted by $\|f \mid L_{\bar{p}}(X)\|$. If $0 < p_i < \infty$ we have in particular:

$$\|f| L_{\bar{p}}(X)\| = \left(\int\limits_{X_n} \cdots \left(\int\limits_{X_2} \left(\int\limits_{X_1} |f(x_1, \ldots, x_n)|^{p_1} \, d\mu_1\right)^{\frac{p_2}{p_1}} d\mu_2\right)^{\frac{p_3}{p_2}} \ldots d\mu_n\right)^{\frac{1}{p_n}}. \tag{2.1}$$

This is the definition of the spaces $L_{\bar{p}}$ with mixed quasinorms given by A. BENEDEK and R. PANZONE [5]. Let us recall that a quasinorm satisfies all conditions of a norm except the triangular inequality. That must be replaced by

$$\|f + g\| \leqq c(\|f\| + \|g\|), \qquad c \geqq 1.$$

For abbreviation we shall put

$$\|f \mid L_{\bar{p}}(X)\| = \|f(x_1, \ldots, x_n) \mid L_{p_1}(X_1) \mid \ldots \mid L_{p_n}(X_n)\|. \tag{2.2}$$

In the case that $p_i = \infty$ we have to modify in the usual way

$$\|f \mid L_{p_i}(X_i)\| = \operatorname*{ess\,sup}_{x_i \in X_i} |f(x_i)|.$$

If $X_i = \mathbf{R}_1$ is the real axis and μ_i is the Lebesgue measure we set $L_{p_i}(X_i) = L_{p_i}$ $(i = 1, 2, \ldots, n)$, $L_{\bar{p}}(X) = L_{\bar{p}}(\mathbf{R}_n) = L_{\bar{p}}$, and

$$\|f \mid L_{\bar{p}}\| = \|f(x_1, \ldots, x_n) \mid L_{p_1} \mid \ldots \mid L_{p_n}\|. \tag{2.3}$$

Our definition also covers the spaces $l_{\bar{p}}$ consisting of all systems of complex numbers $f_{\bar{k}} = f_{(k_1,\ldots,k_n)}$ $(k_i = 0, 1, 2, \ldots)$ with

$$\|f_{\bar{k}} \mid l_{\bar{p}}\| = \left(\sum_{k_n=0}^{\infty} \cdots \left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty} |f_{(k_1,\ldots,k_n)}|^{p_1}\right)^{\frac{p_2}{p_1}}\right)^{\frac{p_3}{p_2}} \cdots\right)^{\frac{1}{p_n}}$$
$$= \|f_{(k_1,\ldots,k_n)} \,|l_{p_1}| \cdots | \,l_{p_n}\| < \infty. \tag{2.4}$$

For $p_1 = \cdots = p_n = p$ these are the usual spaces L_p and l_p. Of special interest to us are certain combinations of (2.3) and (2.4). That means we consider spaces of systems $f_{\bar{k}}(x)$, $\bar{k} = (k_1, \ldots, k_n)$, $x = (x_1, \ldots, x_n)$, where $k_i = 0, 1, \ldots$ and $x_i \in \mathbf{R}_1$. Now we take, in the sense of (2.1), (2.2) successively, L_p and l_q quasinorms, $0 < p, q \leqq \infty$, in different orders. For n-tuples $\bar{p} = (p_1, \ldots, p_n)$ and $\bar{q} = (q_1, \ldots, q_n)$ with $0 < p_i$, $q_i \leqq \infty$ we define the quasinorms

$$\|f_{\bar{k}}(x) \mid Sl_{\bar{q}}(L_{\bar{p}})\| = \|f_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n) \,|L_{p_1} \mid l_{q_1}| \ldots |L_{p_n}| \,l_{q_n}\|, \tag{2.5}$$

$$\|f_{\bar{k}}(x) \mid SL_{\bar{p}}(l_{\bar{q}})\| = \|f_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n) \,|l_{q_1} \mid L_{p_1}| \ldots |l_{q_n}| \,L_{p_n}\|, \tag{2.6}$$

$$\|f_{\bar{k}}(x) \mid L_{\bar{p}}(l_{\bar{q}})\| = \|f_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n) \,|l_{q_1}| \ldots |l_{q_n} \mid L_{p_1}| \ldots \mid L_{p_n}\|. \tag{2.7}$$

The corresponding spaces are denoted by $Sl_{\bar{q}}(L_{\bar{p}})$, $SL_{\bar{p}}(l_{\bar{q}})$, and $L_{\bar{p}}(l_{\bar{q}})$. Note that in (2.5), (2.6), and (2.7), l_{q_i} is connected with summation to the q_i-th power over $k_i = 0, 1, \ldots$ and L_{p_i} with integration to the p_i-th power over $\mathbf{R}_1$ with respect to the Lebesgue measure dx_i.

Let us mention that the above spaces can also be obtained as iterated cases of vector-valued Banach function and sequence spaces.

Finally for the sake of simplicity we want to introduce some abbreviations. If $\bar{r}$ and $\bar{s}$ are n-tuples we write

$$\bar{r} \leqq \bar{s} \; (\bar{r} < \bar{s}, \bar{r} = \bar{s}) \text{ iff } r_i \leqq s_i \; (r_i < s_i, r_i = s_i) \qquad (i = 1, \ldots, n),$$
$$\bar{r} + \bar{s} := (r_1 + s_1, \ldots, r_n + s_n),$$
$$\lambda\bar{r} := (\lambda r_1, \ldots, \lambda r_n) \qquad (\lambda \text{ complex}),$$
$$\bar{r}\bar{s} := (r_1 s_1, \ldots, r_n s_n),$$
$$\bar{r}^{\bar{s}} := r_1^{s_1} \ldots r_n^{s_n} \qquad (r_i > 0;\ i = 1, \ldots, n)$$
$$a^{\bar{s}} := (a^{s_1}, \ldots, a^{s_n}) \qquad (a > 0),$$
$$\bar{r}^a = (r_1^a, \ldots, r_n^a) \qquad (r_i > 0),$$
$$\langle \bar{r}, \bar{s} \rangle := \sum_{i=1}^{n} r_i s_i .$$

Further we agree upon $\bar{0} = (0, \ldots, 0)$, $\bar{1} = (1, \ldots, 1), \ldots, \overline{\infty} = (\infty, \ldots, \infty)$. $\mathbf{Z}_n^+$ denotes the set of all n-tuples $\bar{k} = (k_1, \ldots, k_n)$ with non-negative integer components.

2.2. Inequalities for the Hardy-Littlewood maximal function

In this subsection we want to recall R. J. Bagby's extended maximal inequality [4].

