

The Laguerre-Transform of some Elementary Functions

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In der Arbeit wird die verallgemeinerte Laguerre-Transformation einer Klasse elementarer Funktionen berechnet, die oft in den Anwendungen auftritt. Das Ergebnis wird zur Herleitung einer Lösung einer gewöhnlichen Differentialgleichung verwendet.

В данной работе вычисляется обобщенное преобразование Лагерра одного класса функций, которые часто встречаются в приложениях. С помощью полученного результата решается одно обыкновенное дифференциальное уравнение.

This paper deals with the calculation of the generalized Laguerre transformation of a class of functions, which often appear in applications. The result is used for the derivation of the solution of an ordinary differential equation.

0. In [2, 3] an operational calculus for a generalized Laguerre transformation was developed, which is of importance for the solution of differential equations of the kind

$$[S_a^k x](t) = f(t). \quad (0.1)$$

Here k is a natural number, f a suitable function and S_a the so-called Laguerre differential operator

$$S_a x(t) = e^{-t} t^{-\alpha} D[e^{-t} t^{\alpha+1} D] x(t), \quad D = d/dt. \quad (0.2)$$

In [3] the generalized Laguerre transformation

$$\mathfrak{F}^{(\alpha)}[f](v) = \int_0^{\infty} e^{-t} t^{\alpha} L_v^{(\alpha)}(t) f(t) dt = F_a(v) \quad (0.3)$$

was investigated, where $L_v^{(\alpha)}$ are the Laguerre functions of the first kind

$$L_v^{(\alpha)}(t) = \frac{\Gamma(1 + \alpha + v)}{\Gamma(1 + v) \Gamma(1 + \alpha)} {}_1F_1(-v, \alpha + 1; t), \quad \alpha > -1. \quad (0.4)$$

In [3] under suitable assumptions it was proved, that the generalized Laguerre transformation of the differential equation (0.1) yields

$$X_a(v) = (-v)^{-k} F_a(v), \quad (0.5)$$

where X_a , F_a are the generalized Laguerre transforms of x and f respectively.

The inversion formula of (0.3) is given by

$$f(t) = \int_{(c)} F_a(v) \frac{\Gamma(1 + v) L_v^{(\alpha)}(t)}{\Gamma(1 + \alpha + v) (-1 + e^{-2\pi i v})} dv \quad (0.6)$$

(see [2: Theorem 6.1]), where $\gamma < c < 0$ and γ is the order of growth of f if t tends to infinity, that is

$$f(t) = O(t^\gamma), \quad -1 < \gamma < 0, \quad t \rightarrow +\infty.$$

Applying this to (0.5) we get immediately a formal derivation of a solution of (0.1), namely

$$x(t) = (-1)^k \int_{(c)} v^{-k} F_a(v) \frac{\Gamma(1 + \nu) L_{\nu}^{(a)}(t)}{\Gamma(1 + \alpha + \nu) (e^{-2\pi i \nu} - 1)} d\nu. \quad (0.7)$$

In this paper we derive the transform (0.3) of a class of elementary functions, which often appears in applications and then the result is used for the solution of a special differential equation of the type (1).

Here and in the following terms which are standing on equal places inside the brackets $\{\dots\}$ correspond each other. The set of natural numbers is denoted by \mathbb{N} and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

1. The function, which has to be transformed is

$$f(t) = t^{\lambda-1} e^{-pt} \begin{Bmatrix} \sin bt^r \\ \cos bt^r \end{Bmatrix}, \quad r > 0, \quad b \begin{Bmatrix} > \\ \cong \end{Bmatrix} 0, \quad (1.1)$$

that is we want to calculate the integral

$$\begin{aligned} I &= \int_0^\infty t^{\lambda-1} e^{-pt} \begin{Bmatrix} \sin bt^r \\ \cos bt^r \end{Bmatrix} L_{\nu}^{(a)}(t) dt \\ &= \mathfrak{F}^{(a)} \left[t^{\lambda-1} e^{-pt} \begin{Bmatrix} \sin bt^r \\ \cos bt^r \end{Bmatrix} \right] (\nu). \end{aligned} \quad (1.2)$$

