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A Remark on the Qualitative Spectral Theory of Sturm-Liouville Operators

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Bezeichnet $N(\Lambda)$ die größte Anzahl der Nullstellen nichttrivialer Lösungen der Sturm-Liouvilleschen Gleichung

$$-(p(x)\,u')' + q(x)\,u = \Lambda u, \qquad -\infty \leqq a < x < b \leqq \infty,$$

so ist unter einer gewissen Voraussetzung die Anzahl der unterhalb Λ gelegenen Eigenwerte eines bestimmten selbstadjungierten Operators (Friedrichssche Erweiterung) gleich $N(\Lambda) - 1$.

Если $N(\Lambda)$ — наибольшее число нулей нетривиальных решений уравнения Штурма-Лиувилля

$$-(p(x)\,u')' + q(x)\,u = \Lambda u, \qquad -\infty \leqq a < x < b \leqq \infty,$$

то при некотором предположении число собственных значений одного самосопряженного оператора (расширения Фридрихса), меньших Λ, равно $N(\Lambda) - 1$.

If $N(\Lambda)$ denotes the maximal number of zeros of the non-trivial solutions of the Sturm-Liouville equation

$$-(p(x)\,u')' + q(x)\,u = \Lambda u, \qquad -\infty \leqq a < x < b \leqq \infty,$$

then under some hypothesis the number of eigenvalues of a special selfadjoint operator (Friedrichs extension) is equal to $N(\Lambda) - 1$ below Λ.

Consider the Sturm-Liouville expression

$$\mathscr{A}\varphi \equiv -\big(p(x)\,\varphi'\big)' + q(x)\,\varphi, \qquad -\infty \leqq a < b \leqq \infty,$$

and the symmetric operator

$$A_0\varphi = \mathscr{A}\varphi, \qquad \varphi \in D(A_0) = C_0^\infty(a, b),$$

where

$$p(x) > 0, \quad a < x < b, \quad p \in C^1, \quad q \text{ real} \quad \text{and} \quad q \in C.$$

Concerning the symmetric forms

$$a[\varphi, \psi] = \int_a^b p\varphi'\overline{\psi}'\,dx + \int_a^b q\varphi\overline{\psi}\,dx, \qquad \varphi, \psi \in C_0^\infty(a, b),$$

$$a^+[\varphi, \psi] = \int_a^b p\varphi'\overline{\psi}'\,dx + \int_a^b q^+\varphi\overline{\psi}\,dx, \qquad \varphi, \psi \in C_0^\infty(a, b),$$

$$q^+(x) = \max\big(q(x), 0\big), \qquad q^-(x) = \min\big(q(x), 0\big),$$

$$(q^-\varphi, \psi) = \int_a^b q^-\varphi\overline{\psi}\,dx, \qquad \varphi, \psi \in C_0^\infty(a, b),$$

assume the inequality

$$|(q^-\varphi, \varphi)| \leqq c_1 a^+[\varphi, \varphi] + c_2 \|\varphi\|^2, \qquad \varphi \in C_0^\infty(a, b), \tag{1}$$

with $0 \leqq c_1 < 1$ and $0 \leqq c_2$. $(\cdot, \cdot)$ and $\|\cdot\|$ denote inner product and norm in the Hilbert space $L_2(a, b)$. From (1) it follows that the operator A_0 is bounded from below. Let A be the Friedrichs extension of A_0. We assume in the following that there are eigenvalues λ_k, $k = 1, 2, \ldots, K$ $(K \leqq \infty)$, below the essential spectrum $\sigma_e(A)$ of A. There are connections between the number of zeros of certain solutions of the differential equation

$$\mathscr{A}u = \Lambda u, \quad -\infty < \Lambda < \infty,$$

on the one hand, and the number of eigenvalues below Λ of self-adjoint extensions of A_0 on the other hand (see [1: pp. 1480, 1481]). In the following case where the Friedrichs extension A is considered these connections will be described more precisely.

Theorem: *Consider all solutions of the differential equation*

$$\mathscr{A}u = \Lambda u, \qquad a < x < b, \qquad -\infty < \Lambda < \infty,$$

and let $N(\Lambda)$ be the maximum number of zeros of the different non-trivial solutions on (a, b). If $N(\Lambda)$ is finite, then there exist exactly $N(\Lambda) - 1$ eigenvalues of the operator A on the interval $(-\infty, \Lambda)$. In the case where $N(\Lambda) = \infty$ the interval $(-\infty, \Lambda)$ contains infinitely many points of the spectrum of A.