Let (X, S, μ) be a product measure space as defined in the preceding subsection and let $\mathbf{R}_n$ be the Euclidean n-space. We consider on $\mathbf{R}_n \times X$ locally integrable complex valued functions $f(x, t)$, $x = (x_1, \ldots, x_n) \in \mathbf{R}_n$, $t = (t_1, \ldots, t_m) \in X$ and define the Hardy-Littlewood maximal function with respect to $x \in \mathbf{R}_n$ by

$$\big(Mf(\cdot, t)\big)(x) = \sup_{0<r<\infty} |B(x, r)|^{-1} \int_{B(x,r)} |f(y, t)|\, dy. \tag{2.8}$$

Here $B(x, r)$ denotes a ball with radius r centered at $x \in \mathbf{R}_n$ and $|B(x, r)|$ is its Lebesgue measure (cf. E. M. Stein [18: Chapter 1]). Then we have

Proposition 1 (R. J. Bagby [4]): *Let $\bar{q} = (q_1, \ldots, q_m)$ be a m-tuple with $1 < q_i \leqq \infty$ and let $1 < p < \infty$.*

There exists a constant $c > 0$, independent of $f(x, t)$, such that

$$\|(Mf(\cdot, t))(x) \mid L_{\bar{q}}(X) \mid L_p(\mathbf{R}_n)\| \leqq c\, \|f(x, t) \mid L_{\bar{q}}(X) \mid L_p(\mathbf{R}_n)\|. \tag{2.9}$$

Remark 1: (2.9) is a generalization of the famous vector-valued maximal inequality of C. Fefferman and E. M. Stein [7]. Of course, in (2.9) first the norm

$$\|\cdot \mid L_{\bar{q}}(X)\| = \|\cdot \mid L_{q_1}(X_1) \mid \ldots \mid L_{q_m}(X_m)\|$$

is taken with respect to $t = (t_1, \ldots, t_m) \in X = \prod_{i=1}^{m} X_i$ and the measure $\mu = \prod_{i=1}^{m} \mu_i$. After that the norm in $L_p(\mathbf{R}_n)$ is taken with respect to $x \in \mathbf{R}_n$ and the Lebesgue measure. We are especially interested in the cases that $L_{\bar{q}}(X) = L_{\bar{q}}(\mathbf{R}_m)$ (and $Sl_{\bar{q}}(L_{\bar{p}})$, $SL_{\bar{p}}(l_{\bar{q}})$, $L_{\bar{p}}(l_{\bar{q}})$).

2.3. The spaces $S_{\bar{p}}^{\bar{\varkappa}}H$ and $S_{\bar{p}}^{\bar{m}}W$

Let $\bar{\varkappa} = (\varkappa_1, \ldots, \varkappa_n)$ and $\bar{p} = (p_1, \ldots, p_n)$ be n-tuples with $-\overline{\infty} < \bar{\varkappa} < \overline{\infty}$ and $\bar{1} < \bar{p} < \overline{\infty}$. F and F^{-1} represent the Fourier transform in the space of tempered distributions $S'(\mathbf{R}_n)$ and its inverse, respectively. For $f \in L_1(\mathbf{R}_n)$ we have

$$\begin{aligned} (Ff)(x) &= (2\pi)^{-n/2} \int_{R_n} f(\xi)\, e^{-i\langle x, \xi\rangle}\, d\xi, \\ \langle x, \xi\rangle &= \sum_{i=1}^{n} x_i \xi_i. \end{aligned} \tag{2.10}$$

Then we define, if it makes sense,

$$\|f(x) \mid S_{\bar{p}}^{\bar{\varkappa}}H\| = \left\| F^{-1} \prod_{j=1}^{n} (1 + x_j^2)^{\varkappa_j/2} Ff \mid L_{\bar{p}}(\mathbf{R}_n) \right\| \tag{2.11}$$

and

$$S_{\bar{p}}^{\bar{\varkappa}}H = \{f(x) \in L_{\bar{p}}(\mathbf{R}_n) \mid \|f(x) \mid S_{\bar{p}}^{\bar{\varkappa}}H\| < \infty\}. \tag{2.12}$$

If $\bar{m} = (m_1, \ldots, m_n)$, m_i non-negative integer, we put

$$S_{\bar{p}}^{\bar{m}}W = \{f \in L_{\bar{p}}(\mathbf{R}_n) \mid D^{\bar{\theta}\bar{m}} f(x) \in L_{\bar{p}}(\mathbf{R}_n);\ \theta_i = 0, 1;\ i = 1, \ldots, n\} \tag{2.13}$$

and further

$$\|f(x) \mid S_{\bar{p}}^{\bar{m}}W\| = \sum_{\substack{\theta_i = 0,1 \\ i = 1,\ldots,n}} \|D^{\bar{\theta}\bar{m}} f(x) \mid L_{\bar{p}}(\mathbf{R}_n)\|. \tag{2.14}$$

Here all derivatives have to be understood in the sense of distributions and $D^{(\alpha_1,\ldots,\alpha_n)}f$, α_i non-negative integer, means

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

If all p_i are equal we write $S_p^{\bar{\varkappa}}H$ and $S_p^{\bar{m}}W$, respectively.

Proposition 2: *We have*

$$S_{\bar{p}}^{\bar{m}}W = S_{\bar{p}}^{\bar{m}}H. \tag{2.15}$$

Remark 2: The proof of (2.15) can be carried out analogously to the isotropic case $p_1 = \cdots = p_n$ (see for example H. Triebel [20: Theorem 2.3.3]) using the Fourier multiplier theorem for $L_{\bar{p}}$ of P. I. Lizorkin [8].

The spaces $S_{\bar{p}}^{\bar{m}}W$ and $S_{\bar{p}}^{\bar{\varkappa}}H$ generalize the classical Sobolev-Lebesgue spaces with a dominating mixed derivative treated by P. I. Lizorkin and S. M. Nikol'skij [9] to mixed norms. Now, let us establish an important inequality of Bernstein type, which will be used later on.