Because of the asymptotic behaviour of $L_{\nu}^{(a)}(t)$, (see [1: vol. I, p. 278, (3)])

$$L_{\nu}^{(a)}(t) = -\pi^{-1} \sin \pi \nu \Gamma(1 + \alpha + \nu) e^{t^{\alpha-1}} [1 + O(t^{-1})] \quad (1.3)$$

if $t \rightarrow +\infty$ and of the boundness of $L_{\nu}^{(a)}(t)$ in a neighborhood of the origin the integral I converges under the restrictions

$$\begin{aligned} &b \begin{Bmatrix} > \\ \cong \end{Bmatrix} 0, \quad r > 0, \quad \operatorname{Re}(p) > 0 \\ &\text{or } [\operatorname{Re}(p) = 0 \quad \text{and} \\ &\operatorname{Re}(\nu) > \operatorname{Re}(\lambda) - 1], \quad \operatorname{Re}(\lambda) > \begin{Bmatrix} -\alpha - r \\ -\alpha \end{Bmatrix}. \end{aligned} \quad (1.4)$$

In the language of integral transformations this is: The generalized Laguerre transform of f is an entire function of ν if $\operatorname{Re}(p) > 0$ and a holomorphic function in the halfplane $\operatorname{Re}(\nu) > \operatorname{Re}(\lambda) - 1$ if $\operatorname{Re}(p) = 0$ (under the restrictions (1.4) for the other parameters).

Remark: The integral I may be considered as

$$\begin{aligned} I &= \mathfrak{M} \left[e^{-pt-t} \left\{ \begin{matrix} \sin bt^r \\ \cos bt^r \end{matrix} \right\} L_{r^{(\alpha)}}(t) \right] (\lambda + \alpha) \\ &= r^{-1} \mathfrak{L} \left[t^{(\lambda+\alpha)/r-1} e^{-t^r} \left\{ \begin{matrix} \sin bt \\ \cos bt \end{matrix} \right\} L_{r^{(\alpha)}}(t^{1/r}) \right] (p) \\ &= r^{-1} \left\{ \begin{matrix} \mathfrak{F}_s \\ \mathfrak{F}_c \end{matrix} \left[t^{(\lambda+\alpha)/r-1} e^{-pt-t^r} L_{r^{(\alpha)}}(t^{1/r}) \right] (b) \right\}, \end{aligned}$$

where \mathfrak{M} , \mathfrak{L} , \mathfrak{F}_s , \mathfrak{F}_c are the Mellin-, Laplace-, Fourier-sinus and Fourier-cosinus transform respectively.

For the calculation of I we are substituting (under the condition $p > 0$)

$$\begin{aligned} \cos \beta\pi &:= p(b^2 + p^2)^{-1/2}, \quad x := (p^2 + b^2)^{-1/2r}, \quad \theta := x/t \\ K_1(\theta) &:= e^{-\theta^{-r} \cos \beta\pi} \left\{ \begin{matrix} \sin(\theta^{-r} \sin \beta\pi) \\ \cos(\theta^{-r} \sin \beta\pi) \end{matrix} \right\} \\ K_2(t) &:= t^{\lambda+\alpha} e^{-t} L_{r^{(\alpha)}}(t). \end{aligned} \tag{1.5}$$

With these abbreviations the integral (1.2) can be written as the Mellin-convolution of K_1 and K_2 , that is

$$I = \int_0^\infty K_1(xt^{-1}) K_2(t) t^{-1} dt =: K(x),$$

with

$$\hat{K}_i(s) = \mathfrak{M}[K_i](s) = \int_0^\infty t^{s-1} K_i(t) dt, \quad i = 1, 2$$

and by help of the convolution theorem for the Mellin transformation we have, for the Mellin-transform \hat{K} of K

$$\hat{K} = \hat{K}_1 \cdot \hat{K}_2.$$

By means of the formulas [4: p. 139–140: 3.8(1), 3.9(1), p. 130: 1.4, 1.5 and p. 21: 10.40 (1)] we have

