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Interior Estimates for Singularly Perturbed Problems

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Die Lösung des Dirichletproblems für eine singulär gestörte elliptische Differentialgleichung $2m$-ter Ordnung $\varepsilon L_1 u + L_0 u = h$ konvergiert außerhalb der Grenzschicht für $\varepsilon \to 0$ gleichmäßig gegen eine Lösung der ausgearteten elliptischen Gleichung $L_0 w = h$ niedrigerer Ordnung. Es wird gezeigt, daß sich im Fall nullter Ordnung von L_0 dieser Sachverhalt unmittelbar beweisen läßt, d. h. ohne die übliche Konstruktion der Grenzschicht, und zwar elementar bei geringen Glattheitsforderungen an den Rand des Gebietes.

Решение проблемы Дирихле для сингулярно возмущенного дифференциального уравнения эллиптического типа $\varepsilon L_1 u + L_0 u = h$ порядка $2m$ сходится, для $\varepsilon \to 0$, равномерно вне пограничного слоя к решению вырожденного уравнения эллиптического типа $L_0 u = h$ низшего порядка. Оказывается, что в случае порядка 0 оператора L_0 этот факт можно доказать непосредственно, т. е. без конструкции пограничного слоя, к тому же элементарно и с незначительными предположениями гладкости границы области.

The solution of the Dirichlet problem for a singularly perturbed elliptic differential equation $\varepsilon L_1 u + L_0 u = h$ of order $2m$ converges, for $\varepsilon \to 0$, outside of the boundary layer uniformly to a solution of the degenerate elliptic equation $L_0 w = h$ of lower order. It is shown in the case of order zero of L_0 this assertion may be proved immediately, i.e., without the usual construction of boundary layer terms, but rather elementary and on weak smoothness conditions with respect to the boundary of the domain.

As well known, the solution $u = u_\varepsilon$ of the singularly perturbed Dirichlet problem of order $2m$ in an n-dimensional bounded domain G

$$\begin{cases} G\colon \ \varepsilon L_1 u + L_0 u = h \\ \partial G\colon \qquad\quad D^\gamma u = 0 \qquad (|\gamma| \leqq m-1) \end{cases} \tag{1}$$

behaves — in the case of "regular degeneration" — as follows for $\varepsilon \to +0$ (cf., e.g., [5–7]): In every compact subdomain $\bar{G}' \subseteq G$ we have uniform convergence to the solution w_0 of the degenerate (reduced) elliptic problem of order $2k$ $(k < m)$

$$G\colon L_0 w_0 = h, \qquad \partial G\colon D^\gamma w_0 = 0 \qquad (|\gamma| \leqq k-1) \tag{2}$$

whilest in a narrow strip Γ_ε along the boundary ∂G of G arises a so-called (Prandlt's) boundary layer compensating the supernumerary boundary conditions of the perturbed problem ($\varepsilon > 0$) which the solution of the reduced problem in general will fail to satisfy; the width of Γ_ε is about some power of ε. Usually the asymptotic properties of u are studied by an expansion

$$\begin{aligned} u_\varepsilon &= w_0 + \varepsilon w_1 + \cdots + \varepsilon^r w_r + v_0 + \varepsilon v_1 + \cdots + \varepsilon^s v_s + \varepsilon^t z \\ &= w + v + \varepsilon^t z. \end{aligned}$$

Here the "regular" part w describes the convergence in $G \setminus \Gamma_\varepsilon$, while the functions v_i are of boundary-layer type: they are smaller than any power of ε outside of the neighbourhood Γ_ε of the boundary (exponential decay). This expansion has to be constructed and, after that, one must prove z to be bounded in all of G for some positive t. The main tools to be used for the latter are based, finally, on a-priori estimates as shown by AGMON, DOUGLIS and NIRENBERG [1]; for details — far from being trivial in general — we refer, e.g., to BESJES [2], or to the monograph [3]. But it seems that, until now, only in the simplest case

$$G: -\varepsilon^2 \Delta u + u = h, \qquad \partial G: u = 0 \tag{3}$$

the attempt has been made to prove directly the regular behaviour of u in $G \setminus \Gamma_\varepsilon$: L. TARTAR derived [5: p. 131]

$$\int_{G'} \sum \left[\frac{\partial}{\partial x_i}(u - h)\right]^2 dx \leqq C \|h\|_1^2$$

with

$$G' = \{x \in G: \operatorname{dist}(x, \partial G) \geqq \varepsilon^\alpha, 0 < \alpha < 1\}.$$

In the present paper we will submit a more general procedure, elementary in the main, which enables to prove even uniform pointwise convergence of u in $G \setminus \Gamma_\varepsilon$ and, moreover, does not claim (for itself) higher regularity of the boundary as a-priori estimates do in general.

0. Introduction. In order to give an outline of the method we will sketch it in the simple case (3) (cf. [4]).

Multiplying the equation by u and integrating by parts yield at once $\|u\|$ and $\varepsilon \|Du\|$ bounded (in the L_2-norm of G, D any first order derivative); using, e.g., a-priori estimates just mentioned it is possible to extend this result to derivatives of any order l: $\varepsilon^l \|D^l u\| \leqq C$ (by the way, we cannot expect essential improvements in general!).

Now we introduce "quasi-testfunctions" $\varphi = \varphi(x; \varepsilon, \mathring{x})$ which are equal 1 for $x = \mathring{x}$, the generic point under consideration in the interior of G, and of order $O(\exp(-c/\varepsilon^\delta))$ outside the ball of radius $\varepsilon^{1-\delta}$ centred at $\mathring{x}$ so that $\varphi D^\alpha u$ will there also be small relative to any power of ε. The advantage of φ in comparison with usual testfunctions is the fact that it is, in some sense, reproducing itself:

$$D\varphi = \frac{1}{\varepsilon^{1-\delta}} \cdot C(x)\, \varphi$$

with $C(x)$ smooth and bounded.

Next we set up the equation for $v = D^l u$, multiply by $\varphi^2 \cdot v$, and integration by parts neglecting (boundary-) terms of order $O(\exp(-c/\varepsilon^\delta))$ leads to

$$\|\varphi D^l u\| \leqq C.$$

By means of Sobolev's imbedding theorem for balls we can conclude the uniform boundedness of $D^l u(\mathring{x})$ for all $\mathring{x}$ with distance $\geqq \varepsilon^{1-2\delta}$ to the boundary ∂G, and uniform convergence of u and all its derivatives follows immediately via equation (3).

