Interior Estimates for Singularly Perturbed Problems

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Die Lösung des Dirichletproblems für eine singulär gestörte elliptische Differentialgleichung 2m-ter Ordnung $\epsilon L_1 u + L_0 u = h$ konvergiert außerhalb der Grenzschicht für $\epsilon \to 0$ gleichmäßig gegen eine Lösung der ausgearteten elliptischen Gleichung $L_0w = h$ niedrigerer Ordnung. Es wird gezeigt, daß sich im Fall nullter Ordnung von L_0 dieser Sachverhalt unmittelbar beweisen läßt, d. h. ohne die übliche Konstruktion der Grenzschicht, und zwar elementar bei geringen Glattheitsforderungen an den Rand des Gebietes.

Решение проблемы Дирихле для сингулярно возмущенного дифференциального уравнения эллиптического типа $\varepsilon L_1 u + L_0 u = h$ порядка $2m$ сходится, для $\varepsilon \to 0$, равномерно вне пограничного слоя к решению вырожденного уравнения эллиптического типа $L_0u=h$ низшего порядка. Оказывается, что в случае порядка 0 оператора L_0 этот факт можно доказать непосредственно, т. е. без конструкции пограничного слоя, к тому же элементарно и с незначительными предположениями гладкости границы области.

The solution of the Dirichlet problem for a singularly perturbed elliptic differential equation $\epsilon L_1 u + L_0 u = h$ of order 2*m* converges, for $\epsilon \to 0$, outside of the boundary layer uniformly to a solution of the degenerate elliptic equation $L_0w = h$ of lower order. It is shown in the case of order zero of L_0 this assertion may be proved immediately, i.e., without the usual construction of boundary layer terms, but rather elementary and on weak smoothness conditions with respect to the boundary of the domain.

As well known, the solution $u = u_t$ of the singularly perturbed Dirichlet problem of order $2m$ in an n-dimensional bounded domain G

$$
\begin{cases}\nG: \ \varepsilon L_1 u + L_0 u = h \\
\partial G: \quad D^{\gamma} u = 0 \quad (|\gamma| \leq m - 1)\n\end{cases} \tag{1}
$$

behaves – in the case of "regular degeneration" – as follows for $\varepsilon \to +0$ (cf., e.g., $[5-7]$: In every compact subdomain $\bar{G}' \subseteq G$ we have uniform convergence to the solution w_0 of the degenerate (reduced) elliptic problem of order 2k ($k < m$)

$$
G: L_0 w_0 = h, \qquad \partial G: D^{\gamma} w_0 = 0 \qquad (|\gamma| \le k - 1)
$$
 (2)

whilest in a narrow strip Γ_c along the boundary ∂G of G arises a so-called (Prandlt's) boundary layer compensating the supernumerary boundary conditions of the perturbed problem ($\varepsilon > 0$) which the solution of the reduced problem in general will fail to satisfy; the width of Γ_{ϵ} is about some power of ϵ . Usually the asymptotic properties of u are studied by an expansion

$$
u_{\epsilon} = w_0 + \epsilon w_1 + \cdots + \epsilon^r w_r + v_0 + \epsilon v_1 + \cdots + \epsilon^s v_s + \epsilon^t z
$$

= $w + v + \epsilon^t z$.

Here the "regular" part w describes the convergence in $G \setminus \Gamma_i$, while the functions v_i are of boundary-layer type: they are smaller than any power of ε outside of the

neighbourhood Γ_{ϵ} of the boundary (exponential decay). This expansion has to be constructed and, after that, one must prove *z* to be: hounded in all of *0* for some positive t . The main tools to be used for the latter are based, finally, on a-priori estimates as shown by AGMON, DOUGLIS and NIRENBERG $[1]$; for details $-$ far-from being trivial in general $-$ we refer, e.g., to BESJES $[2]$, or to the monograph $[3]$. But it seems that, until now, only in the simplest case **D.** Gönder
 C: 2 *G: C: C*

$$
G\colon -\ \varepsilon^2\,\varDelta u\,+\,u\,=\,h,\qquad \partial G\colon u\,=\,0
$$

the attempt has been made to prove directly the regular behaviour of *u* in $G \setminus P_i$: L. TARTAR derived [5: p. 131]

Input has been made to prove direct

\n
$$
\mathbf{R} \text{ derived } [5: \text{p. } 131]
$$

\n
$$
\int_{\mathcal{I}} \sum \left[\frac{\partial}{\partial x_i} (u - h) \right]^2 dx \leq C \|h\|_1^2
$$

with

 $G' = \{x \in G : \text{dist}(x, \partial G) \geq \varepsilon^{\alpha}, 0 < \alpha < 1\}.$

In the present paper we will submit a more general procedure, elementary in the main, which enables to prove even uniform pointwise convergence of *u* in $G \setminus T_e$ and, moreover, does not claim (for itself) higher regularity of the boundary as apriori estimates do in general.

0. Introduction. In order to give an outline of the method we will sketch it in the simple case (3) (cf. [41).

Multiplying the equation by *u* and integrating by parts yield at once $||u||$ and ε $||Du||$ bounded (in the L₂-norm of G, D any first order derivative); using, e.g., a-priori estimates just mentioned it is possible to extend this result to derivatives of any order $l: \varepsilon^l ||D^l u|| \leq C$ (by the way, we cannot expect essential improvements in general!).

Now we introduce "quasi-testfunctions" $\varphi = \varphi(x; \varepsilon, \hat{x})$ which are equal 1 for $x = \hat{x}$, the generic point under consideration in the interior of *G*, and of order $O(\exp(-c/\epsilon^{\delta}))$ outside the ball of radius $\epsilon^{1-\delta}$ centred at \hat{x} so that $\varphi D^{\alpha}u$ will there also be small relative to any power of ϵ . The advantage of φ in comparison with usual testfunctions is the fact that it is, in some sense, reproducing itself:

$$
D\varphi\,=\,\frac{1}{\varepsilon^{1-\delta}}\cdot C(x)\ \varphi\,.
$$

with $C(x)$ smooth and bounded.

Next we set up the equation for $v = D^l u$, multiply by $\varphi^2 \cdot v$, and integration by parts neglecting (boundary-) terms of order $O(\exp(-c/\epsilon^{\delta}))$ leads to

$$
||\varphi D^lu||\leqq C\,.
$$

By means of Sobolev's imbedding theorem for balls we can conclude the uniform boundedness of $D^l u(x)$ for all \dot{x} with distance $\geq \varepsilon^{1-2\delta}$ to the boundary ∂G , and uniform convergence of u and all its derivatives follows immediately via equation (3). id bounded.

he equation for $v = D^t u$, multiply by $\varphi^2 \cdot v$, and integration by

undary-) terms of order $O(\exp(-c/e^s))$ leads to

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ness of $D^l u(\hat{x})$ for all \hat{x} with distance $\geq \varepsilon$

1. Position of the problem. In a bounded *n*-dimensional domain G we shall study the Dirichlet problem of order $2m$

$$
\begin{cases}\nG: L_i u = \sum_{i=0}^{2m} \varepsilon^i L_i u = h \\
\partial G: D^r u = 0 \quad (|\gamma| \leq m - 1),\n\end{cases}
$$

*L*_{*i*} being differential operators of order *i*
 $L_i u = \sum_{|\alpha| \leq i} a_{\alpha}^{(i)} D^{\alpha} u$

$$
L_i u = \sum_{|\mathfrak{a}| \leq i} a_{\mathfrak{a}}^{(i)} D^{\mathfrak{a}} u
$$

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L_i being differential operators of order *i*
 $L_i u = \sum_{|\alpha| \leq i} a_{\alpha}^{(i)} D^{\alpha} u$

with principal parts L_i' ; α, γ are the usual multiindices. The ellipticity condition is posed in terms of Gårding's inequality for the operators of even order $i = 2j$: For Singularly Perturbed P
 *L*_i being differential operators of order i
 $L_i u = \sum_{|\alpha| \leq i} a_{\alpha}^{(i)} D^{\circ} u$

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posed in terms of Gårding's inequalit (ii) $\lim_{n \to \infty} L_i'$; α , γ are the usual multiliterms of Gårding's inequality for the op \hat{H}_j there is valid $(L'_{2j}u, u) \geq A_j ||u||_j^2 - B_j ||u||_{j-1}^2$ ($j = L_0u, u$) $\geq A_0 ||u||_2^2$ gularly Perturbed Problems 31

mdices. The ellipticity condition is

erators of even order $i = 2j$: Fo

1, ..., *m*) (5 $u \in H_{2j}$ of H_j there is valid It is usual multiindices. The ellipticity condition is

y for the operators of even order $i = 2j$: For
 $\begin{pmatrix} \n\frac{1}{j-1} & (j = 1, ..., m) \\
\vdots & \vdots & \vdots \\
\frac{1}{j} & \min \{A_j, 0\} \n\end{pmatrix}^2 \leq c \cdot A_0 A_m,$ (6)

y signify the scalar product

$$
(L'_{2j}u, u) \ge A_j ||u||_j^2 - B_j ||u||_{j-1}^2 \qquad (j = 1, ..., m)
$$

\n
$$
L_0u, u) \ge A_0 ||u||^2
$$
\n(5)

with constants A_j , B_j , the first ones are to obey

$$
(L'_{2j}u, u) \ge A_j ||u||_j^2 - B_j ||u||_{j-1}^2 \qquad (j = 1, ..., m)
$$
\n
$$
L_0u, u) \ge A_0 ||u||^2
$$
\nstands A_j , B_j , the first ones are to obey

\n
$$
A_0 > 0, \qquad A_m > 0, \qquad \left(\sum_{j=1}^{m-1} \min \{A_j, 0\}\right)^2 \le c \cdot A_0 A_m,
$$
\n
$$
(6)
$$

 $c = \text{const} < 1$. Here the brackets (\cdot, \cdot) signify the scalar product in $L_2(G)$, and $\|\cdot\|_f$ denotes the norm in the Sobolev space $H_j = W_2(i)(G)$. The reduced problem $(\epsilon = 0)$ is simply terms of Garding's inequality for the operators of even order $i = 2j$: For H_j there is valid
 $(L'_{2j}u, u) \ge A_j ||u||_j^2 - B_j ||u||_{j-1}^2$ $(j = 1, ..., m)$ (5)
 $L_0u, u) \ge A_0 ||u||^2$

stants A_j, B_j , the first ones are to obey
 $A_0 > 0$

$$
L_0w=a_0w=l
$$

in $G -$ no boundary conditions.

