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The Convergence of Galerkin and Collocation Methods with Splines for Pseudodifferential Equations on Closed Curves

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In der vorliegenden Arbeit wird die näherungsweise Lösung von Pseudodifferentialgleichungen auf geschlossenen Kurven mittels Galerkin- und Kollokationsverfahren untersucht, die als Ansatzfunktionen polynomiale Splines benutzen. Es werden hinreichende und im allgemeinen notwendige Bedingungen für die Konvergenz dieser Verfahren in Sobolevräumen angegeben.

В предлагаемой работе рассматривается приближенное решение псевдодифференциальных уравнений на замкнутых кривых методами Галеркина и коллокации, в которых приближенное решение ищется в виде полиномиального сплайна. Даются достаточные и, как правило, также необходимые условия сходимости этих методов в пространствах Соболева.

The present paper studies the approximate solution of pseudodifferential equations on closed curves using Galerkin and nodal collocation methods with polynomial splines. We give sufficient and in general necessary conditions for the convergence of these methods in Sobolev spaces.

1. Introduction

Various physical problems can be reduced to pseudodifferential equations on closed curves. These equations include for example linear differential equations, certain classes of first and second kind Fredholm integral equations, singular integral equations with Cauchy kernel and integrodifferential equations. There is a sizable literature on the numerical treatment of such equations.

It is the purpose of the present paper to investigate the spline approximation of elliptic pseudodifferential equations via standard Galerkin procedures and via nodal collocation methods. We shall give conditions which are sufficient and in special cases also necessary for the convergence of these methods in Sobolev norms and establish quasioptimal error estimates in a range of Sobolev spaces.

Let Γ be a simple closed C^∞-curve in the complex plane given by the equation $z = z(x)$, $x \in [0, 1]$. We identify functions $u(z)$ on Γ with 1-periodic functions $u(x) = u(z(x))$ on the real axis $\mathbf{R}$. Let A be a classical pseudodifferential operator of order $2n \in \mathbf{R}$ on Γ, whose complete symbol $a(x, \xi)$ ($\xi \in \mathbf{R}$) has an asymptotic expansion

$$a(x, \xi) \sim \sum_{l=0}^{\infty} a_{2n-l}(x, \xi) \quad \text{as } |\xi| \to \infty, \tag{1.1}$$

where a_{2n-l} are C^∞ for $\xi \neq 0$ and positively homogeneous of degree $2n - l$ with respect to ξ. Throughout the paper C^∞ denotes the set of all infinitely differentiable 1-periodic functions on $\mathbf{R}$ and H^s the periodic Sobolev space of arbitrary real order s, i.e. the closure of C^∞ with respect to the norm

$$\|u\|_s := \left\{|\hat{u}_0|^2 + \sum_{0 \neq k \in \mathbf{Z}} |\hat{u}_k|^2 \, |2\pi k|^{2s}\right\}^{1/2},$$

where $\hat{u}_k = \int_0^1 u(x)\, e^{-2\pi i k x}\, dx$. Note that the inner product in the Hilbert space H^s

$$\langle u, v\rangle_s = \hat{u}_0\bar{\hat{v}}_0 + \sum_{0 \neq k \in \mathbf{Z}} \hat{u}_k \bar{\hat{v}}_k\, |2\pi k|^{2s} \tag{1.2}$$

extends to a duality between H^{s+r} and H^{s-r} for arbitrary $r \in \mathbf{R}$ and

$$\sup_{v \in H^{s-r}} \frac{\langle u, v\rangle_s}{\|v\|_{s-r}} = \|u\|_{s+r}. \tag{1.3}$$

Clearly, H^s is continuously and compactly imbedded in H^t for $t < s$ and, moreover, imbedded in the space of 1-periodic continuous functions C for $s > 1/2$.

The pseudodifferential operator A with symbol (1.1) has the representation [1]

$$Au(x) = \sum_{0 \neq k \in \mathbf{Z}} \hat{u}_k a(x, 2\pi k)\, e^{2\pi i k x} + \int_0^1 K(x, y)\, u(y)\, dy,$$

where $u \in C^\infty$ and $K(x, y)$ is a smooth kernel. We note, that A is called *elliptic* if its principal symbol $a_{2n}(x, \xi) \neq 0$ on $\mathbf{R} \times \{\pm 1\}$. Then it generates a bounded Fredholm operator in Sobolev spaces, $A \in L(H^{s+n}, H^{s-n})$, $s \in \mathbf{R}$, whose index is equal to the winding number $(2\pi)^{-1} \{\arg a_{2n}(x, -1)/a_{2n}(x, +1)\}_{x=0}^{x=1}$. Our aim is to investigate the approximation of the equation

$$Au = f \tag{1.4}$$

by Galerkin and nodal collocation methods using polynomial splines.

Let $\Delta = \{x_k\}_{k=-\infty}^{\infty}$ be a 1-periodic mesh of the real line, i.e. $x_k < x_{k+1}$, $x_{k+n} = x_k + 1$ for some fixed $n \in \mathbf{N}$ and all $k \in \mathbf{Z}$. By $S_d(\Delta)$ we shall denote the space of all 1-periodic, $d - 1$ times continuously differentiable splines of degree d subordinate to the partition Δ. We have $S_d(\Delta) \subset H^s$ if and only if $s < d + 1/2$. The spline spaces $S_d(\Delta)$ provide the important approximation property (cf. [6]):

Let $h_\Delta = \max_k (x_{k+1} - x_k)$. If $s < d + 1/2$ and $s \leqq r \leqq d + 1$ then for any $u \in H^r$ and any partition Δ there exists $u_\Delta \in S_d(\Delta)$ such that

$$\|u - u_\Delta\|_t \leqq c(t)\, h_\Delta^{r-t}\, \|u\|_r \tag{1.5}$$

for all $t \leqq s$, $c(t)$ denoting constants independent of u and Δ.

The *standard Galerkin method* for approximate solving equation (1.4) can be formulated as to find a spline $u_\Delta \in S_d(\Delta)$ satisfying the Galerkin equations

$$\langle Au_\Delta, v_\Delta\rangle_0 = \langle f, v_\Delta\rangle_0 \quad \text{for all } v_\Delta \in S_d(\Delta). \tag{1.6}$$

The *nodal collocation* of equation (1.4) reads as: Find $u_\Delta \in S_d(\Delta)$ such that the collocation equations

$$Au_\Delta(x_k) = f(x_k) \qquad (k = 1, \ldots, n) \tag{1.7}$$

are satisfied.

We study the problem under which conditions on A, the right-hand side f and the splines equations (1.6) resp. (1.7) are uniquely solvable for all sufficiently fine meshes Δ and the sequences of the approximate solutions $\{u_\Delta\}$ converge to a solution of (1.4) in certain Sobolev norms.