1. Position of the problem. In a bounded n-dimensional domain G we shall study the Dirichlet problem of order $2m$

$$\begin{cases} G: L_\varepsilon u = \sum_{i=0}^{2m} \varepsilon^i L_i u = h \\ \partial G: D^\gamma u = 0 \qquad (|\gamma| \leqq m - 1), \end{cases} \tag{4}$$

L_i being differential operators of order i

$$L_i u = \sum_{|\alpha| \leq i} a_\alpha^{(i)} D^\alpha u$$

with principal parts L_i'; α, γ are the usual multiindices. The ellipticity condition is posed in terms of Gårding's inequality for the operators of even order $i = 2j$: For $u \in H_{2j} \cap \mathring{H}_j$ there is valid

$$\begin{aligned} (L_{2j}'u, u) &\geqq A_j \|u\|_j^2 - B_j \|u\|_{j-1}^2 \qquad (j = 1, \ldots, m) \\ L_0 u, u) &\geqq A_0 \|u\|^2 \end{aligned} \tag{5}$$

with constants A_j, B_j, the first ones are to obey

$$A_0 > 0, \quad A_m > 0, \quad \left(\sum_{j=1}^{m-1} \min \{A_j, 0\}\right)^2 \leqq c \cdot A_0 A_m, \tag{6}$$

$c = \text{const} < 1$. Here the brackets $(\cdot, \cdot)$ signify the scalar product in $L_2(G)$, and $\|\cdot\|_j$ denotes the norm in the Sobolev space $H_j = W_2^{(j)}(G)$. The reduced problem ($\varepsilon = 0$) is simply

$$L_0 w = a_0 w = h \tag{7}$$

in G — no boundary conditions.

Coefficients and right hand side h are assumed sufficiently smooth, and the boundary ∂G to be regular enough in order to guaranty the existence of solutions $u = u_\varepsilon \in H_{2m} \cap \mathring{H}_m$ of (4) for small $\varepsilon > 0$ which are of class $C^\infty(G')$ in each compact subdomain $G' \subseteqq G$; at least, ∂G should be piecewise of class C^1, and Gauss's integration theorem be applicable in G.

2. A preliminary estimate. For a solution u of (4) integration by parts yields

$$\begin{aligned} (h, u) = (L_\varepsilon u, u) \geqq \sum_{j=1}^{m} \varepsilon^{2j} (A_j \|u\|_j^2 - B_j \|u\|_{j-1}^2) + A_0 \|u\|^2 \\ - \sum_{i=1}^{2m} \varepsilon^i \sum_{j=1}^{i} A_j^i \|u\|_{\left[\frac{j}{2}\right]} \|u\|_{\left[\frac{j-1}{2}\right]} \end{aligned} \tag{8}$$

with proper constants A_j^i, and $[x]$ denoting the integer part of x. In deriving the last sum it has been made use of the fact

$$(D^\alpha u, u) = (-1)^{|\alpha|} (u, D^\alpha u) = 0$$

for $|\alpha|$ odd and, consequently, the possibility to substitute for $(a_\alpha^{(i)} D^\alpha u, u)$ terms of total order less than $|\alpha|$. By the help of arithmetic-geometric-mean inequality used for j even

$$2 \|u\|_{\left[\frac{j}{2}\right]} \|u\|_{\left[\frac{j-1}{2}\right]} \leqq \varepsilon \|u\|_{\left[\frac{j}{2}\right]}^2 + \frac{1}{\varepsilon} \|u\|_{\left[\frac{j-1}{2}\right]}^2$$

we may conclude from (8)

$$(h, u) \geqq \sum_{j=0}^{m} \varepsilon^{2j} (A_j - \varepsilon A_j' - \varepsilon^2 B_{j+1}) \|u\|_j^2 \tag{9}$$

with $A_j' = A_j'(\varepsilon)$ uniformly bounded, $B_{m+1} = 0$. Finally, we omit the terms with $A_j > 0$ ($j = 1, \ldots, m-1$), and if $A_j \leqq 0$ we use the

Interpolation Lemma: *For* $u \in \mathring{H}_m$, $\beta \leqq \alpha$, $0 < j = |\beta| < |\alpha| = m$, *and positive* ε, q *it is valid*

$$\|D^\beta u\|^2 \leqq \frac{j}{m} q \varepsilon^{2(m-j)} \|D^\alpha u\|^2 + \frac{m-j}{m} \frac{1}{q} \varepsilon^{-2j} \|u\|^2. \tag{10}$$

This assertion may be proved by induction based on

$$\|D^j u\|^2 = (D^j u, D^j u) = -(D^{j+1} u, D^{j-1} u)$$

and the arithmetic-geometric-mean inequality.

Hence we now have

$$(h, u) \geqq \varepsilon^{2m} \left(A_m - \varepsilon C_m + \sum_{j=1}^{m-1} (\bar{A}_j q - \varepsilon C_j) \right) \|u\|_m^2$$
$$+ \left(A_0 - \varepsilon C_0 + \sum_{j=1}^{m-1} \left(\bar{A}_j \frac{1}{q} - \varepsilon C_j' \right) \right) \|u\|^2$$

where $\bar{A}_j = \min\{A_j, 0\}$, and C_j, C_j' proper constants. If we choose $q^2 = A_m/A_0$ we obtain

$$A_m + q \sum \bar{A}_j \geqq c' A_m, \qquad A_0 + \frac{1}{q} \sum \bar{A}_j \geqq c' A_0$$

with $c' = 1 - \sqrt{c} > 0$ (c the constant in (6)) and, therefore,

$$(h, u) \geqq \varepsilon^{2m} c_m \|u\|_m^2 + c_0 \|u\|^2 \tag{11}$$

for $\varepsilon \leqq \varepsilon_0$ with positive constants c_m, c_0 independent of ε and u. A simple application of Schwarz's inequality shows

$$\|u\| \leqq \frac{1}{c_0} \|h\|, \qquad \varepsilon^m \|u\|_m \leqq \frac{1}{\sqrt{c_0 c_m}} \|h\|$$

which may be extended by interpolation lemma to

$$\varepsilon^j \|u\|_j \leqq c_j' \|h\|, \qquad j = 0, \ldots, m. \tag{12}$$

This result can be further extended to orders of derivation beyond m, and that without additional supposition of smoothness if we restrict our consideration to ε-approximating subdomains

$$G_\varepsilon = \{x \in G : \operatorname{dist}(x, \partial G) > \varepsilon\} \tag{13}$$

of G. As easily to be seen by Lemma 1 of the appendix (cf. (44)) we can state: For $j > m$ there exists a constant c_j' so that for solutions of (4) holds the inequality

$$\varepsilon^j \|u\|_{\varepsilon,j} \leqq c_j' \sum_{i=0}^{j-m} \varepsilon^i \|h\|_i \qquad (j > m) \tag{14}$$

where $\|\cdot\|_{\varepsilon,j}$ denotes the norm of $H_j(G_\varepsilon)$.