$$\hat{K}_1(s) = r^{-1} \Gamma(-s/r) \left\{ \begin{matrix} -\sin \pi\beta s/r \\ \cos \pi\beta s/r \end{matrix} \right\}, \quad |\beta| < 1/2, \quad \text{Re}(s) < \left\{ \begin{matrix} r \\ 0 \end{matrix} \right\}$$

and

$$\begin{aligned} \hat{K}_2(s) &= \frac{1}{\Gamma(\nu+1)} \Gamma \left[\begin{matrix} s + \alpha + \lambda, 1 + \nu - \lambda - s \\ 1 - \lambda - s \end{matrix} \right], \\ &\quad -\text{Re}(\alpha + \lambda) < \text{Re}(s) < 1 + \text{Re}(\nu - \lambda). \end{aligned}$$

Here (see [4]) as usual

$$\Gamma \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \right] = \prod_{i=1}^p \Gamma(a_i) \left[\prod_{j=1}^q \Gamma(b_j) \right]^{-1}. \tag{1.6}$$

The condition $|\beta| < 1/2$ can be fulfilled if $p > 0$. The case $\text{Re}(p) > 0$ after the calculations follows by means of analytical continuation.

With the help of the inversion formula of the Mellin transformation we obtain

$$K(x) = (2\pi ir)^{-1} \int_{(a)} \Gamma \left[\begin{matrix} -s/r, \alpha + \lambda + s, 1 + \nu - \lambda - s \\ \nu + 1, 1 - \lambda - s \end{matrix} \right] \times \left\{ \begin{matrix} -\sin \pi\beta s/r \\ \cos \pi\beta s/r \end{matrix} \right\} x^{-s} ds, \tag{1.7}$$

with the condition for the path of integration

$$-\operatorname{Re}(\alpha + \lambda) < a = \operatorname{Re}(s) < \min \left(1 + \operatorname{Re}(\nu - \lambda), \left\{ \begin{matrix} r \\ 0 \end{matrix} \right\} \right).$$

As usual (a) denotes the vertical straight line from $a - i\infty$ to $a + i\infty$. The calculation of the integral (1.7) is possible by left- or right-shifting of the path (a) according to the convergence of the integral. Using a conclusion of a theorem of SLATER ([4: Theorem 17, p. 48]) we obtain immediately in the case $r > 1$ or $r = 1, |x| < 1$

$$K(x) = \sum_{k=0}^{\infty} \operatorname{Res}_{s=-k-\alpha-\lambda} [\hat{K}(s) x^{-s}].$$

The poles in the points $s = -k - \alpha - \lambda$ ($k \in \mathbb{N}_0$) are simple ones. In the case $r < 1$ or $r = 1, |x| > 1$ we have to shift the path to the right. There we have two series of poles in the points $s = kr$ and $s = 1 + \nu - \lambda + l$ ($k, l \in \mathbb{N}_0$). In common these are simple poles. Double poles do appear only, if for a value of $k \in \mathbb{N}_0$ does exist such a $l \in \mathbb{N}_0$, that the diophantic equation

$$kr = 1 + \nu - \lambda + l$$

is solvable. The corresponding sum of residues we denote by $\sum_{k=0(\exists l)}^{\infty}$. The sums of the residues in the remaining points kr and $1 + \nu - \lambda + l$ we denote by $\sum_{k(\exists l)}$ and $\sum_{l(\exists k)}$ respectively. The calculation of the residues in the simple poles is trivial because of the wellknown behaviour of the Γ -function. For the calculation of $\sum_{k(\exists l)}$ we develop $\hat{K}(s) t^{-s}$ with $s = kr + \varepsilon = 1 + \nu - \lambda + l + \varepsilon$ in the Laurent series in $0 < |\varepsilon| < R$, with a suitable R and look at the coefficient of the power ε^{-1} . This is the residuum which we were searching. After some calculations we obtain the

