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Fredholmness and finite section method for Toeplitz operators in $l^p(\mathbf{Z}_+ \times \mathbf{Z}_+)$ with piecewise continuous symbols II[1])

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In dieser Arbeit beweisen wir hinlängliche Bedingungen dafür, daß ein diskreter Toeplitzoperator mit stückweise stetigem Symbol im Raum l^p über der Viertelebene noethersch ist und ebenso dafür, daß auf einen solchen Operator das Reduktionsverfahren anwendbar ist. Dabei wird entscheidend von Bilokalisierungstechniken und vom lokalen Prinzip von Douglas und Krupnik Gebrauch gemacht. Teil I dieser Arbeit war den Beweisen für die Notwendigkeit der entsprechenden Bedingungen, den nötigen Definitionen und der Formulierung der Hauptresultate gewidmet.

В данной работе доказываются достаточные условия для нетеровости операторов теплица с кусочно-непрерывными символами в пространстве l^p на квадранте, а также для применимости метода редукции к таким операторам. Методы настоящей работы существенно опираются на билокальную технику и на локальный принцип Дугласа-Крупника. Часть I работы была посвящена доказательствам необходимости соответствующих условий, всем нужным определениям и формулировке главных результатов.

In this paper we prove sufficient conditions for Fredholmness of discrete Toeplitz operators with piecewise continuous symbols on the space l^p over the quarter-plane and for the applicability of the finite section method to such operators. The methods used here are based on a bilocalization technique and the local principle of Douglas and Krupnik. Part I of this work contained the proofs of the necessity of the corresponding conditions, the necessary definitions, and the formulation of the main results.

This paper continues the paper [1] and it is devoted to the proof of the sufficiency part of the Theorems 1 and 2 of [1]. All definitions and notations used here and not being explicitly explained were introduced in [1].

§ 5. Further auxiliary propositions on one-dimensional Toeplitz operators

With regard to a theory of the finite section method for two-dimensional Toeplitz operators some one-dimensional results have to be precised. This is the purpose of the present section.

Set

$$\mathbf{F}_p = \left\{\{A_n\}_{n=0}^{\infty} : A_n : \operatorname{Im} P_n \to \operatorname{Im} P_n, \quad \|\{A_n\}\| := \sup_n \|A_n P_n\|_{\mathfrak{L}(l^p)} < \infty\right\},$$

$$\mathbf{G}_p = \left\{\{A_n\}_{n=0}^{\infty} \in \mathbf{F}_p : \|A_n P_n\|_{\mathfrak{L}(l^p)} \to 0 \quad (n \to \infty)\right\}.$$

By $\mathbf{A}_p$ we denote the closure in $\mathbf{F}_p$ of the collection of all sequences of the form

$$\{A_n\} = \left\{\sum_{j=1}^{r} \prod_{k=1}^{s} T_n(a_{jk})\right\},$$

[1]) Part I appeared in No. 2 (1984) of this journal.

where $r, s \in \mathbf{Z}_+$, $a_{jk} \in PC_0$. Finally, let $\mathbf{I}_p$ be the set of all sequences $\{A_n\} \in \mathbf{F}_p$ which are of the form

$$\{A_n\} = \{P_n K P_n + W_n K_1 W_n + C_n\},$$

where $K, K_1 \in \mathfrak{K}_p$ and $\{C_n\} \in \mathbf{G}_p$ (the operators P_n and W_n were defined in Section 2).

As in [2] the inclusion $\mathbf{I}_p \subset \mathbf{A}_p$ can be proved and then in the same fashion as in [8] it can be verified that $\mathbf{I}_p$ forms a closed two-sided ideal in the Banach algebra $\mathbf{A}_p$ and that $\mathbf{A}_p/\mathbf{I}_p$ is commutative. Let τ_p denote the canonical projection of $\mathbf{A}_p$ onto $\mathbf{A}_p/\mathbf{I}_p$, $\mathbf{N}_p$ the maximal ideal space of $\mathbf{A}_p/\mathbf{I}_p$ and $\Gamma_{\mathbf{N}_p}$ the Gelfand map of $\mathbf{A}_p/\mathbf{I}_p$ into $C(\mathbf{N}_p)$. Note that $\{T_n(a)\} \in \mathbf{A}_p$ if $a \in PC_p(\mathbf{T})$.

Proposition 3: *Let $a \in PC_p(\mathbf{T})$. If $T(a) \in G\mathfrak{L}(l^r)$ for every $r \in [p, q]$, $1/p + 1/q = 1$, then $\tau_p\{T_n(a)\} \in G(\mathbf{A}_p/\mathbf{I}_p)$.*

Recall that $a \in PC_p(\mathbf{T})$ implies $T(a) \in \mathfrak{L}(l^r)$ $\forall\, r \in [p, q]$.

Proof: Applying the local principle of I. C. GOHBERG and N. YA. KRUPNIK [4] with the method of [8] we find that $\tau_p\{T_n(a)\}$ is locally equivalent at $t_0 \in \mathbf{T}$ to $\tau_p\{T_n(a_{t_0})\}$, where a_{t_0} is defined by

$$a_{t_0}(t) = \begin{cases} a(t_0 + 0), \arg t_0 < \arg t < \arg t_0 + \pi \\ a(t_0 - 0), \arg t_0 - \pi < \arg t < \arg t_0, \end{cases}$$

$t \in \mathbf{T}$. Let $\mathfrak{O}_p^{t_0}$ denote the closed "lentiform" domain in $\mathbf{C}$ which has as its boundary the two circular arcs

$$\big\{\big(1 - s_p(\mu)\big)\, a(t_0 - 0) + s_p(\mu)\, a(t_0 + 0) \colon \mu \in [0, 1]\big\},$$
$$\big\{\big(1 - s_p(\mu)\big)\, a(t_0 + 0) + s_p(\mu)\, a(t_0 - 0) \colon \mu \in [0, 1]\big\}$$

(recall the notation (2.2) of Section 2). According to Theorem *H* of Section 2 we have $T(a_{t_0}) - \lambda I \in \Pi_p\{P_n\}$ for $\lambda \notin \mathfrak{O}_p^{t_0}$.