Remark: A similar result might be achieved too by utilization of the well-know na-priori estimates for solutions of elliptic boundary value problems (e.g. [1: Chapter 15]) via homothetic transformation $x = \varepsilon \cdot x'$, in the case of smooth boundary even with $\|\cdot\|_j$ instead of $\|\cdot\|_{\varepsilon,j}$; but in order to maintain a self-contained elementary treatment as far as possible we establish (14) by integration by parts as done in the appendix.

3. Quasi-testfunctions. The desired uniform pointwise estimates for the solution u and its derivates shall be set up in the subdomain

$$G_{\bar\eta} = \{x \in G: \text{dist}\,(x, \partial G) > \bar\eta = c \cdot \varepsilon^{1-2\delta}\}; \tag{15}$$

c, δ given positive constants (cf. (13); of course, $\delta < 1/2$). $G_{\bar\eta}$ might also be conceived as an analogically defined subdomain of G_ε (for $\varepsilon \leq \varepsilon_0$ and a proper constant c); we shall sometimes do so in what follows.

Our main tool for analysing u locally will be the "quasi-testfunction" φ:

$$\begin{aligned} &\varphi(x) = \varphi_\varepsilon(x) = \varphi_m\left(\frac{r}{\eta}\right) \quad \text{with} \quad r = |x - \mathring{x}|, \quad \eta = \varepsilon^{1-\delta}; \\ &\varphi_m(\varrho) = \frac{1}{1 + \varrho^m e^\varrho}, \end{aligned} \tag{16}$$

where the point under consideration $\mathring{x}$ is any fixed point in $G_{\bar\eta}$. For $k \leq m$, we have

$$\frac{d^k\varphi_m(\varrho)}{d\varrho^k} = \begin{cases} \varrho^{m-k}B_k(\varrho) \cdot \varphi_m(\varrho) & \text{for} \quad \varrho \leq 1 \\ C_k(\varrho) \cdot \varphi_m(\varrho) & \text{for} \quad \varrho \geq 1 \end{cases}$$

with bounded functions B_k, C_k, whence we obtain for any partial derivative of order k (with respect to the variables x_i)

$$D^\alpha\varphi_\varepsilon(x) = \frac{1}{\eta^k}\, C_\alpha(x, \eta) \cdot \varphi_\varepsilon(x), \qquad |\alpha| = k \leq m \tag{17}$$

with bounded continuous C_α.

Instead of vanishing outside some neighbourhood of $\mathring{x}$ the function φ_ε will only tend to zero exponentially if $\varepsilon \to +0$, and that will do for our purpose. Especially, in a neighbourhood of the boundary ∂G of width $d \cdot \varepsilon$ (d positive constant) it is easily seen from (15) and (16)

$$\varphi_\varepsilon(x) = O\big(\exp\,(-c/\varepsilon^\delta)\big) \quad \text{for} \quad \varepsilon \to +0. \tag{18}$$

4. The L_2 estimate. By the next step, for all derivatives $D^\alpha u$ of the solution, $\|\varphi D^\alpha u\|$ will turn out bounded — uniformly with respect to ε and the choice of $\mathring{x}$ in $G_{\bar\eta}$. Differentiating equation (4) we obtain for any derivative $D^\alpha u = v$ of order $|\alpha| = l$

$$L_\varepsilon v = D^\alpha h + \sum_{\substack{|\beta| \leq l-1 \\ |\gamma| \leq 2m}} \varepsilon^{|\gamma|} c_{\beta\gamma}(x)\, D^{\beta+\gamma} u =: \bar h. \tag{19}$$

The proof of our assertion will now be given by multiplying this equation by $\varphi^2 \cdot v$ and integrating by parts. To this we point out an observation on principle: All integrands (and so all norms) will involve functions of kind $\varphi \cdot D^\alpha u$. If we, additionally, multiply by a testfunction $\psi_\varepsilon \in \mathring{C}^\infty(G_\varepsilon)$ with $\psi_\varepsilon(x) = 1$ for $x \in G_{2\varepsilon}$ as used in the appendix we enforce vanishing at the boundary though we only give rise of an error of the integrals of order $O\big(\exp\,(-c/\varepsilon^\delta)\big)$ as to be inferred from the preliminary estimates (12), (14), the property (18) of φ and the fact that the derivatives of ψ_ε also grow like powers of $1/\varepsilon$. Because $\exp\,(-c/\varepsilon^\delta) = o(\varepsilon^N)$ for any N, while the quantities in consideration in what follows are of orders ε^M only, we shall omit the boundary terms introducing an equivalence relation $\equiv$ and a weakened order relation $\leqq$ with meaning "equal (resp., less or equal) up to additional terms of order $O(\exp(-c/\varepsilon^\delta))$". Then we can formally integrate by parts as if the integrands were exactly equal zero at the boundary, if we relate the integrals and norms to the subdomain G_ε what we will do in this section without special notation.

Remark: Another way to become aware of this fact is to choose a subdomain similar to G_ε so that its image G_ε' under homothetic transformation $x' = x/\varepsilon$ has uniformly smooth boundary and then use the continuity of the trace map $\pi\colon H_{k+1}(G_\varepsilon') \to H_k(\partial G_\varepsilon')$; (12), (14), and (18) will now tell that the boundary values are "very small" in the sense described.