There is a large literature on Galerkin methods and the following fact is well known (cf. [3, 14]):

Assume that A is coercive elliptic, i.e.

$$\operatorname{Re} a_{2n}(x, \xi) > 0 \quad on \ \mathbf{R}\times\{\pm 1\}, \tag{1.8}$$

and $\dim \ker A = 0$. *For $f \in H^{r-2n}$, $n \leqq r \leqq d + 1$, $n < d + 1/2$, the Galerkin equations* (1.6) *are uniquely solvable for any mesh Δ with sufficiently small h_Δ and the approximate solutions $u_\Delta \in S_d(\Delta)$ converge in the norm of H^t, $2n - d - 1 \leqq t \leqq n$, to the exact solution u as $h_\Delta \to 0$ with optimal order $O(h_\Delta^{r-t})$.*

Obviously, this assertion is valid if the principal symbol of A satisfies $\operatorname{Re} \vartheta a_{2n}(x, \xi) > 0$ for some number $\vartheta \in \mathbf{C}$. In Section 4 we shall prove that the assertion holds if the pseudodifferential operator A is *strongly elliptic*, i.e. there exists a functions $\vartheta \in C^\infty$ such that

$$\operatorname{Re} \vartheta(x)\, a_{2n}(x, \xi) > 0 \quad \text{on} \ \mathbf{R}\times\{\pm 1\}. \tag{1.9}$$

Moreover, we shall show that in a special case the strong ellipticity is even necessary for the convergence of Galerkin's method.

Although in practice most numerical computations for solving equation (1.4) employ collocation procedures, hitherto the convergence of these methods is rather completely studied only for certain special equations, as for Fredholm integral equations of the second kind and for ordinary differential equations. Recently it was shown in two papers that the nodal spline collocation converges for strongly elliptic operators. In [11] S. Prössdorf and G. Schmidt state the L^2-convergence of the collocation with linear splines for a singular integral equations on the unit circle if and only if this equation is strongly elliptic. In [2] D. Arnold and W. Wendland developed a new and elegant technique to investigate the nodal collocation with splines of odd degree relating the collocation equations with certain nonstandard Galerkin equations. They proved the convergence of nodal collocation using splines of arbitrary odd degree $d > 2n$ for strongly elliptic pseudodifferential equations and obtained quasioptimal error bounds for any right-hand side $f \in H^{(d+1)/2-n}$. Using these ideas and some facts about projection methods the author considered in [13] the convergence of the nodal collocation using splines of arbitrary degree $d \geqq 1$ for pseudodifferential operators of order zero, i.e. for singular integral operators

$$a(z)\, u(z) + b(z)\, Su(z), \qquad z \in \Gamma.$$

Here

$$Su(z) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{u(\zeta)}{\zeta - z}\, d\zeta \tag{1.10}$$

is the Cauchy singular operator having the symbol $\xi/|\xi|$.

In the present paper we generalize the results of [2] and [13] to pseudodifferential operators of arbitrary real order $2n$ and for spline spaces $S_d(\Delta)$ of degree $d > 2n$. In order to formulate a special case of the results in Section 3 we define: A mesh Δ is said to be *γ-quasiuniform* ($\gamma > 0$) if $\min\limits_k (x_{k+1} - x_k) \geqq \gamma h_\Delta$. The set of all γ-quasiuniform meshes for some fixed γ we shall denote by $\mathcal{D}_\gamma$. Then we can prove:

Let $\dim \ker A = 0$ *and $\gamma > 0$ be fixed. For $f \in H^{r-2n}$, $1/2 + 2n < r \leqq d + 1$, the collocation equations* (1.7) *are uniquely solvable for any $\Delta \in \mathcal{D}_\gamma$ with sufficiently small h_Δ*

and the approximate solutions $u_\Delta \in S_d(\Delta)$ *converge in* H^t, $2n \leqq t < d + 1/2$ *and* $t \leqq r$, *to the exact solution u with optimal order* $O(h_\Delta^{r-t})$ *if d is odd and the operator A is strongly elliptic or if d is even and the operator AS is strongly elliptic.*

Note that in [13] we proved the necessity of these conditions for the convergence of the nodal collocation in special cases.

Comparing these results we can state: The spline collocation converges for a wider class of pseudodifferential equations but for a smaller set of right-hand sides than the standard Galerkin method. Moreover, the Galerkin procedure achieves higher rates of convergence.

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2. Galerkin method in H^s

In this section we collect some results on Galerkin procedures in Sobolev spaces H^s, which will be applied in Sections 3 and 4. As a rule the results can be easily deduced from standard literature [3, 15] and we shall only sketch the proofs.

Let $X_h (0 < h < 1)$ be a sequence of finitedimensional subspaces of H^s for all $s < m \in \mathbf{R}$ having the *approximation property*

A 1: *if* $s < m$ *and* $s \leqq r \leqq m + 1/2$ *then for any* $u \in H^r$ *and any h there exists* $u_h \in X_h$ *such that*

$$\|u - u_h\|_t \leqq c(t)\, h^{r-t}\, \|u\|_r \quad \text{for all } t \leqq s,$$

c(t) denoting constants independent of u and h.

At some places we shall additionally require that the subspaces possess the *inverse property*

A 2: *for* $t \leqq s < m$ *there exists a constant c such that*

$$\|v_h\|_s \leqq c h^{t-s}\, \|v_h\|_t \quad \text{for all } h \text{ and all } v_h \in X_h.$$

The *Galerkin method* in H^s for approximate solving equation (1.4) is defined as to find an element $u_h \in X_h$ satisfying the equation

$$\langle A u_h, v_h \rangle_s = \langle f, v_h \rangle_s \quad \text{for all } v_h \in X_h. \tag{2.1}$$

Obviously, equations (2.1) are well defined if $s < m - n$ and $f \in H^r$ for $r > 2s - m$. As usual we consider the pseudodifferential operator A as a bounded operator from H^{s+n} in H^{s-n} and write (2.1) as projection equations

$$P^*_{s+n,h} A u_h = P^*_{s+n,h} f, \tag{2.2}$$

where $P^*_{s+n,h}$ denotes the formal s-adjoint operator of the orthogonal projection $P_{s+n,h}: H^{s+n} \to X_h$ defined by

$$\langle P_{s+n,h} u, v_h \rangle_{s+n} = \langle u, v_h \rangle_{s+n} \quad \text{for all } v_h \in X_h \tag{2.3}$$

and

$$\langle P^*_{s+n,h} u, v \rangle_s = \langle u, P_{s+n,h} \rangle_s \quad \text{for } u \in H^{s-n},\ v \in H^{s+n}. \tag{2.4}$$

Hence, the Galerkin method in H^s is the *projection method* $\{X_h, P^*_{s+n,h}\}$. We shall write $A \in \Pi(\{X_h, P^*_{s+n,h}\}; H^t)$ if equations (2.2) are uniquely solvable for all sufficiently small h and all $f \in H^{t-2n}$ and if the sequence of solution $\{u_h\}$ converges in the norm of H^t to an exact solution of (1.4) as $h \to 0$.