Proposition 3: *Let $\bar{d} = (d_1, \ldots, d_n)$, $\bar{p} = (p_1, \ldots, p_n)$ be n-tuples with $\bar{0} \leqq \bar{d} < \bar{\infty}$ and $\bar{0} < \bar{p} \leqq \bar{2}$. If $\bar{\varkappa} > \bar{d} + \frac{1}{\bar{p}} - \frac{1}{\bar{2}}$, then there exists a positive constant c, such that for all $f \in S_2^{\bar{\varkappa}}H$*

$$\left\| \prod_{i=1}^{n} (1 + x_i^2)^{d_i/2} Ff \mid L_{\bar{p}} \right\| \leqq c \, \|f(x) \mid S_2^{\bar{\varkappa}}H\|. \tag{2.16}$$

Proof: Let $\bar{l} = (l_1, \ldots, l_n)$ be a n-tuple of non-negative integers. We set

$$Q_{\bar{l}} = I_{l_1} \times \cdots \times I_{l_n},$$

where $I_0 = [-1, 1]$, $I = [-2^l, -2^{l-1}) \cup (2^{l-1}, 2^l]$ $(l = 1, 2, \ldots)$. Further let χ_l be the characteristic function of I_l and

$$\chi_{\bar{l}}(x) = \chi_{l_1}(x_1) \cdots \chi_{l_n}(x_n)$$

the characteristic function of $Q_{\bar{l}}$. Using these notations the equivalence

$$\left\| \prod_{i=1}^{n} (1 + x_i^2)^{d_i/2} Ff \mid L_{\bar{p}} \right\| \sim \left\| \prod_{i=1}^{n} 2^{l_i d_i} \chi_{l_i}(x_i) \, Ff \mid Sl_{\bar{p}}(L_{\bar{p}}) \right\| \tag{2.17}$$

can be derived. Now, Hölders inequality for integrals with respect to $\frac{2}{p_i}$ is successively applied to the right-hand side of (2.17) from $i = 1$ up to $i = n$. Thus, it can be estimated from above by

$$c \, \|2^{\bar{l}(\bar{d} + \bar{p}^{-1} - \bar{2}^{-1})} \chi_{\bar{l}} Ff \mid Sl_{\bar{p}}(L_{\bar{2}})\|.$$

Using Hölder's inequality for series, and because of the assumption $\varepsilon_i = \varkappa_i - \left(d_i + \frac{1}{p_i} - \frac{1}{2}\right) > 0$ $(i = 1, \ldots, n)$ we obtain from (2.17)

$$\left\| \prod_{i=1}^{n} (1 + x_i^2)^{d_i/2} Ff \mid L_{\bar{p}} \right\| \leqq c \, \| 2^{\bar{l}\bar{\varkappa}} \chi_{\bar{l}}(x) \, Ff \mid Sl_{\bar{2}}(L_{\bar{2}})\|$$
$$= c \, \|2^{\bar{l}\bar{\varkappa}} \chi_{\bar{l}}(x) \, Ff \mid L_{\bar{2}}(l_{\bar{2}})\|.$$

It is easy to see that the last expression is an equivalent norm in $S_2^{\bar{\varkappa}}H$. Hence, the Proposition is proved ∎

Remark 3: For $n = 1$ the Proposition follows from H. Triebel [23, Lemma 1.5.5].

3. Maximal inequalities and Fourier multipliers

3.1. Definitions

In view of the desired applications to function spaces, let us introduce the analytic counterparts of the spaces with mixed quasinorms from (2.5)—(2.7). In the following definition $\mathfrak{A}$ is said to be the set of all systems $\{f_{\bar{k}}\}_{\bar{k}\in Z_n^+}$ of tempered distributions from $S'(\mathbf{R}_n)$ with

$$\operatorname{supp} F[f_{\bar{k}}(2^{-\bar{k}}\cdot)] \subset \{\bar{x} \mid |x_i| \leqq 1 \quad (i = 1, \ldots, n)\}, \qquad \bar{k} \in \mathbf{Z}_n^+. \tag{3.1}$$

Then by the Paley-Wiener-Schwartz theorem each $f_{\bar{k}}$ is an entire analytic function of exponential type.

Definition 1: *Let* $\bar{0} < \bar{p}, \bar{q} \leqq \bar{\infty}$. *We put*

$$Sl_{\bar{q}}(L_{\bar{p}}{}^A) = \mathfrak{A} \cap Sl_{\bar{q}}(L_{\bar{p}}), \tag{3.2}$$

$$SL_{\bar{p}}{}^A(l_{\bar{q}}) = \mathfrak{A} \cap SL_{\bar{p}}(l_{\bar{q}}), \tag{3.3}$$

$$L_{\bar{p}}{}^A(l_{\bar{q}}) = \mathfrak{A} \cap L_{\bar{p}}(l_{\bar{q}}). \tag{3.4}$$

These spaces are equipped with the quasinorms from (2.5)—(2.7). We are interested in the following maximal function.

Definition 2: Let $\bar{a} > \bar{0}$ and let $\{f_{\bar{k}}(x)\}_{\bar{k}\in\mathbf{Z}_n^+} \in \mathfrak{A}$. Then we define

$$f_{\bar{k}}{}^*(x) = \sup_{y\in\mathbf{R}_n} \frac{|f_{\bar{k}}(x - y)|}{\prod_{i=1}^{n} (1 + |2^{k_i} y_i|)^{a_i}}. \tag{3.5}$$

Remark 4. This construction of a maximal function is analogous to J. PEETRE [11] and H. TRIEBEL [25, 2.2.3].

The aim of this subsection is the proof of some inequalities for the maximal function (3.5) in the spaces (3.2)—(3.4) and of Fourier multiplier theorems. That means we prove inequalities of the form

$$\|F^{-1}[\varphi_{\bar{k}}(\xi)\, Ff_{\bar{k}}]\,(x) \mid \cdot\| \leqq c\, \|f_{\bar{k}}(x) \mid \cdot\|,$$

where $\{\varphi_{\bar{k}}(x)\}_{\bar{k}\in\mathbf{Z}_n^+}$ is a system of infinitely differentiable functions on $\mathbf{R}_n$. Here the dot denotes one of the spaces from Definition 1.

3.2. The fundamental Lemmata

We follow some ideas of J. PEETRE [11] and H. TRIEBEL [25, 2.3.3]. In the sense of (2.8) we put

$$(M_i f)\,(x) = \big(Mf(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)\big)\,(x_i) \qquad (i = 1, \ldots, n) \tag{3.6}$$

for a locally integrable function $f(x)$, $x \in \mathbf{R}_n$. Further we denote

$$(\nabla f)\,(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right), \qquad |(\nabla f)\,(x)| = \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}(x)\right|,$$

if $f(x)$ is a continuously differentiable function.

