

## Some properties of a new kind of modulus of smoothness

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### Der Glattheitsmodul

$$\omega(f, \delta)_{\varphi, p} = \sup_{0 < h \leq \delta} \|A_{h\varphi}^2\|_{L^p}$$

tritt bei der Untersuchung positiver Operatoren vom Kantorovitsch'schen Typ auf. Wir zeigen, daß  $\omega_{\varphi, p}$  ähnlich dem gewöhnlichen Fall  $\varphi \equiv 1$  ist und charakterisieren jene Funktionen, für die  $\omega(f, \delta)_{\varphi, p} = O(\delta^2)$  gilt. Die erhaltenen Resultate finden Anwendungen in der Theorie der positiven Operatoren.

### Модуль гладкости

$$\omega(f, \delta)_{\varphi, p} = \sup_{0 < h \leq \delta} \|A_{h\varphi}^2\|_{L^p}$$

появился при исследовании положительных операторов типа Канторовича. Мы покажем, что  $\omega(f, \delta)_{\varphi, p}$  похож на обычный модуль с  $\varphi \equiv 1$ , и даем характеризацию тех функций, для которых  $\omega(f, \delta)_{\varphi, p} = O(\delta^2)$ . Полученные результаты имеют приложения в теории положительных операторов.

### The modulus of smoothness

$$\omega(f, \delta)_{\varphi, p} = \sup_{0 < h \leq \delta} \|A_{h\varphi}^2\|_{L^p}$$

has arisen during the investigation of positive operators of the Kantorovich type. Here we show that  $\omega_{\varphi, p}$  resembles the ordinary case  $\varphi \equiv 1$  and we give the characterization of those functions  $f$  for which  $\omega(f, \delta)_{\varphi, p} = O(\delta^2)$ . The results obtained have applications to positive operators.

In connection with approximation by positive operators a new kind of modulus of smoothness has arisen and it has turned out that this new modulus has an intimate connection with the rate of approximation, and that it cannot be replaced by previously used, "ordinary" measurements of smoothness (see [5, 6, 7, 8]). In this paper we investigate the basic properties of this new modulus of smoothness and apply the results to approximation by Kantorovich type operators.

Let  $(a, b)$  be a (finite or infinite) interval, and  $\varphi$  a twice continuously differentiable positive function on  $(a, b)$  with the properties:

- (i)  $\varphi$  is convex or concave to the right (left) of  $a$  ( $b$ ).
- (ii) There is a  $h_0 > 0$  such that  $x \pm h_0\varphi(x) \in (a, b)$  for every  $x \in (a, b)$  provided  $(a, b) \neq (-\infty, \infty)$ , and if  $(a, b) = (-\infty, \infty)$  then we require  $\varphi(x) = O(|x|)$  ( $|x| \rightarrow \infty$ ).
- (iii) Let  $d(x)$  denote the distance of  $x \in (a, b)$  from the nearest endpoint of  $(a, b)$  when  $(a, b) \neq (-\infty, \infty)$ , and let  $d(x) = |x| + 1$  when  $(a, b) = (-\infty, \infty)$ . Our final

assumption is that there is a  $K$  for which

$$|\varphi'(x)| \leq K \frac{\varphi(x)}{d(x)}, \quad |\varphi''(x)| \leq K \frac{\varphi(x)}{d^2(x)} \quad (x \in (a, b))$$

$$\frac{1}{K} \varphi(x) \leq \varphi(y) \leq K\varphi(x) \quad \text{for } y \in \left(x - \frac{d(x)}{2}, x + \frac{d(x)}{2}\right)$$

are satisfied.

It is clear that both  $x + h\varphi(x)$  and  $x - h\varphi(x)$  belong to  $(a, b)$  when  $x \in (a, b)$  and  $0 \leq h \leq h_0$ ; furthermore,  $\varphi(x)$  tends to zero as  $x$  approaches a finite endpoint — if there is any — of  $(a, b)$ .

Let further  $B$  be any of the Banach spaces  $L^p(a, b)$  ( $1 \leq p < \infty$ ),  $C[a, b]$ ,  $C(a, b)$ ,  $C[a, b]$  or  $C(a, b)$  with the corresponding  $L^p$  or supremum norm  $\|\cdot\|_B$ . For the sake of simplicity any of the last four spaces will be denoted by  $C[a, b]$ . The modulus of smoothness for  $f \in B$  mentioned is defined by

$$\omega(f, \delta)_{\varphi, B} = \omega(\delta) = \sup_{0 < h \leq \min(h_0, \delta)} \|\Delta_{h\varphi}^2 f\|_B$$

where

$$\Delta_{h\varphi}^2 f(x) = f(x - h) - 2f(x) + f(x + h)$$

is the usual symmetrical second difference. The important point in the definition of  $\omega(\delta)$  is that in  $\Delta_{h\varphi}^2 f(x)$  the distance  $h\varphi(x)$  varies together with  $x$ . One result is that even the inequality  $\omega(2\delta) \leq K\omega(\delta)$  becomes non-trivial.