Lemma 2.1: *Assume* A *coercive elliptic* (1.8) *and* $\dim\ker A = 0$. *Then* $A \in \Pi(\{X_h, P^*_{s+n,h}\}; H^{s+n})$.

Proof: In light of the results in [3: Th. 9.2] and [8] it suffices to show that A satisfies the inequality

$$\operatorname{Re}\langle Au, u\rangle_s \geqq c\,\|u\|^2_{s+n} + c_1\,\|u\|^2_{s+n-\varepsilon}$$

for some constants $c > 0$, $\varepsilon > 0$ and c_1 and all $u \in C^\infty$. Let us define the mapping Λ^t, $t \in \mathbf{R}$, by

$$\Lambda^t u(x) = \sum_{0 \neq k \in \mathbf{Z}} \hat{u}_k\,|2\pi k|^t\,e^{2\pi ikx} + \hat{u}_0.$$

Obviously, Λ^t is a pseudodifferential operator with symbol $|\xi|^t$. Furthermore, $\Lambda^t\Lambda^s = \Lambda^{t+s}$ and $\langle u, v\rangle_s = \langle \Lambda^t u, \Lambda^t v\rangle_{s-t}$ for all $s \in \mathbf{R}$ and $u, v \in C^\infty$. Hence, $\langle Au, u\rangle_s = \langle \Lambda^{2s}Au, u\rangle_0$. The pseudodifferential operator $\Lambda^{2s}A$ is of order $2(s+n)$ and has the principal symbol $|\xi|^{2s}\,a_{2n}(x, \xi)$. Because of $\operatorname{Re}|\xi|^{2s}\,a_{2n}(x, \xi) > 0$ on $\mathbf{R}\times\{\pm 1\}$ the Gårding inequality [9] yields

$$\operatorname{Re}\langle \Lambda^{2s}Au, u\rangle_0 = \operatorname{Re}\langle Au, u\rangle_s \geqq c\,\|u\|^2_{s+n} + c_1\,\|u\|^2_{s+n-\varepsilon} \quad ∎$$

Next we consider the convergence of the H^s-Galerkin method in a range of Sobolev spaces. To this end we formulate a special case of a general convergence theorem of projection methods $\{X_h, K_h\}$ with uniformly bounded projections K_h.

Lemma 2.2 [13]: *If* $\dim X_h = \dim \operatorname{im} K_h$ *and* $\|K_h u - u\|_{t-2n} \to 0$ *as* $h \to 0$ *for all* $u \in H^{t-2n}$ *then the following two conditions are equivalent:*

(i) $A \in \Pi(\{X_h, K_h\}; H^t)$;

(ii) *there exists* $A^{-1} \in L(H^{t-2n}, H^t)$ *and the finitedimensional operators* $K_hA|_{X_h}$ *are stable in* H^t, *i.e., there exist constants* $c > 0$ *and* $h_0 > 0$ *such that*

$$\|K_hAv_h\|_{t-2n} \geqq c\,\|v_h\|_t \quad \text{for all } h < h_0 \text{ and } v_h \in X_h.$$

If one of the conditions is satisfied then the approximate solutions u_h *converge to the exact solution* u *with quasioptimal rate*

$$\|u - u_h\|_t \leqq c \inf_{v_h \in X_h} \|u - v_h\|_t.$$

In order to use Lemma 2.2 we consider some properties of the projections $P_{s+n,h}$.

Lemma 2.3: *Let the spaces* X_h *satisfy assumptions* A 1 *and* A 2. *Then the projections* $P_{s+n,h}$ *are uniformly bounded in* H^r *for* $2(s+n) - m < r < m$ *and*

$$\|u - P_{s+n,h}u\|_t \leqq ch^{r-t}\,\|u\|_r \tag{2.5}$$

for $2(s+n) - m - 1/2 \leqq t \leqq r \leqq m + 1/2$, $t < m$, $2(s+n) - m < r$ *and for all* $u \in H^r$.

Proof: We use an argument of Nitsche [10]. Let $s + n < r < m$ and $u \in H^r$. Choosing u_h from A 1 and using A 2 we conclude

$$\begin{aligned}\|u - P_{s+n,h}u\|_r &\leqq \|u - u_h\|_r + \|u_h - P_{s+n,h}u\|_r \\ &\leqq c\big(\|u\|_r + h^{s+n-r}(\|u_h - u\|_{s+n} + \|u - P_{s+n,h}u\|_{s+n})\big) \leqq c\,\|u\|_r.\end{aligned}$$

For $2(s+n) - m < r < s + n$ we have

$$\|P_{s+n,h}u\|_r = \sup_{w \neq 0} \frac{\langle P_{s+n,h}u, w\rangle_{s+n}}{\|w\|_{2(s+n)-r}} = \sup_{w \neq 0} \frac{\langle u, P_{s+n,h}w\rangle_{s+n}}{\|w\|_{2(s+n)-r}} \leqq c\,\|u\|_r.$$

Thus, for $2(s+n)-m<t<m$ estimate (2.5) follows immediately from the uniform boundedness and from A 1. For $2(s+n)-m-1/2 \leqq t \leqq 2(s+n)-m$ we obtain

$$\|u - P_{s+n,h}u\|_t = \sup_{w \neq 0} \frac{\langle u - P_{s+n,h}u, w\rangle_{s+n}}{\|w\|_{2(s+n)-t}}$$

$$= \sup_{w\neq 0} \inf_{w_h \in X_h} \frac{\langle u - P_{s+n,h}u, w - w_h\rangle_{s+n}}{\|w\|_{2(s+n)-t}} \leqq \|u - P_{s+n,h}u\|_{t_1}$$

$$\times \sup_{w \neq 0} \inf_{w_h \in X_h} \frac{\|w - w_h\|_{2(s+n)-t_1}}{\|w\|_{2(s+n)-t}} \leqq ch^{t_1 - t} \|u - P_{s+n,h}u\|_{t_1}$$

$$\leqq ch^{r-t} \|u\|_r$$

where $2(s+n)-m<t_1<m$, $t_1 \leqq r$ ∎

By duality (1.3) and (2.4) we derive

Lemma 2.4: *Let X_h $(0<h<1)$ satisfy assumptions* A 1 *and* A 2. *Then the projections $P^*_{s+n,h}$ are uniformly bounded in H^r for $2s-m<r<m-2n$ and for $2s-m-1/2 \leqq t \leqq r \leqq m-2n+1/2$, $t<m-2m$, $r>2s-m$ we have*

$$\|u - P^*_{s+n,h}u\|_t \leqq ch^{r-t} \|u\|_r \quad \textit{for all } u \in H^r. \tag{2.6}$$