Lemma 1: *Let $g(x)$ be a continuously differentiable function on $\mathbf{R}_n$ and let $0 < \bar{r} < \overline{\infty}$, $\bar{a} = \bar{r}^{-1}$. Then we have for $0 < \delta \leqq 1$*

$$\sup_{y \in \mathbf{R}_n} \frac{|g(x-y)|}{\prod_{i=1}^{n} (1 + |y_i|)^{a_i}}$$

$$\leqq c \sup_{y \in \mathbf{R}_n} \frac{|(\nabla g)(x+y)|}{\prod_{i=1}^{n} (1+|y_i|)^{a_i}} + \bar{\delta}^{-\bar{a}} \left(M_n \left(\dots M_2 (M_1 |g|^{r_1})^{\frac{r_2}{r_1}} \dots\right)^{\frac{r_n}{r_{n-1}}}\right)^{\frac{1}{r_n}}. \tag{3.7}$$

Proof: *Step* 1: Let $Q = [-1, 1] \times \cdots \times [-1, 1]$ and let us put

$$\|g(x) \mid C^1(Q)\| = \sup_{x \in Q} |g(x)| + \sup_{x \in Q} |(\nabla g)(x)|.$$

Then we prove

$$\|g(x) \mid C^1(Q)\| \sim \sup_{x \in Q} |(\nabla g)(x)| + \|g(x) \mid L_{\bar{r}}(Q)\|. \tag{3.8}$$

If $w \in Q$ is chosen, such that $|g(w)| = \min_{x \in Q} |g(x)|$, we obtain as a consequence of the mean value theorem

$$|g(v)| \leqq c \left(|g(w)| + \sup_{x \in Q} |\nabla g(x)|\right) \leqq c \left(\|g(x) \mid L_{\bar{r}}(Q)\| + \sup_{x \in Q} |\nabla g(x)|\right)$$

for arbitrary $v \in Q$. The other direction is obvious because of $|Q| < \infty$.

Step 2: Let $v \in Q_\delta = [-\delta, \delta] \times \cdots \times [-\delta, \delta] \subset \mathbf{R}_n$ and let $g(x)$ be continuously differentiable. Then it follows from (3.8)

$$|g(v)| = c\delta^{-\frac{1}{r_1}} \dots \delta^{-\frac{1}{r_n}} \|g(y) \mid L_{\bar{r}}(Q)\| + c\delta \sup_{x \in Q_\delta} |(\nabla g)(y)|,$$

where c is independent of δ. This estimate applied to the function $g(x - z - v)$ with fixed x and z, and $v \in Q$ yields for $v = 0$

$$|g(x-z)| \leqq c\bar{\delta}^{-\bar{r}^{-1}} \|g(x - z - \cdot) \mid L_{\bar{r}}(Q_\delta)\| + c\delta \sup_{y \in Q_\delta} |(\nabla g)(x - y - z)|. \tag{3.9}$$

The substitutions $z_i + y_i = u_i$ $(i = 1, \dots, n)$ give

$$\|g(x - z - \cdot) \mid L_{\bar{r}}(Q)\| \leqq \|f(x - \cdot) \mid L_{\bar{r}}(Z_1 \times \cdots \times Z_n)\|$$

with $Z_i = [-(1 + |z_i|), 1 + |z_i|]$. It holds that

$$\left(\int_{Z_i} |f(\dots, x_i - u_i, \dots)|^{r_i} du_i\right)^{\frac{1}{r_i}} \leqq c(1 + |z_i|)^{\frac{1}{r_i}} (M_i |g|^{r_i})^{\frac{1}{r_i}} (x_i)$$

and thus

$$\|g(x - z - \cdot) \mid L_{\bar{r}}(Q_\delta)\| \leqq c \prod_{i=1}^{n} (1 + |z_i|)^{\frac{1}{r_i}} \left(M_n \left(\dots (M_1 |g|^{r_1})^{\frac{r_2}{r_1}} \dots\right)^{\frac{r_n}{r_{n-1}}}\right)^{\frac{1}{r_n}} (x).$$

Together with (3.9) this inequality leads to (3.7). Hence, the Lemma is proved ∎

Lemma 2: *Let $\{f_{\bar{k}}(x)\} \in \mathfrak{A}$ and let $\{\varphi_{\bar{k}}(x)\}_{\bar{k} \in Z_n^+}$ be a system of infinitely differentiable functions. If $\bar{\varkappa} > \bar{a} + 2^{-1}$ then we have*

$$(F^{-1} \varphi_{\bar{k}} F f_{\bar{k}})^* (x) \leqq c \|\varphi_{\bar{k}}(2^{\bar{k}} y) \mid S_2^{\bar{\varkappa}} H\| f_{\bar{k}}^*(x), \tag{3.10}$$

where c is independent of $k \in Z_n^+$ and $x \in \mathbf{R}_n$.

Proof: We choose a function $\psi(x) \in S(\mathbf{R}_n)$ with the properties

$$\psi(x) = 1 \quad \text{if} \quad |x_i| \leqq 1 \quad (i = 1, \ldots, n), \quad \operatorname{supp} \psi(x) \subset \{x \mid |x_i| \leqq 2 \; (i = 1, \ldots, n)\}$$

and put $\varrho_{\bar{k}}(x) = \varphi_{\bar{k}}(x)\, \psi(2^{-\bar{k}}x)$. Recall that $2^{-\bar{k}}x = (2^{-k_1}x_1, \ldots, 2^{-k_n}x_n)$. Then it holds that

$$|(F^{-1}\varphi_{\bar{k}}Ff_{\bar{k}})(x-y)| \leqq c \int_{\mathbf{R}_n} |(F^{-1}\varrho_{\bar{k}})(x-y-z)|\, |f_{\bar{k}}(z)|\, dz$$

$$\leqq c f_{\bar{k}}^*(x) \int_{\mathbf{R}_n} |(F^{-1}\varrho_{\bar{k}})(x-y-z)| \left| \prod_{i=1}^{n} (1 + |2^{k_i}(x_i - z_i)|^{a_i} \right| dz$$

$$\leqq c f_{\bar{k}}^*(x) \prod_{i=1}^{n} (1 + |2^{k_i}y_i|^{a_i}) \int_{\mathbf{R}_n} |(F^{-1}\varrho_{\bar{k}})(x-y-z)|$$

$$\times \left| \prod_{i=1}^{n} \left(1 + |2^{k_i}(x_i - y_i - z_i)|\right)^{a_i} \right| dz.$$

Division by $\prod_{i=1}^{n} (1 + |2^{k_i}y_i|)^{a_i}$ yields

$$(F^{-1}\varrho_{\bar{k}}Ff_{\bar{k}})^*(x) \leqq c f_{\bar{k}}^*(x) \left\| \prod_{i=1}^{n} (1 + |z_i|)^{a_i} \left(F^{-1}\varrho_{\bar{k}}(2^{\bar{k}}\cdot)\right)(z) \mid L_1(\mathbf{R}_n) \right\|.$$

The right-hand side can be estimated with the help of Proposition 3, (2.16). This gives

$$(F^{-1}\varphi_{\bar{k}}Ff_{\bar{k}})^*(x) \leqq c f_{\bar{k}}^*(x)\, \|\varphi_{\bar{k}}(2^{\bar{k}}y)\, \psi(y) \mid S_2^{\bar{\varkappa}}H\|,$$

with $\bar{\varkappa} > \bar{a} + 2^{-1}$. Furthermore it holds that

$$\|\varphi_{\bar{k}}(2^{\bar{k}}y)\, \psi(y) \mid S_2^{\bar{\varkappa}}H\| \leqq c(\psi)\, \|\varphi_{\bar{k}}(2^{\bar{k}}y) \mid S_2^{\bar{\varkappa}}H\|.$$