Coefficients and right hand side *h* are assumed sufficiently smooth, and the boundary $\partial \tilde{G}$ to be regular enough in order to guaranty the existence of solutions $u = u_{\epsilon} \epsilon H_{2m} \cap \dot{H}_{m}$ of (4) for small $\epsilon > 0$ which are of class $C^{\infty}(G')$ in each compact in G \rightarrow no boundary conditions.
Coefficients and right hand side h are assumed sufficiently smooth, and the
boundary $\partial \tilde{G}$ to be regular enough in order to guaranty the existence of solutions
 $u = u_{\epsilon} \in H_{2m} \cap \$ $A_0 > 0$, $A_m > 0$, $\left(\sum_{j=1}^{m-1} \min \{A_j, 0\}\right)^2 \leq c \cdot A_0 A_m$,
 $c = \text{const} < 1$. Here the brackets $(.,.)$ signify the scalar product in L ,

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omain $G' \subseteq G$; at least, ∂G

on theorem be applicable in

preliminary estimate. For ϵ
 $\overline{h}, u) = (L_{\epsilon}u, u) \geq \sum_{j=1}^{m}$

$$
L_0w = a_0w = h
$$
\n(7)

\nin G — no boundary conditions.

\nCoefficients and right hand side h are assumed sufficiently smooth, and the boundary ∂G to be regular enough in order to guarantee of solutions $u = u_t \in H_{2m} \cap \hat{H}_m$ of (4) for small $\varepsilon > 0$ which are of class $C^{\infty}(G')$ in each compact subdomain $G' \subseteq G$; at least, ∂G should be piecewise of class C^1 , and Gauss's integration theorem be applicable in G .

\n2. A preliminary estimate. For a solution u of (4) integration by parts yields

\n
$$
\overline{h}_v u = (L_t u, u) \geq \sum_{j=1}^m \varepsilon^{2j} (A_j ||u||_j^2 - B_j ||u||_{j-1}^2) + A_0 ||u||^2
$$
\n
$$
= \sum_{i=1}^{\frac{\varepsilon}{2m}} \varepsilon^i \sum_{j=1}^{\varepsilon} A_j^i ||u||_{[\frac{\varepsilon}{2}]} ||u||_{[\frac{\varepsilon-1}{2}]} \qquad (8)
$$
\nwith proper constants A_j , and $[x]$ denoting the integer part of x . In deriving the last sum it has been made use of the fact

\n
$$
(D^u u, u) = (-1)^{|s|} (u, D^s u) = 0
$$
\nfor $|\alpha|$ odd and, consequently, the possibility to substitute for $(a_s^{(i)} D^s u, u)$ terms of total order less than $|\alpha|$. By the help of arithmetic-geometric-mean inequality used for j even

\n
$$
2 ||u||_{[\frac{\varepsilon}{2}]} ||u||_{[\frac{\varepsilon-1}{2}]} \leq \varepsilon ||u||_{[\frac{\varepsilon}{2}]}^2 + \frac{1}{\varepsilon} ||u||_{[\frac{\varepsilon-1}{2}]}^2
$$
\nwe may conclude from (8)

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$$
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$$

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2 $\|u\|_{\left[\frac{j}{2}\right]}$ $\|u\|_{\left[\frac{j-1}{2}\right]} \le$ ber constants A_j , and [x] denoting the integer part of x. In deriving the
it has been made use of the fact
 $(D^{\circ}u, u) = (-1)^{|s|} (u, D^{\circ}u) = 0$
id and, consequently, the possibility to substitute for $(a_a^{(i)}D^{\circ}u, u)$ terms consequently, the possibility to substitute for (a_i)

nan $|\alpha|$. By the help of arithmetic-geometric-means
 $||u||_{\left[\frac{j-1}{2}\right]} \leq \varepsilon ||u||_{\left[\frac{j}{2}\right]}^2 + \frac{1}{\varepsilon} ||u||_{\left[\frac{j-1}{2}\right]}^2$

from (8)
 $\sum_{j=0}^{m} \varepsilon^{2j} (A_j - \varepsilon A_j'$

$$
2\|u\|_{\left[\frac{j}{2}\right]}\|u\|_{\left[\frac{j-1}{2}\right]}\leq \varepsilon\|u\|_{\left[\frac{j}{2}\right]}^2+\frac{1}{\varepsilon}\|u\|_{\left[\frac{j-1}{2}\right]}^2.
$$

we.may conclude from (8)

$$
(h, u) \geq \sum_{j=0}^m \varepsilon^{2j} (A_j - \varepsilon A_j' - \varepsilon^2 B_{j+1}) ||u||_j^2
$$

-

-

with $A_j' = A_j(\varepsilon)$ uniformly bounded, $B_{m+1} = 0$. Finally, we omit the terms with $A_i > 0$ $(i = 1, ..., m - 1)$, and if $A_i \leq 0$ we use the

Interpolation Lemma: For $u \in \mathring{H}_m$, $\beta \leq \alpha$, $0 < j = |\beta| < |\alpha| = m$, and positive ε , q it is valid

$$
||D^{\beta}u||^{2} \leq \frac{j}{m} q \varepsilon^{2(m-j)} ||D^{\alpha}u||^{2} + \frac{m-j}{m} \frac{1}{q} \varepsilon^{\frac{1}{2}2j} ||u||^{2}.
$$
 (10)

This assertion may be proved by induction based on

$$
||D^{j}u||^{2} = (D^{j}u, D^{j}u) = -(D^{j+1}u, D^{j-1}u)
$$

and the arithmetic-geometric-mean inequality.

Hence we now have

$$
(h, u) \geq \varepsilon^{2m} \left(A_m - \varepsilon C_m + \sum_{j=1}^{m-1} (\overline{A}_j q - \varepsilon C_j) \right) ||u||_m^2
$$

+
$$
\left(A_0 - \varepsilon C_0 + \sum_{j=1}^{m-1} \left(\overline{A}_j \frac{1}{q} - \varepsilon C_j' \right) \right) ||u||^2
$$

where $\overline{A}_j = \min \{A_j, 0\}$, and C_j, C_j proper constants. If we choose $q^2 = A_m/A_0$ we obtain.

$$
A_m + q \sum \bar{A}_j \ge c' A_m, \qquad A_0 + \frac{1}{q} \sum \bar{A}_j \ge c' A_0
$$

with $c' = 1 - \sqrt{c} > 0$ (c the constant in (6)) and, therefore,

$$
(h, u) \ge \varepsilon^{2m} c_m \, ||u||_m^2 + c_0 \, ||u||^2 \tag{11}
$$

for $\varepsilon \leq \varepsilon_0$ with positive constants c_m , c_0 independent of ε and u . A simple application of Schwarz's inequality shows

$$
||u|| \leqq \frac{1}{c_0} ||h||, \qquad \varepsilon^m ||u||_m \leqq \frac{1}{\sqrt{c_0 c_m}} ||h||
$$

which may be extended by interpolation lemma to

$$
\varepsilon^{j} \|u\|_{j} \leq c_{j}' \|h\|, \qquad j = 0, ..., m. \tag{12}
$$

This result can be further extended to orders of derivation beyond m , and that without additional supposition of smoothness if we restrict our consideration to ε -approximating subdomains

$$
G_{\epsilon} = \{x \in G : \text{dist}(x, \partial G) > \epsilon\} \tag{13}
$$

of G. As easily to be seen by Lemma 1 of the appendix (cf. (44)) we can state: For $j > m$ there exists a constant c_i so that for solutions of (4) holds the inequality

$$
\varepsilon^j \|u\|_{\varepsilon,j} \leqq c_j' \sum_{i=0}^{j-m} \varepsilon^i \|h\|_i \qquad (j > m)
$$
 (14)

where $\lVert \cdot \rVert_{\epsilon,i}$ denotes the norm of $H_i(G_\epsilon)$.

Remark: A similar result might be achieved too by utilization of the well-know napriori estimates for solutions of elliptic boundary value problem's (e.g. [1: Chapter 15]) via homothetic transformation $x = \varepsilon \cdot x'$, in the case of smooth boundary even with $\|\cdot\|_j$ instead of $\|\cdot\|_{\epsilon,j}$; but in order to maintain a self-contained elementary treatment as far as possible we establish (14) by integration by parts as done in the appendix.

3. Quasi-test functions. The desired uniform pointwise estimates for the solution u and its derivates shall be set up in the subdomain

$$
G_{\bar{\eta}} = \{x \in G : \text{dist}(x, \partial G) > \bar{\eta} = c \cdot \varepsilon^{1-2\delta}\},\tag{15}
$$

c, δ given positive constants (cf. (13); of course, $\delta < 1/2$). $G_{\vec{n}}$ might also be conceived as an analogically defined subdomain of G_{ϵ} (for $\epsilon \leq \epsilon_0$ and a proper constant c); we shall sometimes do so in what follows.

Our main tool for analysing u locally will be the "quasi-test function" φ :

$$
\varphi(x) = \varphi_{\epsilon}(x) = \varphi_m\left(\frac{r}{\eta}\right) \quad \text{with} \quad r = |x - \hat{x}|, \quad \eta = \epsilon^{1-\delta},
$$
\n
$$
\varphi_m(\varrho) = \frac{1}{1 + \varrho^m \epsilon^{\rho}}, \tag{16}
$$

where the point under consideration \hat{x} is any fixed point in $G_{\bar{x}}$. For $k \leq m$, we have

$$
\frac{d^k \varphi_m(\varrho)}{d\varrho^k} = \begin{cases} \varrho^{m-k} B_k(\varrho) \cdot \varphi_m(\varrho) & \text{for} \quad \varrho \le 1 \\ C_k(\varrho) \cdot \varphi_m(\varrho) & \text{for} \quad \varrho, \ge 1 \end{cases}
$$

with bounded functions B_k , C_k , whence we obtain for any partial derivative of order k (with respect to the variables x_i)

$$
D^{\alpha}\varphi_{\epsilon}(x) = \frac{1}{\eta^{k}} C_{\alpha}(x, \eta) \cdot \varphi_{\epsilon}(x), \qquad |\alpha| = k \leq m \qquad (17)
$$

with bounded continuous C_{α} .

- Instead of vanishing outside some neighbourhood of \hat{x} the function φ , will only tend to zero exponentially if $\varepsilon \to +0$, and that will do for our purpose. Especially, in a neighbourhood of the boundary ∂G of width $d \cdot \varepsilon$ (d positive constant) it is easily seen from (15) and (16)

$$
\varphi_{\epsilon}(x) = O(\exp(-c/\epsilon^{\delta})) \quad \text{for} \quad \epsilon \to +0. \tag{18}
$$

4. The L₂ estimate. By the next step, for all derivatives $D^{\alpha}u$ of the solution, $\|\varphi D^{\alpha}u\|$ will turn out bounded – uniformly with respect to ε and the choice of \hat{x} in $G_{\bar{\pi}}$. Differentiating equation (4) we obtain for any derivative $D^{\alpha}u = v$ of order $|\alpha| = l$

$$
L_{\iota}v = D^{\circ}h + \sum_{\substack{|\beta| \leq l-1 \\ |\gamma| \leq 2m}} \varepsilon^{|\gamma|} c_{\beta\gamma}(x) D^{\beta+\gamma}u =: \bar{h}.
$$
 (19)

The proof of our assertion will now be given by multiplying this equation by $\varphi^2 \cdot v$ and integrating by parts. To this we point out an observation on principle: All integrands (and so all norms) will involve functions of kind $\varphi \cdot D^{\bullet}u$. If we, additionally, multiply by a test unction $\psi_{\epsilon} \in C^{\infty}(G_{\epsilon})$ with $\psi_{\epsilon}(x) = 1$ for $x \in G_{2\epsilon}$ as used in the appendix we enforce vanishing at the boundary though we only give rise of an error of the integrals of order $\tilde{O}(\exp(-c/\varepsilon^d))$ as to be inferred from the preliminary estimates (12), (14), the property (18) of φ and the fact that the derivatives of ψ_{ϵ} also grow like powers of $1/\varepsilon$. Because $\exp(-c/\varepsilon^{\delta}) = o(\varepsilon^{N})$ for any N, while the quantities in consideration in what follows are of orders ε^M only, we shall omit the

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boundary terms introducing an equivalence relation \equiv and a weakened order relation \leq with meaning "equal (resp., less or equal) up to additional terms of order $O(\exp(-c/\epsilon^b))$ ". Then we can formally integrate by parts as if the integrands were exactly equal zero at the boundary, if we relate the integrals and norms to the subdomain G_{ϵ} what we will do in this section without special notation.