We list some properties of  $\omega$  in

**Theorem 1:** (i) *Defining the so called  $K$ -functional by*

$$K(t^2, f) = \inf_{\substack{g \in B \\ g, g' \text{ absolutely continuous}}} (\|f - g\|_B + t^2 \|\varphi^2 g''\|_B),$$

*then there is a constant  $K_1$ , independent of  $f$  and  $0 \leq t \leq h_0$ , such that there holds the inequality*

$$\frac{1}{K_1} \omega(f, t) \leq K(t^2, f) \leq K_1 \omega(f, t).$$

(ii) *There is a constant  $K$  for which*

$$\omega(f, \lambda\delta) \leq K\lambda^2\omega(\delta)$$

*is satisfied for all  $f \in B$ ,  $\lambda \geq 1$  and  $\delta > 0$ .*

(iii) *If  $\varphi_1$  and  $\varphi_2$  are two functions with the above properties, then  $\varphi_1 \leq K\varphi_2$  implies  $\omega(f, \delta)_{\varphi_1, B} \leq K_1\omega(f, \delta)_{\varphi_2, B}$  with a  $K_1$  independent of  $f \in B$  and  $\delta > 0$ ,*

(iv) *If  $\omega(f, \delta_n) = o(\delta_n^2)$  for a sequence  $\delta_n \rightarrow 0$  then  $f$  is linear.*

**Proof:** (i) was proved in [8; Theorem 1], and (ii), (iii) are obvious consequences of (i). Finally, to prove (iv) we can argue as follows. Let  $[a_1, b_1] \subseteq (a, b)$  be any finite interval. Then there is a  $\varphi_1$  such that  $\varphi_1(x) = 1$  for  $x \in (a_1, b_1)$ , and  $\varphi_1(x) \leq K\varphi(x)$  for some  $K$  and all  $x \in (a, b)$ . If  $\omega(f, \delta_n)_{\varphi, B} = o(\delta_n^2)$ , then the comparison assertion given in (iii) yields that  $\omega(f, \delta_n)_{\varphi_1, B} = o(\delta_n^2)$  is also satisfied and thus, since  $\varphi_1(x) = 1$  on  $(a_1, b_1)$ , we obtain that the ordinary modulus of smoothness of  $f$  on  $(a_1, b_1)$  is  $o(\delta_n^2)$  as  $n \rightarrow \infty$ ; it is well known that this implies the linearity of  $f$  on  $(a_1, b_1)$ . Since  $[a_1, b_1] \subseteq (a, b)$  was arbitrary, (iv) follows ■

By (iv)  $\omega(\delta)$  cannot tend to zero faster than  $\delta^2$ . Our next concern is the characterization of those functions  $f$  for which  $\omega(\delta)$  has order  $O(\delta^2)$ .

**Theorem 2:** *Let  $f \in B$ .*

(i) *If  $B = C\{a, b\}$  then*

$$\omega(f, \delta)_{\varphi, B} = O(\delta^2) \quad (\delta \rightarrow 0) \tag{1}$$

*if and only if  $f$  has an absolutely continuous derivative on  $(a, b)$  and  $\varphi^2 f''$  is bounded.*

(ii) *If  $B = L^p(a, b)$ ,  $1 < p < \infty$ , then (1) holds if and only if  $f$  has an absolutely continuous derivative on  $(a, b)$  with  $\varphi^2 f'' \in L^p(a, b)$ .*

(iii) *For  $B = L^1(a, b)$  (1) holds if and only if  $f$  is absolutely continuous and  $\varphi^2 f'$  coincides a.e. with a function of bounded variation on  $(a, b)$ .*

Naturally (ii) and (iii) mean that  $f$  coincides almost everywhere with a function having the stated properties.

It will be important to extend Theorem 2 to a larger class of the  $\varphi$ 's, namely, assume that  $(a, b) \neq (-\infty, \infty)$  and all of our assumptions on  $\varphi$  are satisfied except (ii), and instead of (ii) we require the following:

(ii)' a)  $\varphi(x)$  tends to zero as  $x$  tends to a finite endpoint of  $(a, b)$ .

b) If  $a(b)$  is finite, then there is a  $\gamma < 1$  such that  $\varphi(x)/(x - a)^\gamma$  ( $\varphi(x)/(b - x)^\gamma$ ) decreases (increases) as  $x$  increases in a neighbourhood of  $a$  ( $b$ ).

c) If  $(a, b)$  is infinite, then  $\varphi(x)/|x|$  remains bounded as  $|x| \rightarrow \infty$ .

Roughly speaking, (ii)' allows  $\varphi$  to tend more slowly to zero at finite endpoints than (ii) did; at infinite endpoints there is no difference between (ii) and (ii)'. E.g.  $\varphi(x) = \sqrt{x(1-x)}$  satisfies (ii)' but does not satisfy (ii) on  $(0, 1)$ . Clearly, we cannot assure any more that the numbers  $x \pm h\varphi(x)$  will belong to  $(a, b)$  together with  $x$ ; so the definition of  $\omega$  needs some correction. In any case the fact that  $\varphi$  is convex or concave around the endpoints yields that for sufficiently small positive  $h$  there are a smallest and a largest (possibly infinite) number, say  $h^*$  and  $h^{**}$ , with  $x \pm h\varphi(x) \in (a, b)$  for  $h^* < x < h^{**}$  (see also [8]). Now for small  $\delta$  let

$$\omega(f, \delta)_{\varphi, B} = \sup_{0 \leq h \leq \delta} \|\Delta_{h\varphi}^2 f\|_{B_{(h^*, h^{**})}}, \tag{2}$$

where  $B_{(h^*, h^{**})}$  is the restriction of  $B$  to  $(h^*, h^{**})$ , i.e.,

$$\|g\|_{B_{(h^*, h^{**})}} = \sup_{x \in (h^*, h^{**})} |g(x)|, \quad \text{when } B = C\{a, b\}$$

and

$$\|g\|_{B_{(h^*, h^{**})}} = \|g\|_{L^p(h^*, h^{**})}, \quad \text{when } B = L^p(a, b).$$