Now we will implement the integration by parts of (19) after scalar multiplication by $\varphi^2 \cdot v$ according to our rule:

$$\begin{aligned}(L_\varepsilon v, \varphi^2 v) = (\varphi \bar{h}, \varphi v) &\equiv \sum_{j=0}^{m} \varepsilon^{2j}\big(L'_{2j}(\varphi v), \varphi v\big) \\ &+ \sum_{i=1}^{2m} \varepsilon^i \sum_{\substack{|\alpha|+|\beta|+|\gamma_1|+|\gamma_2|=i \\ |\gamma_1|+|\gamma_2|\geqq 1}} \int b_{\alpha\beta\gamma_1\gamma_2} D^\alpha v D^\beta v D^{\gamma_1}\varphi D^{\gamma_2}\varphi \, dx,\end{aligned} \tag{20}$$

where may be assumed $|\alpha|, |\beta| \leqq m$. By means of (17), the integrals appearing here in the second sum are to be seen majorized by expressions of type

$$\frac{C}{\eta^r} \|\varphi D^p v\| \, \|\varphi D^q v\|, \qquad q \leqq p \leqq q+1, \quad p+q+r = i, \quad r \geqq 1;$$

here D^p, D^q denote any derivatives of total order p or q. Application of the inequality $ab \leqq \frac{1}{2}\left(\varrho a^2 + \frac{1}{\varrho} b^2\right)$ with $\varrho = \varepsilon^{2p+r-i}$ leads to bounds

$$C_i\{\varepsilon^{2p} \|\varphi D^p v\|^2 + \varepsilon^{2q} \|\varphi D^q v\|^2\} \cdot \varepsilon^{r\delta}, \qquad r \geqq 1, \tag{21}$$

for the integrals in (20) multiplied by ε^i (remember $\eta = \varepsilon^{1-\delta}$). On the other hand, the principal terms in (20) — fisrt sum — will obey, in our weakened sense, Gårding's inequality (5), i.e.

$$\big(L'_{2j}(\varphi v), \varphi v\big) \geqq A_j \|\varphi v\|_j^2 - B_j \|\varphi v\|_{j-1}^2 .$$

Finally, in order to adapt this estimation to (21), we shall express the norms of $D^\alpha(\varphi v)$ here by those of $\varphi D^\alpha v$ in (21). Using triangle inequality and (17) we obtain

$$\|D^\alpha(\varphi v)\| \geqq \|\varphi D^\alpha v\| - \sum_{|\beta| \leqq |\alpha|-1}' \frac{c_\beta}{\eta^{|\alpha|-|\beta|}} \|\varphi D^\beta v\|$$

and therefore, for $j \geqq 1$

$$\begin{aligned}&\varepsilon^{2j}\big(L'_{2j}(\varphi v), \varphi v\big) \\ &\geqq \varepsilon^{2j} A_j \sum_{|\alpha|=j} \|\varphi D^\alpha v\|^2 - \sum_{i=0}^{j-1} \sum_{|\beta|=i} c_{j\beta} \varepsilon^{2i+2(j-i)\delta} \|\varphi D^\beta v\|^2 .\end{aligned} \tag{22}$$

Estimating now the right hand side of (20) by means of (21) and (22) we arrive at the desired result

$$\sum_{j=0}^{m} \varepsilon^{2j} \sum_{|\alpha|=j} C_\alpha \|\varphi D^\alpha v\|^2 \leqq (\varphi \bar{h}, \varphi v) \leqq \|\varphi \bar{h}\| \, \|\varphi v\| \tag{23}$$

for ε small ($\varepsilon \leqq \varepsilon_1$) with positive constants C_α independent of ε and $\mathring{x} \in G_{\bar{\eta}}$, and especially

$$\|\varphi v\| \leqq C \|\varphi \bar{h}\| \tag{24}$$

for any derivative v of order l of the solution u. Taking in consideration the structure of $\bar{h}$ (cf. (19)) and the preliminary estimates (12), (14) we observe

$$\|\varphi v\| \leqq \frac{C}{\varepsilon^{l-1}},$$

i.e., $\varphi v = \varphi D^\alpha u$ obeys a general estimation of type (14), but improved by a factor ε compared with v itself according (14). Now, in turn, the improved estimate could at once be applied to $\bar{h}$ again, for u and its derivatives occur in $\bar{h}$ only multiplied by φ. This entails a further lifting of the power of ε in the last inequality, and finally we can conclude, in this way,

$$\|\varphi D^\alpha u\| \leqq C_\alpha \tag{25}$$

for all derivatives of u in G_ε with constants C_α independent of ε ($\varepsilon \leqq \varepsilon_1$) and $\mathring{x} \in G_{\bar{\eta}}$.

5. Uniform bounds. From Sobolev's imbedding theorem for the n-dimensional unit ball B_1

$$\sup_{B_1} |u(x)| \leqq C_1 \|u\|_{l_0}^{B_1}, \qquad l_0 = \left[\frac{n}{2}\right] + 1,$$

it follows

$$\sup_{B_\varrho} |u(x)| \leqq C_1 \varrho^{-n/2} \sum_{|\gamma|=0}^{l_0} \varrho^{|\gamma|} \|D^\gamma u\|^{B_\varrho}$$

for a ball B_ϱ with radius ϱ. Application to $v = D^\alpha u$, $|\alpha| = l \geqq l_0$, in B_η (centred at $\mathring{x}$) yields

$$\sup_{B_\eta} |v(x)| \leqq C_1{}' \eta^{-n/2} \sum_{|\gamma|=0}^{l_0} \eta^{|\gamma|} C_{\alpha+\gamma}$$

because of (25) and $\varphi \geqq \dfrac{1}{1 + \mathrm{e}}$ in B_η; therefore

$$\sup_{x \in G_{\bar{\eta}}} |D^\alpha u(x)| \leqq \frac{C}{\sqrt{\eta}^{\,n}} \tag{26}$$

for every derivative of the solution of (4). By means of the differential equation of (4) and with regard to the reduced equation (7) we see that in $G_{\bar{\eta}}$ at least

$$|u(x)| \leqq \left\{\max \frac{C_\alpha}{\sqrt{\eta}^{\,n}} \cdot \varepsilon, \frac{1}{A_0} \sup |h|\right\}$$

($a_0 \geqq A_0 > 0$). Differentiating (4) we obtain analogous estimates for all derivatives of u (of course, h replaced by its corresponding derivative), so that we successively can improve the result until we arrive at

$$\sup_{x \in G_{\bar{\eta}}} |D^\alpha u(x)| \leqq C_\alpha, \tag{27}$$

i.e., the uniform boundedness of all derivatives of u in the expanding and, for $\varepsilon \to 0$, exhausting subdomain $G_{\bar{\eta}}$ ($\bar{\eta} = \varepsilon^{1-2\delta}$).

6. Uniform convergence. Based on (27) we immediately perceive the uniform pointwise convergence of $u = u_\varepsilon$ to w, the solution of the reduced problem (7): After division by $a_0(x)$ the equation of (4) reads

$$\varepsilon B(x) + u_\varepsilon = \frac{h}{a_0} = w$$

with $B(x)$ uniformly bounded in $G_{\bar\eta}$ on account of (27).