Remark: Another way to become aware of this fact is to choose a subdomain similar to G_e so that its image G_e' under homothetic transformation $x' = x/e$ has uniformly smooth boundary and then use the continuity of the trace map π : \rightarrow $H_k(\partial G_{\epsilon})$; (12), (14), and (18) will now tell that the boundary values are "very small?' in the sense described. boundary terms introducing an equiva

relation \leq with meaning "equal (resp.,

order $O(\exp(-c/\epsilon^{\delta}))$ ". Then we can form

were exactly equal zero at the boundary,

subdomain G_{ϵ} what we will do in this sec

Remark: An FR: Another way to becomple that the conduct of G_{ϵ} is that its image G_{ϵ}' , smooth boundary and the sense described.
 G_{ϵ} , (12) , (14) , and (18) with the sense described.
 F_{ϵ} will implement the integr

Now we will implement the integration by parts of (19) after scalar multiplication

e2i(Lj(9,v), 9v) 2m (20) *± ' f dx,* **i—i** I+IPI+h'aI+Iy,Ii - Iy,I+IyII where may be assumed jaj, *ml* By means of (17), the integrals appearing here

where may be assumed $|\alpha|, |\beta| \leq m$. By means of (17), the integrals appearing here in the second sum are to be seen majorized by expressions of type

$$
\frac{C}{\eta^r}\|\varphi D^p v\| \|\varphi D^q v\|,\qquad q\leqq p\leqq q+1,\quad p+q+r=i,\quad r\geqq 1;
$$

here *D^p*, *D*^q, denote any derivatives of total order p or q. Application of the inequality $\frac{C}{\eta^r} \|\varphi D^p v\| \|\varphi D^q v\|, \qquad q \leq p \leq q+1, \quad p+1$

here D^p , D^q denote any derivatives of total order p or q.
 $ab \leq \frac{1}{2} \left(\varrho a^2 + \frac{1}{\varrho} b^2 \right)$ with $\varrho = \varepsilon^{2p+r-i}$ leads to bounds *, D*^{*s*} denote any deriva
 $\left(\varrho a^2 + \frac{1}{\varrho} b^2\right)$ with ϱ
 $\left\{\frac{C_1}{\varrho^2} e^{2p} \frac{\|\varphi D^p v\|^2}{r} + \frac{\varrho^2 q}{r}\right\}$ $\sum_{j=0}^{n} \sum_{\{r|p|+|y_1|+|y_2|=i}^{n} \int b_{\alpha\beta j_1 y_1} D^*v D^{\beta}v D^{\gamma}v \phi dx,$ (20)
 $\sum_{|y_1|+|y_2| \ge 1}^{n} \int b_{\alpha\beta j_1 y_1} D^*v D^{\beta}v D^{\gamma}v \phi dx,$
 $|\le m$. By means of (17), the integrals appearing here

een majorized by expressi $\frac{C}{\eta^r} \|\varphi D^p v\| \|\varphi D^q v\|, \qquad q \leq p \leq q+1, \quad p+q+r = i, \quad r \geq 1;$

here D^p , D^q denote any derivatives of total order p or q . Application of the inequality
 $ab \leq \frac{1}{2} \left(\varrho a^2 + \frac{1}{\varrho} b^2 \right)$ with $\varrho = \varepsilon^{$ where may be assumed $|\alpha|$, $|\beta| \leq m$.

in the second sum are to be seen majorities of

in the second sum are to be seen majorities of
 $\frac{C}{\eta^r} ||\varphi D^p v|| ||\varphi D^q v||$, $q \leq p$

here D^p , D^q denote any derivatives of

the principal terms in (20) — fisrt sum — will obey, in our weakened sense, Gårding's for the integrals in (20) multiplied by ε^i (remember $\eta = \varepsilon^{1-\delta}$). On the other hand, the principal terms in (20) — fisrt sum — will obey, in our weakened sense, Gårding's inequality (5), i.e.

$$
(L'_{2j}(\varphi v), \varphi v) \geqq A_j ||\varphi v||_j^2 - B_j ||\varphi v||_{j-1}^2.
$$

Finally, in order to adapt this estimation to (21), we shall express the norms of $D^{\alpha}(qv)$ here by those of $\varphi D^{\alpha}v$ in (21). Using triangle inequality and (17) we obtain $\left\langle L'_{2j}(\varphi v), \varphi v \right\rangle \geq A_j \|\varphi v\|_j^2$ —
in order to adapt this es
ree by those of $\varphi D^a v$ in $(2 \|D^a(\varphi v)\| \geq \|\varphi D^a v\| - \sum_{|\beta| \leq |a|}$

$$
||D^{\mathfrak{a}}(p v)|| \geq ||\varphi D^{\mathfrak{a}} v|| - \sum_{|\beta| \leq |\alpha|-1} \frac{c_{\beta}}{\eta^{|\alpha|+|\beta|}} ||\varphi D^{\beta} v||
$$

and therefore, for $j \geq 1$.

$$
C_i\{\varepsilon^{2p} \|\varphi D^p v\|^2 + \varepsilon^{2q} \|\varphi D^q v\|^2\} \cdot \varepsilon^{r\delta}, \qquad r \ge 1,
$$
\n(21)

\nntegrals in (20) multiplied by ε^i (remember $\eta = \varepsilon^{1-\delta}$). On the other hand, ε is given by (5), i.e.

\n
$$
(L'_{2j}(\varphi v), \varphi v) \geq A_j \|\varphi v\|_j^2 - B_j \|\varphi v\|_{j-1}^2.
$$
\nin order to adapt this estimation to (21), we shall express the norms of ε by those of $\varphi D^s v$ in (21). Using triangle inequality and (17) we obtain $\|D^a(\varphi v)\| \geq \|\varphi D^a v\| - \sum_{|\beta| \leq |a|-1} \frac{c_\beta}{\eta^{|a|+|\beta|}} \|\varphi D^\beta v\|$

\nrefore, for $j \ge 1$.

\n
$$
\varepsilon^{2j} (L'_{2j}(\varphi v), \varphi v)
$$
\n
$$
\geq \varepsilon^{2j} A_j \sum_{|\alpha|=j} \|\varphi D^s v\|^2 - \sum_{i=0}^{j-1} \sum_{|\beta|=i} c_{j\beta} \varepsilon^{2i+2(j-i)\delta} \|\varphi D^\beta v\|^2.
$$
\n(22)

\ning now the right hand side of (20) by means of (21) and (22) we arrive at $\sum_{j=0}^{\infty} \varepsilon^{2j} \sum_{|\alpha|=j} C_{\alpha} \|\varphi D^a v\|^2 \leq (\varphi \overline{h}, \varphi v) \leq \|\varphi \overline{h}\| \|\varphi v\|$

\n(23)

Estimating now the right hand side of (20) by means of (21) and (22) we arrive at the desired result

$$
\sum_{i=0}^{m} \varepsilon^{2j} \sum_{|\alpha|=j} C_{\alpha} ||\varphi D^{\alpha} v||^{2} \leqq (\varphi \overline{h}, \varphi v) \leq ||\varphi \overline{h}|| \, ||\varphi v|| \tag{23}
$$

/

Singularly Perturbed Problems 321
for ε small $(\varepsilon \leq \varepsilon_1)$ with positive constants C_a independent of ε and $\hat{x} \in G_{\overline{\eta}}$, and
especially for ε small $(\varepsilon \leq \varepsilon_1)$ with positive constants
especially
 $\|\varphi v\| \leq C \|\varphi \bar{h}\|$
for any derivative v of order l of the solution

I

$$
\|\varphi v\| \leq C \|\varphi \bar{h}\| \tag{24}
$$

Singularly Perturbed Problems 321
 ε_1) with positive constants C_a independent of ε and $\hat{x} \in G_{\overline{\eta}}$, and
 $C ||\varphi \overline{h}||$ (24)
 \vdots ψ of order *l* of the solution *u*. Taking in consideration the struct for—any derivative *v* of order *1* of the solution *u.* Taking in consideration the structure of \hbar (cf. (19)) and the preliminary estimates (12), (14) we observe all $(\varepsilon \leq \varepsilon_1)$ wi
 $\|\varphi v\| \leq C \|\varphi \bar{h}\|$

erivative v of (19) and the p
 $\|\varphi v\| \leq \frac{C}{\varepsilon^{l-1}}$,
 $\varphi D^{\alpha}u$ obeys a

durith cuitedfuction Singularly Perturbed 1

stants C_a independent of ε

ution u . Taking in considerat

mates (12), (14) we observe

tion of type (14), but improve

Now, in turn, the improve

$$
|\varphi v\|\leqq \frac{C}{\varepsilon^{l-1}}
$$

i.e., $\varphi v = \varphi D^2 u$ obeys a general estimation of type (14), but improved by a factor ε compared with *v* itself according (14). Now, in turn, the improved estimate, could at once be applied to \bar{h} again, for *u* and its derivatives occur in \bar{h} only multiplied by φ . This entails a further lifting of the power of ε in the last inequality, and finally we can conclude, in this way, $\|\varphi v\| \leq C \|\varphi \overline{h}\|$ (24)

erivative v of order l of the solution u. Taking in consideration the structure

19)) and the preliminary estimates (12), (14) we observe
 $\|\varphi v\| \leq \frac{C}{\epsilon^{l-1}}$,
 $\varphi D^s u$ obeys a genera

$$
\|\varphi D^{\circ}u\| \leq C_{\alpha} \tag{25}
$$

we can conclude, in this way,
 $\|\varphi D^*u\| \leq C_{\alpha}$ (25)

for all derivatives of u in G_{ϵ} with constants C_{α} independent of ε ($\varepsilon \leq \varepsilon_1$) and $\hat{x} \in G_{\overline{\eta}}$.