**Theorem 3:** *If  $B = C\{a, b\}$  or  $B = L^p(a, b)$  then the assertions of Theorem 2 remain valid for the  $\omega$  defined in (2). In the case  $B = L^1(a, b)$ , however,  $\omega(f, \delta)_{\varphi, B} = O(\delta^2)$  ( $\varphi$  being as in (2)) if and only if  $f$  is (a.e.) equal to the indefinite integral of a function  $v$  which is of bounded variation on compact subintervals of  $(a, b)$  and for which*

$$\int_a^b \varphi^2(\tau) d|v(\tau)| < \infty.$$

Here  $|v|$  denotes the total variation of  $v$ .

We mention that condition (ii) of Theorem 2 does not hold in general for (2). E.g. if  $(a, b) = (0, 1)$ ,  $\varphi(x) = \sqrt{x(1-x)}$  and  $f(x) = \log x$ , then  $\varphi^2 f'$  is of bounded vari-

ation on  $(0, 1)$ , but for small  $h$

$$\| \Delta_{h\varphi}^2 f \|_{L^1(h^*, 1-h^*)} \geq c_1 h^2 \| \varphi^2 f'' \|_{L^1(2h^*, 1/2)} \geq c_1 h^2 \log \frac{1}{h},$$

i.e.,  $\omega(f, \delta)_{\varphi, L^1} \neq O(\delta^2)$ .

Remark: If  $a$  is finite, then we could define, instead of (2)

$$\omega(f, \delta)_{\varphi, B} = \sup_{h \leq \delta} \| \Delta_{h\varphi}^2 f \|_{B(h^* + C(h^* - a), h^{**})} \tag{2}$$

$C > 0$  being fixed. Theorem 3 would then hold just as well. The same applies for finite  $b$  (see [8: Remark 1]).

The remark justifies our right to write convenient bounds in the norms (see also [8: Remark 1]). E.g., if  $(a, b) = (0, 1)$ ,  $B = L^p(0, 1)$  and  $\varphi(x) = \sqrt{x(1-x)}$  then  $h^* = h^2/(1+h^2)$ , and  $h^{**} = 1 - h^2/(1+h^2)$ , so that (2) defines

$$\omega(f, \delta) = \sup_{h \leq \delta} \| \Delta_{h\varphi}^2 f \|_{L^p\left(\frac{h^2}{1+h^2}, 1 - \frac{h^2}{1+h^2}\right)},$$

but the remark permits us to put

$$\omega(f, \delta) = \sup_{h \leq \delta} \| \Delta_{h\varphi}^2 f \|_{L^p(h^*, 1-h^*)},$$

and for this last  $\omega$  the conclusion of Theorem 3 remains valid.

Before proving Theorems 2 and 3 we give an application. Let

$$K_n f(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \int_{k/(n+1)}^{(k+1)/(n+1)} f(u) du$$

be the so called Kantorovitch polynomials associated with  $f \in L^p(0, 1)$ ,  $1 \leq p < \infty$ . These converge to  $f$  in  $L^p$ -norm and it was proved by MAIER [2] that for  $p = 1$   $\{K_n\}$  is saturated with order  $O\left(\frac{1}{n}\right)$ , and saturation class

$$\{f \mid f \text{ abs. cont., } x(1-x)f'(x) \text{ is of bounded variation on } (0, 1)\}.$$

For  $p > 1$  the saturation order is again  $O\left(\frac{1}{n}\right)$  but the saturation class is then (see [3, 4])

$$\{f \mid f' \text{ abs. cont., } (x(1-x)f'(x)) \in L^p(0, 1)\},$$

and it was proved in [5] that this coincides with

$$\{f \mid f' \text{ abs. cont., } x(1-x)f''(x) \in L^p(0, 1)\}.$$

On the other hand, for  $0 < \alpha < 1$  the non-optimal order of approximation

$$\|K_n f - f\|_{L^p(0,1)} = O(n^{-\alpha}) \tag{3}$$

is characterized for every  $1 \leq p < \infty$  by the Lipschitz condition

$$\| \Delta_{h\varphi}^2 f \|_{L^p(h^*, 1-h^*)} = O(h^{2\alpha}) \quad (\varphi(x) = \sqrt{x(1-x)}) \tag{4}$$

(see [5, 6, 8]). Now an application of Theorem 3 yields

Theorem 4: For all  $1 \leq p < \infty$  and  $0 < \alpha \leq 1$ , except for the case  $p = \alpha = 1$ , conditions (3) and (4) are equivalent.