This becomes clear for $\bar{\varkappa}$ with integer components by means of Proposition 2, (2.15). The general case follows by the interpolation theorem of H. Triebel [21, p. 239/ formula (49)]. Thus the proof is complete ∎

Lemma 3: *Let* $\{f_{\bar{k}}(x)\}_{\bar{k} \in Z_n^+} \in \mathfrak{A}$ *such that* $\sup_{\bar{k} \in Z_n^+} \|f_{\bar{k}}(x) \mid L_\infty(\mathbf{R}_n)\| < \infty$, *for all* $k \in Z_n^+$. *If* $\bar{a} = \bar{r}^{-1}$, *then we have*

$$f_{\bar{k}}^*(x) \leqq c \left(M_n \left(\ldots (M_1 |f_{\bar{k}}|^{r_1})^{\frac{r_2}{r_1}} \ldots \right)^{\frac{r_n}{r_{n-1}}}\right)^{\frac{1}{r_n}}(x). \tag{3.11}$$

Proof: From Lemma 1 it follows, with $g(x) = f_{\bar{k}}(2^{-\bar{k}}x)$, $\bar{k} \in Z_n^+$,

$$f_{\bar{k}}^*(2^{-\bar{k}}x) \leqq c\delta \sum_{i=1}^{n} 2^{-k_i} \left(\frac{\partial f_{\bar{k}}}{\partial x_i}\right)^* (2^{-\bar{k}}x) \tag{3.12}$$

$$+ \bar{\delta}^{-\bar{a}} \left(M_n \ldots \left(M_1 |f_{\bar{k}}(2^{-\bar{k}}\cdot)|^{r_1}\right)^{\frac{r_2}{r_1}} \ldots \right)^{\frac{r_n}{r_{n-1}}}\Bigg)^{\frac{1}{r_n}}(x).$$

It is easy to see that

$$\left(M_i g(\ldots, 2^{-k_i}, \ldots)\right)(x) = (M_i g)(x_1, \ldots, 2^{-k_i}x, \ldots, x_n)$$

and hence by iteration

$$\left(M_n \left(\ldots (M_1 \mid f_{\bar{k}}(2^{-\bar{k}}\cdot)^{r_1})^{\frac{r_2}{r_1}} \ldots \right)^{\frac{r_n}{r_{n-1}}}\right)^{\frac{1}{r_n}}(x)$$

$$= \left(M_n \left(\ldots (M_1 |f_{\bar{k}}(\cdot)|^{r_1})^{\frac{r_2}{r_1}} \ldots \right)^{\frac{r_n}{r_{n-1}}}\right)^{\frac{1}{r_n}}(2^{-\bar{k}}x).$$

This identity applied to (3.12) implies

$$f_{\bar k}^*(x) \leqq c\delta \sum_{i=1}^n 2^{-k_i} \left(\frac{\partial f_{\bar k}}{\partial x_i}\right)^* (x) + c\bar\delta^{-\bar a} \left(M_n \ldots \left(M_1 |f_{\bar k}|^{r_1}\right)^{\frac{r_2}{r_1}} \ldots\right)^{\frac{1}{r_n}} (x).$$

Now, we choose a function $\psi(t)$ ($t \in \mathbf{R}_1$), infinitely differentiable, with $\psi(t) = 1$ if $t \in [-1, 1]$, supp $\psi(t) \subset [-2, 2]$. Clearly, by definition $f_{\bar k}(x) = \left(F^{-1} \prod_{j=1}^n \psi(2^{-k_j} y_j) F f_{\bar k}\right)(x)$ and therefore

$$2^{-k_i} \left(\frac{\partial f_{\bar k}}{\partial x_i}\right)^* (x) = \left(F^{-1} 2^{-k_i} y_i \prod_{j=1}^n \psi(2^{-k_j} y_j)\right) F f_{\bar k}) (x).$$

Setting $\sigma_{\bar k}^{(i)}(y) = 2^{-k_i} y_i \prod_{j=1}^n \psi(2^{-k_j} y_j)$ it follows with the help of Lemma 2 that

$$2^{-k_i} \left(\frac{\partial f_{\bar k}}{\partial x_i}\right)^* (x) \leqq c \left\| y_i \prod_{j=1}^n \psi(y_j) \mid S_2^{\bar\varkappa} H \right\| f_{\bar k}^*(x),$$

where $\bar\varkappa > \bar a + 2^{-1}$. The first factor on the right-hand side can be estimated by a constant. Next we use the inequality (3.13) and obtain

$$f_{\bar k}^*(x) \leqq c\delta f_{\bar k}^*(x) + c\bar\delta^{-\bar a} \left(M_n \ldots (M_1 |f_{\bar k}|^{r_1})^{\frac{r_2}{r_1}} \ldots\right)^{\frac{1}{r_n}} (x), \qquad 0 < \delta < 1.$$

If we choose δ sufficiently small this gives the assertion of the Lemma, taking into account that $\sup_{\bar k, x} |f_{\bar k}(x)| < \infty$ ∎

3.3. Maximal inequalities

Theorem 1: *Let* $\bar 0 < \bar p < \bar\infty$, $\bar 0 < \bar q \leqq \bar\infty$.

(i) *If* $a_i > \min(p_1, \ldots, p_i; q_1, \ldots, q_{i-1})^{-1}$, *then*

$$\|f_{\bar k}^*(x) \mid Sl_{\bar q}(L_{\bar p})\| \leqq c \|f_{\bar k}(x) \mid Sl_{\bar q}(L_{\bar p})\| \tag{3.14}$$

for all $\{f_{\bar k}(x)\} \in Sl_{\bar q}(L_{\bar p}^A)$.

(ii) *If* $a_i > \min(p_1, \ldots, p_i; q_1, \ldots, q_i)^{-1}$, *then*

$$\|f_{\bar k}^*(x) \mid SL_{\bar p}(l_{\bar q})\| \leqq c \|f_{\bar k}(x) \mid SL_{\bar p}(l_{\bar q})\| \tag{3.15}$$

for all $\{f_{\bar k}(x)\} \in SL_{\bar p}^A(l_{\bar q})$.

(iii) *If* $a_i > \min(p_1, \ldots, p_i; q_1, \ldots, q_n)^{-1}$, *then*

$$\|f_{\bar k}^*(x) \mid L_{\bar p}(l_{\bar q})\| \leqq c \|f_{\bar k}(x) \mid L_{\bar p}(l_{\bar q})\| \tag{3.16}$$

for all $\{f_{\bar k}(x)\} \in L_{\bar p}^A(l_{\bar q})$.