5. Uniform bounds. From Sobolev's imbedding theorem for the n -dimensional unit ball B_1 **B.** Uniform bounds. From Sobolev's imbedding theorem for the *n*-dimensional unit

ball B_1
 \therefore sup $|u(x)| \le C_1 ||u||_{l_1}^{B_1}$, $l_0 = \left[\frac{n}{2}\right] + 1$,

it follows

sup $|u(x)| \le C_1 e^{-n/2} \sum_{|y|=0}^{l_0} e^{|y|} ||D^y u||_{l_0}^{B_0}$ the constants C_a independent of ε (
 $\forall v$'s imbedding theorem for the
 $l_0 = \left[\frac{n}{2}\right] + 1,$
 $\varrho^{|y|} ||D^{\gamma}u||^{B_{\varrho}}$

cation to $v = D^a u$, $|\alpha| = l \geq l_0,$
 $\varrho^{|y|} C_{a+\gamma}$
 ϱ
 $\exists B_n$; therefore

derivatives of
$$
u
$$
 in G_{ϵ} with constants C_a ind
\nform bounds. From Sobolev's imbedding the
\n
$$
\sup_{B_1} |u(x)| \leq C_1 ||u||_{l_{\bullet}}^{B_1}, \qquad l_0 = \left[\frac{n}{2}\right] + 1,
$$
\n
$$
\sup_{B_2} |u(x)| \leq C_1 e^{-n/2} \sum_{|\gamma|=0}^{l_0} e^{|\gamma|} ||D^{\gamma}u||^{B_{\epsilon}}
$$

it follows

$$
\sup_{B_1} |u(x)| \leq C_1 ||u||_h^{-1}, \qquad t_0 = \boxed{2}
$$
\n
$$
\sup_{B_2} |u(x)| \leq C_1 e^{-n/2} \sum_{|\gamma|=0}^{t_0} e^{|\gamma|} ||D^{\gamma}u||_{B_2}
$$
\n
$$
B_e \text{ with radius } \rho. \text{ Application to } v = \sup_{B_{\eta}} |v(x)| \leq C_1' \eta^{-n/2} \sum_{|\gamma|=0}^{t_0} \eta^{|\gamma|} C_{\alpha+\gamma}
$$
\n
$$
\text{for (25) and } \varphi \geq \frac{1}{1-\gamma} \text{ in } B_{\eta}; \text{ therefore}
$$

yields **X**
 XEGUARE $|u(x)| \leq C_1 e^{-n/2} \sum_{|\gamma|=0}^{l_0} e^{|\gamma|} ||D^{\gamma}u||^{B_{\rho}}$ **

XEGUARE** $|u(x)| \leq C_1 e^{-n/2} \sum_{|\gamma|=0}^{l_0} e^{|\gamma|} ||D^{\gamma}u||^{B_{\rho}}$ **

XEGUARE** $|v(x)| \leq C_1' \eta^{-n/2} \sum_{|\gamma|=0}^{l_0} \eta^{|\gamma|} C_{\alpha+\gamma}$ **

SEGUARE** η η of (25) and $\$

$$
\sup_{B_{\rho}} |u(x)| \leq C_1 e^{-n/2} \sum_{|\gamma|=0} e^{|\gamma|} ||D^{\gamma}u||
$$

\n
$$
|B_{\rho} \text{ with radius } \rho. \text{ Application to}
$$

\n
$$
\sup_{B_{\eta}} |v(x)| \leq C_1' \eta^{-n/2} \sum_{|\gamma|=0}^{l_2} \eta^{|\gamma|} C_{\alpha+\gamma}
$$

yields
 $\sup_{B_{\eta}} |v(x)| \leqq C_1' \eta^{-n/2} \sum_{|\gamma|=0}^{l_0} \eta^{|\gamma|} C_{\alpha+\gamma}$

because of (25) and $\varphi \geqq \frac{1}{1+e}$ in B_{η} ; therefore of (25) and $\varphi \ge \frac{1}{1+e}$ in B_{η} ; therefore

sup $|D^{\circ}u(x)| \le \frac{C}{\sqrt{\eta}}$ (26).

$$
\sup_{x \in G_{\overline{n}}} |D^{\alpha} u(x)| \leqq \frac{C}{\sqrt{\eta}^n}
$$

'for every derivative of the solution of (4). By means of the differential equation of (4) and with regard to the reduced equation (7) we see that in $G_{\bar{\eta}}$ at least $|u(x)| \le \left\{ \max \frac{C_a}{\sqrt{n^n}} \cdot \varepsilon, \frac{1}{A_0} \sup |h| \right\}$

$$
|u(x)| \leq \left\{ \max \frac{C_a}{\sqrt{\eta^n}} \cdot \varepsilon, \frac{1}{A_0} \sup |h| \right\}
$$

 $(a_0 \geq A_0 > 0)$. Differentiating (4) we obtain analogous estimates for all derivatives of u (of course, *h* replaced by its corresponding derivative), so that we successively can improve the result until we arrive at of (25) and $\varphi \geq \frac{1}{1+e}$ in B_{η} ; to
 $\sup_{x \in G_{\overline{\eta}}} |D^{\circ}u(x)| \leq \frac{C}{\sqrt{\eta}}$

derivative of the solution of (

derivative of the solution of (
 $|u(x)| \leq \left\{ \max \frac{C_{\mathfrak{a}}}{\sqrt{\eta}} \cdot \varepsilon, \frac{1}{A_0} \text{ sup} \right\}$
 > 0). D $\frac{1}{2}$ derivation derivation of $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation of $\frac{1}{2}$ derivation $|u(x)| \ge \begin{cases} \max \overline{\sqrt{\eta}}^n & \text{ϵ}, \overline{A_0} \text{ sup } |a| \ a_0 \ge A_0 > 0. \text{ Differentiating (4) we obtain} \end{cases}$

($a_0 \ge A_0 > 0$). Differentiating (4) we obthorn u (of course, h replaced by its correspondent matrix at sup $|D^{\alpha}u(x)| \le C_{\alpha}$,
 $\text{Re } \over$ $(a_0 \geq A_0 > 0)$. Differentiating (4) we obtain a
of u (of course, h replaced by its correspondin
can improve the result until we arrive at
 $\sup_{x \in G_{\overline{\eta}}} |D^{\alpha}u(x)| \leq C_{\alpha}$,
i.e., the uniform boundedness of all derivative

$$
\sup_{x\in G_{\overline{n}}} |D^{\alpha}u(x)| \leq C_{\alpha}
$$

i.e., the uniform boundedness of all derivatives of u in the expanding and, for $\varepsilon \to 0$, exhausting subdomain $G_{\bar{v}}(\bar{\eta} = \varepsilon^{1-2\delta})$.

(27)

6. Uniform convergence. Based on (27) we immediately perceive the uniform pointwise convergence of $u = u_t$ to *w*, the solution of the reduced problem (7): After division by $a_0(x)$ the equation of (4) reads **2** *D. GOHDE*
 D. GOHDE
 and *****and and a_{a***_{** *a***_{** *a***_{** *a***_{** *a a a a a a a a a a a a a a a a a a a*}}}}}

$$
\dot{\epsilon}B(x) + u_{\epsilon} = \frac{h}{a_0} = w
$$

with $B(x)$ uniformly bounded in $G_{\overline{n}}$ on account of (27).

Summarizing, we may establish the

Theorem 1: Coefficients and right hand side of equation (4) shall be smooth in the n-dimensional bounded domain 0; the boundary)G is supposed regular enough. (at least piecewise C^1 *in order the problem* (4) be solvable for all positive $\varepsilon \leq \varepsilon_0'$, the solution $u = u_{\epsilon}$ being smooth in the interior of G. Then, for $\varepsilon \to +0$, u_{ϵ} converges in each point Theorem 1: Coefficients and right hand side of equation (4) shall be smooth in the n-dimensional bounded domain G; the boundary ∂G is supposed regular enough. (least piecewise C¹) in order the problem (4) be solvable *iergence* is *uniform in the sense*

$$
\lim_{\epsilon \to 0} \sup_{x \in G_{\overline{n}}} |D^{\alpha} u_{\epsilon}(x) - D^{\alpha} w(x)| = 0,
$$

vergence is uniform in the sense
 $\lim_{\epsilon \to 0} \sup_{x \in G_{\overline{\eta}}} |D^{\circ}u_{\epsilon}(x) - D^{\circ}w(x)| = 0,$
 i.e., in the set $G_{\overline{\eta}}$ of all points of G with distance at least $\overline{\eta} = \partial G$ (c, δ positive constants arbitrarily chosen). $c \cdot \varepsilon^{1-2\delta}$ *to the boundary G (c, O positive constants arbitrarily, chosen).*

7. Extension in the case $k = 0.0$ "totally degenerating" problem. In equation (4) the operators L_i are multiplied by that power of ε which exactly coincides with their orders. As easily to be seen, Theorem I also comprehends the situation of additional regular perturbations by continuous dependence of coefficients and right hand side of ε or, especially, if $\varepsilon^{i}L_{i}$ in (4) is replaced by $\varepsilon^{i+k(i)}L_{i}$ with $k(i) \geq 0$ for $0 < i < 2m$ (but $k(0) = k(2m) = 0$); condition (6) will not be violated. *r* and the open

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But it, is of some interest that our method will work also in-the case of superposing singular perturbations, that is, if the higher derivatives are multiplied by another ε -power besides ε^i . After some strengthening the premises concerning the ellipticity constants A_j , the proof of the following assertion will run just as explained in 2. to 6. above, at least, if we, for simplicity, assume the coefficients to be constant. interest that our method will work also in the case of superposing
ons, that is, if the higher derivatives are multiplied by another
After some strengthening the premises concerning the ellipticity
oof of the following as

assuming constant coefficients, the solution $u = u_{\epsilon}$ of

power besides
$$
\varepsilon^i
$$
. After some strengthening the premises concerning the ellipticity instants A_j the proof of the following assertion will run just as explained in 2. to 6.

\nove, at least, if we, for simplicity, assume the coefficients to be constant.

\nTheorem 2: With notations and under regularity conditions of Theorem 1, moreover suming constant coefficients, the solution $u = u_i$ of

\n
$$
\begin{cases}\nG: L_i u := \sum_{i=0}^{2m} \varepsilon^{k_i} L_i u = h \\
\frac{\partial G}{\partial u} = 0 & \text{if } |y| \leq m - 1 \\
\frac{\partial G}{\partial v} = 0 & \text{if } |v| \leq m - 1\n\end{cases}
$$
\n(28)

with $k_0 = 0$, $k_{i+1} \geq k_i + 1$ $(i = 0, 1, ..., 2m - 1)$, and ellipticity condition (5) *sharpened by claiming all ellipticity constants* $A_i > 0$ ($j = 0, 1, ..., m$), will converge *quasi-uniformly to the soltition w of the reduced problem (7), i.e.,*

$$
\lim_{\epsilon \to 0} \sup_{x \in G_{\overline{\eta}}} |D^a u_{\epsilon}(x) - D^a w(x)| = 0
$$

as in Theorem 1.'

As to the proof, we only mention that the preliminary estimate can be derived as shown in 2. but, on account of the more stringent condition $A_i > 0$ for all j, without use of the interpolation lemma, and with aid of Lemma 2' of the appendix stating

> **• 'I**

 $\frac{1}{\sqrt{2}}\sum_{i=1}^{n-1}\frac{1}{i} \sum_{j=1}^{n-1} \frac{1}{j} \sum$

that also in this case the derivatives of the solution will grow, for $\varepsilon \to 0$, at the most as powers of ε^{-1} , and that will do, because in the constant coefficient case the essential step 4. will be finished already at (24).