Summarizing, we may establish the

Theorem 1: *Coefficients and right hand side of equation* (4) *shall be smooth in the n-dimensional bounded domain G; the boundary ∂G is supposed regular enough (at least piecewise C^1) in order the problem* (4) *be solvable for all positive $\varepsilon \leqq \varepsilon_0'$, the solution $u = u_\varepsilon$ being smooth in the interior of G. Then, for $\varepsilon \to +0$, u_ε converges in each point $x \in G$ to the solution w of the reduced problem* (7), *and so do all derivatives. This convergence is uniform in the sense*

$$\lim_{\varepsilon\to 0} \sup_{x\in G_{\bar\eta}} |D^\alpha u_\varepsilon(x) - D^\alpha w(x)| = 0,$$

i.e., in the set $G_{\bar\eta}$ of all points of G with distance at least $\bar\eta = c \cdot \varepsilon^{1-2\delta}$ to the boundary ∂G (c, δ positive constants arbitrarily chosen).

7. Extension in the case k = 0 of "totally degenerating" problem. In equation (4) the operators L_i are multiplied by that power of ε which exactly coincides with their orders. As easily to be seen, Theorem 1 also comprehends the situation of additional regular perturbations by continuous dependence of coefficients and right hand side of ε or, especially, if $\varepsilon^i L_i$ in (4) is replaced by $\varepsilon^{i+k(i)} L_i$ with $k(i) \geqq 0$ for $0 < i < 2m$ (but $k(0) = k(2m) = 0$); condition (6) will not be violated.

But it is of some interest that our method will work also in the case of superposing singular perturbations, that is, if the higher derivatives are multiplied by another ε-power besides ε^i. After some strengthening the premises concerning the ellipticity constants A_j the proof of the following assertion will run just as explained in **2.** to **6.** above, at least, if we, for simplicity, assume the coefficients to be constant.

Theorem 2: *With notations and under regularity conditions of Theorem* 1, *moreover assuming constant coefficients, the solution $u = u_\varepsilon$ of*

$$\begin{cases} G\colon L_\varepsilon u := \sum_{i=0}^{2m} \varepsilon^{k_i} L_i u = h \\ \partial G\colon D^\gamma u = 0 \qquad (|\gamma| \leqq m-1) \end{cases} \tag{28}$$

with $k_0 = 0$, $k_{i+1} \geqq k_i + 1$ $(i = 0, 1, \ldots, 2m-1)$, and ellipticity condition (5) *sharpened by claiming all ellipticity constants $A_j > 0$ $(j = 0, 1, \ldots, m)$, will converge quasi-uniformly to the solution w of the reduced problem* (7), *i.e.,*

$$\lim_{\varepsilon\to 0} \sup_{x\in G_{\bar\eta}} |D^\alpha u_\varepsilon(x) - D^\alpha w(x)| = 0$$

as in Theorem 1.

As to the proof, we only mention that the preliminary estimate can be derived as shown in **2.** but, on account of the more stringent condition $A_j > 0$ for all j, without use of the interpolation lemma, and with aid of Lemma 2 of the appendix stating that also in this case the derivatives of the solution will grow, for $\varepsilon \to 0$, at the most as powers of ε^{-1}, and that will do, because in the constant coefficient case the essential step **4.** will be finished already at (24).

8. Remarks to the case k $\geqq$ 1. Unfortunately, our simple method will not work off-hand in the case $k \geqq 1$ in general where the reduced problem is of positive order. Some exemplifications will indicate the range of it rather restricted. Amongst other things, additional smoothness conditions of the boundary must be imposed. In the following we shall denote by G' a subdomain of G with sufficiently smooth boundary $\partial G'$ (corners and edges rounded off), and by $\|\cdot\|_k'$ the Sobolev norms with respect to G'.

We will occupy ourselves with fourth order problems which are (nearly) factorable. Let Δ and L be elliptic operators of second order with smooth coefficients in G so that

$$(-\Delta v, v) \geqq c_1 \|v\|_1^2 - c_0 \|v\|^2, \qquad (-Lv, v) \geqq d_1 \|v\|_1^2 - d_0 \|v\|^2 \tag{29}$$

for v vanishing at ∂G, with positive constants c_1, d_1. The formal adjoint operator Δ^* of Δ will obey such a Garding's inequality too.

Proposition 1: *Let be $c_0 = 0$ in (29). The solution $u = u_\varepsilon$ of*

$$\begin{cases} G: \varepsilon^2 \Delta L u - \Delta u = h \\ \partial G: u = Lu = 0 \end{cases} \tag{30}$$

converges to the solution of the degenerate problem

$$G: -\Delta w = h, \qquad \partial G: w = 0$$

in the sense of Theorem 1:

$$\lim_{\varepsilon \to 0} \sup_{x \in G_{\bar\eta}'} |D^\alpha u(x) - D^\alpha w(x)| = 0$$

for every α ($G_{\bar\eta}'$ is the subset of $x \in G'$ with distance to the boundary $\partial G'$ more than $\bar\eta = \varepsilon^{1-2\delta}$).

If $n = 2$, and $\Delta = L$ the Laplacian operator, the problem could be considered as a model for a supported membrane with small stiffness.

Proof: As $\varepsilon^2 \Delta L u - \Delta u = -\Delta(-\varepsilon^2 L u + u)$ we shall, of course, set $-\varepsilon^2 L u + u = w$, and this function w must then be the solution of the degenerate problem. Because of $c_0 = 0$ (29) implies $\|w\|_1 \leqq C \|h\|$, and by the well-known a-priori estimates for subdomains G' we obtain for any l $\|w\|_{2+l}' \leqq C_l \|h\|_l$. We now pass to the "interior" problem

$$\begin{cases} G: -\varepsilon^2 L u + u = w \\ \partial G: \qquad u = 0 \end{cases} \tag{31}$$

and derive first, using (29), $\|u\| + \varepsilon \|u\|_1 \leqq C \|w\| \leqq C' \|h\|$ and, according to the appendix, $\|u\|_{1+p}' \leqq \dfrac{C_p}{\varepsilon^{1+p}}$ ($p = 0, 1, \ldots$). All derivatives of the right hand side w are bounded, and preliminary estimates of type (12), (14) for the solution u of (31) are valid, so that the procedure leading to Theorem 1 pursuant steps **3.** to **6.** may work with respect to problem (31) in G' too ∎

Proposition 2: *If $u = u_\varepsilon$ is the solution of*

$$\begin{cases} G: \varepsilon^2 \Delta \Delta^* u - \Delta u + cu = h \\ \partial G: u = \Delta^* z = 0, \end{cases} \tag{32}$$