Proof: We use Lemma 3. All assumptions are satisfied because of the Nikol'skij inequality for mixed quasinorms of B. Stöckert [19]. Therefore (3.11) yields

$$\begin{aligned}\|f_{\bar k}^*(x) \mid L_{\bar p}(l_{\bar q})\| &\leqq c \left\| M_n(\ldots M_1 |f_{\bar k}|^{r_1} \ldots)^{\frac{r_n}{r_{n-1}}} (x) \mid L_{\frac{\bar p}{r_n}} \left(l_{\frac{\bar q}{r_n}}\right) \right\|^{\frac{1}{r_n}} \\ &= c \left\| M_n(\ldots M_1 |f_{\bar k}|^{r_1} \ldots)^{\frac{r_n}{r_{n-1}}} (x) \left| L_{\frac{\bar p'}{r_n}} \left(l_{\frac{\bar q}{r_n}}\right) \right| L_{\frac{p_n}{r_n}} \right\|^{\frac{1}{r_n}}.\end{aligned}$$

Here $\bar{p}' = (p_1, \ldots, p_{n-1})$. We have $r_i = a_i^{-1}$ and by assumption

$$\infty > \frac{p_n}{r_n} > 1, \quad \infty > \frac{p_i}{r_n} > 1 \quad (i = 1, \ldots, n-1),$$

$$\infty \geqq \frac{q_i}{r_i} > 1 \quad (i = 1, \ldots, n).$$

Therefore we can apply Bagby's maximal inequality from Proposition 1 to the right-hand side. This yields

$$\|f_{\bar{k}}^*(x) \mid L_{\bar{p}}(l_{\bar{q}})\| \leqq c \left\| (M_{n-1} \ldots M_1 |f_{\bar{k}}|^{r_1} \ldots)^{\frac{r_n}{r_{n-1}}}(x) \mid L_{\frac{\bar{p}}{r_n}}\left(l_{\frac{\bar{q}}{r_n}}\right)\right\|^{\frac{1}{r_n}}$$

$$= c \left\| \left(M_{n-1}(\ldots M_1 |f_{\bar{k}}|^{r_1} \ldots)^{\frac{r_{n-1}}{r_{n-2}}}\right)^{\frac{1}{r_{n-1}}}(x) \, |L_{\bar{p}}(l_{\bar{q}})| \, L_{p_n} \right\|.$$

Successive applications of this procedure lead to the desired inequality (3.16). In the same way we can prove the other cases (3.14) and (3.15) ∎

3.4. Fourier multipliers

First let us introduce some notation. If $\bar{0} < \bar{p} < \bar{\infty}$, $\bar{0} < \bar{q} \leqq \bar{\infty}$ we set

$$\varkappa_i^B(p, q) = \min (p_1, \ldots, p_i; q_1, \ldots, q_{i-1})^{-1} + \frac{1}{2}$$

$$\varkappa_i^F(p, q) = \min (p_1, \ldots, p_i; q_1, \ldots, q_i)^{-1} + \frac{1}{2} \qquad (i = 1, \ldots, n)$$

$$\varkappa_i^H(p, q) = \min (p_1, \ldots, p_i; q_1, \ldots, q_n)^{-1} + \frac{1}{2}$$

and $\bar{\varkappa}^B = (\varkappa_1^B, \ldots, \varkappa_n^B)$ etc.

Theorem 2: *Let $\bar{0} < \bar{p} < \bar{\infty}$, $\bar{0} < \bar{q} \leqq \bar{\infty}$ and let $\{\varphi_{\bar{k}}(x)\}_{\bar{k} \in Z_n^+}$ be a system of infinitely differentiable functions.*

(i) *If $\bar{\varkappa} > \bar{\varkappa}^B$, then*

$$\|(F^{-1}\varphi_{\bar{k}} F f_{\bar{k}})(x) \mid Sl_{\bar{q}}(L_{\bar{p}})\| \leqq c \sup_{\bar{k} \in Z_n^+} \|\varphi_{\bar{k}}(2^{\bar{k}} y) \mid S_2^{\bar{\varkappa}} H\| \, \|f_{\bar{k}}(x) \mid Sl_{\bar{q}}(L_{\bar{p}})\| \tag{3.17}$$

for all $\{f_{\bar{k}}(x)\} \in Sl_{\bar{q}}(L_{\bar{p}}^A)$.

(ii) *If $\bar{\varkappa} > \bar{\varkappa}^F$, then*

$$\|(F^{-1}\varphi_{\bar{k}} F f_{\bar{k}})(x) \mid SL_{\bar{p}}(l_{\bar{q}})\| \leqq c \sup_{\bar{k} \in Z_n^+} \|\varphi_{\bar{k}}(2^{\bar{k}} y) \mid S_2^{\bar{\varkappa}} H\| \, \|f_{\bar{k}}(x) \mid SL_{\bar{p}}(l_{\bar{q}})\| \tag{3.18}$$

for all $\{f_{\bar{k}}(x)\} \in SL_{\bar{p}}^A(l_{\bar{q}})$.

(iii) *If $\bar{\varkappa} > \bar{\varkappa}^H$, then*

$$\|(F^{-1}\varphi_{\bar{k}} F f_{\bar{k}})(x) \mid L_{\bar{p}}(l_{\bar{q}})\| \leqq c \sup_{\bar{k} \in Z_n^+} \|\varphi_{\bar{k}}(2^{\bar{k}} y) \mid S_2^{\bar{\varkappa}} H\| \, \|f_{\bar{k}}(x) \mid L_{\bar{p}}(l_{\bar{q}})\| \tag{3.19}$$

for all $\{f_{\bar{k}}(x)\} \in L_{\bar{p}}^A(l_{\bar{q}})$.

Proof: The proof is an immediate consequence of Lemma 2, Theorem 1, and $|(F^{-1}\varphi_{\bar{k}} F f_{\bar{k}})(x)| = (F^{-1}\varphi_{\bar{k}} F f_{\bar{k}})^*(x)$ ∎

4. Applications

4.1. Spaces with dominating mixed smoothness of the Besov-Triebel-Lizorkin type

Let Φ be the class of systems $\varphi = \{\varphi_l(t)\}_{l=0}^{\infty}$ of infinitely differentiable functions, having the properties:

$$\operatorname{supp} \varphi_0 \subset [-2, 2], \quad \operatorname{supp} \varphi_l \subset [-2^{l+1}, -2^{l-1}] \cup [2^{l-1}, 2^{l+1}] \quad (l = 1, 2, \ldots), \tag{4.1}$$

$$\left|\frac{d^m}{dt^m} \varphi_l(t)\right| \leqq c2^{-lm} \qquad (l = 0, 1, 2, \ldots), \tag{4.2}$$

$$\sum_{l=0}^{\infty} \varphi_l(t) \equiv 1. \tag{4.3}$$

These are the systems of test functions widely used in the theory of function spaces (cf. H. Triebel [20—25]). If $\{\varphi_l(t)\}_{l=0}^{\infty} \in \Phi$ we put

$$\varphi_{\bar{k}}(x) = \prod_{i=1}^{n} \varphi_{k_i}(x_i), \qquad x \in \mathbf{R}_n, \quad \bar{k} \in \mathbf{Z}_n^+.$$