If  $p = \alpha = 1$ , then (3) and (4) are not equivalent, e.g., for  $f(x) = \log x$  we have by Maier's result (3) but as we have already mentioned (4) is not satisfied. This raises the following problem:

**Problem:** Find a smoothness condition (similar to (4)) which is equivalent to (3) for all  $1 \leq p < \infty$  and  $0 < \alpha \leq 1$ !

In the same way it can be proved that if

$$S_n^* f(x) = \sum_{k=0}^{\infty} \binom{(k+1)/n}{k/n} \left( n \int_{k/n}^{(k+1)/n} f(u) du \right) e^{-nx} \frac{(nx)^k}{k!} \quad (x \geq 0)$$

and

$$V_n^* f(x) = \sum_{k=0}^{\infty} \binom{(k+1)/n}{k/n} \binom{n+k-1}{k} x^k (1+x)^{-n-k} \quad (x \geq 0)$$

are the modified Szász-Mirakjan and Baskakov operators (see [8]), respectively, then for every  $1 \leq p < \infty$ ,  $f \in L^p(0, \infty)$  and  $0 < \alpha \leq 1$ , except for the case  $p = \alpha = 1$ , the conditions

$$\|S_n^* f - f\|_{L^p(0, \infty)} = O(n^{-\alpha}), \quad \|V_n^* f - f\|_{L^p(0, \infty)} = O(n^{-\alpha})$$

and

$$\|\Delta_{h\varphi}^2 f\|_{L^p(2h^2, \infty)} = O(h^{2\alpha}) \quad \left( 0 \leq h \leq \frac{1}{2} \right)$$

are equivalent where  $\varphi(x) = \sqrt{x}$  for  $S_n^*$ , and  $\varphi(x) = \sqrt{x(1+x)}$  for  $V_n^*$ , respectively.

**Proof of Theorem 2: Necessity.** Let  $f \in \dot{B}$  and  $\omega(f, \delta)_{\varphi, B} = O(\delta^2)$ .

(i) *The case  $B = C[a, b]$ .* Exactly as in the proof of Theorem 1 (iv) it can be proved that if  $[a', b'] \subseteq (a, b)$  and

$$v(f, \delta) = \sup_{\substack{0 \leq h \leq \delta \\ x \in [a', b']}} |\Delta_h^2 f(x)|$$

is the ordinary modulus of smoothness of  $f$  on  $[a', b']$ , then  $v(f, \delta) = O(\delta^2)$  ( $\delta \rightarrow 0$ ); it is well known that this implies the absolute continuity of  $f'$  on  $(a', b')$  (see e.g. [1: p. 5.]). Since  $a', b' \in (a, b)$  were arbitrary, we can conclude that  $f'$  is (locally) absolutely continuous on  $(a, b)$ . But then

$$\lim_{h \rightarrow 0} \frac{1}{h^2} \Delta_{h\varphi(x)} f(x) = \varphi^2(x) f''(x)$$

almost everywhere, and so the boundedness of  $\frac{1}{h^2} \Delta_{h\varphi(x)}^2 f(x)$  implies that of  $\varphi^2 f''$ .

(ii) *The case  $B = L^p(a, b)$ ,  $1 < p < \infty$ .* By Theorem 1 (i)  $K(t^2, f) = O(t^2)$ , and hence there is a sequence  $\{g_n\}$  such that the derivative of each  $g_n$  is absolutely continuous, and

$$\|f - g_n\|_{L^p(a, b)} = o(1), \quad \|\varphi^2 g_n''\|_{L^p(a, b)} = O(1).$$

By weak compactness there is a subsequence  $\{g_{n_k}\}$  and a function  $g \in L^p(a, b)$  such that  $\varphi^2 g_{n_k}$  converges weakly to  $g$ . Let  $\xi \in (a, b)$  be fixed. For every  $x \in (a, b)$

$$\lim_{k \rightarrow \infty} \int_{\xi}^x (x - \tau) g_{n_k}''(\tau) d\tau = \int_{\xi}^x \frac{x - \tau}{\varphi^2(\tau)} g(\tau) d\tau,$$

and since the left hand side is equal to

$$\lim_{k \rightarrow \infty} (g_{n_k}(x) - g_{n_k}(\xi) - g'_{n_k}(\xi)(x - \xi)) = f(x) - \lim_{k \rightarrow \infty} (g_{n_k}(\xi) + g'_{n_k}(\xi)(x - \xi))$$

a.e., we can infer that  $f$  has the form

$$f(x) = cx + d + \int_{\xi}^x \frac{x - \tau}{\varphi^2(\tau)} g(\tau) d\tau = cx + d + \int_{\xi}^x \int_{\xi}^u \frac{g(\tau)}{\varphi^2(\tau)} d\tau du. \quad (\text{a.e.})$$

Thus  $f$  coincides a.e. with a function having an absolutely continuous derivative; furthermore,  $\varphi^2(x) f''(x) = g(x)$  a.e., i.e.,  $\varphi^2 f'' \in L^p(a, b)$ .