8. Remarks to the case $k \geq 1$. Unfortunately, our simple method will not work off-hand in the case $k \ge 1$ in general where the reduced problem is of positive order. Some exemplifications will indicate the range of it rather restricted. Amongst other things, additional smoothness conditions of the boundary must be imposed. In the following *we* shall denote by G' a subdomain of G with sufficiently smooth boundary $\partial G'$ (corners and edges rounded off), and by $|| \cdot ||_k'$ the Sobolev norms with respect to G' . Singularly Perturbed Problems 32:

that also in this case the derivatives of the solution will grow, for $\varepsilon \to 0$, at the most

as powers of ε^{-1} , and that will do, because in the constant coefficient case the essent **.** Unfortunately, our simple method will not weneral where the reduced problem is of positive or cate the range of it rather restricted. Amongst of conditions of the boundary must be imposed. In a subdomain of G with s *A* in the case $u \equiv 1$ in general wind in the tended problem in the interactions will indicate the range of it rather restrict diditional smoothness conditions of the boundary must lwe shall denote by G' a subdomain of

Let Δ and L be elliptic operators of second order with smooth coefficients in G so that

$$
(-\Delta v, v) \geq c_1 \|v\|_1^2 - c_0 \|v\|^2, \qquad (-Lv, v) \geq d_1 \|v\|_1^2 - d_0 \|v\|^2. \tag{29}
$$

for *v* vanishing at ∂G , with positive constants c_1 , d_1 . The formal adjoint operator Δ^* of Δ will obey such a Garding's inequality too.

Proposition 1: Let be $c_0 = 0$ in (29). The solution $u = u_t$ of

$$
\begin{cases}\nG: \varepsilon^2 \Delta L u - \Delta u = h \\
\partial G: u = L u = 0\n\end{cases}
$$
\n(30)

converges to the solution of the degenerate problem

 $G: -\Delta w = h, \quad \partial G: w = 0$

in the sense of Theorem 1:

 $\lim_{\epsilon \to 0} \sup_{x \in G_{\pi'}} |D^{\epsilon}u(x) - D^{\epsilon}w(x)| = 0$

for every α ($G_{\overline{p}}'$ *is the subset of* $x \in G'$ *with distance to the boundary* $\partial G'$ *more than* $\bar{\eta} = \varepsilon^{1-2\delta}$.

If $n = 2$, and $\forall 1 = L$ the Laplacian operator, the problem could be considered as a

converges to the solution of the degenerate problem
 $G: -\Delta w = h$, $\partial G: w = 0$

in the sense of Theorem 1:
 $\lim_{\epsilon \to 0} \sup_{x \in G_{\overline{n}}} |D^{\circ}u(x) - D^{\circ}w(x)| = 0$
 $\longleftrightarrow_{\epsilon \to 0} \sum_{\epsilon \in G_{\epsilon}} |G^{\sigma}u(x) - G^{\sigma}u(x)| = 0$

for every α (G_{\over Proof: As $\varepsilon^2 \Delta Lu - \Delta u = -\Delta(-\varepsilon^2 Lu + u)$ we shall, of course, set $-\varepsilon^2 Lu + u = w$. and this function *w* must then be the solution of the degenerate problem. Because of $c_0 = 0$ (29) implies $||w||_1 \leq C ||h||$, and by the well-known a-priori estimates for in in the supported membrane with small stiffness.

Proof: $As \varepsilon^2 \Delta Lu - \Delta u = -\Delta(-\varepsilon^2 Lu + u)$ we shall, of course, set $-\varepsilon^2 Lu + u = w$,

and this function w must then be the solution of the degenerate problem. Because

of c_0 Proof: $\text{As } \varepsilon^2 \Delta Lu - \Delta u = -\Delta(-\varepsilon^2 Lu + u)$ we shall, of course, set $-\varepsilon^2 Lu + u = w$, and this function w must then be the solution of the degenerate problem. Because of $c_0 = 0$ (29) implies $||w||_1 \leq C ||h||$, and by the well-*Jg:* $\lim_{\epsilon \to 0} \sup_{x \in G_{\eta}} |D^s u(x) - D^s w(x)| = 0$
 $\int_{-\infty}^{\infty} \frac{z e^{C_{\eta}}}{z}$
 α (G_{η} ' is the subset of $x \in G'$ w
 \int).

2, and $\Delta = L$ the Laplacian operation
 $\int_{-\infty}^{\infty} \frac{z^2}{L} du - \Delta u = -\Delta(-\varepsilon^2 L u + \frac{1}{2})$
 $\$ lim sup $|D^{\circ}u(x) - D^{\circ}w(x)| = 0$
 $\epsilon \to 0$ $x \epsilon G_{\overline{\eta}}'$ is the subset of $x \in G'$ u

2. and $\forall 1 = L$ the Laplacian operal supported membrane with sn
 \therefore As $\epsilon^2 \Delta Lu - \Delta u = -\Delta(-\epsilon^2 Lu -$

function w must then be the so

(29) *he boundary* $\partial G'$ *more the*
 m could be considered as

course, set $-\varepsilon^2 Lu + u =$

generate problem. Becau

own a-priori estimates f

now pass to the "interio"

(3)
 $||h||$ and, according to t

of the right hand side

$$
\begin{cases} G: -\varepsilon^2 Lu + u = w \\ \partial G: u = 0 \end{cases}
$$

problem
 $\left\{\n\begin{aligned}\nG: & -e^2Lu + u = w \\
\partial G: & u = 0\n\end{aligned}\n\right.$ (31)

and derive first, using (29), $||u|| + \varepsilon ||u||_1 \leq C ||w|| \leq C' ||h||$ and, according to the

appendix, $||u||'_{1+p} \leq \frac{C_p}{e^{1+p}}$ $(p = 0, 1, ...).$ All derivatives of the right are bounded, and preliminary estimates of type (12), (14) for the solution *u* of (31) are valid, so that the procedure leading to Theorem 1 pursuant steps **3.** to 6. may. work with respect to problem (31) in *0'* too *^g E2*^{*I*} in the set of $\begin{cases} G: -\varepsilon^2 Lu + u = w \\ \partial G: u = 0 \end{cases}$
ve first, using (29), $||u|| + \varepsilon ||u||_1 \leq C ||w|| \leq$
i, $||u||'_{1+p} \leq \frac{C_p}{\varepsilon^{1+p}} (p = 0, 1, \ldots)$. All derivative
ded, and preliminary estimates of type (12),
l, so that the procedure leading

5
5
5

(32)

Proposition 2: If $u = u_i$ is the solution of

$$
\begin{cases} G: \varepsilon^2 \Delta \Delta^* u - \Delta u + cu = h \\ \partial G: u = \Delta^* z = 0, \end{cases}
$$

21*

 $c = c(x) \ge c_0' > c_0$ (cf. (29)), then the assertion of Proposition 1 holds with respect *low given by the degenerate problem* **D.** GÖNDE
 $\geq c_0' > c_0$ (cf. (29)), then the assertion of Proposition 1 holds with respect
 ∂t by the degenerate problem
 $G: -\Delta w + cw = h$, $\partial G: w = 0$. (32₀)
 \therefore At first we must look for preliminary estimates as se B24 D. Gönue
 $c = c(x) \ge c_0' > c_0$ (cf. (29)), then the assertion of Proposition 1 holds with
 ω w given by the degenerate problem
 $G: -\Delta w + cw = h,$ $\partial G: w = 0$.

Proof: At first we must look for preliminary estimates as set **j**, $G\ddot{\sigma}nD\ddot{\sigma}$
 $\geq c_0' > c_0$ (cf. (29)), then the assertion of Proposition 1 holds with respect
 by the degenerate problem

7: $-2w + cw = h$, $\partial G : w = 0$. (32₀)

At first we must look for preliminary estimates as

$$
G: -\Delta w + cw = h, \qquad \partial G: w = 0. \tag{32.}
$$

Proof: At first we must look for preliminary estimates as set up in 2. We multiply by u , integrate by parts, and use (29) : $2! - 2iw + cw = h,$ $\partial G : w = 0.$
At first we must look for preliminary estimate by parts, and use (29):
 $2 \frac{1}{2} |\Delta^* u|^2 + c_1 ||u||_1^2 - c_0 ||u||^2 + c_0' ||u||^2 \leq 1$ 11 It is the must look for preliminary estimates as

grate by parts, and use (29):
 $\varepsilon^2 ||\Delta^* u||^2 + c_1 ||u||_1^2 - c_0 ||u||^2 + c_0' ||u||^2 \le ||h|| ||u||$

efore
 $||u||_1 \le C ||h||$, $\varepsilon ||\Delta^* u|| \le C ||h||$.

inequality gives rise to
 $||u||_2' \le C(||\$

$$
e^{2}||A^*u||^2 + c_1||u||_1^2 - c_0||\dot{u}||^2 + c_0'||u||^2 \le ||h|| ||u||^2
$$

$$
||u||_1 \leq C ||h||, \qquad \varepsilon ||\Delta^* u|| \leq C ||h||.
$$

The last inequality gives rise to

$$
\|u\|_2{'}\leqq C\{\|{\mathcal{A}}^*u\|\mathrel{{+}}_{\mathbb{Q}}\|u\|\}\leqq \frac{1}{\varepsilon}C\,\|h\|
$$

in G'. Since $\Delta\Delta^*$ is elliptic too, the premises of Lemma 2 of the appendix are satisfied, hence

$$
||u||_{p}'' \leqq \frac{C_p}{\varepsilon^{(p-1)^*}} \quad (p = 0, 1, \ldots) \tag{34}
$$

 $\begin{aligned} &\text{d} \text{ use } (29): \ &- c_0 \ \lVert u \rVert^2 + c_0' \ \lVert u \rVert^2 \leq \lVert h \rVert \ &\text{d}^* u \rVert \leq C \ \lVert h \rVert. \ &\text{e} \text{ to } \ &\text{d} u \rVert \} \leq \frac{1}{\varepsilon} \ C \ \lVert h \rVert \ &\text{oo, the premises of Lem} \ &\text{or, for } p \geq 2, G(p-2)\varepsilon, \ &\text{e result is valid for the set} \end{aligned}$ where $|| \cdot ||''$ is to be taken over G or, for $p \ge 2$, $G'_{(p-2)\epsilon}$, respectively; $[x]^+$ = max $\{x, 0\}$.
As easily to be seen, the same result is valid for the solution \overline{u} of the slightly altered problem As easily to be seen, the same result is valid for the solution \bar{u} of the slightly altered $||u||_2' \leq C||$

in G'. Since $\Delta \Delta^*$ is

fied, hence
 $||u||_p'' \leq \frac{C}{\varepsilon^{\lceil p \rceil}}$

where $||\cdot||'$ is to be taken,

As easily to be seen,

problem
 $\left\{\begin{aligned}\nG: \bar{L}_c \bar{u} := \frac{C}{\varepsilon} \end{aligned}\right\}$ *±ci=h* in G' . Since $\Delta\Delta^*$ is elliptic too, the premises of Lemma 2 of the apper
fied, hence
 $||u||_{p}'' \leqq \frac{C_p}{\epsilon^{(p-1)^*}}$ ($p = 0, 1, ...$)
where $||\cdot||''$ is to be taken over G or, for $p \geqq 2$, $G'_{(p-2)t}$, respectively; $[x]^*$ 2.