Definition 3: Let $\bar{0} < \bar{p} < \bar{\infty}$, $\bar{0} < \bar{q} \leqq \bar{\infty}$, $-\bar{\infty} < \bar{r} < \bar{\infty}$. We define:

(i) $S_{\bar{p},\bar{q}}^{\bar{r}}B = \{f \in S'(\mathbf{R}_n) \mid \exists\, \varphi \in \Phi$, such that

$$\|f \mid S_{\bar{p},\bar{q}}^{\bar{r}}B\| = \|2^{\bar{k}\bar{r}}(F^{-1}\varphi_{\bar{k}}Ff)\,(x)\; |L_{\bar{p}}|\; l_{\bar{q}}\| < \infty\}, \tag{4.4}$$

(ii) $SB_{\bar{p},\bar{q}}^{\bar{r}} = \{f \in S'(\mathbf{R}_n) \mid \exists\, \varphi \in \Phi$, such that

$$\|f \mid SB_{\bar{p},\bar{q}}^{\bar{r}}\| = \|2^{\bar{k}\bar{r}}(F^{-1}\varphi_{k}Ff)\,(x) \mid Sl_{\bar{q}}(L_{\bar{p}})\| < \infty\}, \tag{4.5}$$

(iii) $SF_{\bar{p},\bar{q}}^{\bar{r}} = \{f \in S'(\mathbf{R}_n) \mid \exists\, \varphi \in \Phi$, such that

$$\|f \mid SF_{\bar{p},\bar{q}}^{\bar{r}}\| = \|2^{\bar{k}\bar{r}}(F^{-1}\varphi_{\bar{k}}Ff)\,(x) \mid SL_{\bar{p}}(l_{\bar{q}})\| < \infty\}, \tag{4.6}$$

(iv) $S_{\bar{p},\bar{q}}^{\bar{r}}F = \{f \in S'(\mathbf{R}_n) \mid \exists\, \varphi \in \Phi$, such that

$$\|f \mid S_{\bar{p},\bar{q}}^{\bar{r}}F\| = \|2^{\bar{k}\bar{r}}(F^{-1}\varphi_{\bar{k}}Ff)\,(x) \mid L_{\bar{p}}(l_{\bar{q}})\| < \infty\}. \tag{4.7}$$

Remark 5: The spaces (i)—(iv) are spaces of functions and distributions with dominating mixed smoothness properties. All spaces are quasi-Banach spaces (Banach if $\bar{1} \leqq \bar{p}, \bar{q}$). For different φ's the quasinorms (4.4)—(4.7) are equivalent to each other. This and the other basic properties can be derived from Theorem 2. We refer to [14—17].

Remark 6: The spaces $S_{\bar{p},\bar{q}}^{\bar{r}}B$ generalize the spaces $S_{p,q}^{(r)}B$ $(1 \leqq p, q \leqq \infty, \bar{0} \leqq \bar{r} < \bar{\infty})$ of T. I. Amanov [2, 3] and the spaces $B_{p,q}^{g(x)}$ $(0 < p,\ q \leqq \infty,\ g(x) = \prod_{i=1}^{n} (1 + x_i^2)^{r_i/2})$ of H. Triebel [21—23] to mixed quasinorms. For the details we refer the reader to H.-J. Schmeisser [16]. The spaces $SB_{\bar{p},\bar{q}}^{\bar{r}}$ were introduced by H. Triebel [26] in the case $n = 2$. They are investigated together with the spaces $SF_{\bar{p},\bar{q}}^{\bar{r}}$, which are the counterparts of the Triebel-Lizorkin type, in H.-J. Schmeisser [14, 15]. These two papers deal with the main properties such as Fourier multipliers and representation theorems. For imbeddings we refer to a forthcoming paper (see also [17]). The spaces $S_{\bar{p},\bar{q}}^{\bar{r}}F$ generalize the spaces $F_{p,q}^{g(x)}$ $(0 < p < \infty,\ 0 < q \leqq \infty,\ g(x) = \prod_{i=1}^{n} (1 + x_i^2)^{r_i/2})$ of H. Triebel [21—23] to mixed quasinorms. The relations to the spaces $S_{\bar{p}}^{\bar{r}}H$ (see 2.3, (2.11)) are considered in the next subsection.

4.2. Sobolev-Lebesgue spaces

Looking at the theory of the isotropic Triebel-Lizorkin spaces $F^r_{p,q}$ ($0 < p < \infty$, $0 < q \leqq \infty$, $-\infty < r < \infty$) and at the definitions of $S^{\bar r}_{\bar p,\bar q}F$, $SF^{\bar r}_{\bar p,\bar q}$, it is natural to ask about connections to mixed L_p-spaces and to Sobolev spaces. The answer is given in the following theorem.

Theorem 3: *Let $\bar 1 < \bar p < \bar\infty$, $-\bar\infty < \bar r < \bar\infty$. Further let $\bar m$ be a n-tuple of non-negative integers. Then we have*

$$S^{\bar 0}_{\bar p,\bar 2}F = L_{\bar p}, \tag{4.8}$$

$$S^{\bar m}_{\bar p,\bar 2}F = S_{\bar p}^{\bar m}W, \tag{4.9}$$

$$S^{\bar r}_{\bar p,\bar 2}F = S_{\bar p}^{\bar r}H. \tag{4.10}$$

Further it holds that

$$SF^{\bar 0}_{\bar p,\min(\bar p,\bar 2)} \subset L_{\bar p} \subset SF^{\bar 0}_{\bar p,\max(\bar p,\bar 2)}, \tag{4.11}$$

$$SF^{\bar m}_{\bar p,\min(\bar p,\bar 2)} \subset S_{\bar p}^{\bar m}W \subset SF^{\bar m}_{\bar p,\max(\bar p,\bar 2)}, \tag{4.12}$$

$$SF^{\bar r}_{\bar p,\min(\bar p,\bar 2)} \subset S_{\bar p}^{\bar r}H \subset SF^{\bar r}_{\bar p,\max(\bar p,\bar 2)}, \tag{4.13}$$

where

$$\min(\bar p, \bar 2) = \big(2, \min(p_1, 2), \ldots, \min(p_1, \ldots, p_{n-1}, 2)\big)$$

$$\max(\bar p, \bar 2) = \big(2, \max(p_1, 2), \ldots, \max(p_1, \ldots, p_{n-1}, 2)\big).$$