Now, let $\mathfrak{A}_p$ be the Banach algebra playing the dominant part in [8], i.e. $\mathfrak{A}_p$ consists of all sequences $\{A_n\}_{n=0}^{\infty}$, $A_n\colon \operatorname{Im} P_n \to \operatorname{Im} P_n$ for which there exist operators $A, A_1 \in \mathfrak{L}(l^p)$ such that $A_n P_n \to A$, $A_n^* P_n \to A^*$, $W_n A_n W_n \to A_1$, $W_n A_n^* W_n \to A_1^*$ (the convergence in the strong sense, the asterisk denoting the Hermitian conjugate). $\mathbf{I}_p$ again forms a closed two-sided ideal in $\mathfrak{A}_p$. Let us by τ_p' denote the canonical projection of $\mathfrak{A}_p$ onto $\mathfrak{A}_p/\mathbf{I}_p$.

The results of [8] now imply that $\tau_p'\{T_n(a_{t_0}) - \lambda P_n\} \in G(\mathfrak{A}_p/\mathbf{I}_p)$ if only $T(a_{t_0}) - \lambda I \in \Pi_p\{P_n\}$. Then λ does not belong to the spectrum of $\tau_p'\{T_n(a_{t_0})\}$ in $\mathfrak{A}_p/\mathbf{I}_p$, i.e.

$$\lambda \notin \operatorname{spec}_{\mathfrak{A}_p/\mathbf{I}_p}\big(\tau_p'\{T_n(a_{t_0})\}\big).$$

Hence

$$\operatorname{spec}_{\mathfrak{A}_p/\mathbf{I}_p}\big(\tau_p'\{T_n(a_{t_0})\}\big) \subset \mathfrak{O}_p^{t_0}.$$

By [7: 10.18] we conclude

$$\operatorname{spec}_{\mathfrak{A}_p/\mathbf{I}_p}\big(\tau_p\{T_n(a_{t_0})\}\big) \subset \mathfrak{O}_p^{t_0}. \tag{1}$$

On condition that $T(a) \in G\mathfrak{L}(l^r)$ for every $r \in [p, q]$, $1/p + 1/q = 1$, from Theorem *G* of Section 2 follows that $0 \notin \mathfrak{O}_p^{t_0}$ for every $t_0 \in \mathbf{T}$. Thus (1) shows that $\tau_p\{T_n(a_{t_0})\} \in G(\mathbf{A}_p/\mathbf{I}_p)$ and application of the local principle of [4] gives $\tau_p\{T_n(a)\} \in G(\mathbf{A}_p/\mathbf{I}_p)$ ∎

Proposition 4: *Let $N \in \mathbf{N}_p$. Then there exist $r \in [p, q]$ $(1/p + 1/q = 1)$, $\zeta \in \mathbf{T}$, and $\mu \in [0, 1]$ such that*

$$\big(\Gamma_{\mathbf{N}_p}\tau_p\{T_n(a)\}\big)(N) = \big(1 - s_r(\mu)\big)\, a(\zeta - 0) + s_r(\mu)\, a(\zeta + 0)$$

for every $a \in PC_p(\mathbf{T})$.

Proof: Let $\mathbf{W}$ be the Wiener algebra of all functions on $\mathbf{T}$ with absolutely convergent Fourier series. Define the map ω by

$$\omega\colon \mathbf{W} \to \mathbf{A}_p/\mathbf{I}_p,\ a \mapsto \tau_p\{T_n(a)\}.$$

Obviously, ω is a continuous algebraic homomorphism. If φ_N is the complex homomorphism (continuous multiplicative linear functional) associated with $N \in \mathbf{N}_p$, i.e. $N = \operatorname{Ker} \varphi_N$, then $\varphi_N \circ \omega$ is a complex homomorphism on $\mathbf{W}$. Consequently, there is a $\zeta \in \mathbf{T}$ with

$$\varphi_N\big(\tau_p\{T_n(a)\}\big) = a(\zeta) \tag{2}$$

for every $a \in \mathbf{W}$. For $\vartheta \in \mathbf{T}$ define the function $a_\vartheta \in PC_0$ by

$$a_\vartheta(t) = \begin{cases} 1, & \arg \vartheta < \arg t < \arg \vartheta + \pi \\ 0, & \arg \vartheta - \pi < \arg t < \arg \vartheta \end{cases}$$

and the complex number c by

$$c = \varphi_N\big(\tau_p\{T_n(a_\zeta)\}\big) \tag{3}$$

(ζ given by (2)). From the inclusion (1) we get the existence of an $r \in [p, q]$ and of a $\mu \in [0, 1]$ such that

$$c = \big(1 - s_r(\mu)\big) \cdot 0 + s_r(\mu) \cdot 1 = s_r(\mu). \tag{4}$$

Now, the identity $a_\zeta + a_{-\zeta} = 1$ gives

$$\varphi_N\big(\tau_p\{T_n(a_{-\zeta})\}\big) = 1 - s_r(\mu). \tag{5}$$

Thus we have evaluated φ_N at $\tau_p\{T_n(a_\zeta)\}$ and at $\tau_p\{T_n(a_{-\zeta})\}$. Let $c(\vartheta)$ be the value of φ_N at $\tau_p\{T_n(a_\vartheta)\}$ for $\vartheta \in \mathbf{T}$, $\vartheta \neq \pm\zeta$, i.e. define $c(\vartheta)$ by

$$c(\vartheta) = \varphi_N\big(\tau_p\{T_n(a_\vartheta)\}\big). \tag{6}$$

We are going to prove that

$$c(\vartheta) = \big(1 - s_r(\mu)\big)\, a_\vartheta(\zeta - 0) + s_r(\mu)\, a_\vartheta(\zeta + 0). \tag{7}$$

For this purpose we choose a function $b \in \mathbf{W}$ being identically 1 in a neighborhood of ζ and identically 0 in a neighborhood of ϑ and $-\vartheta$. Then, obviously, $b \cdot a_\vartheta \in \mathbf{W}$ and we have

$$\begin{aligned} a_\vartheta(\zeta) &= b(\zeta)\, a_\vartheta(\zeta) && \text{(since } b(\zeta) = 1) \\ &= \varphi_N\big(\tau_p\{T_n(b a_\vartheta)\}\big) && \text{(because of (2))} \\ &= \varphi_N\big(\tau_p\{T_n(b)\}\big) \cdot \varphi_N\big(\tau_p\{T_n(a_\vartheta)\}\big) \\ &= b(\zeta)\, c(\vartheta) && \text{(because of (2) and (6))} \\ &= c(\vartheta) && \text{(since } b(\zeta) = 1). \end{aligned}$$