We are now in position to prove

Theorem 2.1: *Let $A \in \Pi(\{X_h, P^*_{s+n,h}\}; H^{s+n})$ and X_h $(0<h<1)$ satisfy assumptions* A 1 *and* A 2. *Then $A \in \Pi(\{X_h, P^*_{s+n,h}\}; H^t)$ for $2(s+n)-m<t<m$. Moreover, for $f \in H^{r-2n}$, $2(s+n)-m<r \leqq m+1/2$ the approximate solutions converge in the norm of H^t, $2(s+n)-m-1/2 \leqq t<m$ and $t \leqq r$, with optimal order to the exact solution:*

$$\|u - u_h\|_t \leqq ch^{r-t} \|u\|_r \leqq ch^{r-t} \|f\|_{r-2n}. \tag{2.7}$$

Proof: The main step is to show that the operators $P^*_{s+n,h}A|_{X_h}$ are stable in H^t. Let $2(s+n)-m-1/2 \leqq t<s+n$. Since $A \in L(H^t, H^{t-2n})$ is invertible we obtain error estimates in spaces of lower order by applying Nitsche's trick [15]. There exists $v \in H^{2(s+n)-t}$ with $\|A^*v\|_{2s+t}=1$, where $A^* \in L(H^{2(s+n)-t}, H^{2s-t})$ is the formal s-adjoint of A, such that $\|u-u_h\|_t = \langle u-u_h, A^*v\rangle_s$. Hence

$$\|u - u_h\|_t = \langle A(u-u_h), v\rangle_s = \inf_{v_h \in X_h} \langle A(u-u_h), v - v_h\rangle_t$$

$$\leqq \|A(u-u_h)\|_{s-n} \inf_{v_h \in X_h} \|v - v_h\|_{s+n} \leqq ch^{s+n-t} \|u - u_h\|_{s+n}$$

and, consequently,

$$\|A^{-1}f - (P^*_{s+n,h}A|_{X_h})^{-1} P^*_{s+n,h}f\|_t \leqq ch^{s+n-t} \|f\|_{s-n}$$

for all $f \in H^{s-n}$. We set $f = \varphi_h \in \operatorname{im} P^*_{s+n,h}$ and remark that

$$\|\varphi_h\|_{s-n} = \sup_{w \neq 0} \frac{\langle \varphi_h, w\rangle_s}{\|w\|_{s+n}} = \sup_{w\neq 0} \frac{\langle \varphi_h, P_{s+n,h}w\rangle s}{\|w\|_{s+n}}$$

$$\leqq \|\varphi_h\|_{t-2n} \sup_{w \neq 0} \frac{\|P_{s+n,h}w\|_{2(s+n)-t}}{\|w\|_{s+n}}$$

for $2(s+n)-m<t \leqq s+n$. Applying the inverse property A 2 we obtain

$$\|P_{s+n,h}w\|_{2(s+n)-t} \leqq ch^{t-(s+n)} \|P_{s+n,h}w\|_{s+n} \leqq ch^{t-(s+n)} \|w\|_{s+n}$$

and, hence

$$\|\varphi_h\|_{s-n} \leqq c h^{(t-2n)-(s-n)} \|\varphi_h\|_{t-2n}.$$

The results of [4: Th. 4.1.3] imply that the last inequality holds for all $t < s + n$. Hence,

$$\|(A^{-1} - (P^*_{s+n,h} A|_{X_h})^{-1} P^*_{s+n,h}) \varphi_h\|_t \leqq c \|\varphi_h\|_{t-2n}$$

and, consequently,

$$\|P^*_{s+n,h} A v_h\|_{t-2n} \geqq c \|v_h\|_t \quad \text{for } 2(s+n) - m - 1/2 \leqq t \leqq s + n.$$

In order to prove the stability of the operators $P^*_{s+n,h} A|_{X_h}$ for $s + n < t < m$ we assume that the exact solution $z \in H^t$. Then, using (2.5) and A 2 we derive

$$\begin{aligned}\|u - u_h\|_t &\leqq \|u - P_{s+n,h} u\|_t + \|P_{s+n,h} u - u_h\|_t \\ &\leqq c(\|u\|_t + h^{t-(s+n)}(\|P_{s+n,h} u - u\|_{s+n} + \|u - u_h\|_{s+n}) \leqq c \|u\|_t.\end{aligned}$$

Hence

$$\|(A^{-1} - (P^*_{s+n,h} A|_{X_h})^{-1} P^*_{s+n,h}) f\|_t \leqq c \|f\|_{t-2n}$$

for all $f \in H^{t-2n}$. Since due to Lemma 2.4 im $P^*_{s+n,h} \subset H^{t-2n}$ for $t < m$ we have shown that the operators

$$P^*_{s+n,h} A|_{X_h} : H^t \to H^{t-2n}, \qquad 2(s+n) - m - 1/2 \leqq t < m$$

are stable. In view of Lemma 2.2 this implies together with Lemma 2.4 the first assertion and estimate (2.7) for $2(s+n) - m < t < m$. Using once more Nitsche's trick we obtain (2.7) for $2(s+n) - m - 1/2 \leqq t \leqq 2(s+n) - m$, which completes the proof ∎

3. Convergence of the collocation method

In this section we connect the nodal collocation using splines of degree d with a Galerkin method in H^j, where $j = (d+1)/2$. Thus, we can apply results of Section 2 in order to study the spline collocation.

First we introduce some mappings. By S_0 we denote the pseudodifferential operator with symbol $\xi/|\xi|$ defined by

$$S_0 u(s) := \sum_{k=0}^{\infty} \hat{u}_k \, e^{2\pi i k x} - \sum_{k=-\infty}^{-1} \hat{u}_k \, e^{2\pi i k x}.$$

Obviously, $S_0 \in L(H^s)$, $s \in \mathbf{R}$, and $S_0^{-1} = S_0$[1]. Further, we define onedimensional operators J and J_Δ by

$$Ju := \int_0^1 u(x)\, dx \quad \text{and} \quad J_\Delta u := \sum_{k=1}^{n} u(x_k)\,(x_{k+1} - x_{k-1})/2.$$

The following theorem goes back to D. ARNOLD and W. WENDLAND and is fundamental for our further considerations.