Proof: Let the spectrum of A be denoted by $\sigma(A)$ and suppose

$$\lambda_k < \Lambda \leqq \inf \sigma_e(A), \qquad (\lambda_k, \Lambda) \cap \sigma(A) = \emptyset. \tag{2}$$

The eigenvalues λ_k $(k = 1, 2, \ldots)$ are to be denumerated in ascending order. Note that each λ_k is a simple eigenvalue of A. Λ can be a regular point of A, an eigenvalue $(\Lambda = \lambda_{k+1})$, or Λ can be equal to the lowest point of $\sigma_e(A)$. The eigenfunction u_k belonging to λ_k has exactly $k - 1$ zeros on (a, b) [1: p. 1480]. By these zeros x_j, $j = 1, \ldots, k - 1$, the interval (a, b) is divided into k subintervals (x_{j-1}, x_j), $j = 1, \ldots, k$, $x_0 = a$, $x_k = b$. By Sturm's comparison theorem, that can also be applied to the intervals (a, x_1) and (x_{k-1}, b) under the hypothesis (1) [2], a solution of $\mathscr{A}u = \Lambda u$ realizing the maximum number $N(\Lambda)$ of zeros has at least k zeros on (a, b), because u_Λ has at least one zero on each interval (x_{j-1}, x_j). By using a non-trivial solution u_Λ vanishing at x_1, for instance, one easily can see that u_Λ has at least $k + 1$ zeros. Let us now assume that a not identically vanishing solution u_Λ has at least $k + 2$ zeros on (a, b) and let $\xi_1, \ldots, \xi_{k+2}$ be the first $k + 2$ of these zeros such that $a < \xi_1 < \cdots < \xi_{k+2} < b$. Then on the interval (ξ_1, ξ_{k+2}) there are k zeros of u_Λ. The restriction $\tilde{u}$ of u_Λ to (ξ_1, ξ_{k+2}) is an eigenfunction of the Friedrichs extension $\tilde{A}$ of the operator

$$\tilde{A}_0\varphi = \mathscr{A}\varphi, \quad \varphi \in D(\tilde{A}_0) = C_0^\infty(\xi_1, \xi_{k+2}),$$

belonging to the eigenvalue Λ. Λ is the $(k + 1)$th eigenvalue of $\tilde{A}$. Hence to the left of Λ there are k eigenvalues of $\tilde{A}$. We set $\xi_1 = \alpha$ and $\xi_{k+2} = \beta$ and consider α and β as parameters. If the endpoints α and β of the interval (α, β) tend strictly monotone to a and b, respectively, then the eigenvalues of the Friedrichs extension $A_{\alpha,\beta}$ of the operator

$$A_{\alpha,\beta,0}\varphi = \mathscr{A}\varphi, \quad \varphi \in D(A_{\alpha,\beta,0}) = C_0^\infty(\alpha, \beta),$$

are strictly decreasing (the spectrum of $A_{\alpha,\beta}$ is discrete) [3]. Thus, it follows that there exist at least $k + 1$ eigenvalues of the operator A to the left of Λ. In view of (2), however, we have only k eigenvalues of A to the left of Λ. Consequently, a solution u_Λ realizing the maximum number $N(\Lambda)$ has exactly $k + 1$ zeros on (a, b). Hence, the equality $k = N(\Lambda) - 1$ is proved.

To handle the case

$$(-\infty, \Lambda) \cap \sigma(A) = \emptyset$$

a non-trivial solution u_Λ of $\mathscr{A}u = \Lambda u$ will be compared with the eigenfunction u_1 not having any zero on (a, b). By assuming that u_Λ has two zeros on (a, b) the Sturm comparison theorem implies that u_1 has at least one zero between the zeros of u_Λ. Since this situation is impossible, the solution u_Λ has at most one zero on (a, b). Of course, a zero of a non-trivial solution u_Λ of $\mathscr{A}u = \Lambda u$ can be realized on (a, b). Thus, we have $N(\Lambda) - 1 = 0$.

Finally, let $N(\Lambda) = \infty$ and consider a non-trivial solution u_Λ of $\mathscr{A}u = \Lambda u$. Assume that there are only finite points of the spectrum of A below Λ. These points are eigenvalues of A, say $\lambda_1, \ldots, \lambda_k$. Choose $k + 2$ zeros $\xi_1, \ldots, \xi_{k+2}$ of u_Λ such that $a < \xi_1 < \cdots < \xi_{k+2} < b$ and consider the interval (ξ_1, ξ_{k+2}). Now, we have the situation as above and, analogously, we can conclude that there are at least $k + 1$ eigenvalues of A below Λ. This contradicts the hypothesis that there are only k eigenvalues to the left of Λ. This proves the Theorem.

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