Hence

$$\begin{aligned} c(\vartheta) = a_\vartheta(\zeta) &= \begin{cases} 1, & \arg \vartheta < \arg \zeta < \arg \vartheta + \pi \\ 0, & \arg \vartheta - \pi < \arg \zeta < \arg \vartheta \end{cases} \\ &= \big(1 - s_r(\mu)\big)\, a_\vartheta(\zeta - 0) + s_r(\mu)\, a_\vartheta(\zeta + 0), \end{aligned}$$

since $a_\vartheta(\zeta - 0) = a_\vartheta(\zeta + 0)$ for $\vartheta \neq \pm\zeta$. Thus by (3–7) we have expressed $\varphi_N\big(\tau_p\{T_n(a_\vartheta)\}\big)$ for every $\vartheta \in \mathbf{T}$ in terms of r, μ and ζ such as it is desired. Considering finite linear combinations we get

$$\varphi_N(\tau_p\{T_n(\chi)\}) = (1 - s_r(\mu))\,\chi(\zeta - 0) + s_r(\mu)\,\chi(\zeta + 0)$$

for every $\chi \in PC_0$.

Given an arbitrary function $a \in PC_p(\mathbf{T})$ we can find $a_j \in PC_0$ such that

$$\|\{T_n(a)\} - \{T_n(a_j)\}\|_{\mathbf{A}_p} \to 0 \qquad (j \to \infty).$$

Hence, first of all,

$$\begin{aligned}&\left|\varphi_N(\tau_p\{T_n(a)\}) - \varphi_N(\tau_p\{T_n(a_j)\})\right| \\ &\leqq \|\{T_n(a)\} - \{T_n(a_j)\}\|_{\mathbf{A}_p} \to 0 \qquad (j \to \infty).\end{aligned} \tag{8}$$

Furthermore

$$\begin{aligned}\|\{T_n(a)\} - \{T_n(a_j)\}\|_{\mathbf{A}_p} &= \sup_n \|T_n(a - a_j)\| \\ &\geqq \liminf_{n\to\infty} \|T_n(a - a_j)\| = \|T(a - a_j)\|_{\mathfrak{L}(l^p)} = \|T(a - a_j)\|_{\mathfrak{L}(l^q)}\end{aligned}$$

and

$$\begin{aligned}&\left|(1 - s_r(\mu))\,(a - a_j)\,(\zeta - 0) + s_r(\mu)\,(a - a_j)\,(\zeta + 0)\right| \\ &\leqq \max_{\substack{(\xi,\lambda)\in\mathbf{T}\times[0,1] \\ w\in[p,q]}} \left|(1 - s_w(\lambda))\,(a - a_j)\,(\xi - 0) + s_w(\lambda)\,(a - a_j)\,(\xi + 0)\right| \\ &= \max_{\substack{(\xi,\lambda)\in\mathbf{T}\times[0,1] \\ w\in\{p,q\}}} \left|(1 - s_w(\lambda))\,(a - a_j)\,(\xi - 0) + s_w(\lambda)\,(a - a_j)\,(\xi + 0)\right| \\ &= \max\,\{\|\Gamma_{\mathfrak{N}_p}\sigma_p T(a - a_j)\|_{C(\mathfrak{N}_p)}, \|\Gamma_{\mathfrak{N}_q}\sigma_q T(a - a_j)\|_{C(\mathfrak{N}_q)}\} \\ &\leqq \max\,\{\|T(a - a_j)\|_{\mathfrak{B}_p}, \|T(a - a_j)\|_{\mathfrak{B}_q}\} \\ &= \|T(a - a_j)\|_{\mathfrak{L}(l^p)} = \|T(a - a_j)\|_{\mathfrak{L}(l^q)}.\end{aligned}$$

Thus

$$\left|(1 - s_r(\mu))\,(a - a_j)\,(\zeta - 0) + s_r(\mu)\,(a - a_j)\,(\zeta + 0)\right| \to 0 \tag{9}$$

as $j \to \infty$. Combining (8) and (9) we get the assertion in full generality ∎

§ 6. Some lemmas on Banach algebras

The simple facts stated here will be applied in the Sections 8 and 9.

Let $\mathfrak{A}$ be a Banach algebra with unit e and $\mathfrak{J} \subset \mathfrak{A}$ a closed two-sided ideal. Suppose that $\mathfrak{A}/\mathfrak{J}$ is *commutative*. By j we denote the canonical projection of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$, by N the maximal ideal space of $\mathfrak{A}/\mathfrak{J}$ and by Γ_N the Gelfand map of $\mathfrak{A}/\mathfrak{J}$ into $C(N)$. For concrete examples put $\mathfrak{A} = \mathfrak{B}_p$, $\mathfrak{J} = \mathfrak{K}_p$ or $\mathfrak{A} = \mathbf{A}_p$, $\mathfrak{J} = \mathbf{I}_p$. In what follows $\otimes$ always denotes the projective tensor product. Put $\mathfrak{A}^2 = \mathfrak{A} \otimes \mathfrak{A}/\mathfrak{A} \otimes \mathfrak{J}$ and denote by j_2 the canonical projection of $\mathfrak{A} \otimes \mathfrak{A}$ onto $\mathfrak{A}^2$. Then $U^2 = \{e\} \otimes \mathfrak{A}/\mathfrak{A} \otimes \mathfrak{J}$ is naturally embedded in $\mathfrak{A}^2$ and let clos U^2 denote the closure of U^2 in $\mathfrak{A}^2$.