[1]) It is well known [7] that S_0 represents the Cauchy singular operator (1.10) on the unit circle $\Gamma = \{z : |z| = 1\}$.

Theorem 3.1: *Let $u \in H^s$ for $s > 1/2$. Then the following conditions are equivalent:*
(i) $u(x_k) = 0$ *for* $k = 1, \dots, n$;
(ii) $\langle (I - J + J_\Delta)\, u, v_\Delta \rangle_j = 0$ *for all odd* $d \geqq 1$ *and all* $v_\Delta \in S_d(\Delta)$;
(iii) $\langle (I - J + J_\Delta)\, u, S_0 v_\Delta \rangle_j = 0$ *for all even* $d \geqq 0$ *and all* $v_\Delta \in S_d(\Delta)$.

Proof: (i) $\leftrightarrow$ (ii) see [2], (i) $\leftrightarrow$ (iii) see [13] ∎

We see that the collocation equations (1.7) can be written as equations using inner products of Sobolev spaces. Since $A \in L(H^{s+n}, H^{s-n})$ the equations (1.7) are well defined for $s + n < d + 1/2$ and $s - n > 1/2$, which imply in particular relation $d > 2n$ between the order of pseudodifferential operator A and the degree of the splines. Setting $s = j = (d + 1)/2$ all requirements are fulfilled and we obtain from Theorem 3.1 that the collocation equations (1.7) are equivalent to

$$\langle (I - J + J_\Delta)\, A u_\Delta, v_\Delta \rangle_j = \langle (I - J + J_\Delta)\, f, v_\Delta \rangle_j$$

for $u_\Delta, v_\Delta \in S_d(\Delta)$ and odd $d > 2n$

and to

$$\langle (I - J + J_\Delta)\, A u_\Delta, S_0 v_\Delta \rangle_j = \langle (I - J + J_\Delta)\, f, S_0 v_\Delta \rangle_j$$

for $u_\Delta, v_\Delta \in S_d(\Delta)$ and even $d > 2n$.

We note the *error estimate for the trapezoidal rule*

$$|(J - J_\Delta)\, u| \leqq c h_\Delta^s \, \|u\|_s \tag{3.1}$$

for all $u \in H^s$ and all meshes Δ if $1 \leqq s \leqq 2$, and for all γ-quasiuniform meshes with fixed $\gamma > 0$ if $1/2 < s < 1$ (see [6]). For arbitrary meshes the error estimate

$$|(J - J_\Delta)\, u| \leqq c h_\Delta^\varepsilon \, \|u\|_s \tag{3.2}$$

is easily established, where $0 < \varepsilon < s - 1/2$. Thus, one can expect that equations (1.7) are uniquely solvable for all sufficiently small h_Δ if for odd d equations

$$\langle A u_\Delta, v_\Delta \rangle_j = \langle f, v_\Delta \rangle_j \quad \text{for all } v_\Delta \in S_d(\Delta) \tag{3.3}$$

or if for even d equations

$$\langle A u_\Delta, S_0 v_\Delta \rangle_j = \langle f, S_0 v_\Delta \rangle_j \quad \text{for all } v_\Delta \in S_d(\Delta) \tag{3.4}$$

are uniquely solvable.

We remark that (3.3) is the Galerkin method $\{S_d(\Delta), P_{j+n,\Delta}^*\}$ applied to operator A, where $P_{j+n,\Delta}$ is the orthogonal projection of H^{j+n} onto $S_d(\Delta)$. Introducing $\tilde{S}_d(\Delta) := S_0 S_d(\Delta)$ and denoting by $\tilde{P}_{j+n,\Delta}$ the orthogonal projection of H^{j+n} onto $\tilde{S}_d(\Delta)$ we see, that (3.4) is the Galerkin method $\{\tilde{S}_d(\Delta), \tilde{P}_{j+n,\Delta}^*\}$ applied to the operator $A S_0$.

In order to prove the hypothesis we shall write equations (1.7) as projection equations using some interpolation projections $Q_{d,\Delta}$ with

$$\operatorname{im} Q_{d,\Delta} = \operatorname{im} P_{j+n,\Delta}^* \quad \text{for odd } d,$$

resp.

$$\operatorname{im} Q_{d,\Delta} = \operatorname{im} \tilde{P}_{j+n,\Delta}^* \quad \text{for even } d.$$

The existence of these interpolation projections follows from Theorem 3.1.

Let d be odd. Then the interpolating element $Q_{d,\Delta} u \in \operatorname{im} P_{j+n,\Delta}^*$ has to satisfy

$$\langle (I - J + J_\Delta)\, Q_{d,\Delta}\, u, v_\Delta \rangle_j = \langle (I - J + J_\Delta)\, u, v_\Delta \rangle_j$$

for all $v_\Delta \in S_d(\Delta)$. Hence

$$P^*_{j+n,\Delta}(I - J + J_\Delta)\, Q_{d,\Delta} u = P^*_{j+n,\Delta}(I - J + J_\Delta)\, u$$

for all $u \in H^{j-n}$. Since im $P^*_{j+n,\Delta}$ contains the constant functions and $(I - J + J_\Delta)^{-1} = (I + J - J_\Delta)$ [2] we obtain

$$Q_{d,\Delta} = (I + J - J_\Delta)\, P^*_{j+n,\Delta}(I - J + J_\Delta) = P^*_{j+n,\Delta} + (J - J_\Delta)\,(P^*_{j+n,\Delta} - I). \tag{3.5}$$

Analogously one obtains for even d

$$Q_{d,\Delta} = \tilde{P}^*_{j+n,\Delta} + (J - J_\Delta)\,(\tilde{P}^*_{j+n,\Delta} - I). \tag{3.6}$$

Thus, the nodal spline collocation for equation (1.4) is the *projection method* $\{S_d(\Delta), Q_{d,\Delta}\}$.

We are now in position to prove

Lemma 3.1: *Let the degree of the splines d be odd. Then*

$$A \in \Pi(\{S_d(\Delta), Q_{d,\Delta}\};\ H^{j+n}) \text{ if and only if } A \in \Pi(\{S_d(\Delta), P^*_{j+n,\Delta}\};\ H^{j+n}).$$

If d is even then

$$A \in \Pi(\{S_d(\Delta), Q_{d,\Delta}\};\ H^{j+n}) \text{ if and only if } AS_0 \in \Pi(\{\tilde{S}_d(\Delta), \tilde{P}^*_{j+n,\Delta}\};\ H^{j+n}).$$

Proof: Since the finitedimensional spaces $S_d(\Delta)$ and $\tilde{S}_d(\Delta)$ satisfy approximation property A 1 with $m = d + 1/2$ and $h = h_\Delta$ the projections $P^*_{j+n,\Delta}$, $\tilde{P}^*_{j+n,\Delta}$ and $Q_{d,\Delta}$ strongly converge in H^{j-n} to the identity operator. Thus, we can apply Lemma 2.2 and have to study the stability of the finitedimensional operators as $h_\Delta \to 0$.

