On Strongly Nonlinear Poinearé Boundary Value Problems for Harmonic Functions

L. V. WOLFERSDORF

Es wird eine Klasse stark linearer Poincaré-Probleme für harmonische Funktionen im Einheitskreis durch Zurückführung auf ein neues System von Integralgleichungen untersucht, auf das der Schaudersche Fixpunktsatz angewendet wird. Für verschiedene Spezialfälle werden konkrete Existenzaussagen gemacht, insbesondere wird der quasilineare Fall im Detail behandelt.

Исследуется класс сильно нелинейных задач Пуанкаре для гармонических функций в единичном круге. Задачи сводятся к новой системе интегральных уравнений, к которой применяется теорема Шаудера о неподвижных точках. Для разных частных случаев паются конкретные теоремы существования. В частности, подробно рассматривается квазилинейный случай.

A class of strongly nonlinear Poincaré problems for harmonic functions in the unit disk is studied by reducing them to a new integral equation system to which Schauder's fixed point theorem is applied. Specific existence results are given for several special cases, in particular the quasilinear. case is dealt with in detail.

Introduction

The Poincaré boundary value problem is a basic problem of the theory of harmonic functions posed by H. POINCARÉ in his investigation of the mathematical theory of tides in 1910. The plane linear problem of this type containing the problem of A class of strongly nonlinear Poincaré problems for harmonic functions in the unit distudied by reducing them to a new integral equation system to which Schauder's fixed theorem is applied. Specific existence results are g

Existence theorems for nonlinear generalizations of the Poincaré problem are derived by W. POGORZELSKI [14], J. WOLSKA-BOCHENEK [20, 21], and M. SCHLEIFF [16] in the case of a linear main part of.a boundary operator of Steklov or Poincaré type and a strong nonlinearity in the tangential (and also the normal) derivative of the unknown function *u* and in *u* itself satisfying a Holder-Lipschitz condition with sufficiently small constant. Corresponding problems in sufficiently small neighborhood of the Neumann and Steklov boundary value problem of potential theory are already investigated by K. MARUHN $[12]$, too. On the other hand there are global existence assertions for this problem with linear main part of oblique derivative or Poincaré type and nonlinearities in the unknown function u alone given by M. SCHLEIFF [16] again and in more general form by H. AMANN [1, 2], F. INKMANN [6], F. ROTHE [15], and P. WILDENAUER [17].

More intensively, because of its importance in physics, the special case, where the linear main part is the normal derivative of *u* and the nonlinearity depends only on *u*, has been investigated beginning with the classical paper by T. CARLEMAN [3]. In this context we only mention the papers by K. KLINGELHÖFER $[7-11]$, where the method of Hammerstein integral equations is used, for proving the existence of solutions for diverse types of such problems and the paper of J . M. Cusning $[4]$, where a corresponding eigenvalue problem is studied.

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• In this paper we make an attempt to investigate nonlinear Poincaré, problems for harmonic functions in the unit disk by another integral equation m'ethod developed in our paper [18] for the nonlinear Riemann-Hilbert problem of analytic functions. This method works with the differentiated boundary condition on the circumference and reduces the problem to an integral equation system of the type of Villat's equation in the theory of jets. To this system the classical Schauder fixed point theorem is applied.' As a result we obtain existence theorems for some classes of strong nonlinearities in *u* and the tangential derivative of *u* satisfying a constraint on the oscillation of the ascent with respect to the tangential derivative of *u* and depending in some sense weakly on the function u itself.

, **1. Statement of problem**

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Let $G: |z| <$
 $t = e^{is}$ $(-\pi \leq$ 1 be the unit, disk in the complex *z* plane with boundary $\Gamma : |t| = 1$, 1. Statement of problem
 $\text{et } G: |z| < 1$ be the unit disk in the complex z plane with boundary $\Gamma: |t|$
 $= e^{is} \, (-\pi \le s \le \pi)$. We deal with the following nonlinear Poincaré problem. *t* = e^{is} ($-\pi \le s \le \pi$). We deal with the following nonlinear Poincaré problem.
Problem P: Find a regular harmonic function $u(z)$, $z = x + iy$, in G which has

continuous partial derivatives in $\bar{G} = G + \Gamma$, i.e. $u(z) \in C^1(\bar{G})$, and satisfies the boundary condition e^{is} $\left(-\right)$
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$$
\frac{\partial u}{\partial r} + \Phi(s, u(e^{is}), \frac{\partial u}{\partial s}) = f(s) \quad \text{on } \Gamma,
$$
 (1)

where *r* is the polar radius.