$c = c(x) \geqq c_0' > c_0$ (cf. (29)), then the assertion of Proposition 1 holds with respect to w given by the degenerate problem

$$G\colon -\Delta w + cw = h, \qquad \partial G\colon w = 0. \tag{32_0}$$

Proof: At first we must look for preliminary estimates as set up in 2. We multiply by u, integrate by parts, and use (29):

$$\varepsilon^2 \|\Delta^* u\|^2 + c_1 \|u\|_1^2 - c_0 \|u\|^2 + c_0' \|u\|^2 \leqq \|h\| \, \|u\|$$

and therefore

$$\|u\|_1 \leqq C \|h\|, \qquad \varepsilon \|\Delta^* u\| \leqq C \|h\|. \tag{33}$$

The last inequality gives rise to

$$\|u\|_2' \leqq C\{\|\Delta^* u\| + \|u\|\} \leqq \frac{1}{\varepsilon} C \|h\|$$

in G'. Since $\Delta\Delta^*$ is elliptic too, the premises of Lemma 2 of the appendix are satisfied, hence

$$\|u\|_p'' \leqq \frac{C_p}{\varepsilon^{[p-1]^+}} \quad (p = 0, 1, \ldots) \tag{34}$$

where $\|\cdot\|''$ is to be taken over G or, for $p \geqq 2$, $G'_{(p-2)\varepsilon}$, respectively; $[x]^+ = \max\{x, 0\}$. As easily to be seen, the same result is valid for the solution $\bar{u}$ of the slightly altered problem

$$\begin{cases} G\colon \bar{L}_\varepsilon \bar{u} := \varepsilon^2 \Delta\Delta^* \bar{u} - \Delta \bar{u} - \varepsilon^2 c \Delta^* \bar{u} + c\bar{u} = h \\ \partial G\colon \bar{u} = \Delta^* \bar{u} = 0. \end{cases} \tag{35}$$

But this problem proves factorable, for $L_\varepsilon u = (-\Delta + c)(-\varepsilon^2 \Delta^* + 1)\, \bar{u}$. As in the first case we set

$$-\varepsilon^2 \Delta^* \bar{u} + \bar{u} = w \tag{36}$$

which is the solution of the degenerate problem (32_0). The conditions on $-\Delta$ and c let us obtain

$$\|w\|_{2+p}' \leqq C_p \|h\|_p \tag{37}$$

and, consequently, for the solution $\bar{u}$ of equation (36) the assertion of theorem will hold — we remember the fact that its proof dont make use of boundary values immediately, but only of a preliminary estimate as (34) here. Thus

$$\lim_{\varepsilon \to 0} \sup_{x \in G'_{\bar{\eta}}} |D^\alpha \bar{u}(x) - D^\alpha w(x)| = 0. \tag{38}$$

Yet we have to become sure the additional regular perturbation by $-\varepsilon^2 c \Delta^* \bar{u}$ does not affect this property of u itself. To this end we shall, of course, set $u - \bar{u} = z_1$ so that

$$\begin{cases} G\colon L z_1 := \varepsilon^2 \Delta\Delta^* z_1 - \Delta z_1 + c z_1 = -\varepsilon^2 c \Delta^* \bar{u} \\ \partial G\colon z_1 = \Delta^* z_1 = 0. \end{cases}$$

We replace z_1 by $\bar{z}_1$ which is defined as the solution of

$$\begin{cases} G\colon \bar{L}_\varepsilon \bar{z}_1 = -\varepsilon^2 c \Delta^* \bar{u} \\ \partial G\colon \bar{z}_1 = \Delta^* \bar{z}_1 = 0 \end{cases}$$

with $\|-\varepsilon^2 c\Delta^*\bar{u}\| \leqq \varepsilon C\,\|h\|$ according to (33), and therefore this estimate will be true also for $\bar{z}_1$ but with $\varepsilon\,\|h\|$ instead of $\|h\|$. Apparently $\bar{z}_1$ obeys the assertion of Theorem 1 with 0 as solution of the reduced problem.

Now we construct recursively z_i, $\bar{z}_i$ by $z_{i+1} = z_i - \bar{z}_i$,

$$\begin{cases} G: \bar{L}_\varepsilon \bar{z}_{i+1} = -\varepsilon^2 c\Delta^*\bar{z}_i \\ \partial G: \bar{z}_{i+1} = \Delta^*\bar{z}_{i+1} = 0 \end{cases}$$

beginning with $z_0 = u$, $\bar{z}_0 = \bar{u}$, and we obtain successively

$$\|\bar{z}_i\|_1 \leqq C\,\|h\|\,\varepsilon^i, \qquad \|\Delta^*\bar{z}_i\| \leqq C\,\|h\|\,\varepsilon^{i-1},$$

and this is valid also for the z_i themselves (they satisfy the problem with the original operator L_ε instead of $\bar{L}_\varepsilon$, but with the same right hand side), while for the $\bar{z}_i$ even

$$\lim_{\varepsilon\to 0}\ \sup_{x\in G_{\bar{\eta}}'} |D^\alpha \bar{z}_i(x)| = 0 \tag{39}$$

for every α. Thus we have obtained

$$u = \bar{u} + \bar{z}_1 + \cdots + \bar{z}_r + z_{r+1} \tag{40}$$

with (38), (39), and, by (34),

$$\|z_{r+1}\|_{1+p}'' \leqq C_p \varepsilon^{r-p+1}$$

whence the assertion for r sufficiently large ∎

Appendix. As just inducated in Sections 2 and 7, we will complete the discussion there by the proofs of two lemmata concerning the rough basic estimates (cf. (14)) affirming the derivatives of the solution to grow at most as (negative) powers of ε.

Lemma 1: *Let $u = u_\varepsilon$ be a solution of the elliptic equation of order $2m = 2k + 2l$*

$$L_\varepsilon u = \sum_{j=0}^{2l} \varepsilon^j L_{2k+j} u + \sum_{i=0}^{2k} L_i u = h \tag{41}$$

in the domain G (L_r denoting a differential operator of order r) with the property

$$\|u\|_i \leqq \frac{C_i}{\varepsilon^i}\,\|h\| \quad \text{for} \quad 0 \leqq i \leqq m \tag{42}$$

or

$$\|u\|_i \leqq \frac{C_i}{\varepsilon^{[i-1]^+}}\,\|h\| \quad \text{for} \quad 0 \leqq i \leqq m \tag{42'}$$

($[x]^+ = \max\{x, 0\}$). *Furthermore the operator L_ε is assumed positive in the sense*

$$(L_\varepsilon v, v) \geqq \varepsilon^{2l}(A\,\|v\|_m{}^2 - B\,\|v\|^2), \qquad A > 0, \tag{43}$$

for all v with compact support in G.