Theorem: *With the abbreviations (1.5) and the conditions (1.4) the integral I in (1.2) can be calculated to*

$$I = r^{-1} \sum_{k=0}^{\infty} (-1)^k \Gamma \left[\begin{matrix} (k + \alpha + \lambda)/r, k + 1 + \nu + \alpha \\ k + 1, \nu + 1, k + \alpha + 1 \end{matrix} \right] \times \left\{ \begin{matrix} \sin [\pi\beta(k + \alpha + \lambda)/r] \\ \cos [\pi\beta(k + \alpha + \lambda)/r] \end{matrix} \right\} x^{k+\alpha+1}, \tag{1.8}$$

if $r > 1$ or $r = 1, |x| < 1$

and

$$\begin{aligned}
 I = & \sum_{k(\neq l)} (-1)^k \Gamma \left[\begin{matrix} kr + \alpha + \lambda, 1 + \nu - \lambda - kr \\ k + 1, \nu + 1, -kr + 1 - \lambda \end{matrix} \right] \cdot \left\{ \begin{matrix} -\sin \beta \pi k \\ \cos \beta \pi k \end{matrix} \right\} x^{-kr} \\
 & + r^{-1} \sum_{l(\neq k)} (-1)^l \Gamma \left[\begin{matrix} (\lambda - \nu - l - 1)/r, 1 + \nu + \alpha + l \\ l + 1, \nu + 1, -\nu - l \end{matrix} \right] \\
 & \times \left\{ \begin{matrix} -\sin [\pi \beta (1 + \nu - \lambda + l)/r] \\ \cos [\pi \beta (1 + \nu - \lambda + l)/r] \end{matrix} \right\} x^{l-\nu+1-l} \\
 & + \sum_{k(\neq l)} (-1)^{k(r+1)+l-\nu} \Gamma \left[\begin{matrix} \alpha + \lambda + kr \\ k + 1, kr - \nu + \lambda, 1 - \lambda - kr, \nu + 1 \end{matrix} \right] \\
 & \times \left\{ \begin{matrix} -\sin \beta \pi k \\ \cos \beta \pi k \end{matrix} \right\} x^{-kr} \left[\Psi(\alpha + \lambda + kr) - r^{-1} \Psi(k + 1) - \Psi(kr + \lambda - \nu) \right. \\
 & \left. + \Psi(1 - \lambda - kr) + \left\{ \begin{matrix} \pi \beta / r \cot \pi \beta k \\ -\pi \beta / r \tan \pi \beta k \end{matrix} \right\} - r^{-1} \log x^r \right]
 \end{aligned} \tag{1.8'}$$

if $r < 1$ or $r = 1, |x| > 1$ and $\Psi = \Gamma'/\Gamma$.

2. With help of the theorem above one can calculate as well tabulated integral transforms as new ones. The conditions (1.4) are fit for the Laguerre transformation, but they are not necessary for the convergence of the integral (1.2). As an application we consider two special cases.

1. $r = \lambda = 1/2, b = 0$:

In this case the integral I is well known. With the remark of Section 1 and [1: vol. I, 6.10(11)] we have

$$\begin{aligned}
 I &= 2\Omega[t^{2a} e^{-t^2} L_{\nu}^{(\alpha)}(t^2)](p) \\
 &= 2^{-2a} \Gamma \left[\begin{matrix} 2\alpha + 1, \nu + \alpha + 1 \\ \nu + 1, \alpha + 1 \end{matrix} \right] G(\alpha + 1/2, -\nu + 1/2; p^2/4), \\
 &\alpha > -1/2, \text{Re}(p) > 0.
 \end{aligned} \tag{2.1}$$

Here G is the confluent hypergeometric function of second kind, sometimes denoted by Ψ . We want to sketch a proof of this formula by means of the theorem. From (1.8') we get immediately

$$\begin{aligned}
 I &= \sum_{k=0}^{\infty} (-1)^k \Gamma \left[\begin{matrix} (k + 1)/2 + \alpha, \nu + (1 - k)/2 \\ k + 1, \nu + 1, (1 - k)/2 \end{matrix} \right] p^k \\
 &+ 2 \sum_{l=0}^{\infty} (-1)^l \Gamma \left[\begin{matrix} -2\nu - 2l - 1, 1 + \nu + \alpha + l \\ l + 1, \nu + 1, -\nu - l \end{matrix} \right] p^{2(\nu+l)+1}.
 \end{aligned} \tag{*}$$

Because of $\Gamma[(1 - k)/2] = \infty$ if k is an odd integer we substitute $k \rightarrow 2k$ in the first sum. Using repeatedly the duplication formula of the Gamma function