The following basic *Assumption* A on the data is made.

 $\partial u/\partial r + \Phi(s, u(e^{is}), \partial u/\partial s) = f(s)$ on *I*, (1)
where *r* is the polar radius.
The following basic *Assumption* A on the data is made.
(i) $\Phi(s, u, \omega)$ is a real-valued continuous function on $[-\pi, \pi] \times \mathbb{R} \times \mathbb{R}$ which is
vatives $\boldsymbol{\varPhi}_{s}$ and $\boldsymbol{\varPhi}_{u}$ satisfying the Carathéodory conditions and estimations of the Let $G: |z| < 1$ be the unit disk in the complex z plan
 $t = e^{is}$ $(-\pi \le s \le \pi)$. We deal with the following nonl
 Problem P: Find a regular harmonic function $u(z)$,

continuous partial derivatives in $\overline{G} = G + \Gamma$, i.e. continue bound
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vatives
form $\frac{\partial u}{\partial r}$ is the pollowing
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 (s, u, ω)
odic in
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 $|\Phi_s(s,$
 $|\Phi_u(s,$ *u*, $w \geq E(x) \in L_q(\Gamma)$, $\varphi > 1$,
 $u, \omega \geq E(x) \in L_q(\Gamma)$, $\varphi > 1$,
 $u, \omega \geq E(x) \in L_q(\Gamma)$, $\varphi > 1$, $\omega \geq 0$, $\varphi > 1$, $\omega \geq 0$, $\omega \$ Problem P: Find a regular harmonic function $u(z)$, $z = x + iy$, in G which has

attinuous partial derivatives in $\bar{G} = G + \Gamma$, i.e. $u(z) \in C^1(\bar{G})$, and satisfies the

undary condition
 $\partial u/\partial r + \Phi(s, u(e^{is}), \partial u/\partial s) = f(s)$ on Γ ,
 continuous partial derivatives in $\bar{G} = G + \Gamma$, i.e. $u(z)$
boundary condition
 $\partial u/\partial r + \Phi(s, u(e^{is}), \partial u/\partial s) = f(s)$ on Γ ,
where r is the polar radius.
The following basic Assumption A on the data is made.
(i) $\Phi(s, u, \omega)$ is a $\partial u/\partial r + \Phi(s, u(e^{i\theta}), \partial u/\partial s) = f(s)$ on Γ ,

where r is the polar radius.

The following basic Assumption A on the data is made.

(i) $\Phi(s, u, \omega)$ is a real-valued continuous function on $[-\pi, \pi] \times \mathbb{R} \times \mathbb{R}$ which is
 2 2π-periodic in *s* and possesses a continuous partial derivative $Φ_ω$ and partial de

vatives $Φ_θ$, and $Φ_ψ$ satisfying the Carathéodory conditions and estimations of t

form
 $|Φ_θ(s, u, ω)| ≤ E(s) ∈ L_θ(Γ), ρ > 1,$
 $|Φ_ψ(s,$ veriodic in s and possesses a continuous partial der

ves Φ_s and Φ_u satisfying the Carathéodory condit
 $|\Phi_u(s, u, \omega)| \leq E(s) \in L_e(\Gamma)$, $\varrho > 1$,
 $|\Phi_u(s, u, \omega)| \leq G(s) \in L_e(\Gamma)$, $\varrho > 1$,
 u, ω from bounded intervals of **R**

$$
|\Phi_s(s, u, \omega)| \leq E(s) \in L_e(\Gamma), \qquad \rho > 1,
$$
\n(2)

$$
|\varPhi_u(s, u, \omega)| \le G(s) \in L_{\varrho}(\Gamma), \qquad \varrho > 1,
$$
\n(3)

(ii) $f(s)$ is a real-valued absolutely continuous 2π -periodic function on $[-\pi, \pi]$ possessing a derivative $f'(s) \in L_{\rho}(T)$, $\rho > 1$. (ii) $f(s)$ is a real-val