Lemma 4: *clos U^2 is contained in the centre of $\mathfrak{A}^2$.*

Proof: It suffices to prove that

$$j_2(e \otimes a)\,j_2(b \otimes c) = j_2(b \otimes c)\,j_2(e \otimes a) \tag{1}$$

for arbitrary $a, b, c \in \mathfrak{A}$. But since $\mathfrak{A}/\mathfrak{J}$ has been supposed to be commutative, we have $ac - ca \in \mathfrak{J}$ from what (1) results ∎

Lemma 5: *Define the map* $\varrho\colon \mathfrak{A}/\mathfrak{J} \to \operatorname{clos} U^2$ *by* $\varrho\colon ja \mapsto j_2(e \otimes a)$, $a \in \mathfrak{A}$. *Then*
(i) ϱ *is defined correctly,*
(ii) ϱ *is an algebraic homomorphism,*
(iii) ϱ *is continuous.*

Proof: (i) Let $ja = jb$. Then $a - b \in \mathfrak{J}$, hence $e \otimes (a - b) \in \mathfrak{A} \otimes \mathfrak{J}$, i.e. $j_2(e \otimes a) = j_2(e \otimes b)$.
(ii) is obvious.
(iii) We have

$$\|j_2(e \otimes a)\| = \inf \{\|e \otimes a + c\| : c \in \mathfrak{A} \otimes \mathfrak{J}\}$$
$$\leqq \inf \{\|e \otimes a + e \otimes k\| : k \in \mathfrak{J}\} = \inf \{\|a + k\| : k \in \mathfrak{J}\} = \|ja\| \ ∎$$

Thus $\operatorname{clos} U^2$ is a commutative Banach algebra with unit. Let M be the maximal ideal space of $\operatorname{clos} U^2$ and Γ_M the Gelfand map. Then

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{j} & \mathfrak{A}/\mathfrak{J} & \xrightarrow{\Gamma_N} & C(N) \\ & & \downarrow \varrho & & \\ \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{j_2} & \mathfrak{A}^2 \supset \operatorname{clos} U^2 & \xrightarrow{\Gamma_M} & C(M). \end{array}$$

Lemma 6: *If* $m \in M$ *then* $n = \varrho^{-1}(m) \underset{\text{def}}{=} \{a \in \mathfrak{A}/\mathfrak{J} : \varrho a \in m\}$ *belongs to* N. *If* φ_m *denotes the complex homomorphism on* $\operatorname{clos} U^2$ *associated with* m, *i.e.* $m = \operatorname{Ker} \varphi_m$, *then* $\varphi_m \circ \varrho$ *is a complex homomorphism on* $\mathfrak{A}/\mathfrak{J}$ *and* $\varrho^{-1}(m) = \operatorname{Ker} (\varphi_m \circ \varrho)$.

Remark: It is easily shown that, vice versa, for every $n \in N$ there exists an $m \in M$ such that $n = \varrho^{-1}(m)$, but this fact is not needed for our purposes.

Proof of Lemma 6: Let $m \in M$, $m = \operatorname{Ker} \varphi_m$, $\varphi = \varphi_m \circ \varrho$. By Lemma 5, φ is a continuous algebraic homomorphism of $\mathfrak{A}/\mathfrak{J}$ into $\mathbf{C}$. From

$$\varphi(je) = \varphi_m \varrho(je) = \varphi_m\big(j_2(e \otimes e)\big) = 1$$

we get $\varphi \neq 0$, i.e. $n = \operatorname{Ker} \varphi \in N$. The equality $n = \varrho^{-1}(m)$ follows from the equivalences

$$ja \in \varrho^{-1}(m) \Leftrightarrow \varrho ja \in m \Leftrightarrow \varphi_m \varrho ja = 0 \Leftrightarrow \varphi ja = 0 \Leftrightarrow ja \in \operatorname{Ker} \varphi = n \ ∎$$

Lemma 7: *Let* $a \in \mathfrak{A}$. *Then for every* $m \in M$

$$(\Gamma_M \varrho ja)(m) = (\Gamma_N ja)\big(\varrho^{-1}(m)\big).$$

Proof: If $m = \operatorname{Ker} \varphi_m$, then, by Lemma 6, $n = \varrho^{-1}(m) = \operatorname{Ker} (\varphi_m \circ \varrho) \in N$, hence

$$(\Gamma_M \varrho ja)(m) = \varphi_m \varrho ja = (\varphi_m \circ \varrho)\, ja = (\Gamma_N ja)(n) \ ∎$$

§ 7. The local principle of R. G. Douglas and N. Ya. Krupnik

Our proofs of the sufficiency of the conditions of the Theorems 1 and 2 are based upon the local principle of R. G. Douglas and N. Ya. Krupnik (see [3] for the case of C*-algebras and [5] for the case of Banach algebras). This local principle reads as follows:

Let $\mathfrak{C}$ *be the centre of a Banach algebra* $\mathfrak{A}$, $\mathfrak{A}_0$ *a closed subalgebra of* $\mathfrak{C}$ *and* $\mathfrak{R}$ *the* *maximal ideal space of* $\mathfrak{A}_0$. *For* $M \in \mathfrak{N}$ *we denote by* $\mathfrak{J}_M$ *the closed two-sided ideal generated by* M *in* $\mathfrak{A}$, *i.e.*

$$\mathfrak{J}_M = \operatorname*{clos}_{\mathfrak{A}} \{\textstyle\sum A_k X_k : A_k \in \mathfrak{A},\ X_k \in M\}.$$

Finally, let π_M *denote the canonical projection of* $\mathfrak{A}$ *onto* $\mathfrak{A}/\mathfrak{J}_M$. *Then for* $A \in \mathfrak{A}$

$$A \in G\mathfrak{A} \Leftrightarrow \pi_M A \in G(\mathfrak{A}/\mathfrak{J}_M)\ \forall\ M \in \mathfrak{N}.$$