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2) \tag{2.2}$$

and the functional equation

$$\Gamma(z) \Gamma(1 - z) = \pi / \sin \pi z \tag{2.3}$$

we get after a short calculation

$$\begin{aligned}
 I &= 2^{-2\alpha} \Gamma \left[\begin{matrix} 2\alpha + 1, \nu + \alpha + 1 \\ \nu + 1, \alpha + 1 \end{matrix} \right] \\
 &\times \left\{ \Gamma \left[\begin{matrix} \nu + 1/2 \\ \nu + \alpha + 1 \end{matrix} \right] {}_1F_1(\alpha + 1/6; -\nu + 1/6; p^2/4) \right. \\
 &\left. + (p^2/4)^{\nu+1/2} \Gamma \left[\begin{matrix} -\nu - 1/2 \\ \alpha + 1/2 \end{matrix} \right] {}_1F_1(\nu + \alpha + 1; \nu + 3/2; p^2/4) \right\}.
 \end{aligned}$$

Using the connection [1: vol. I, 6.5 (7)] between the confluent hypergeometric functions of the second and of the first kind respectively

$$\begin{aligned}
 G(a, c; x) &= \Gamma \left[\begin{matrix} 1 - c \\ a - c + 1 \end{matrix} \right] {}_1F_1(a; c; x) \\
 &+ \Gamma \left[\begin{matrix} c - 1 \\ a \end{matrix} \right] x^{1-c} {}_1F_1(a - c + 1; 2 - c; x)
 \end{aligned}$$

we have with $a = \alpha + 1/2, c = -\nu + 1/2, x = p^2/4$ at once (2.1).

2. $\lambda = 1, r = -1, b = 0:$

In this case we want to calculate the integral

$$I = -\mathfrak{L}[t^{-\alpha-2} e^{-1/t} L_{\nu}^{(\alpha)}(1/t)](p), \tag{2.4}$$

which is not tabulated yet. Here we will not use our theorem, but calculate directly the integral with help of the method of the proof of the theorem (because $r = -1$). The integral (2.4) may be written in form of a Mellin-convolution

$$I = \int_0^\infty K_1(p/t) K_2(t) t^{-1} dt$$

with $K_1(\eta) = e^{-\eta}, K_2(t) = L_{\nu}^{(\alpha)}(t) e^{-t^{\alpha+1}}$. Using [4: 3.1 (1)] we have

$$\hat{K}_1(s) = \Gamma(s), \quad \text{Re}(s) > 0$$

and using [4: 10.40 (1)] and the multiplication theorem of the Mellin transform we get

$$\hat{K}_2(s) = \Gamma \left[\begin{matrix} s + \alpha + 1, \nu - s \\ -s \end{matrix} \right], \quad -\alpha - 1 < \text{Re}(s) < \text{Re}(\nu).$$

Because of the convolution theorem of the Mellin transform we get with (1.7)

$$I = -(2\pi i)^{-1} \int_{(a)} \Gamma \left[\begin{matrix} s, s + \alpha + 1, \nu - s \\ \nu + 1, -s \end{matrix} \right] p^{-s} ds, \quad 0 < a < \text{Re}(\nu).$$

Left-shifting of the path of integration yields

$$\begin{aligned}
 I &= - \sum_{k=1}^\infty (-1)^k \Gamma \left[\begin{matrix} -k + \alpha + 1, \nu + k \\ k + 1, k, \nu + 1 \end{matrix} \right] p^k \\
 &- \sum_{l=0}^\infty (-1)^l \Gamma \left[\begin{matrix} -\alpha - 1 - l, \nu + \alpha + l + 1 \\ l + \alpha + 1, l + 1, \nu + 1 \end{matrix} \right] p^{l+\alpha+1}.
 \end{aligned}$$