- possessing a derivative

Under these assump

respect to *s* obtaining t
 $u_{rs} + \Phi_s(s, u, \cdot)$

where derivatives of *u*

(1) is equivalent to the
 π
 $\int_a^{\pi} \Phi(s, u(e^{is}), u - \pi)$

following from (1)

' possessing a derivative $f'(s) \in L_{\rho}(r)$, $\rho > 1$.
Under these assumptions we can differentiate the borespect to *s* obtaining the condition •,

$$
|\Phi_u(s, u, \omega)| \leq G(s) \in L_e(\Gamma), \qquad \varrho > 1,
$$
\nfor u, ω from bounded intervals of **R**.

\n(ii) $f(s)$ is a real-valued absolutely continuous 2π -periodic function on $[-\pi, \pi]$ - possessing a derivative $f'(s) \in L_e(\Gamma), \varrho > 1$.

\nUnder these assumptions we can differentiate the boundary condition (1) with respect to s obtaining the condition

\n
$$
u_{rs} + \Phi_s(s, u, u_s) + \Phi_u(s, u, u_s) \cdot u_s + \Phi_\omega(s, u, u_s) \cdot u_{ss} = f'(s) \quad \text{a.e. on } \Gamma,
$$
\nwhere derivatives of u are now denoted by subscripts, too. The boundary condition (1) is equivalent to the condition (4) together with the integral condition

\n
$$
\int_a^{\pi} \Phi(s, u(e^{is}), u_s) ds = \int_a^{\pi} f(s) ds
$$
\n(5)

form
\n
$$
|\Phi_s(s, u, \omega)| \leq E(s) \in L_e(\Gamma), \quad \varrho > 1,
$$
\n(2)
\n
$$
|\Phi_u(s, u, \omega)| \leq G(s) \in L_e(\Gamma), \quad \varrho > 1,
$$
\n(3)
\nfor *u*, ω from bounded intervals of **R**.
\n(ii) $f(s)$ is a real-valued absolutely continuous 2π -periodic function on $[-\pi, \pi]$
\npossessing a derivative $f'(s) \in L_e(\Gamma), \varrho > 1$.
\nUnder these assumptions we can differentiate the boundary condition (1) with
\nrespect to *s* obtaining the condition
\n $u_{rs} + \Phi_s(s, u, u_s) + \Phi_u(s, u, u_s) \cdot u_s + \Phi_\omega(s, u, u_s) \cdot u_{ss} = f'(s) \quad \text{a.e. on } \Gamma,$
\nwhere derivatives of *u* are now denoted by subscripts, too. The boundary condition
\n(1) is equivalent to the condition (4) together with the integral condition
\n
$$
\int_{-\pi}^{\pi} \Phi(s, u(e^{is}), u_s) ds = \int_{-\pi}^{\pi} f(s) ds
$$
\n(5)
\n
$$
-\int_{-\pi}^{\pi} g(s, u(e^{is}), u_s) ds = \int_{-\pi}^{\pi} f(s) ds
$$
\n(6)
\nWe introduce the holomorphic functions in *G*
\n $w(z) = u(z) + iv(z), \quad W(z) = U(z) + iV(z) = zw'(z) = ru, -iu_s,$ \n(6)

We introduce the holomorphic functions in G

Example 21. The following equations are can differentiate the boundary condition (1) with respect to *s* obtaining the condition

\n
$$
u_{rs} + \Phi_s(s, u, u_s) + \Phi_u(s, u, u_s) \cdot u_s + \Phi_\omega(s, u, u_s) \cdot u_{ss} = f'(s) \text{ a.e. on } \Gamma,
$$
\nwhere derivatives of *u* are now denoted by subscripts, too. The boundary condition

\n(1) is equivalent to the condition (4) together with the integral condition

\n
$$
\int_{-\pi}^{\pi} \Phi(s, u(e^{is}), u_s) ds = \int_{-\pi}^{\pi} f(s) ds
$$
\n(5)

\nSolving from (1) by integration over Γ .