2. **i** to be taken over G or, for $p \ge 2$, $G'_{(p-2)\epsilon}$, respectively; $[x]^+ = \max \{x, 0\}$.

be seen, the same result is valid for the solution \overline{u} of the slightly altered
 $\overline{L}_\epsilon \overline{u} := \epsilon^2 \Delta \Delta^* \overline{u} - \Delta \overline{u} ||u||_p'' \leqq \frac{C_p}{\varepsilon^{(p-1)^*}}$
where $||\cdot||''$ is to be taken
As easily to be seen, the
problem
 $\begin{cases} G: \overline{L}_{\epsilon} \overline{u} := \varepsilon^2 \Delta \\ \partial G: \overline{u} = \Delta^* \overline{u} \end{cases}$
But this problem prove:
first case we set
 $-\varepsilon^2 \Delta^* \overline{u} + \overline{u} =$ taken over *G* or, for $p \ge 2$, $G'_{(p-2)\epsilon}$, respectively; $[x]^+ = \max \{x, 0\}$,
 *C*_j a, the same result is valid for the solution \overline{u} of the slightly altered
 $\epsilon^2 \Delta 4^* \overline{u} - \Delta \overline{u} - \epsilon^2 c \Delta^* \overline{u} + c \overline{u} = h$ (3

$$
\begin{cases} G: \bar{L}_\iota \bar{u} := \varepsilon^2 \Delta \Delta^* \bar{u} - \Delta \bar{u} - \varepsilon^2 c \Delta^* \bar{u} + c \bar{u} = h \\ \partial G: \bar{u} = \Delta^* \bar{u} = 0. \end{cases} \tag{35}
$$

But this problem proves factorable, for $L_i u = (-\Delta + c) (-\varepsilon^2 \Delta^* + 1) \overline{u}$. As in the first case we set

$$
-\varepsilon^2 \Delta^* \overline{u} + \overline{u} = w \qquad (36)
$$

which is the solution of the degenerate problem $(32₀)$. The conditions on -4 and *c* $-\varepsilon^2 \Delta^* \bar{u}$ +
the solution
ain
 $||w||'_{2+p} \le$

$$
||w||_{2+p} \le C_p ||h||_p \tag{37}
$$

and, consequently, for the solution \bar{u} of equation (36) the assertion of theorem will hold - we remember the fact that its proof dont make use of boundary values immediately, but only of a preliminary estimate as (34) here. Thus lim sup 1D91(x) - Daw(x)I ⁼0. (38) $||w||'_{2+p} \leq C_p ||h||_p$ (37)

and, consequently, for the solution \overline{u} of equation (36) the assertion of theorem will

hold — we remember the fact that its proof dont make use of boundary values

immediately, but only of $-\varepsilon^2 \Delta^* \overline{u} + \overline{u} = w$

is the solution of the degeneration
 $||w||'_{2+p} \leq C_p ||h||_p$

is the solution of the degeneration
 $||w||'_{2+p} \leq C_p ||h||_p$

is measured by the solution

in sup $|D^* \overline{u}(x) - D^* v|$
 $\leftrightarrow 0$ $x \in G_{\overline$

$$
\lim_{\epsilon \to 0} \sup_{x \in G_{\pi}'} |D^{\epsilon} \overline{u}(x) - D^{\epsilon} w(x)| = 0. \tag{38}
$$

not affect this property of u itself. To this end we shall, of course, set $u - \bar{u} = z_1$ so that we remember the fact that its proof
tely, but only of a preliminary estimated in the sum of $L^{\infty} \bar{u}(x) - D^{\alpha}u(x)| = 0$.
 $L^{\infty} 2\epsilon G' \frac{\pi}{\eta}$

have to become sure the additional referrition this property of *u* itself.

$$
\begin{cases} G: Lz_1 := \varepsilon^2 \Delta \Delta^* z_1 - \Delta z_1 + cz_1 = -\varepsilon^2 c \Delta^* \overline{u} \\ \partial G: z_1 = \Delta^* z_1 = 0. \end{cases}
$$

We replace z_1 by \overline{z}_1 which is defined as the solution of

$$
\begin{aligned}\n\text{ce } z_1 \text{ by } \bar{z}_1 \text{ which is de:} \\
\begin{cases}\nG: \bar{L}_{\epsilon} \bar{z}_1 &= -\epsilon^2 c \Delta^* \bar{u} \\
\partial G: \bar{z}_1 &= \Delta^* \bar{z}_1 = 0\n\end{cases}\n\end{aligned}
$$

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with $\Vert -\varepsilon^2 c \Delta^* \bar{u} \Vert \leq \varepsilon C \Vert h \Vert$ according to (33), and therefore this estimate will be true with $||- \varepsilon |z|^2$ $||z||$ instead of $||h||$. Apparently \overline{z}_1 obeys the assertion of Theorem 1
also for \overline{z}_1 but with $\varepsilon ||h||$ instead of $||h||$. Apparently \overline{z}_1 obeys the assertion of Theorem 1
with 0 as soluti with 0 as solution of the reduced problem. $\|\Delta^* \bar{u}\| \leq \varepsilon C \|\vec{h}\|$ accord

out with $\varepsilon \|\vec{h}\|$ instead colution of the reduced

construct recursively
 $G: \bar{L}_{\varepsilon} \bar{z}_{i+1} = -\varepsilon^2 c \Delta^* \bar{z}_i$
 $\partial G: \bar{z}_{i+1} = \Delta^* \bar{z}_{i+1} = 0$

with $z_0 = u, \bar{z}_0 = \bar{u}$, a where $\|\cdot\|$
also for \overline{z}
with 0 as
Now w
beginning
beginning $\begin{aligned} &\|z_c \Delta^* \bar{u}\| \leq \varepsilon C \, \|h\| \text{ according to (33)} \\ &\text{but with } \varepsilon \, \|h\| \text{ instead of } \|h\|. \text{ App}\n \text{ solution of the reduced problem.} \\ &\text{e construct recursively } z_i, \bar{z}_i \text{ by } z \\ &\text{if } \{G : \bar{L}_\varepsilon \bar{z}_{i+1} = -\varepsilon^2 c \Delta^* \bar{z}_i\} \\ &\text{if } \partial G : \bar{z}_{i+1} = \Delta^* \bar{z}_{i+1} = 0 \end{aligned}$

Now we construct recursively z_i , \overline{z}_i by $z_{i+1} = z_i - \overline{z}_i$,

$$
\begin{cases} G: \bar{L}_{\epsilon} \bar{z}_{i+1} = -\epsilon^2 c \Delta^* \bar{z}_{i} \\ \delta G: \bar{z}_{i+1} = \Delta^* \bar{z}_{i+1} = 0 \end{cases}
$$

beginning with $z_0 = u$, $\overline{z}_0 = \overline{u}$, and we obtain successively

$$
\|\bar{z}_i\|_1 \leq C \, \|h\| \, \varepsilon^i, \qquad \|A^* \bar{z}_i\| \leq C \, \|h\| \, \varepsilon^{i-1},
$$

and this is valid also for the z_i themselves (they satisfy the problem with the original operator L_i instead of \bar{L}_i , but with the same right hand side), while for the \bar{z}_i even also for \overline{z}_1 but with ε ||h|| instead of ||h||. Apparent
with 0 as solution of the reduced problem.
Now we construct recursively z_i , \overline{z}_i by $z_{i+1} =$
 $\left\{ G : \overline{L}_i \overline{z}_{i+1} = -\varepsilon^2 c \Delta^* \overline{z}_i : \right\}$
 $\$ *y* with $z_0 = u$, $\bar{z}_0 = \bar{u}$, and we obtain successively
 $||\bar{z}_i||_1 \leq C ||h|| \varepsilon^i$, $||\Delta^* \bar{z}_i|| \leq C ||h|| \varepsilon^{i-1}$,

is valid also for the z_i themselves (they satisfy the problem with the orig
 *L*_{*t*} instead of

$$
\lim_{\epsilon \to 0} \sup_{x \in G_{\overline{n}}} |D^* \overline{z}_i(x)| = 0 \tag{39}
$$

$$
u=\overline{u}+\overline{z}_1+\cdots+\overline{z}_r+z_{r+1}
$$

 $u = \bar{u} + \bar{z}_1 + \cdots +$
with (38), (39), and, by (34),.
 $||z_{r+1}||_{1+p}^{\prime\prime} \leq C_p e^{r-p+1}$

$$
||z_{r+1}||_{1+n}^{\prime\prime} \leq C_p \varepsilon^{r-p+1}
$$

Now we construct recursively z_i , \bar{z}_i by $z_{i+1} = z_i - \bar{z}_i$,
 $\left\{ G: \bar{L}_i \bar{z}_{i+1} = -\varepsilon^2 c A^* \bar{z}_i \right\}$
 $\left\{ \partial G: \bar{z}_{i+1} = A^* \bar{z}_{i+1} = 0 \right\}$

beginning with $z_0 = u$, $\bar{z}_0 = \bar{u}$, and we obtain successivel Appendix. As just inducated in Sections 2 and 7, we will complete the discussion there by the proofs of two lemmata concerning the rough basic estimates (cf. (14)) affirming the derivatives of the solution to grow at most as (negative) powers of ε **21**
 22
 24
 α . Thus we have obtained
 $u = \overline{u} + \overline{z}_1 + \cdots + \overline{z}_r + z_{r+1}$ (40)

(39), and, by (34),
 $||z_{r+1}||'_{1+p} \leq C_p e^{r-p+1}$

the assertion for r sufficiently large \blacksquare

iix. As just inducated in Sections 2 and 7, we will $\mathcal{L}_{r+1}||_{1+p} \cong \mathbb{U}_p e$

is assertion for r sufficities

is a signal inducated if

the proofs of two lemm

the derivatives of the

integral in the astronomy
 $\mathcal{L}_{r}u = \sum_{j=0}^{2l} \varepsilon^j L_{2k+j}u + \sum_{i=0}^{2k}$

ain G (L *h h h h* **c** *n* **sufficiently large a**

inducated in Sections 2 and 7, we will complete the discussion

of two lemmata concerning the rough basic estimates (cf. (14))

it we of the solution to grow at most as (n pmplete the discussion

sic estimates (cf. (14))

negative) powers of ε .

of order $2m = 2k + 2l$
 (41)

with the property
 (42)

itive in the sense

Lemma 1: Let $u = u_t$ *be a solution of the elliptic equation of order* $2m = 2k + 2l$