Then, for $p \geqq 0$ it is valid

$$\|u\|_{m+p}^{(p)} \leqq C_{m+p}\left\{\frac{1}{\varepsilon^{m+p}}\,\|h\| + \frac{1}{\varepsilon^{l-k}} \sum_{\nu=0}^{p-1} \frac{1}{\varepsilon^\nu}\,\|h\|_{p-\nu}^{(p-1-\nu)}\right\} \tag{44}$$

where $\|\cdot\|_q^{(\nu)}$ denotes the usual Sobolev norm of order q in the subdomain $G^{(\nu)} = G_{\nu d\varepsilon}$ of all points of G with distance at least $\nu d\varepsilon$ to the boundary ∂G (cf. (13)); C_{m+p} is indepen-

dent of ε *(but may depend on the — arbitrarily chosen — positive constant* d*). In the case* (42′) *the denominator* ε^{m+p} *in* (44) *may be replaced by* ε^{m+p-1}.

For $p = 0$ the sum in (44) is to be set zero, of course.

Proof: For $p = 0$ the assertion (44) coincides with supposition (42) (resp. (42′)), $i = m$, and we will assume it to be valid for all $p < q$ for some $q \geqq 1$.

We shall use test functions ψ_i ($i = 1, 2, \ldots$) which are equal to 1 in $G^{(i)}$, vanish outside $G^{(i-1)}$, and have the property $|D^\alpha \psi_i(x)| \leqq C_\alpha \cdot \varepsilon^{-|\alpha|}$ with appropriate constants C_α.

A remark concerning the construction of such $\psi = \psi_i$: The homothetic transformation $x'' = x/\varepsilon$ will take the domains G and $G_{d\varepsilon}$ to G'' and $G_d'' = \{x'' \in G'' : \operatorname{dist}(x'', \partial G'') > d\}$. We choose a finite covering of the closure of G_d'' by balls B_i ($i = 1, \ldots, n$; $n = n(\varepsilon)$) of diameter $d/2$ so that the concentric balls B_i'' constitute a covering with order bounded independently of ε. If η_i are the usual local test functions which are equal 1 in B_i and equal 0 out of B_i'', then

$$\psi_1(x) = 1 - \prod_{i=1}^{n} (1 - \eta_i(x))$$

will possess the claimed properties concerning $G = G^{(0)}$ and $G^{(1)}$.

In order to prove now (44) for $p = q$ we first set up a differential equation for $v = D^q u$ where, for simplicity, D^q denotes any derivative of order q, and apply D^q to equation (41):

$$D^q L_\varepsilon u = L_\varepsilon v + \cdots = D^q h, \text{ resp.}$$

$$L_\varepsilon v = D^q h + \sum_{j=1}^{2l} \varepsilon^j \sum_{|\alpha| = j+2k} \sum_{\substack{\beta \leqq \alpha \\ |\beta| \geqq 1}} D^\beta a_\alpha D^{\alpha-\beta} u + \sum_{|\alpha| \leqq 2k} \sum_{\substack{\beta \leqq \alpha \\ |\beta| \geqq 1}} D^\beta a_\alpha D^{\alpha-\beta} u.$$

Secondly, with $\psi = \psi_q$ we have

$$L_\varepsilon(\psi v) = \psi L_\varepsilon v + \sum_{j=1}^{2l} \varepsilon^j \sum_{|\alpha| = 2k+j} \sum_{\substack{\gamma \leqq \alpha \\ |\gamma| \geqq 1}} a_\alpha D^\gamma \psi D^{\alpha-\gamma} v + \sum_{|\alpha| \leqq 2k} a_\alpha \sum_{\substack{\gamma \leqq \alpha \\ |\gamma| \geqq 1}} D^\gamma \psi D^{\alpha-\gamma} v,$$

and here we can substitute $\psi L_\varepsilon v$ by the previous equation. After that we shall multiply the last equation by ψv and integrate by parts in such a manner that the derivatives of u appearing in each scalar product are of minimal orders, i.e., they differ in order at most by one. Thus we will attain to

$$\big(L_\varepsilon(\psi v), \psi v\big) = (\psi D^q h, \psi v) + \varepsilon^{2l} \sum_{\substack{|\alpha| = |\beta| = k+l \\ |\gamma| = 1,\, \gamma \leqq \alpha}} (a_{\alpha+\beta} D^\gamma \psi D^{\alpha-\gamma} v, \psi D^\beta v) + \cdots$$

plus lower order terms. Applying Schwarz's inequality and, additionally in the case of different orders in a scalar product, the arithmetic-geometric-mean inequality with appropriate weights, we are abel to estimate the right hand side by

$$\varepsilon^{2k} \|\psi D^q h\|^2 + \varepsilon^{-2k} \|\psi v\|^2 + \varepsilon^{2l} \frac{A}{2} \sum_{|\beta| = k+l} \|\psi D^\beta v\|^2 + \varepsilon^{2l} C \sum_{\substack{|\alpha| = k+l \\ |\gamma| = 1}} \|D^\gamma \psi D^{\alpha-\gamma} v\|^2$$

$$+ \sum_{j=1}^{2l} \varepsilon^j \sum_{\substack{|\alpha_1| + |\alpha_2| + |\gamma_1| + |\gamma_2| = 2k+j \\ |\gamma_1| + |\gamma_2| \geqq 1,\ ||\alpha_1| - |\alpha_2|| \leqq 1}} C_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \|D^{\gamma_1} \psi D^{\alpha_1} v\| \, \|D^{\gamma_2} \psi D^{\alpha_2} v\|$$

$$+ \sum_{i=1}^{2k} \sum_{\substack{|\alpha_1| + |\alpha_2| + |\gamma_1| + |\gamma_2| = i \\ |\gamma_1| - |\gamma_2| \geqq 1,\ ||\alpha_1| - |\alpha_2|| \leqq 1}} C'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \|D^{\gamma_1} \psi D^{\alpha_1} v\| \, \|D^{\gamma_2} \psi D^{\alpha_2} v\| + \cdots$$

where the terms omitted are formed analogously by those of the expression for $L_\varepsilon v$ above; compared with the corresponding ones indicated here they are of minor order (because v is at least a derivative of order one). Taking into account $|D^\gamma\psi| \leqq v_\gamma \varepsilon^{-|\gamma|}$ and $v = D^q u$ we shall see