§ 8. Sufficiency of the conditions of Theorem 1

Put $\mathfrak{B}_p{}^0 = \mathfrak{B}_p \otimes \mathfrak{B}_p/\mathfrak{K}_p \otimes \mathfrak{K}_p$, $\mathfrak{B}_p{}^1 = \mathfrak{B}_p \otimes \mathfrak{B}_p/\mathfrak{K}_p \otimes \mathfrak{B}_p$, $\mathfrak{B}_p{}^2 = \mathfrak{B}_p \otimes \mathfrak{B}_p/\mathfrak{B}_p \otimes \mathfrak{K}_p$ and denote by $\alpha_0, \alpha_1, \alpha_2$ the canonical projections of $\mathfrak{B}_p \otimes \mathfrak{B}_p$ onto $\mathfrak{B}_p{}^0$, $\mathfrak{B}_p{}^1$, $\mathfrak{B}_p{}^2$ respectively. For $a \in PC_p(\mathbf{T}^2)$ we have $W(a) \in \mathfrak{B}_p \otimes \mathfrak{B}_p$. In order to show that $W(a) \in \Phi(l^p \otimes l^p)$ it is sufficient to prove that $\alpha_0 W(a) \in G\mathfrak{B}_p{}^0$. On the other hand, $\alpha_0 W(a) \in G\mathfrak{B}_p{}^0$ follows from $\alpha_1 W(a) \in G\mathfrak{B}_p{}^1$ and $\alpha_2 W(a) \in G\mathfrak{B}_p{}^2$ (cf. [6]).

Let us, for example, prove that $\alpha_2 W(a) \in G\mathfrak{B}_p{}^2$. We set $\mathfrak{U}_p{}^2 = \{I\} \otimes \mathfrak{B}_p/\mathfrak{B}_p \otimes \mathfrak{K}_p$ and denote by clos $\mathfrak{U}_p{}^2$ the closure of $\mathfrak{U}_p{}^2$ in $\mathfrak{B}_p{}^2$. By Lemma 4 clos $\mathfrak{U}_p{}^2$ is contained in the centre of $\mathfrak{B}_p{}^2$ and is therefore a commutative Banach algebra. Let $\mathfrak{M}_p$ be the maximal ideal space of clos $\mathfrak{U}_p{}^2$ and $\Gamma_{\mathfrak{M}_p}$ the Gelfand map of clos $\mathfrak{U}_p{}^2$ into $C(\mathfrak{M}_p)$. Finally, we define the map $\gamma\colon \mathfrak{B}_p/\mathfrak{K}_p \to \operatorname{clos} \mathfrak{U}_p{}^2$ by $\gamma\colon \sigma_p A \mapsto \alpha_2(I \otimes A)$, $A \in \mathfrak{B}_p$, (cf. Lemma 5). Thus

$$\begin{array}{ccccc}
\mathfrak{B}_p & \xrightarrow{\sigma_p} & \mathfrak{B}_p/\mathfrak{K}_p & \xrightarrow{\Gamma_{\mathfrak{N}_p}} & C(\mathfrak{N}_p) \\
 & & \downarrow{\scriptstyle\gamma} & & \\
\mathfrak{B}_p \otimes \mathfrak{B}_p & \xrightarrow{\alpha_2} & \mathfrak{B}_p{}^2 \supset \operatorname{clos} \mathfrak{U}_p{}^2 & \xrightarrow{\Gamma_{\mathfrak{M}_p}} & C(\mathfrak{M}_p).
\end{array}$$

For $M \in \mathfrak{M}_p$ define J_M to be the closed two-sided ideal generated by M in $\mathfrak{B}_p{}^2$, i.e.

$$J_M = \operatorname*{clos}_{\mathfrak{B}_p{}^2} \{\textstyle\sum A_j X_j : A_j \in \mathfrak{B}_p{}^2,\ X_j \in M\}.$$

Let π_M denote the canonical projection of $\mathfrak{B}_p{}^2$ onto $\mathfrak{B}_p{}^2/J_M$. In Proposition 5 below we shall prove that

$$\alpha_2 W(a) - \alpha_2\big(T(a^1_{\zeta_0,\mu_0}) \otimes I\big) \in J_M,$$

where $(\zeta_0, \mu_0) \in \mathbf{T} \times [0, 1]$ has to be chosen in accordance with the identification of $\mathfrak{N}_p$ with $\mathbf{T} \times [0, 1]$ as that point on $\mathbf{T} \times [0, 1]$ which corresponds to $N = \gamma^{-1}(M)$ (cf. Lemma 6). Finally, in Proposition 6 it will be proved that

$$\pi_M \alpha_2\big(T(a^1_{\zeta_0,\mu_0}) \otimes I\big) \in G(\mathfrak{B}_p{}^2/J_M)$$

is a consequence of $T(a^1_{\zeta_0,\mu_0}) \in G\mathfrak{L}(l^p)$. Application of the local principle of R. G. Douglas and N. Ya. Krupnik (with $\mathfrak{A} = \mathfrak{B}_p{}^2$, and $\mathfrak{A}_0 = \operatorname{clos} \mathfrak{U}_p{}^2$) then gives $\alpha_2 W(a) \in G\mathfrak{B}_p{}^2$, if only $T(a^1_{\zeta_0,\mu_0}) \in G\mathfrak{L}(l^p)$ for all $(\zeta_0, \mu_0) \in \mathbf{T} \times [0, 1]$.

Proposition 5: *Let* $a \in PC_p(\mathbf{T}^2)$, $M \in \mathfrak{M}_p$ *and* $N = \gamma^{-1}(M) \in \mathfrak{N}_p$. *Let* (ζ_0, μ_0) *be the point on the cylinder* $\mathbf{T} \times [0, 1]$ *which corresponds to* $N \in \mathfrak{N}_p$ *via the homeomorphism* $\mathfrak{N}_p \cong \mathbf{T} \times [0, 1]$. *Then*

$$\alpha_2 W(a) - \alpha_2\big(T(a^1_{\zeta_0,\mu_0}) \otimes I\big) \in J_M. \qquad (1)$$

Proof: At first we consider the case that a is a finite sum of the form

$$a(\xi, \eta) = \sum_i b_i(\xi)\, c_i(\eta), \qquad (\xi, \eta) \in \mathbf{T}^2, \qquad (2)$$

where $b_i, c_i \in PC_p(\mathbf{T})$. Then $W(a) = \sum_i T(b_i) \otimes T(c_i)$. By formula (4.2)

$$a^1_{\zeta_0,\mu_0}(t) = \sum_i \left(\Gamma_{\mathfrak{N}_p} \sigma_p T(c_i)\right)(\zeta_0, \mu_0)\, b_i(t), \qquad t \in \mathbf{T}.$$