(iii) *The case  $B = L^1(a, b)$ .* Just as above there is a sequence  $\{g_n\}$  and a finite Borel measure  $d\mu$  on  $(a, b)$  such that  $\{g_n\}$  converges in  $L^1(a, b)$  to  $f$ , and  $\varphi^2 g_n''$  converges weakly to  $d\mu$ . Let  $\xi \in (a, b)$  be a continuity point of  $\mu$ , and with a fixed  $\xi_1 \in (a, b)$ ,  $\xi_1 < \xi$ , set for  $x \geq \xi$

$$h_x(t) = \begin{cases} \frac{x - \xi}{\xi - \xi_1} (t - \xi_1) & \text{for } \xi_1 \leq t \leq \xi \\ x - t & \text{for } \xi < t \leq x \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $h_x(t) \in C[a, b]$ , and so

$$\lim_{n \rightarrow \infty} \int_a^b h_x(t) g_n''(t) dt = \int_a^b \frac{h_x(t)}{\varphi^2(t)} d\mu(t),$$

i.e.,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[ g_n(x) - g_n(\xi) - g'_n(\xi)(x - \xi) - \frac{x - \xi}{\xi - \xi_1} (g_n(\xi_1) - g_n(\xi) - g'_n(\xi)(\xi - \xi_1)) \right] \\ &= \frac{x - \xi}{\xi - \xi_1} \int_{\xi_1}^{\xi} \frac{\tau - \xi_1}{\varphi^2(\tau)} d\mu(\tau) + \int_{\xi}^x \frac{x - \tau}{\varphi^2(\tau)} d\mu(\tau). \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} g_n(x) = f(x)$  a.e., we can infer that, for  $x \geq \xi$ ,  $f$  is of the form

$$f(x) = cx + d + \int_{\xi}^x \frac{x - \tau}{\varphi^2(\tau)} d\mu(\tau) = cx + d + \int_{\xi}^x \int_{\xi}^y \frac{d\mu(\tau)}{\varphi^2(\tau)} dy. \quad (\text{a.e.})$$

It follows readily that we may consider  $f$  to be absolutely continuous, that

$$f'(x) = c + \int_{\xi}^x \frac{d\mu(\tau)}{\varphi^2(\tau)} \quad (\text{a.e.}), \tag{5}$$

and that  $\varphi^2 f'$  agrees a.e. with a function having bounded variation on every finite interval  $[\xi, b']$ ,  $b' < b$ . In the following we identify  $f'$  with the right side of (5). Let  $b'$  be so large that  $\varphi$  is monotone on  $(b', b)$ , and we distinguish two cases according to whether  $\varphi$  decreases or increases on  $(b', b)$ .

a)  $\varphi$  is decreasing on  $(b', b)$ . Let  $x_0 < x_1 < \dots < x_{n+1} < b$ ,  $x_0 = b'$  be arbitrary points. Denoting the total variation of  $\mu$  by  $|\mu|$  we can write

$$\begin{aligned} & \sum_{i=0}^n |\varphi^2(x_{i+1}) f'(x_{i+1}) - \varphi^2(x_i) f'(x_i)| \\ & \leq |C| \sum_{i=0}^n (\varphi^2(x_i) - \varphi^2(x_{i+1})) + \sum_{i=0}^n \varphi^2(x_{i+1}) \int_{x_i}^{x_{i+1}} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \\ & \quad + \sum_{i=0}^n (\varphi^2(x_i) - \varphi^2(x_{i+1})) \int_{\xi}^{x_i} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \\ & \leq |C| \varphi^2(b') + \sum_{i=0}^n \int_{x_i}^{x_{i+1}} d|\mu(\tau)| + \sum_{i=1}^n \varphi^2(x_i) \left( \int_{\xi}^{x_i} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} - \int_{\xi}^{x_{i-1}} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \right) \\ & \quad + \varphi^2(b') \int_{\xi}^{b'} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \\ & \leq \left( |C| + \int_{\xi}^{b'} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \right) \varphi^2(b') + 2 \int_{b'}^b d|\mu(\tau)| < \infty. \end{aligned}$$

Therefore  $\varphi^2 f'$  is of bounded variation on  $(b', b)$ .

b)  $\varphi$  is increasing on  $(b', b)$ . By our assumption this may occur only when  $b = \infty$ . Since

$$|f'(x) - f'(y)| = \left| \int_x^y \frac{d\mu(\tau)}{\varphi^2(\tau)} \right| \leq K \int_x^y d|\mu(\tau)|$$

tends to zero as  $x$  and  $y$  tend to infinity, and since  $f \in L^1(b', \infty)$ , we can conclude that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} f'(x) = 0, \text{ i.e., } \lim_{x \rightarrow \infty} \left( c + \int_{\xi}^x \frac{d\mu(\tau)}{\varphi^2(\tau)} \right) = c + \int_{\xi}^{\infty} \frac{d\mu(\tau)}{\varphi^2(\tau)} = 0. \text{ This last}$$

equality and (5) show that  $f'(x) = - \int_x^{\infty} \frac{d\mu(\tau)}{\varphi^2(\tau)}$ . Hence we have for every system

$x_0 < x_1 < \dots < x_{n+1}$ ,  $x_0 = b'$  the estimate

$$\begin{aligned} & \sum_{i=0}^n |\varphi^2(x_{i+1}) f'(x_{i+1}) - \varphi^2(x_i) f'(x_i)| \\ & \leq \sum_{i=0}^n \varphi^2(x_i) \int_{x_i}^{x_{i+1}} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} + \sum_{i=0}^n (\varphi^2(x_{i+1}) - \varphi^2(x_i)) \int_{x_{i+1}}^{\infty} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} \leq 2 \int_{b'}^{\infty} d|\mu(\tau)| < \infty, \end{aligned}$$

so that  $\varphi^2 f'$  is of bounded variation on  $(b', \infty)$ .