The proofs of two lemmata concerning the rough basic estimates (cf. (14))
\ng the derivatives of the solution to grow at most as (negative) powers of
$$
\varepsilon
$$
.
\nna 1: Let $u = u_{\varepsilon}$ be a solution of the elliptic equation of order $2m = 2k + 2l$
\n
$$
L_{\varepsilon}u = \sum_{j=0}^{2l} \varepsilon^{j}L_{2k+j}u + \sum_{i=0}^{2k} L_{i}u = h
$$
\n(41)
\n
$$
\text{main } G (L_{\varepsilon} \text{ denoting a differential operator of order } r) \text{ with the property}
$$
\n
$$
||u||_{i} \leq \frac{C_{i}}{\varepsilon^{i}} ||h|| \text{ for } 0 \leq i \leq m
$$
\n
$$
||u||_{i} \leq \frac{C_{i}}{\varepsilon^{(i-1)}} ||h|| \text{ for } 0 \leq i \leq m
$$
\n(42)
\n
$$
\text{max } \{x, 0\}. \text{ Furthermore the operator } L_{\varepsilon} \text{ is assumed positive in the sense}
$$
\n
$$
(L_{\varepsilon}v, v) \geq \varepsilon^{2l} (A ||v||_{m}^{2} - B ||v||^{2}), \qquad A > 0,
$$
\n
$$
\text{with compact support in } G.
$$
\n
$$
\text{for } p \geq 0 \text{ it is valid}
$$

in the domain C (Lr denoting a differential operator of order r) with the property

Domain of
$$
(L_r
$$
 denoting a differentual operator of order r) with the property

\n
$$
||u||_i \leq \frac{C_i}{\varepsilon^{i-1}} ||h|| \quad \text{for } 0 \leq i \leq m \qquad (42)
$$
\n
$$
||u||_i \leq \frac{C_i}{\varepsilon^{(i-1)^+}} ||h|| \quad \text{for } 0 \leq i \leq m \qquad (42')
$$
\n
$$
\text{max } \{x, 0\}. \text{ Furthermore, the operator } L_\varepsilon \text{ is assumed positive in the sense}
$$
\n
$$
(L_\varepsilon v, v) \geq \varepsilon^{2i} (A ||v||_m^2 - B ||v||^2), \qquad A > 0, \qquad (43)
$$
\n
$$
v \text{ with compact support in } G.
$$
\n
$$
v, \text{for } p \geq 0 \text{ it is valid}
$$
\n
$$
||u||_{m+p}^{(p)} \leq C_{m+p} \left\{ \frac{1}{\varepsilon^{m+p}} ||h|| + \frac{1}{\varepsilon^{i-k}} \sum_{\nu=0}^{p-1} \frac{1}{\varepsilon^{\nu}} ||h||_{p-\nu}^{(p-1-\nu)} \right\} \qquad (44)
$$
\n
$$
||u||_{m+p}^{(p)} \leq C_{m+p} \left\{ \frac{1}{\varepsilon^{m+p}} ||h|| + \frac{1}{\varepsilon^{i-k}} \sum_{\nu=0}^{p-1} \frac{1}{\varepsilon^{\nu}} ||h||_{p-\nu}^{(p-1-\nu)} \right\} \qquad (45)
$$
\n
$$
||u||_{m+p}^{(p)} \leq C_{m+p} \left\{ \frac{1}{\varepsilon^{m+p}} ||h|| + \frac{1}{\varepsilon^{i-k}} \sum_{\nu=0}^{p-1} \frac{1}{\varepsilon^{\nu}} ||h||_{p-\nu}^{(p-1-\nu)} \right\} \qquad (46)
$$

or

$$
||u||_{i} \leq \frac{C_{i}}{\varepsilon^{(i-1)^{+}}} ||\hbar|| \quad \text{for} \quad 0 \leq i \leq m \tag{42'}
$$
\n
$$
\text{max } \{x, 0\}. \text{ Furthermore, the operator } L_{\varepsilon} \text{ is assumed positive in the sense}
$$
\n
$$
(L_{\varepsilon}v, v) \geq \varepsilon^{2l} (A ||v||_{m}^{2} - B ||v||^{2}), \qquad A > 0,
$$
\n
$$
\text{with compact support in } G.
$$

 $f(x)^{+} = \max\{x, 0\}$. Furthermore the operator L_{i} is assumed positive in the sense

$$
(L_{\epsilon}v, v) \ge \epsilon^{2l}(A \|v\|_{m}^{2} - B \|v\|^{2}), \qquad A > 0,
$$
\n(43)

for all v with compact support in C. $(L_{\ell}v, v) \geq \varepsilon^{2l}(A ||v||)$
 Then, for $p \geq 0$ *it is valid*

or
\n
$$
||u||_i \leq \frac{C_i}{\varepsilon^{(i-1)!}} ||h|| \text{ for } 0 \leq i \leq m
$$
\n
$$
(|x|^+ = \max \{x, 0\}). \text{ Furthermore the operator } L_{\varepsilon} \text{ is assumed positive in the sense}
$$
\n
$$
(L_{\varepsilon}v, v) \geq \varepsilon^{2l} (A ||v||_m^2 - B ||v||^2), \qquad A > 0,
$$
\n
$$
\text{for all } v \text{ with compact support in } G.
$$
\nThen, for $p \geq 0$ it is valid
\n
$$
||u||_{m+p}^{(p)} \leq C_{m+p} \left\{ \frac{1}{\varepsilon^{m+p}} ||h|| + \frac{1}{\varepsilon^{l-k}} \sum_{\nu=0}^{p-1} \frac{1}{\varepsilon^{\nu}} ||h||_{p-\nu}^{(p-1-\nu)} \right\}
$$
\n
$$
\text{where } ||\cdot||_{q}^{(r)} \text{ denotes the usual Sobolev norm of order } q \text{ in the subdomain } G^{(r)} = G_{\text{vde}} \text{ of all points of } G \text{ with distance at least } v \text{de to the boundary } \partial G \text{ (cf. (13)): } C_{m+p} \text{ is independent}
$$

all points of G with distance at least vde to the boundary ∂G *(cf.* (13)); C_{m+p} *is indepen*

(40)

 $\label{eq:1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

dent of ε *(but may depend on the* $-$ *arbitrarily chosen* $-$ *positive constant d). In the case* (42') the denominator ε^{m+p} in (44) may be replaced by ε^{m+p-1} . *^I*= *m,* and we will assume it to be valid for all *p* <q for some *q 1.*

For $p = 0$ the sum in (44) is to be set zero, of course.

Proof: For $p = 0$ the assertion (44) coincides with supposition (42) (resp. $(42')$),

dent of ε (but may depend on the $-$ arbitrarily chosen $-$ positive constant d). In the case (42') the denominator ε^{m+p} in (44) may be replaced by ε^{m+p-1} .

For $p = 0$ the sum in (44) is to be set zero, of c dent of ε (but may
case (42') the denotor

For $p = 0$ the ε

Proof: For $p = i = m$, and we will

We shall use to

outside $G^{(i-1)}$, ar

stants C_a .

A remark concer
 $x'' = x/\varepsilon$ will take *Froot: For* $p = 0$ *the assertion* (44) conclues with supposition (42) (resp. (42)),
 $= m$, and we will assume it to be valid for all $p < q$ for some $q \ge 1$.

We shall use test functions ψ_i $(i = 1, 2, ...)$ which are equal t

We choose a finite covering of the closure of G_d " by balls B_i $(i = 1, ..., n; n = n(\varepsilon))$ of diameter $d/2$ so that the concentric balls B_i'' constitute a covering with order bounded independently of ε . If η_i are the usual local test functions which are equal 1 in B_i and equal 0 out of B_i ["], then

$$
\psi_1(x) = 1 - \prod_{i=1}^n (1 - \eta_i(x)).
$$

will possess the claimed properties concerning $G = G^{(0)}$ and $G^{(1)}$.

In order to prove now (44) for $p = q$ we first set up a differential equation for $v = D^qu$ where, for simplicity, D^q denotes any derivative of order *q*, and apply D^q to equation (41) :

 $D^q L_i u = L_i v + \cdots = D^q h$, resp. *21 Lv* π *Lv* π β | \geq 1 $\nu_p(x) = 1 - \prod_{i=1}^{n} (1 - \eta_i(x))$

as the claimed properties concerning $G = G^{(0)}$ and $G^{(1)}$.

are to prove now (44) for $p = q$ we first set up a differential equation

where, for simplicity, D^q denotes any derivative of o *i***i** $p = q$ we first set up a differentiative, D^q denotes any derivative of order q,
 \therefore D^q denotes any derivative of order q,
 \therefore $D^q h$, resp.
 \therefore $D^q h$, resp.
 \therefore $D^q h$, resp.
 \therefore $D^q h$, resp.

Secondly, with $\psi=\psi_{0}$ we have

$$
L_{\epsilon}(\psi v)=\psi L_{\epsilon}v+\sum_{j=1}^{2l}\epsilon^{j}\sum_{|\alpha|=2k+j}\sum_{\substack{\gamma\leq\alpha\\|\gamma|\geq 1}}a_{\alpha}D^{\gamma}\psi D^{\alpha-\gamma}v+\sum_{|\alpha|\leq 2k}a_{\alpha}\sum_{\substack{\gamma\leq\alpha\\|\gamma|\geq 1}}D^{\gamma}\psi D^{\alpha-\gamma}v,
$$

and here we can substitute $\psi L_i v$ by the previous equation. After that we shall
multiply the last equation by ψv and integrate by parts in such a manner that the
derivatives of u appearing in each scalar product are o multiply the last equation by ψv and integrate by parts in such a manner that the derivatives of *u* appearing in each scalar product are of minimal orders, i.e., they differ in order at most by one. Thus we will attain to

$$
(L_{\epsilon}(\psi v), \psi v) = (\psi D^g h, \psi v) + \varepsilon^{2l} \sum_{\substack{|\alpha| = |\beta| = k+l \\ |\gamma| = 1, \gamma \leq \alpha}} (a_{\alpha+\beta} D^{\gamma} \psi D^{\alpha-\gamma} v, \psi D^{\beta} v) + \cdots
$$

plus lower order terms. Applying Schwarz's inequality and, additionally in the case for different orders in a scalar product, the arithmetic-geometric-mean inequality
with appropriate weights, we are abel to estimate the right hand side by
 $\epsilon^{2k} ||\psi D^q h||^2 + \epsilon^{-2k} ||\psi v||^2 + \epsilon^{2l} \frac{A}{2} \sum_{|\beta| = k + l} ||\psi D^{\beta} v$ with appropriate weights, we are abel to estimate the right hand side by