Thus (the subscript p will now be dropped for the sake of convenience)

$$\begin{aligned} &\alpha_2 W(a) - \alpha_2\left(T(a^1_{\zeta_0,\mu_0}) \otimes I\right) \\ &= \alpha_2 \sum_i T(b_i) \otimes T(c_i) - \alpha_2 \sum_i T(b_i) \otimes \left(\Gamma_{\mathfrak{N}} \sigma T(c_i)\right)(\zeta_0, \mu_0)\, I \\ &= \sum_i \alpha_2\left(T(b_i) \otimes I\right) \cdot \alpha_2\left(I \otimes \left[T(c_i) - (\Gamma_{\mathfrak{N}} \sigma T(c_i))(\zeta_0, \mu_0)\, I\right]\right). \end{aligned}$$

Because $\alpha_2\left(T(b_i) \otimes I\right) \in \mathfrak{B}_p{}^2$, it remains to show that

$$\alpha_2\left(I \otimes \left[T(c_i) - (\Gamma_{\mathfrak{N}} \sigma T(c_i))(\zeta_0, \mu_0)\, I\right]\right) \in M.$$

This on its hand is in view of $\alpha_2(I \otimes A) = \gamma \sigma A$ equivalent to

$$\gamma \sigma\left(T(c_i) - (\Gamma_{\mathfrak{N}} \sigma T(c_i))(\zeta_0, \mu_0)\, I\right) \in M,$$

i.e. to

$$\left(\Gamma_{\mathfrak{M}} \gamma \sigma T(c_i)\right)(M) - \left(\Gamma_{\mathfrak{M}} \gamma \sigma (\Gamma_{\mathfrak{N}} \sigma T(c_i))(N)\, I\right)(M) = 0$$

or to

$$\left(\Gamma_{\mathfrak{M}} \gamma \sigma T(c_i)\right)(M) = \left(\Gamma_{\mathfrak{N}} \sigma T(c_i)\right)(N).$$

But the latter equality immediately follows from Lemma 7. Thus, for functions of the form (2) the relation (1) has been proved.

For an arbitrary function $a \in PC_p(\mathbf{T}^2)$ we can find functions $a_j(\xi, \eta) = \sum_i b_i^{(j)}(\xi)\, c_i^{(j)}(\eta)$, $(\xi, \eta) \in \mathbf{T}^2$, of the form (2) such that

$$\|W(a) - W(a_j)\|_{\mathfrak{L}(l^p \otimes l^p)} \to 0 \qquad (j \to \infty). \tag{3}$$

From (3) we get

$$\|\alpha_2 W(a) - \alpha_2 W(a_j)\|_{\mathfrak{B}_p{}^2} \to 0 \qquad (j \to \infty)$$

and in order to prove (1) for the general case it remains to show that

$$\|T(a^1_{\zeta_0,\mu_0}) - T[(a_j)^1_{\zeta_0,\mu_0}]\|_{\mathfrak{B}_p} \to 0 \qquad (j \to \infty).$$

This follows by an argument used already in the proof of Lemma 3: with the help of Lemma 2 we can show that $\{T[(a_j)^1_{\zeta_0,\mu_0}]\}_{j=1}^\infty$ forms a Cauchy sequence in $\mathfrak{B}_p$ and then from (3) we can conclude that its limit is just $T(a^1_{\zeta_0,\mu_0})$ ∎

Proposition 6: *If* $T(a^1_{\zeta,\mu}) \in G\mathfrak{L}(l^p)$ *then*

$$\pi_M \alpha_2\left(T(a^1_{\zeta,\mu}) \otimes I\right) \in G(\mathfrak{B}_p{}^2/J_M).$$

Proof: First of all, we show that $T^{-1}(a^1_{\zeta,\mu})$ belongs not only to $\mathfrak{L}(l^p)$, but even to $\mathfrak{B}_p$. Indeed, the spectrum of $\sigma_p T(a^1_{\zeta,\mu}) \in \mathfrak{B}_p/\mathfrak{K}_p$ is in virtue of (2.1) just the curve

$$\left\{\left(1 - s_p(\mu)\right) a^1_{\zeta,\mu}(t - 0) + s_p(\mu)\, a^1_{\zeta,\mu}(t + 0) : t \in \mathbf{T}, \quad \mu \in [0, 1]\right\}.$$

Since $T(a^1_{\zeta,\mu})$ has been supposed to be invertible in $\mathfrak{L}(l^p)$, Theorem G of Section 2 shows that the origin cannot lie on this curve, i.e. $\sigma_p T(a^1_{\zeta,\mu}) \in G(\mathfrak{B}_p/\mathfrak{K}_p)$. Thus there is a regularisizer $R \in \mathfrak{B}_p$ (modulo $\mathfrak{K}_p$). From $T^{-1}(a^1_{\zeta,\mu}) - R \in \mathfrak{K}_p \subset \mathfrak{B}_p$ we get $T^{-1}(a^1_{\zeta,\mu}) \in \mathfrak{B}_p$. Now $\left(T^{-1}(a^1_{\zeta,\mu}) \otimes I\right)\left(T(a^1_{\zeta,\mu}) \otimes I\right) = I \otimes I$ implies

$$\pi_M\alpha_2(T^{-1}(a^1_{\zeta,\mu}) \otimes I) \cdot \pi_M\alpha_2\left(T(a^1_{\zeta,\mu}) \otimes I\right) = \pi_M\alpha_2(I \otimes I),$$

i.e. $\pi_M\alpha_2\left(T(a^1_{\zeta,\mu}) \otimes I\right) \in G(\mathfrak{B}_p{}^2/J_M)$ ∎

§ 9. Sufficiency of the conditions of Theorem 2

The Theorems Φ and G of Section 2 and a little geometrical consideration give the following result.

Lemma 8: *Let $a \in PC_p(\mathbf{T})$. If $T(a) \in G\mathfrak{L}(l^p)$ and $T(a) \in G\mathfrak{L}(l^q)$ then $T(a) \in G\mathfrak{L}(l^r)$ for every $r \in [p, q]$. Furthermore, $T(a) \in G\mathfrak{L}(l^p)$ if and only if $T(\tilde{a}) \in G\mathfrak{L}(l^q)$,* where *$1/p + 1/q = 1$ and $\tilde{a}(t) = a(1/t)$, $t \in \mathbf{T}$.*

Before proceeding to the subject of this section itself, it is necessary to prove still one auxiliary fact.