Proof: The Littlewood-Paley theorem (4.8) is a consequence of P. I. LIZORKIN [8, Theorem 2 and Lemma 2] (see also [3, Lemma 15.2]). Formula (4.8) and Definition 3 (iv) give

$$\left\| F^{-1} \prod_{i=1}^n (1 + \xi_i^2)^{\frac{r_i}{2}} Ff \mid L_{\bar p} \right\| \sim \left\| F^{-1} \prod_{i=1}^n (1 + \xi_i^2)^{\frac{r_i}{2}} \varphi_{k_i}(\xi_i)\, Ff \mid L_{\bar p}(l_{\bar 2}) \right\|. \tag{4.14}$$

We put

$$\psi_{k_i}(\xi_i) = 2^{-r_ik_i}(1 + \xi_i^2)^{\frac{r_i}{2}} \varphi^*_{k_i}(\xi_i),$$

$$\varrho_{k_i}(\xi_i) = 2^{r_ik_i}(1 + \xi_i^2)^{-\frac{r_i}{2}} \varphi_{k_i}(\xi_i) \qquad (k_i = 0, 1, 2, \ldots; i = 1, \ldots, n),$$

$$f_{\bar k}(x) = F^{-1} \prod_{i=1}^n (1 + \xi_i^2)^{\frac{r_i}{2}} \varphi_{k_i}(\xi_i)\, Ff.$$

Because of (4.1) and (4.3) the following two inequalities hold:

$$\|f_{\bar k}(x) \mid L_{\bar p}(l_{\bar 2})\| \leqq \sum_{\bar l} \left\| F^{-1} \prod_{i=1}^n \psi_{k_i} F(2^{\bar r\bar k} F^{-1}\varphi_{\bar k+\bar l}Ff)\,(x) \mid L_{\bar p}(l_{\bar 2}) \right\|$$

and

$$\|(2^{\bar r\bar k}F^{-1}\varphi_{\bar k}Ff)\,(x) \mid L_{\bar p}(l_{\bar 2})\| \leqq \sum_{\bar l} \|F^{-1}\varrho_{\bar k+\bar l}Ff_{\bar k})\,(x) \mid L_{\bar p}(l_{\bar 2})\|,$$

where the sum over $\bar l \in \mathbf{Z}_n^+$ is finite. Now we apply Theorem 2 (iii) to both inequalities with ψ and ϱ instead of φ, and considering (4.14), we obtain formula (4.10). Then (4.9) is a consequence of Proposition 2.

The imbeddings (4.11)—(4.13) follow from the more general ones

$$SF^{\bar{r}}_{\bar{p},\min(\bar{p},\bar{q})} \subset S^{\bar{r}}_{\bar{p},\bar{q}}F \subset SF^{\bar{r}}_{\bar{p},\max(\bar{p},\bar{q})},$$

$$\min(\bar{p}, \bar{q}) = \big(q_1, \ldots, \min(p_1, \ldots, p_{i-1}, q_i), \ldots, \min(p_1, \ldots, p_{n-1}, q_n)\big), \quad (4.15)$$

$$\max(\bar{p}, \bar{q}) = \big(q_1, \ldots, \max(p_1, \ldots, p_{i-1}, q_i), \ldots, \max(p_1, \ldots, p_{n-1}, q_n)\big),$$

which are a consequence of elementary imbedding properties of the spaces $L_{\bar{p}}(l_{\bar{q}})$, and from the just proved formulas (4.8)—(4.10). Thus the proof is complete ∎

Remark 7: The theorem states that the Sobolev-Lebesgue spaces $S_{\bar{p}}^{\bar{r}}H$ with dominating mixed derivatives are contained in $S^{\bar{r}}_{\bar{p},\bar{q}}F$ as a special case. However note that this is not true for the spaces $SF^{\bar{r}}_{\bar{p},\bar{q}}$. This s personal information from Prof. A. Pietsch, Jena, who has constructed a counterexample. In fact he has proved that $SF^{\bar{0}}_{\bar{p},\bar{2}} = L_{\bar{p}}$ if and only if $\bar{p} = \bar{2}$. We can only prove the useful imbeddings (4.11 to 4.13).

Let us further mention that Littlewood-Paley theorems for mixed $L_{\bar{p}}$-spaces in the sense of (4.8) are not new. We refer to P. I. Lizorkin [8], Besov, Il'jin, Nikol'skij [6], and S. I. Agalarov [1] (weighted case).

4.3. Remarks on integral operators

H. Triebel [26, 27] found that the spaces $SB^{\bar{r}}_{\bar{p},\bar{q}}$ and also $SF^{\bar{r}}_{\bar{p},\bar{q}}$ are very suitable for the description of smoothness properties of kernels of integral operators (see also [14] for the F-case). But this is also true for $S^{\bar{r}}_{\bar{p},\bar{q}}$ and $S^{\bar{r}}_{\bar{p},\bar{q}}F$.

Let $B^r_{p,q}(\mathbf{R}_1)$, $F^r_{p,q}(\mathbf{R}_1)$ be the usual Besov and Triebel-Lizorkin spaces, respectively (cf. [25]). Here $-\infty < r < \infty$, $1 \leqq p, q \leqq \infty$. If $K(x, y) \in S^{(-t,r)}_{(u',p)(v',q)}B(\mathbf{R}_1 \times \mathbf{R}_1)$ ($1 \leqq u, p \leqq \infty$, $1 \leqq v, q \leqq \infty$, $-\infty < t, r < \infty$) then

$$\mathfrak{K}: B^t_{u,v}(\mathbf{R}_1) \to B^r_{p,q}(\mathbf{R}_1). \quad (4.16)$$

If $K(x, y) \in S^{(-t,r)}_{(u',p),(v',q)}F(\mathbf{R}_1 \times \mathbf{R}_1)$ ($1 < u \leqq \infty$, $1 \leqq p < \infty$, $1 \leqq v$, $q \leqq \infty$, $-\infty < t, r < \infty$) then

$$\mathfrak{K}: F^t_{u,v}(\mathbf{R}_1) \to F^r_{p,q}(\mathbf{R}_1). \quad (4.17)$$

Here $\mathfrak{K}$,

$$(\mathfrak{K}f)(y) = \int_{-\infty}^{\infty} K(x, y) f(x)\, dx$$

is the corresponding integral operator. Of course $\frac{1}{u} + \frac{1}{u'} = 1$. The proofs of (4.16) and (4.17) are obvious modifications of Triebel's proof in [26], see also [17, 8.1]. We omit the details.

As a corollary we obtain from (4.17) and Theorem 3 mapping properties of integral operators between L_p-spaces and Sobolev-Lebesgue spaces.

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