Let d be odd. Since $j - n > 1/2$ we obtain from (3.5), (3.1) and (3.2) $\|P^*_{j+n,\Delta} - Q_{d,\Delta}\|_{j-n} \to 0$, hence the operators $Q_{d,\Delta}A|_{S_d(\Delta)}$ are stable in H^{j+n} if and only if $P^*_{j+n,\Delta}A|_{S_d(\Delta)}$ are stable. For even d we derive from (3.6), (3.1) and (3.2) $\|\tilde{P}^*_{j+n,\Delta} - Q_{d,\Delta}\|_{j-n} \to 0$ and hence, the operators $Q_{d,\Delta}A|_{S_d(\Delta)} = Q_{d,\Delta}AS_0|_{\tilde{S}_d(\Delta)}$ are stable in H^{j+n} if and only if $\tilde{P}^*_{j+n,\Delta}AS_0|_{\tilde{S}_d(\Delta)}$ are stable ∎

We can now give conditions for the convergence of the spline collocation. We remark that in the case of odd degree d this condition was already established in [2]. Here we give a different proof.

Theorem 3.2: *If* dim ker $A = 0$ *and for odd $d > 2n$ the operator A is strongly elliptic or for even $d > 2n$ the operator AS is strongly elliptic then $A \in \Pi(\{S_d(\Delta), Q_{d,\Delta}\};\ H^{j+n})$. Furthermore, for $f \in H^{r-2n}$, $j + n \leqq r \leqq d + 1$, we have*

$$\|u - u_\Delta\|_{j+n} \leqq c h_\Delta^{r-(j+n)}\, \|f\|_{r-2n}. \tag{3.7}$$

(S denotes the Cauchy singular operator (1.10).)

Proof: By Θ we denote the operator of multiplication with a nonzero function $\vartheta(x) \in C^\infty$. Obviously, $Q_{d,\Delta}\Theta^{-1}Q_{d,\Delta}\Theta A = Q_{d,\Delta}A$ and $\Theta^{-1} \in \Pi(\{\text{im}\, Q_{d,\Delta}, Q_{d,\Delta}\};\ H^{j-n})$. Hence, $A \in \Pi\big(\{S_d(\Delta), Q_{d,\Delta}\};\ H^{j+n}\big)$ if and only if $\Theta A \in \Pi\big(\{S_d(\Delta), Q_{d,\Delta}\};\ H^{j+n}\big)$, and because of Lemma 3.1, if and only if

$$\Theta A \in \Pi(\{S_d(\Delta), P^*_{j+n,\Delta}\};\ H^{j+n}) \text{ for odd } d \tag{3.8}$$

for

$$\Theta AS_0 \in \Pi\big(\{\tilde{S}_d(\Delta), \tilde{P}^*_{j+n,\Delta}\};\ H^{j+n}\big) \text{ for even } d \tag{3.9}$$

and some nonzero function ϑ. Due to Lemma 2.1 (3.8) holds if $\operatorname{Re}\vartheta(x)\,a_{2n}(x,\xi) > 0$ on $\mathbf{R}\times\{\pm 1\}$, i.e., if A is strongly elliptic. (3.9) is valid if $\operatorname{Re}\vartheta(x)\,a_{2n}(x,\xi)\,\xi/|\xi| > 0$ on $\mathbf{R}\times\{\pm 1\}$, i.e., if the operator AS is strongly elliptic. Estimate (3.7) follows from Lemma 2.2 and A 1 ∎

Let us derive equivalent conditions for the strong ellipticity of A and AS. Since $a_{2n}(x,\xi)$ is positively homogeneous of degree $2n$ with respect to ξ, there holds

$$\begin{aligned} a_{2n}(x,\xi) &= |\xi|^{2n}\big(a_{2n}(x,+1)\,(1+\xi/|\xi|)/2 + a_{2n}(x,-1)\,(1-\xi/|\xi|)/2\big) \\ &= |\xi|^{2n}\big(c(x)+d(x)\,\xi/|\xi|\big) \end{aligned}$$

with

$$c(x) = \frac{a_{2n}(x,+1)+a_{2n}(x,-1)}{2}, \quad d(x) = \frac{a_{2n}(x,+1)-a_{2n}(x,-1)}{2}.$$

Hence, $\operatorname{Re}\vartheta(x)\,a_{2n}(x,\xi) > 0$ on $\mathbf{R}\times\{\pm 1\}$ iff $\operatorname{Re}\vartheta(x)\big(c(x)\pm d(x)\big) > 0$ on $\mathbf{R}$. The last relation holds for some function $\vartheta\in C^\infty$ if and only if $c(x)+\lambda d(x) \neq 0$ for $x\in\mathbf{R}$, $\lambda\in[-1,1]$ [11: Lemma 4.4]. Thus, we obtain that A is strongly elliptic if and only if

$$\mu a_{2n}(x,+1) + (1-\mu)\,a_{2n}(x,-1) \neq 0 \quad \text{for} \quad x\in\mathbf{R}, \quad \mu\in[0,1]. \tag{3.10}$$

Analogeously, operator AS is strongly elliptic if and only if $d(x)+\lambda c(x) \neq 0$ for $x\in\mathbf{R}$, $\lambda\in[-1,1]$, or equivalently,

$$\mu a_{2n}(x,+1) - (1-\mu)\,a_{2n}(x,-1) \neq 0 \quad \text{for} \quad x\in\mathbf{R}, \quad \mu\in[0,1]. \tag{3.11}$$

We now investigate the convergence of the nodal spline collocation in a range of Sobolew spaces. To this end we remark that the spaces $S_d(\Delta)$ and $\tilde{S}_d(\Delta)$ satisfy the *inverse property* A 2 with $m = d + 1/2$ and $h = h_\Delta$ if all meshes under consideration are γ-quasiuniform with fixed $\gamma > 0$ (cf. [6]).