In the same way it can be proved that  $\varphi^2 f'$  is of bounded variation on  $(a, \xi)$  and so the first part of the proof is complete.

Sufficiency: (i) The case  $B = C[a, b]$ . Let  $f \in C[a, b]$  be such that  $f'$  is absolutely continuous with  $|\varphi^2(x) f''(x)| \leq K$  a.e. Our assumptions clearly imply that for small  $h$  and  $y \in (x - h\varphi(x), x + h\varphi(x))$  the estimate

$$\frac{1}{K_1} \varphi(x) \leq \varphi(y) \leq K_1 \varphi(x) \quad (K_1 > 0) \tag{6}$$

holds. But then for small  $h$  and  $x \in (a, b)$  we have

$$\begin{aligned} |\Delta_{h\varphi(x)}^2 f(x)| &= \left| \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} f''(x + u + v) \, du \, dv \right| \leq K \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} \frac{1}{\varphi^2(x + u + v)} \, du \, dv \\ &\leq KK_1^2 \frac{1}{\varphi^2(x)} \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} \int_{\frac{-h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} \, du \, dv \leq KK_1^2 h^2, \end{aligned}$$

and this is what we wanted to prove.

(ii) The case  $B = L^p(a, b)$ ,  $1 < p < \infty$ . Let  $f \in L^p(a, b)$  with absolutely continuous derivative and with  $\varphi^2 f'' \in L^p(a, b)$ . Using the Hardy-Littlewood maximal function  $M(\cdot)$  and the maximal inequality we obtain for small  $h$  (see also the above consideration)

$$\begin{aligned} \|\Delta_{h\varphi}^2 f\|_{L^p(a,b)} &\leq \left\| \int_{-h\varphi/2}^{h\varphi/2} \int_{-h\varphi/2}^{h\varphi/2} \frac{1}{\varphi^2(\cdot + u + v)} |\varphi^2(\cdot + u + v) f''(\cdot + u + v)| \, du \, dv \right\|_{L^p(a,b)} \\ &\leq K_1^2 \left\| \int_{-h\varphi/2}^{h\varphi/2} \int_{-h\varphi/2}^{h\varphi/2} |\varphi^2(\cdot + u + v) f''(\cdot + u + v)| \, du \, dv \right\|_{L^p(a,b)} \\ &\leq K_1^2 h^2 \|M(\varphi^2 f'')\|_{L^p(a,b)} \leq K_2 h^2 \|\varphi^2 f''\|_{L^p(a,b)} \end{aligned}$$

and we are through with the proof.

(iii) The case  $B = L^1(a, b)$ . By assumption  $f$  has (a.e.) the form

$$f(x) = C + \int_{\xi}^x \frac{\nu(\tau)}{\varphi^2(\tau)} \, d\tau \quad (\xi \in (a, b)) \tag{7}$$

with a function  $\nu$  having bounded variation on  $(a, b)$ . We shall need another representation for  $f$  and to this end we first prove that  $(\varphi'/\varphi) \nu \in L^1(a, b)$ . Clearly,  $(\varphi'/\varphi) \nu \in L^1(a', b')$  for every finite interval  $[a', b'] \subseteq (a, b)$ . So it is sufficient to show that  $(\varphi'/\varphi) \nu \in L^1(\xi, b)$  for some  $\xi \in (a, b)$  (the relation  $(\varphi'/\varphi) \nu \in L^1(a, \xi')$  for some  $\xi' \in (a, b)$  can be proved similarly).

First let us consider the case  $b = \infty$ . Let  $\xi > 0$ ,  $\xi \in (a, \infty)$  be arbitrary, and let

$$g(x) = \int_x^{\infty} \frac{\nu(\tau)}{\tau^2} \, d\tau \quad (x > 0).$$

Since (7) holds with this  $\xi$ , we obtain by integration by parts

$$g(x) = \int_x^{\infty} \frac{\varphi^2(\tau)}{\tau^2} \frac{\nu(\tau)}{\varphi^2(\tau)} \, d\tau = \frac{\varphi^2(\tau)}{\tau^2} f(\tau) \Big|_x^{\infty} - \int_x^{\infty} \frac{2\varphi(\tau) \varphi'(\tau) \tau^2 - 2\varphi^2(\tau) \tau}{\tau^4} f(\tau) \, d\tau.$$