We can substitute
$$
\psi L_i v
$$
 by the previous equation. After that we shall
be last equation by ψv and integrate by parts in such a manner that the
es of *u* appearing in each scalar product are of minimal orders, i.e., the
order at most by one. Thus we will attain to
 $(L_i(\psi v), \psi v) = (\psi D^g h, \psi v) + \varepsilon^{2l} \sum_{\substack{|a| = |\beta| = k+l \\ |v| = 1, \gamma \leq a}} (a_{a+\beta}D^{\gamma}\psi D^{a-\gamma}v, \psi D^{\beta}v) + \cdots$
or order terms. Applying Schwarz's inequality and, additionally in the case
ent orders in a scalar product, the arithmetic-geometric-mean inequality
ropriate weights, we are able to estimate the right hand side by
 $\varepsilon^{2k} ||\psi D^g h||^2 + \varepsilon^{-2k} ||\psi v||^2 + \varepsilon^{2l} \frac{A}{2} \sum_{|\beta| = k+l} ||\psi D^{\beta} v||^2 + \varepsilon^{2l} C \sum_{\substack{|a| = k+l \\ |y| = 1}} ||D^{\gamma} \psi D^{a-\gamma} v||^2$
 $+ \sum_{j=1}^{2l} \varepsilon^{j} \sum_{|\gamma_1| + |\gamma_2| = 2k+j \\ |\gamma_2| + |\gamma_3| \geq 1, |\alpha_1| - |\alpha_3| \leq 1} |C_{\alpha_1 \alpha_1 \gamma_1 \gamma_1} ||D^{\gamma_1} \psi D^{\alpha_1} v|| ||D^{\gamma_2} \psi D^{\alpha_1} v|| + \cdots$
 $+ \sum_{j=1}^{2k} \sum_{|\alpha_1| + |\alpha_j| + |\gamma_1| + |\gamma_2| = 2k+j \\ |\gamma_1| - |\gamma_2| \geq 1, |\alpha_1| - |\alpha_3| \leq 1} |C_{\alpha_1 \alpha_1 \gamma_1 \gamma_1} ||D^{\gamma_1} \psi D^{\alpha_1} v|| ||D^{\gamma_2} \psi D^{\alpha_1} v|| + \cdots$

where the terms omitted are formed analogously by those of the expression for $L_{\ell}v$. above; compared with the corresponding ones indicated here they are of minor order (because v is at least a derivative of order one). Taking into account $|D^{\gamma}y|$ $\leq v_* \varepsilon^{-|\gamma|}$ and $v' = D^q u$ we shall see

$$
\varepsilon^{j} ||D^{\gamma_1} \psi D^{\alpha_1} v|| ||D^{\gamma_1} \psi D^{\alpha_1} v||
$$
\n
$$
\leq \begin{cases}\nC \frac{\varepsilon^{j}}{\varepsilon^{2k+j-2s}} ||u||_{q+s}^{(q-1)^{2}} & \text{for} \quad |\alpha_1| = |\alpha_2| = s \\
C \frac{\varepsilon^{j}}{\varepsilon^{2k+j-2s-1}} \left(\varepsilon ||u||_{q+s+1}^{(q-1)^{2}} + \frac{1}{\varepsilon} ||u||_{q+s}^{(q-1)^{2}} \right) & \text{for} \quad |\alpha_1| = |\alpha_2| + 1 = s + \n\end{cases}
$$

with $s = 0, 1, ..., k + \left[\frac{j}{2}\right], j = 1, ..., 2l - 1$, and, for the second sum, mD^a 125 || || D 2120 D^a 225 ||

$$
\mathbb{E}\left\{\n\begin{aligned}\nC & \frac{1}{\epsilon^{i-2s}} \left\|u\right\|_{q+s}^{[(q-1)^{2}} \text{ for } |x_{1}| = |\alpha_{2}| = s \\
C & \frac{1}{\epsilon^{i-2s-1}} \left(\n\epsilon \left\|u\right\|_{q+s+1}^{[(q-1)^{2}]} + \frac{1}{\epsilon} \left\|u\right\|_{q+s}^{[(q-1)^{2}]} \right) \text{ for } |\alpha_{1}| = |\alpha_{2}| + 1 = s+1\n\end{aligned}\n\right\}
$$
\nwith $s = 0, 1, \ldots, \left[\frac{i}{2}\right], i = 1, \ldots, 2k.$

Summarizing we obtain

$$
\left\langle L_{\epsilon}(\psi v), \psi v \right\rangle - \varepsilon^{2l} \frac{A}{2} \sum_{|\beta| = k + l} ||\psi D^{\beta} v||^{2}
$$

\n
$$
\leq C \left\{ \sum_{s=0}^{m-1} \frac{1}{\varepsilon^{2k-2s}} ||u||_{q+s}^{(p-1)^{2}} + \frac{1}{\varepsilon^{2k}} ||u||_{q}^{(q-1)^{2}} + \varepsilon^{2k} ||h||_{q}^{(q-1)^{2}} \right\}
$$

and, due to (43) ,

$$
\| \psi D^q u \|_m \leq \frac{C}{\varepsilon^m} \sum_{s=0}^{m-1} \varepsilon^s \| u \|_{q+s}^{(q-1)} + \frac{C'}{\varepsilon^{l-k}} \| h \|_q^{(q-1)}.
$$
 (45)

Using now the proposition (44) for the norms of u here and (42) we have

$$
||u||_{m+q}^{(q)} \leqq \frac{C}{\varepsilon^m} \sum_{s=0}^{m-1} \frac{\varepsilon^s}{\varepsilon^{q+s}} ||h|| + \frac{C}{\varepsilon^m} \sum_{s=m-q+1}^{m-1} \frac{\varepsilon^s}{\varepsilon^{r-k}} + \sum_{\nu=0}^{s+q-m-1} \frac{1}{\varepsilon^r} ||h||_{s+q-m-1-\nu}^{(s+q-m-2-\nu)} + \frac{C'}{\varepsilon^{r-k}} ||h||_{q}^{(q-1)},
$$

whence

$$
||u||_{m+q}^{(q)} \leqq \frac{C}{\varepsilon^{m+q}} ||h|| + \frac{C}{\varepsilon^{l-k}} \sum_{\mu=0}^{q-1} \frac{1}{\varepsilon^{\mu}} ||h||_{q-\mu}^{(q-\mu-1)},
$$

the assertion (44) for q instead of $p < q$. In the case of (42') instead of (42) the relevant assertion will be reproduced in the same way I

Remark: It might be conjectured that in the case where in (42) the order ε^{-1} could be replaced by $\varepsilon^{-(i-k)^+}$ (e.g. if for u are given homogeneous Dirichlet data) in the assertion (44) likewise $\varepsilon^{-m-\bar{p}} ||h||$ could be replaced by $\varepsilon^{-l-p} ||h||$; but our proof will fail for the first $k-1$ steps of induction.

Lemma 2: Let $u = u_t$ be the solution of the problem of Theorem 2

$$
G: L_{\epsilon}u = \sum_{i=0}^{2m} \epsilon^{k_i} L_i u = h, \qquad \partial G: D^{\gamma}u = 0 \qquad (|\gamma| \leq m-1)
$$

$$
^{327}
$$

 (28)

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 \cdot with coefficients and right hand side sufficiently smooth and ellipticity constants of all L_{2i} *positive. Them there is valid a preliminary estimate*

D. GÖHDE
\n*ficients and right hand side sufficiently smooth and ellipticity constants of all*
$$
L_2
$$
,
\nThen there is valid a preliminary estimate
\n $||u||_i^{(1i-m)^*} \leq \frac{C_i}{\varepsilon^{ri}}$ $(i = 0, 1, ...)$ (46)
\nmay depend on h and d but not on ε ; $k_0 = 0$, and
\n $r = \max_{1 \leq i \leq 2m} \{k_i - k_{i-1}\} \geq 1.$ (47)
\nremember the norm $||\cdot||^{(i)}$ refers to subdomain G_{id_i} . The proof starts with
\n $||u||_i \leq c_i \frac{||h||}{\varepsilon^{k_i}} \leq \frac{C_i}{\varepsilon^{ri}}$
\n0, ..., m, by scalar multiplying the equation by u and integration by parts.
\nr $i = m + p, p \geq 0$, we follow up the induction argument of the proof of

where C_i may depend on h and d but not on ε ; $k_0 = 0$, and

$$
r = \max \{k_i - k_{i-1}\} \geq 1. \tag{47}
$$

 $1 \leq i \leq 2m$
We remember the norm $\|\cdot\|^{(i)}$ refers to subdomain G_{id_i} . The proof starts with establishing the inequality

$$
||u||_i \leqq c_i \frac{||h||}{\varepsilon^{k_i}} \leqq \frac{C_i}{\varepsilon^{r_i}}
$$

for $i = 0, ..., m$, by scalar multiplying the equation by *u* and integration by parts. Then, for $i = \hat{m} + p$, $p \geq 0$, we follow up the induction argument of the proof of the preceding lemma, replacing ε^j by ε^{k_j} . The essential inequality (45) will then read

(observe $k = 0$)
 $\|\varphi D^q u\|_{m} \leq \frac{C}{\sqrt{\varepsilon^{k_{\text{tm}}}}}\left\{\sum_{s=0}^{m-1}\sum_{j=2s}^{2m-1} \left|\frac{\varepsilon^{k_j}}{\varepsilon^{j-2s}}\right|\|u\|_{q+s}^{\lfloor q-$ (observe $k = 0$) in $G_{id\epsilon}$. The proof starts with
by *u* and integration by parts.
ction argument of the proof of
al inequality (45) will then read
 $+ ||h||_q (q-1)$ $r = \frac{1}{16}$
 $\text{e} \text{ rememb}$
 $\text{d} \text{ishing th}$
 $\text{d} \text{inj} \text{ the right, } \text{d} \text{ is the right, }$ plying the equation

follow up the index

of by ε^{k_j} . The essent
 $\sum_{\substack{\epsilon=1 \\ \epsilon_j \neq 0}}^{\infty} \frac{1}{\varepsilon^{j-2s}} \frac{||u||_{q+s}^{(q-1)}}{||u||_{q+s}^{(q-1)}}$

\n- 1.
$$
\ldots
$$
, m , by scalar multiplying the equation by u and in r $i = m + p$, $p \geq 0$, we follow up the induction argument, replacing ε^j by ε^{k_j} . The essential inequality $k = 0$
\n- $||\psi D^q u||_m \leq \frac{C}{\sqrt{\varepsilon^{k_{\text{tm}}}}}\left\{\sum_{s=0}^{m-1} \sum_{j=2s}^{2m-1} \sqrt{\frac{\varepsilon^{k_j}}{\varepsilon^{j-2s}}}\right\} ||u||_{q+s}^{(q-1)} + ||h||_q(q-1)\right\}$
\n- $\leq \frac{C}{\varepsilon^{rm}}\left\{\sum_{s=0}^{m-1} \varepsilon^{rs} ||u||_{q+1}^{(q-1)} + ||h||_q(q-1)\right\}$
\n- 1. \ldots \ldots

if we pay regard to the fact that $\varepsilon^{k}/\varepsilon^{j-2s}$ *is maximal for* $j = 2s$ *, and* $k_{2s} \geq k_{2m}$ $-2(m-s)$ *r*. Now the assertion (46) is at once to be seen reproducing itself.

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we pay regard to the fact that $\epsilon^{k_j} / \epsilon^{j-2s}$ is maximal for $j = 2s$, and $k_{2s} \geq 2(m - s) r$. Now the assertion (46) is at once to be seen reprod
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