Theorem 3.3: *Suppose* $\dim\ker A = 0$ *and* $A\in\mathscr{D}_\gamma$. *If* A *satisfies* (3.10) *and* d *is odd or if* A *satisfies* (3.11) *and* d *is even then* $A\in\Pi(\{S_d(\Delta), Q_{d,\Delta}\}; H^t)$ *for* $2n+1/2 < t < d+1/2$. *Moreover, for any right-hand side* $f\in H^{r-2n}$, $2n+1/2 < r \leqq d+1$, *the approximate solutions* u_Δ *converge in the norm of* H^t, $2n \leqq t < d+1/2$, $t\leqq r$, *to the exact solution* u *with*

$$\|u-u_\Delta\|_t \leqq c h_\Delta^{r-t}\,\|f\|_{r-2n}. \tag{3.12}$$

Proof: Due to Lemma 2.4 the projections $P^*_{j+n,\Delta}$ and $\tilde{P}^*_{j+n,\Delta}$ are uniformly bounded in H^{t-2n} for $2n+1/2 < t < d+1/2$. Hence, using error estimate (3.1) and representations (3.5), (3.6) we conclude that the interpolation projections $Q_{d,\Delta}$ strongly converge to the identity operator in H^{t-2n} as $h_\Delta\to 0$. Thus, the first assertion is proved if we show that the operators $Q_{d,\Delta}A|_{S_d(\Delta)}$ are stable in H^t.

Let d be odd. Because of Theorem 3.2, Lemmas 2.2 and 3.1 condition (3.10) yields the stability of $P^*_{j+n,\Delta}A|_{S_d(\Delta)}$ in H^{j+n} as $h_\Delta\to 0$. The proof of Theorem 2.1 shows that

$$\|P^*_{j+n,\Delta}Av_\Delta\|_{t-2n} \geqq c\,\|v_\Delta\|_t \tag{3.13}$$

for all γ-quasiuniform meshes Δ with $h_\Delta < h$, all $v_\Delta\in S_d(\Delta)$ and $2n\leqq t < d+1/2$. Formula (3.5) implies

$$\|(Q_{d,\Delta}-P^*_{j+n,\Delta})\,v\|_{t-2n} = |(J-J_\Delta)\,(P^*_{j+n,\Delta}-I)\,v| \leqq ch^r\,\|v\|_r$$

for $v\in H^r$, $r > 1/2$, such that

$$\|Q_{d,\Delta}-P^*_{j+n,\Delta}\|_{t-2n}\to 0 \quad \text{for} \quad 2n+1/2 < t < d+1/2,$$

which proves the stability of $Q_{d,\Delta}A|_{S_d(\Delta)}$ in H^t. If d is even then condition (3.11) implies the stability of $\tilde{P}^*_{j+n}AS_0|_{\tilde{S}_d(\Delta)}$ in H^t for γ-quasiuniform meshes and $2n \leqq t < d + 1/2$. (3.6) yields $\|Q_{d,\Delta} - \tilde{P}^*_{j+n,\Delta}\|_{t-2n} \to 0$ for $2n + 1/2 < t < d + 1/2$ and, consequently, the stability of

$$Q_{d,\Delta}AS_0|_{\tilde{S}_d(\Delta)} = Q_{d,\Delta}A|_{S_d(\Delta)} \quad \text{in } H^t.$$

In view of Lemma 2.2 $A \in \Pi(\{S_d(\Delta), Q_{d,\Delta}\}; H^t)$ and estimate (3.12) holds for $2n + 1/2 < t < d + 1/2$.

In order to establish (3.12) in the remaining case $2n \leqq t \leqq 2n + 1/2$ we first consider the case of odd d and remark that because of (3.5) u_Δ solves the equation

$$P^*_{j+n,\Delta}Au_\Delta + (J - J_\Delta)(P^*_{j+n,\Delta} - I)Au_\Delta = P^*_{j+n,\Delta}f - (J - J_\Delta)(P^*_{j+n} - I)f.$$

Denoting by u_Δ' the solution of the Galerkin equation

$$P^*_{j+n,\Delta}Au_\Delta' = P^*_{j+n,\Delta}f$$

we obtain

$$P^*_{j+n,\Delta}A(u_\Delta - u_\Delta') = (J - J_\Delta)(P^*_{j+n,\Delta} - I)(f - Au_\Delta).$$

Hence, using (3.13), (3.1) and Lemma 2.4 we derive

$$\begin{aligned}\|u_\Delta - u_\Delta'\|_t &\leqq c\,\|P^*_{j+n}A(u_\Delta - u_\Delta')\|_{t-2n} \leqq ch_\Delta^{r-2n}\,\|Au_\Delta - f\|_{r-2n}\\ &\leqq ch_\Delta^{r-2n}\,\|f\|_{r-2n}.\end{aligned}$$

Theorem 2.1 states that $\|u - u_\Delta'\|_t \leqq ch_\Delta^{r-t}\,\|f\|_{r-2n}$ and therefore

$$\|u - u_\Delta\|_t \leqq c(h_\Delta^{r-t}\,\|f\|_{r-2n} + h_\Delta^{r-2n}\,\|f\|_{r-2n}) \leqq ch_\Delta^{r-t}\,\|f\|_{r-2n}.$$

The same arguments prove estimate (3.12) in the case of even d ∎

4. Convergence of the standard Galerkin method

The example of the Galerkin method using trigonometrical polynomials, which converges for the operator of multiplication with a nonzero function if and only if this function has the winding number zero [7], shows that in general Galerkin methods do not converge for strongly elliptic operators. In this section we shall demonstrate that the standard Galerkin procedure with splines converges for strongly elliptic pseudodifferential equations as it is stated without proof in some papers. Our proof utilizes essentially results of Section 3.

First we consider relations between Galerkin methods in different Sobolev spaces and using splines of different degree.

Let us denote by D^l, $l \in \mathbf{Z}$, the pseudodifferential operator

$$D^lu = \sum_{0 \neq k \in Z} \hat{u}_k(2\pi k)^l\, e^{2\pi ikx} + \hat{u}_0$$

having the symbol ξ^l. Obviously, for sufficiently smooth u and $l \in \mathbf{N}$ we have $D^lu = \left(\frac{1}{i}\,\frac{d}{dx}\right)^l u + Ju$. Furthermore, D^l maps H^s ($s \in \mathbf{R}$) isomorphically onto H^{s-l} and $\|D^lu\|_{s-l} = \|u\|_s$. Using the periodicity of the splines we obtain

$$S_{d-l}(\Delta) = D^lS_d(\Delta) \quad \text{for } l \leqq d.$$

In difference to Chap. 3 the orthogonal projection of H^s, $s < d + 1/2$, onto the subspaces $S_d(\Delta)$ is now denoted by $P_{s,\Delta,d}$. It is easily seen that

$$P_{s-l,\Delta,d-l} = D^lP_{s,\Delta,d}D^{-l}, \qquad l \leqq d. \tag{4.1}$$

Lemma 4.1: *Let $s + n < d + 1/2$ and the integer $l \leqq d$. Then $A \in \Pi(\{S_{d-l}(\Delta), P^*_{s-l+n,\Delta,d-l}\}; H^{s-l+n})$ if and only if $A \in \Pi(\{S_d(\Delta), P^*_{s+n,\Delta,d}\}; H^{s+n})$.*