Proposition 7: *Let $a \in PC_p(\mathbf{T}^2)$ and suppose that $W(a)$, $W(a_1)$, $W(a_2)$, $W(a_{12}) \in \Phi(l^p \otimes l^p)$. Then $T(a^1_{v,\zeta,\mu}) \in G\mathfrak{L}(l^r)$ and $T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^r)$ for every $r, v \in [p, q]$ $(1/p + 1/q = 1)$, $\zeta \in \mathbf{T}$ and $\mu \in [0, 1]$.*

Proof: In accordance with Theorem 1, from $W(a)$, $W(a_1) \in \Phi(l^p \otimes l^p)$ we get $T(a^1_{p,\zeta,\mu})$, $T\left((a_1)^1_{p,\zeta,\mu}\right) \in G\mathfrak{L}(l^p)$. Since $(a_1)^1_{p,\zeta,\mu} = (a^1_{p,\zeta,\mu})^\sim$ (recall the notation $\tilde{a}(t) = a(1/t)$, $t \in \mathbf{T}$), we get $T(a^1_{p,\zeta,\mu})$, $T\left((a^1_{p,\zeta,\mu})^\sim\right) \in G\mathfrak{L}(l^p)$. Applying Lemma 8, what results is

$$T(a^1_{p,\zeta,\mu}) \in G\mathfrak{L}(l^v) \quad \forall\, v \in [p, q] \quad \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{1}$$

Furthermore, $W(a_2) \in \Phi(l^p \otimes l^p)$ implies by Theorem 1 $T\left((a_2)^1_{p,\zeta,\mu}\right) \in G\mathfrak{L}(l^p)$ for every $(\zeta, \mu) \in \mathbf{T} \times [0, 1]$ and from $(a_2)^1_{p,\zeta,\mu} = a^1_{q,\zeta,1-\mu}$ we obtain

$$T(a^1_{q,\zeta,\mu}) \in G\mathfrak{L}(l^p) \quad \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{2}$$

Due to $W(a_{12}) \in \Phi(l^p \otimes l^p)$ we have, again by Theorem 1, $T\left((a_{12})^1_{p,\zeta,\mu}\right) \in G\mathfrak{L}(l^p)$. But $(a_{12})^1_{p,\zeta,\mu} = (a^1_{p,\zeta,1-\mu})^\sim$, thus by Lemma 8

$$T(a^1_{q,\zeta,\mu}) \in G\mathfrak{L}(l^q) \quad \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{3}$$

From (2), (3), and Lemma 8 we get

$$T(a^1_{q,\zeta,\mu}) \in G\mathfrak{L}(l^v) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{4}$$

Analogously one can prove that

$$T(a^2_{p,\zeta,\mu}) \in G\mathfrak{L}(l^v) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1], \tag{5}$$

$$T(a^2_{q,\zeta,\mu}) \in G\mathfrak{L}(l^v) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{6}$$

Now, recalling (3.4), it is easy to show that the following equivalence holds

$$T(a^1_{p,\zeta,\mu}) \in G\mathfrak{L}(l^v) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]$$
$$\Leftrightarrow T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^p) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1].$$

Thus from (1), we get

$$T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^p) \quad \forall\, v \in [p, q]\, \forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1] \tag{7}$$

and from (4)

$$T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^q) \quad \forall\, v \in [p, q] \,\forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1]. \tag{8}$$

Combining (7), (8), and Lemma 8 we arrive at

$$T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^r) \quad \forall\, r, v \in [p, q] \,\forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1].$$

Similarly, (5) and (6) give

$$T(a^2_{v,\zeta,\mu}) \in G\mathfrak{L}(l^r) \quad \forall\, r, v \in [p, q] \,\forall\, (\zeta, \mu) \in \mathbf{T} \times [0, 1] \;\blacksquare$$

Now we are in position to proceed in analogy to the Fredholm theory considered in the preceding section. Put

$$\mathbf{A}_p^0 = \mathbf{A}_p \otimes \mathbf{A}_p/\mathbf{I}_p \otimes \mathbf{I}_p, \qquad \mathbf{A}_p^1 = \mathbf{A}_p \otimes \mathbf{A}_p/\mathbf{I}_p \otimes \mathbf{A}_p,$$
$$\mathbf{A}_p^2 = \mathbf{A}_p \otimes \mathbf{A}_p/\mathbf{A}_p \otimes \mathbf{I}_p$$

and denote the corresponding canonical projections by β_0, β_1, β_2, respectively. Note that $\{W_n(a)\} \in \mathbf{A}_p \otimes \mathbf{A}_p$ for $a \in PC_p(\mathbf{T}^2)$ ($W_n(a)$ are the "finite sections" of $W(a)$ defined by (3.5)). In the same way as in [2] we can prove that $W(a) \in \Pi_p\{P_n \otimes P_n\}$ if only $\beta_0\{W_n(a)\} \in G\mathbf{A}_p^0$ and $W(a)$, $W(a_1)$, $W(a_2)$, $W(a_{12}) \in G\mathfrak{L}(l^p \otimes l^p)$. The usual standard trick [6] may be applied to derive $\beta_0\{W_n(a)\} \in G\mathbf{A}_p^0$ from $\beta_1\{W_n(a)\} \in G\mathbf{A}_p^1$ and $\beta_2\{W_n(a)\} \in G\mathbf{A}_p^2$.