Analogous to the first example by substituting $k \rightarrow k + 1$ and using (2.3) for $\Gamma(\alpha - k)$ and $\Gamma(-\alpha - 1 - l)$ we get after a short calculation (because of (2.4))

$$\begin{aligned} \mathfrak{L}[t^{-\alpha-2} e^{-1/t} L_{\nu}^{(\alpha)}(1/t)](p) &= \Gamma(\alpha) {}_1F_2(\nu + 1; 1 - \alpha; 2; p) p \\ &+ \Gamma\left[\begin{matrix} -\alpha, \nu + \alpha + 1 \\ \nu + 1, \alpha + 2 \end{matrix}\right] p^{\alpha+1} {}_1F_2(\nu + \alpha + 1; \alpha + 1; \alpha + 2; p), \\ \operatorname{Re}(p) &> 0. \end{aligned} \tag{2.4'}$$

Here, as usual

$${}_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!}, \quad |z| < \infty \tag{2.5}$$

and

$$(a)_0 = 1, \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad n \in \mathbb{N} \tag{2.6}$$

are the Pochhammer symbols.

3. Finally we look for a particular solution of the inhomogeneous Laguerre differential equation (of order $\alpha = 0$)

$$S_0 y(t) = e^t D(e^{-t} t D) y(t) = e^{-2t} \tag{3.1}$$

in the interval $0 < t < \infty$. Taking the transformation $\mathfrak{F}^{(0)}$ (see (0.3)) on both sides of this equation and using the result (0.5) with $k = 1$ and $\alpha = 0$ for the calculation of the $\mathfrak{F}^{(0)}$ -transform of the left hand side and the formula (1.8) for the calculation of the $\mathfrak{F}^{(0)}$ -transform of the right hand side (with $\alpha = 0, \lambda = 1, r = 1, p = 2, b = 0$) we obtain immediately ((1.5) yields $\cos \beta\pi = 1$, that is $\beta = 0$ because of $|\beta| < 1/2$ and $x = 1/2$):

$$-\nu Y(\nu) = 2^{-1} \sum_{k=0}^{\infty} (-1)^k \Gamma\left[\begin{matrix} k + \nu + 1 \\ k + 1, \nu + 1 \end{matrix}\right] 2^{-k} = \sum_{k=0}^{\infty} \binom{-\nu - 1}{k} 2^{-k-1},$$

that is

$$Y(\nu) = -(2\nu)^{-1} (2/3)^{\nu+1}. \tag{3.2}$$

Using (0.6) with $\alpha = 0$ we obtain a solution of (2.1) as the inversion of the image Y with respect to the transformation $\mathfrak{F}^{(0)}$:

$$y(t) = 3^{-1} \int_{(c)} \nu^{-1} (2/3)^{\nu} L_{\nu}(t) (1 - e^{-2t\nu})^{-1} d\nu, \quad -1 < c < 0. \tag{3.3}$$

By means of right-shifting of the path of integration we get

$$y(t) = -(2\pi i/3) \sum_{\nu=0}^{\infty} \operatorname{Res} [\nu^{-1} (2/3)^{\nu} L_{\nu}(t) (1 - e^{-2t\nu})^{-1}]. \tag{3.3'}$$

The poles in the points $\nu = n \in \mathbb{N}$ are simple ones. Only in the point $\nu = 0$ we have a pole of second order. Developing the integrand of (3.3) in a neighbourhood of $\nu = 0$ we obtain with

$$L_{\nu}(t) = \sum_{k=0}^{\infty} (-\nu)_k (k!)^{-2} t^k = 1 - \varphi(t) \nu + O(\nu^2),$$

where

$$\varphi(t) = \sum_{k=1}^{\infty} k^{-1}(k!)^{-1} t^k = \text{Ei}(t) - C - \log t$$

(see [6: 1.3.2, 33.]), C is the Euler constant, after a short calculation the result

$$y(t) = -3^{-1}[\log(2/3)t + \pi i + C - \text{Ei}(t) + \sum_{n=1}^{\infty} n^{-1}(2/3)^n L_n(t)]. \quad (3.4)$$

Here $\text{Ei}(t)$ is, as usual, the exponential integral

$$\text{Ei}(t) = \int_{-\infty}^t t^{-1} e^t dt, \quad t \neq 0, \quad (3.5)$$

where the integral takes its Cauchy principal value, when t is positive. One quickly verifies, that (3.4) indeed is a particular solution of (3.1).

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