If  $\varphi(\tau) \nearrow \infty$  as  $\tau \rightarrow \infty$ , then  $f'(\tau) = v(\tau)/\varphi^2(\tau) \rightarrow 0$ , and so  $f(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$  ( $f \in L^1(a, \infty)$ !). If, however,  $\varphi(\tau)$  decreases, or increases to a finite limit, as  $\tau \rightarrow \infty$ , then we have by (7)  $|f(\tau)| \leq K + K \frac{\tau}{\varphi^2(\tau)}$ . Since we assumed  $\varphi(\tau) = O(\tau)$  ( $\tau \rightarrow \infty$ ), we have in any case  $(\varphi^2(\tau)/\tau^2) f(\tau) = o(1)$  as  $\tau \rightarrow \infty$ . By our assumptions on  $\varphi$

$$\left| \frac{2\varphi(\tau) \varphi'(\tau) \tau^2 - 2\varphi^2(\tau) \tau}{\tau^4} \right| \leq K_1 \frac{\varphi^2(\tau)}{\tau^3} \leq K_2 \frac{1}{\tau},$$

hence  $|g(x)| \leq K_3 |f(x)| + K_3 \int_x^\infty \frac{|f(\tau)|}{\tau} d\tau$ , and in view of the inequality  $\int_\xi^\infty \int_x^\infty \frac{|f(\tau)|}{\tau} d\tau dx \leq \int_\xi^\infty |f(\tau)| d\tau$  we have  $g \in L^1(\xi, \infty)$ . But then

$$\begin{aligned} \int_\xi^x g(u) du &= \int_\xi^x \frac{v(\tau)}{\tau^2} (\tau - \xi) d\tau + (x - \xi) \int_x^\infty \frac{v(\tau)}{\tau^2} d\tau \\ &= \int_\xi^x \frac{v(\tau)}{\tau} d\tau + x \int_x^\infty \frac{v(\tau)}{\tau^2} d\tau - \xi \int_\xi^\infty \frac{v(\tau)}{\tau^2} d\tau \end{aligned} \tag{8}$$

is of bounded variation on  $(\xi, \infty)$ . An integration by parts now gives that

$$x \int_x^\infty \frac{v(\tau)}{\tau^2} d\tau = x \left( -\frac{1}{\tau} v(\tau) \right) \Big|_x^\infty + x \int_x^\infty \frac{dv(\tau)}{\tau} = -v(x) + x \int_x^\infty \frac{dv(\tau)}{\tau}.$$

The argument used in point (iii) of the necessity part of our proof shows that the function  $x \int_x^\infty \frac{dv(\tau)}{\tau}$  is of bounded variation on  $(\xi, \infty)$ , and since  $v$  also has this property we can conclude (see (8)) that  $h(x) = \int_\xi^x \frac{v(\tau)}{\tau} d\tau$  is of bounded variation on  $(\xi, \infty)$ .

Since  $h$  is absolutely continuous this can happen only if  $v(\tau)/\tau \in L^1(\xi, \infty)$ . Thus,  $v(\tau)/\tau \in L^1(\xi, \infty)$ , and if we multiply this function by the bounded function  $\tau\varphi'(\tau)/\varphi(\tau)$  we obtain  $(\varphi'/\varphi) v \in L^1(\xi, \infty)$  as was stated above.

Now let  $b$  be finite, and let  $\xi \in (a, b)$  be so close to  $b$  that

$$d(x) \stackrel{\text{def}}{=} \min(x - a, b - x) = b - x$$

is satisfied for all  $x \in (\xi, b)$ . If

$$\begin{aligned} g(x) &= \int_\xi^x \frac{v(\tau)}{d^2(\tau)} d\tau \\ &= \frac{\varphi^2(x)}{d^2(x)} f(x) - \frac{\varphi^2(\xi)}{d^2(\xi)} f(\xi) - \int_\xi^x \frac{2\varphi(\tau) \varphi'(\tau) d^2(\tau) + 2d(\tau) \varphi^2(\tau)}{d^4(\tau)} f(\tau) d\tau, \end{aligned}$$

then we can show as above that  $g \in L^1(\xi, b)$  (notice that  $\tilde{\varphi}(\tau) = O(d(\tau))$  as  $\tau \rightarrow b$ ). But then the function

$$\begin{aligned} \int_x^b g(\tau) d\tau &= (b-x) \int_x^b \frac{v(u)}{d^2(u)} du + \int_x^b \int_x^u \frac{v(u)}{d^2(u)} du d\tau \\ &= v(x) - (b-x) \frac{v(\xi)}{d(\xi)} + (b-x) \int_x^b \frac{dv(u)}{b-u} + \int_x^b \frac{v(u)}{d(u)} du \end{aligned}$$

is of bounded variation on  $(\xi, b)$ , and we can deduce exactly as above that  $v(u)/d(u) \in L^1(\xi, b)$ . Since  $\varphi'(u) = O(\varphi(u)/d(u))$  as  $u \rightarrow b$ , we can again conclude that  $(\varphi'/\varphi)v \in L^1(\xi, b)$ .