Proof: It suffices to prove the assertion in one direction. Let $A \in \Pi(\{S_d(\Delta), P^*_{s+n,\Delta,d}\}; H^{s+n})$ and suppose that the operators $P^*_{s-l+n,\Delta,d-l}A|_{S_{d-l}(\Delta)}$ are not stable in H^{s-l+n} as $h_\Delta \to 0$. Then there exists a sequence $\{v_\Delta \in S_{d-l}(\Delta)\}$ with $\|v_\Delta\|_{s-l+n} = 1$ weakly converging to some $v \in H^{s-l+n}$ such that

$$\|P^*_{s-l+n,\Delta,d-l}Av_\Delta\|_{s-l-n} \to 0.$$

Since $P^*_{s-l+n,\Delta,d-l}Av_\Delta$ weakly converge to Av in H^{s-l-n} we obtain $v = 0$.

Using (4.1) we get on the other hand

$$P^*_{s-l+n,\Delta,d-l}Av_\Delta = D^l P^*_{s+n,\Delta,d} D^{-l}Av_\Delta = D^l P^*_{s+n,\Delta,d} AD^{-l}v_\Delta + D^l P^*_{s+n,\Delta,d} Kv_\Delta,$$

where $K = D^{-l}A - AD^{-l}$ is a pseudodifferential operator of order $2n - l - 1$. Hence, $\|Kv_\Delta\|_{s-n} \to 0$. Because of

$$\|P^*_{s-l+n,\Delta,d-l}Av_\Delta\|_{s-l-n} = \|P^*_{s+n,\Delta,d}(AD^{-l}v_\Delta + Kv_\Delta)\|_{s-n}$$

we derive $\|P^*_{s+n,\Delta,d}AD^{-l}v_\Delta\|_{s-n} \to 0$, which is impossible since $D^{-l}v_\Delta \in S_d(\Delta)$ and $\|D^{-l}v_\Delta\|_{s+n} = \|v_\Delta\|_{s-l+n} = 1$ ∎

Theorem 4.1: *Let the pseudodifferential operator A be strongly elliptic and $\dim \ker A = 0$. Then $A \in \Pi(\{S_d(\Delta), P^*_{n,\Delta,d}\}; H^n)$ for any $d > n - 1/2$. For $f \in H^{r-2n}$, $n \leqq r \leqq d + 1$, the approximate solutions u_Δ converge in H^t, $2n - d - 1 \leqq t \leqq n$ to the exact solution with the rate*

$$\|u - u_\Delta\|_t \leqq ch_\Delta^{r-t} \|f\|_{r-2n}. \tag{4.2}$$

If $\Delta \in \mathcal{D}_\gamma$ then estimate (4.2) holds for $2n - d - 1 \leqq t \leqq r \leqq d + 1$, $t < d + 1/2$, $r > 2n - d - 1/2$.

Proof: Setting in Lemma 4.1 $s = 0$, $l = -(d + 1)$ we obtain that $A \in \Pi(\{S_d(\Delta), P^*_{n,\Delta,d}\}; H^n)$ if $A \in \Pi(\{S_{2d+1}(\Delta), P^*_{d+1+n,\Delta,2d+1}\}; H^{d+1+n})$. From Lemma 3.1 and Theorem 3.2 we know that the Galerkin method in H^{d+1} with splines of arbitrary odd degree $2d + 1 > 2n$ converges for strongly elliptic pseudodifferential operators. Estimate (4.2) follows from Theorem 2.1 ∎

Using Lemma 4.2 we can also show that in a special case the *strong ellipticity is necessary for the convergence of the standard Galerkin method.*

Let A be the singular integral operator

$$Au(x) = a(x)\, u(x) + b(x)\, Su(x)$$

and suppose that $A \in \Pi(\{S_0(\Delta), P^*_{0,\Delta,0}\}; H^0)$, i.e., Galerkin's method with piecewise constant functions converges in $H^0 = L^2$. Due to Lemmas 4.1 and 3.1 we obtain $A \in \Pi(\{S_1(\Delta), Q_{1,\Delta}\}; H^1)$. But [13: Th. 4.8] states that A must be strongly elliptic.

We want to mention that the Galerkin method can converge for degenerate and, in particular, for not strongly elliptic operators if the subprincipal symbol $a'_{2n-1}(x, \xi) = a_{2n-1}(x, \xi) - (2i)^{-1}\, \partial^2 a_{2n}/\partial x\, \partial\xi$ satisfies additional requirements. The convergence of Galerkin's method with splines for such operators was considered by ELSCHNER [5] and as a special case of the obtained results one can formulate:

Suppose that A is elliptic with $\dim \ker A = 0$, but not strongly elliptic and satisfies

$$\operatorname{Re} a_{2n}(x, \xi) \geqq 0 \quad \text{on } \mathbf{R} \times \{\pm 1\},$$

$$\operatorname{Re} a'_{2n}(x, \xi) > 0 \quad \text{on } \textstyle\sum_{\operatorname{Re} a_{2n}} := \{(x, \pm 1): \operatorname{Re} a_{2n}(x, \pm 1) = 0\}.$$

Then for $f \in H^{r-2n}$, $n + 1/2 \leqq r \leqq d + 1$, $n < d$, the standard Galerkin equations are uniquely solvable and the approximate solutions u_Δ converge in the norm of $H^{n-1/2}$ to the exact solution with

$$\|u - u_\Delta\|_{n-1/2} \leqq ch_\Delta^{r-1-(n-1/2)} \|f\|_{r-2n}.$$

Consequently, the set of the right-hand sides, for which Galerkin's method converges, and the order of convergence is smaller than in the strongly elliptic case and this cannot be improved in general (cf. [5]).

Finally we remark that all statements remain valid for the more general case of systems of pseudodifferential equations of the same order on a system of mutually disjoint C^∞-curves. We only mention the equivalent conditions of the strong ellipticity for A and AS (cf. [12]):

$$\det \left(\mu a_{2n}(x, +1) + (1 - \mu)\, a_{2n}(x, -1)\right) \neq 0$$

and respectively

$$\det \left(\mu a_{2n}(x, +1) - (1 - \mu)\, a_{2n}(x, -1)\right) \neq 0$$

for $x \in \mathbf{R}$, $\mu \in [0, 1]$, where the matrix function $a_{2n}(x, \xi)$ is the principal symbol of the system of equations.

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