$$\varepsilon^j \|D^{\gamma_1}\psi D^{\alpha_1}v\| \, \|D^{\gamma_2}\psi D^{\alpha_2}v\| \leqq \begin{cases} C\dfrac{\varepsilon^j}{\varepsilon^{2k+j-2s}} \|u\|_{q+s}^{(q-1)^2} & \text{for } |\alpha_1| = |\alpha_2| = s \\ C\dfrac{\varepsilon^j}{\varepsilon^{2k+j-2s-1}}\left(\varepsilon \|u\|_{q+s+1}^{(q-1)^2} + \dfrac{1}{\varepsilon}\|u\|_{q+s}^{(q-1)^2}\right) & \text{for } |\alpha_1| = |\alpha_2| + 1 = s+1 \end{cases}$$

with $s = 0, 1, \ldots, k + \left[\dfrac{j}{2}\right]$, $j = 1, \ldots, 2l - 1$, and, for the second sum,

$$\|D^{\gamma_1}\psi D^{\alpha_1}v\| \, \|D^{\gamma_2}\psi D^{\alpha_2}v\| \leqq \begin{cases} C\dfrac{1}{\varepsilon^{i-2s}} \|u\|_{q+s}^{(q-1)^2} & \text{for } |\alpha_1| = |\alpha_2| = s \\ C\dfrac{1}{\varepsilon^{i-2s-1}}\left(\varepsilon \|u\|_{q+s+1}^{(q-1)^2} + \dfrac{1}{\varepsilon}\|u\|_{q+s}^{(q-1)^2}\right) & \text{for } |\alpha_1| = |\alpha_2| + 1 = s+1 \end{cases}$$

with $s = 0, 1, \ldots, \left[\dfrac{i}{2}\right]$, $i = 1, \ldots, 2k$.

Summarizing we obtain

$$\left(L_\varepsilon(\psi v), \psi v\right) - \varepsilon^{2l}\frac{A}{2}\sum_{|\beta|=k+l} \|\psi D^\beta v\|^2 \leqq C\left\{\sum_{s=0}^{m-1} \frac{1}{\varepsilon^{2k-2s}} \|u\|_{q+s}^{(p-1)^2} + \frac{1}{\varepsilon^{2k}}\|u\|_q^{(q-1)^2} + \varepsilon^{2k}\|h\|_q^{(q-1)^2}\right\}$$

and, due to (43),

$$\|\psi D^q u\|_m \leqq \frac{C}{\varepsilon^m}\sum_{s=0}^{m-1} \varepsilon^s \|u\|_{q+s}^{(q-1)} + \frac{C'}{\varepsilon^{l-k}}\|h\|_q^{(q-1)}. \tag{45}$$

Using now the proposition (44) for the norms of u here and (42) we have

$$\|u\|_{m+q}^{(q)} \leqq \frac{C}{\varepsilon^m}\sum_{s=0}^{m-1}\frac{\varepsilon^s}{\varepsilon^{q+s}}\|h\| + \frac{C}{\varepsilon^m}\sum_{s=m-q+1}^{m-1}\frac{\varepsilon^s}{\varepsilon^{l-k}}\sum_{\nu=0}^{s+q-m-1}\frac{1}{\varepsilon^\nu}\|h\|_{s+q-m-1-\nu}^{(s+q-m-2-\nu)} + \frac{C'}{\varepsilon^{l-k}}\|h\|_q^{(q-1)},$$

whence

$$\|u\|_{m+q}^{(q)} \leqq \frac{C}{\varepsilon^{m+q}}\|h\| + \frac{C}{\varepsilon^{l-k}}\sum_{\mu=0}^{q-1}\frac{1}{\varepsilon^\mu}\|h\|_{q-\mu}^{(q-\mu-1)},$$

the assertion (44) for q instead of $p < q$. In the case of (42') instead of (42) the relevant assertion will be reproduced in the same way ∎

Remark: It might be conjectured that in the case where in (42) the order ε^{-i} could be replaced by $\varepsilon^{-[i-k]^+}$ (e.g. if for u are given homogeneous Dirichlet data) in the assertion (44) likewise $\varepsilon^{-m-p}\|h\|$ could be replaced by $\varepsilon^{-l-p}\|h\|$; but our proof will fail for the first $k - 1$ steps of induction.

Lemma 2: *Let $u = u_\varepsilon$ be the solution of the problem of Theorem 2*

$$G: L_\varepsilon u = \sum_{i=0}^{2m} \varepsilon^{k_i} L_i u = h, \qquad \partial G: D^\gamma u = 0 \quad (|\gamma| \leqq m-1) \tag{28}$$

with coefficients and right hand side sufficiently smooth and ellipticity constants of all L_{2j} positive. Then there is valid a preliminary estimate

$$\|u\|_i^{(|i-m|^+)} \leqq \frac{C_i}{\varepsilon^{ri}} \qquad (i = 0, 1, \ldots) \tag{46}$$

where C_i may depend on h and d but not on ε; $k_0 = 0$, and

$$r = \max_{1 \leqq i \leqq 2m} \{k_i - k_{i-1}\} \geqq 1. \tag{47}$$

We remember the norm $\|\cdot\|^{(i)}$ refers to subdomain $G_{id\varepsilon}$. The proof starts with establishing the inequality

$$\|u\|_i \leqq c_i \frac{\|h\|}{\varepsilon^{k_i}} \leqq \frac{C_i}{\varepsilon^{ri}}$$

for $i = 0, \ldots, m$, by scalar multiplying the equation by u and integration by parts. Then, for $i = m + p$, $p \geqq 0$, we follow up the induction argument of the proof of the preceding lemma, replacing ε^j by ε^{k_j}. The essential inequality (45) will then read (observe $k = 0$)

$$\begin{aligned} \|\psi D^q u\|_m &\leqq \frac{C}{\sqrt{\varepsilon^{k_{2m}}}} \left\{ \sum_{s=0}^{m-1} \sum_{j=2s}^{2m-1} \sqrt{\frac{\varepsilon^{k_j}}{\varepsilon^{j-2s}}} \|u\|_{q+s}^{(q-1)} + \|h\|_q^{(q-1)} \right\} \\ &\leqq \frac{C}{\varepsilon^{rm}} \left\{ \sum_{s=0}^{m-1} \varepsilon^{rs} \|u\|_{q+1}^{(q-1)} + \|h\|_q^{(q-1)} \right\} \end{aligned}$$

if we pay regard to the fact that $\varepsilon^{k_j}/\varepsilon^{j-2s}$ is maximal for $j = 2s$, and $k_{2s} \geqq k_{2m} - 2(m - s)\,r$. Now the assertion (46) is at once to be seen reproducing itself.

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