We shall prove that $\beta_2\{W_n(a)\} \in G\mathbf{A}_p^2$ by means of the local principle of R. G. Douglas and N. Ya. Krupnik. Lemma 4 yields that the closure clos $\mathbf{U}_p^2$ of $\mathbf{U}_p^2 \underset{\text{def.}}{=} \{\{P_n\}\} \otimes \mathbf{A}_p/\mathbf{A}_p \otimes \mathbf{I}_p$ in $\mathbf{A}_p^2$ is contained in the centre of $\mathbf{A}_p^2$ and is therefore a commutative Banach algebra. Let $\mathbf{M}_p$ denote the maximal ideal space of clos $\mathbf{U}_p^2$, $\Gamma_{\mathbf{M}_p}$ the Gelfand map and δ the map of $\mathbf{A}_p/\mathbf{I}_p$ into clos $\mathbf{U}_p^2$ defined by $\delta: \tau_p\{A_n\} \mapsto \beta_2(\{P_n \otimes A_n\})$ (cf. Lemma 5). Thus

$$\begin{array}{ccccc} \mathbf{A}_p & \xrightarrow{\tau_p} & \mathbf{A}_p/\mathbf{I}_p & \xrightarrow{\Gamma_{\mathbf{N}_p}} & C(\mathbf{N}_p) \\ & & \downarrow{\scriptstyle\delta} & & \\ \mathbf{A}_p \otimes \mathbf{A}_p & \xrightarrow{\beta_2} & \mathbf{A}_p^2 \supset \text{clos}\, \mathbf{U}_p^2 & \xrightarrow{\Gamma_{\mathbf{M}_p}} & C(\mathbf{M}_p). \end{array}$$

For $M \in \mathbf{M}_p$ let

$$J_M = \operatorname*{clos}_{\mathbf{A}_p^2} \{\textstyle\sum A_j X_j : A_j \in \mathbf{A}_p^2, X_j \in M\}.$$

Then J_M is a closed two-sided ideal in $\mathbf{A}_p^2$. We denote by π_M the canonical projection of $\mathbf{A}_p^2$ onto $\mathbf{A}_p^2/J_M$. For $M \in \mathbf{M}_p$ we have, by Lemma 6, $N = \delta^{-1}(M) \in \mathbf{N}_p$. Due to Proposition 4 there exist $r \in [p, q]$ $(1/p + 1/q = 1)$, $\mu \in [0, 1]$ and $\zeta \in \mathbf{T}$ such that

$$\left(\Gamma_{\mathbf{N}_p}\tau_p\{T_n(a)\}\right)(N) = \left(1 - s_r(\mu)\right) a(\zeta - 0) + s_r(\mu)\, a(\zeta + 0)$$

for every $a \in PC_p(\mathbf{T})$. Now in the same way as Proposition 5 was proved, one may show that

$$\beta_2\{W_n(a)\} - \beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\} \in J_M.$$

Combining the just proved Proposition 7 with Proposition 8 proved below, we obtain that under the conditions of Theorem 2

$$\pi_M\beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\} \in G(\mathbf{A}_p^2/J_M)$$

for every $r \in [p, q]$, $\zeta \in \mathbf{T}$, $\mu \in [0, 1]$, $M \in \mathbf{M}_p$. Applying the local principle quoted in Section 7 (with $\mathfrak{A} = \mathbf{A}_p^2$, $\mathfrak{A}_0 = \text{clos}\, \mathbf{U}_p^2$) we get $\beta_2\{W_n(a)\} \in G\mathbf{A}_p^2$.

Proposition 8: *Let $a \in PC_p(\mathbf{T}^2)$, $M \in \mathbf{M}_p$ and $(r; \zeta, \mu)$ be the triplet corresponding to M by Proposition 4. If $T(a^1_{r,\zeta,\mu}) \in G\mathfrak{L}(l^v)$ for every $v \in [p, q]$ $(1/p + 1/q = 1)$ then*

$$\pi_M \beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\} \in G(\mathbf{A}_p{}^2/J_M).$$

Proof: By Proposition 3 we have

$$\tau_p\{T_n(a^1_{r,\zeta,\mu})\} \in G(\mathbf{A}_p/\mathbf{I}_p). \tag{9}$$

Furthermore, $T(a^1_{r,\zeta,\mu}) \in G\mathfrak{L}(l^v)$ for every $v \in [p, q]$ and Lemma 8 gives that

$$T(a^1_{r,\zeta,\mu}),\ T\big((a^1_{r,\zeta,\mu})^\sim\big) \in G\mathfrak{L}(l^p). \tag{10}$$

From (9) and (10) we may with the method of the proof of Satz 3 in [8] derive that there exists a $\{R_n'\} \in \mathbf{A}_p$ such that $R_n' T_n(a^1_{r,\zeta,\mu}) = P_n + C_n'$, where $\{C_n'\} \in \mathbf{A}_p$ and $\|C_n'\| \to 0$ $(n \to \infty)$.
Thus

$$\begin{aligned} &\beta_2\{R_n' \otimes P_n\} \cdot \beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\} \\ &= \beta_2\{(P_n + C_n') \otimes P_n\} = \beta_2\{P_n \otimes P_n\} + \beta_2\{C_n' \otimes P_n\}. \end{aligned} \tag{11}$$

In case $\|C_n'\| > 0$ we have $C_n' \otimes P_n = C_n' \|C_n'\|^{-1/2} \otimes \|C_n'\|^{1/2} P_n$. Put

$$D_n = \begin{cases} 0 & \|C_n'\| = 0 \\ C_n' \|C_n'\|^{-1/2}, & \|C_n'\| > 0, \end{cases}$$

$$E_n = \begin{cases} 0, & \|C_n'\| = 0 \\ \|C_n'\|^{1/2} P_n, & \|C_n'\| > 0. \end{cases}$$

Similarly as in [2] the inclusion $\mathbf{G}_p \subset \mathbf{A}_p$ can be proved. Consequently, $\{D_n \otimes E_n\} \in \mathbf{A}_p \otimes \mathbf{A}_p$. Since, moreover, $\{D_n \otimes E_n\} \in \mathbf{I}_p \otimes \mathbf{I}_p \subset \mathbf{A}_p \otimes \mathbf{I}_p$ and $C_n' \otimes P_n = D_n \otimes E_n$, we get $\beta_2\{C_n' \otimes P_n\} = \beta_2\{D_n \otimes E_n\} = 0$. Now (11) gives the invertibility of $\beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\}$ in $\mathbf{A}_p{}^2$, implying, of course, the invertibility of $\pi_M \beta_2\{T_n(a^1_{r,\zeta,\mu}) \otimes P_n\}$ in $\mathbf{A}_p{}^2/\mathfrak{J}_M$ ∎

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