So far we have proved that  $(\varphi'/\varphi)v \in L^1(a, b)$ ; hence the function

$$\mu(x) = v(x) - 2 \int_x^b \frac{\varphi'(\tau)}{\varphi(\tau)} v(\tau) d\tau$$

is of bounded variation on  $(a, b)$  ( $\xi \in (a, b)$  is fixed). Clearly,

$$\int_x^y \int_x^y \frac{d\mu(\tau)}{\varphi^2(\tau)} dy = \int_x^y \int_x^y d\left(\frac{v(\tau)}{\varphi^2(\tau)}\right) dy = \int_x^y \frac{v(y)}{\varphi^2(y)} dy - (x-\xi) \frac{v(\xi)}{\varphi^2(\xi)}$$

from which it follows that  $f$  has the form

$$f(x) = c'x + d' + \int_x^y \int_x^y \frac{d\mu(\tau)}{\varphi^2(\tau)} dy = c'x + d' + \int_x^y \frac{x-\tau}{\varphi^2(\tau)} d\mu(\tau).$$

This is the desired representation.

From here on the estimate of the  $L^1$ -norm of

$$\Delta_{h\varphi(x)}^2 f(x) = \int_x^{x+h\varphi(x)} \frac{x+h\varphi(x)-\tau}{\varphi^2(\tau)} d\mu(\tau) - \int_{x-h\varphi(x)}^x \frac{x-h\varphi(x)-\tau}{\varphi^2(\tau)} d\mu(\tau)$$

is easy: for small  $h$ , say for  $h \leq h_1$ ,  $\tau \in (x-h\varphi(x), x+h\varphi(x))$  implies that

$$\frac{1}{K} \varphi(x) < \varphi(\tau) < K\varphi(x)$$

for some constant  $K$ . So for  $h \leq h_1/K$  we have

$$\tau - Kh\varphi(\tau) + h\varphi(\tau - Kh\varphi(\tau)) < \tau,$$

$$\tau + Kh\varphi(\tau) - h\varphi(\tau + Kh\varphi(\tau)) > \tau.$$

These results imply at once the estimate

$$\begin{aligned} \int_a^b |\Delta_{h\varphi(x)}^2 f(x)| dx &\leq h \int_a^b \varphi(x) \int_{x-h\varphi(x)}^{x+h\varphi(x)} \frac{d|\mu(\tau)|}{\varphi^2(\tau)} dx \leq Kh \int_a^b \int_{x-h\varphi(x)}^{x+h\varphi(x)} \frac{d|\mu(\tau)|}{\varphi(\tau)} dx \\ &= Kh \int_a^b \frac{1}{\varphi(\tau)} \left( \int_{x-h\varphi(x) \leq \tau \leq x+h\varphi(x)} dx \right) d|\mu(\tau)| \\ &\leq Kh \int_a^b \frac{2Kh\varphi(\tau)}{\varphi(\tau)} d|\mu(\tau)| \leq 2K^2h^2 \int_a^b d|\mu(\tau)| \end{aligned}$$

for all sufficiently small  $h$ , and the proof is complete ■

**Proof of Theorem 3:** (i) *The case  $B = C\{a, b\}$ .* It was proved in [8: Theorem 7] that Theorem 1 holds for  $B = C\{a, b\}$  even for the more general functions  $\varphi$  of Theorem 3; so the necessity part can be proved as in Theorem 2. Again, the above-mentioned proof shows that if  $f'$  is absolutely continuous and  $|\varphi^2 f''| \leq K$ , then

$$|\Delta_{h\varphi(x)}^2 f(x)| \leq K_1 h^2 \tag{9}$$

for  $x \in (h^* + (h^* - a), h^{**} - (b - h^{**}))$  (notice that the assumed monotonicity of  $\varphi(x)/(d(x))^p$  around the endpoints obviously implies (6) for  $x \in (h^* + (h^* - a), h^{**} - (b - h^{**}))$  and  $y \in (x - h\varphi(x), x + h\varphi(x))$ . By the analogue of [8: Remark 1] for the continuous case, this already implies (9) for all  $x \in (h^*, h^{**})$ .

(ii) *The case  $B = L^p(a, b)$ ,  $1 < p < \infty$ .* By [8: Theorem 2] for the  $\varphi$ 's under consideration the relations  $\omega(f, \delta) = O(\delta^2)$  and  $K(t^2, f) = O(t^2)$  are equivalent, so the proof of Theorem 2 work almost word by word (see also point (i) above and [8: Remark 1]).

(iii) *The case  $B = L^1(a, b)$ .* Here we have again the equivalence of  $\omega(f, \delta) = O(\delta^2)$  and  $K(t^2, f) = O(t^2)$ , so by the proof of Theorem 2  $\omega(f, \delta) = O(\delta^2)$  implies that  $f$  has the form

$$f(x) = cx + d + \int_{\xi}^x \int_{\xi}^y \frac{d\mu(\tau)}{\varphi^2(\tau)} dy \quad (\text{a.e.}) \tag{10}$$

Putting

$$v(y) = c + \int_{\xi}^y \frac{d\mu(\tau)}{\varphi^2(\tau)},$$

we obtain the necessity of our condition. Conversely, if  $f$  has the form (10), then, exactly as at the end of the sufficiency part of the proof of Theorem 2, one can show that for small  $h$

$$\|\Delta_{h\varphi}^2 f\|_{L^1(h^* + (h^* - a), h^{**} - (b - h^{**}))} \leq Kh^2,$$

and this already implies  $\omega(f, \delta) = O(\delta^2)$  (see [8: Theorem 2]) ■

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