\nWe introduce the holomorphic functions in G

\n
$$
w(z) = u(z) + iv(z), \quad W(z) = U(z) + iV(z) = zw'(z) = ru, -iu_s,
$$
\n
$$
X(z) = \varphi(z) + i\psi(z) = zw'(z) = rU, -iU_s = V_s + irV,
$$
\n(6)

 $\frac{d}{dt} \sum_{i=1}^{n} \frac{d}{dt} \left(\frac{d}{dt} \right) \left(\frac{d}{dt} \right)$

$$
X(t) = \varphi(t) + i\psi(t) = -u_{ss} - iu_{rs} \quad \text{on} \quad T.
$$
 (7)

Then (4) takes the form of a Riemann-Hilbert condition for $X(z)$:

$$
\begin{array}{ll}\n\lambda & \text{Poincaré Boundary Value Problems} & 387 \\
\text{with} & X(t) = \varphi(t) + i\psi(t) = -u_{ss} - iu_{rs} \quad \text{on} \quad \Gamma. \\
\text{Then (4) takes the form of a Riemann-Hilbert condition for } X(z):\n\end{array}\n\tag{7} \text{Re}\left[\left(A(s) - i\right)X(t)\right] = A(s)\varphi(t) + \psi(t) = g(s) \quad \text{a.e. on } \Gamma, \tag{8} \\
A(s) = \Phi_{\omega}(s, u, u_s), \\
g(s) = \Phi_s(s, u, u_s) + \Phi_u(s, u, u_s) \cdot u_s - f'(s).\n\tag{9} \\
\text{The solution } X(z) \text{ of the Riemann-Hilbert problem: (8) has to fulfill the additional condition}\n\end{array}
$$

 $\mathbf w$ her

$$
A(s) = \Phi_{\omega}(s, u, u_s), \tag{9}
$$

$$
g(s) = \boldsymbol{\Phi}_s(s, u, u_s) + \boldsymbol{\Phi}_u(s, u, u_s) \cdot u_s - f'(s). \qquad (10)
$$

The solution $X(z)$ of the Riemann-Hilbert problem (8) has to fulfil the additional •àondition

$$
\bar{X}(0) = 0 \quad \text{in } z = 0. \tag{11}
$$

From $X(z)$ the boundary values $u(e^{i\theta})$ of the sought function $u(z)$ follow by the

$$
X(t) = \varphi(t) + i\psi(t) = -u_{ss} - iu_{rs} \text{ on } T.
$$
\n(7)

\nThen (4) takes the form of a Riemann-Hilbert condition for $\overline{X}(z)$:

\n
$$
\text{Re}[(A(s) - i) \overline{X}(t)] \equiv A(s) \varphi(t) + \psi(t) = \overline{g}(s) \text{ a.e. on } T,
$$
\n(8)

\nwhere

\n
$$
A(s) = \Phi_{\omega}(s, u, u_s),
$$
\n(9)

\n
$$
g(s) = \Phi_s(s, u, u_s) + \Phi_u(s, u, u_s) \cdot u_s - f'(s).
$$
\n(10)

\nThe solution $X(z)$ of the Riemann-Hilbert problem: (8) has to fulfill the additional condition

\n
$$
\overline{X}(0) = 0 \text{ in } z = 0.
$$
\n(11)

\nFrom $X(z)$ the boundary values $u(e^{is})$ of the sought function $u(z)$ follow by the relations

\n
$$
u(e^{is}) = \int_0^s u_s(e^{is}) \, d\sigma + k,
$$
\n
$$
u_s(e^{is}) = -\int_0^s \varphi(e^{is}) \, d\sigma + k_1,
$$
\n(12)

\nwhere

\n
$$
k_1 = \frac{1}{2\pi} \int_0^s \int_0^s \varphi(e^{is}) \, d\sigma
$$
\nand the constant $k = u(1)$ has to be determined in fulfilling the integral condition (5).

\nThe harmonic function $u(z)$ itself is then given by the Poisson interval of the

$$
k_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\delta} \varphi(e^{i\sigma}) d\sigma \qquad (14)
$$

and the constant $k = u(1)$ has to be determined in fulfilling the integral condition (5). The harmonic function $u(z)$ itself is then given by the Poisson integral of the boundary values $u(e^{i\theta})$. Thus, Problem P is equivalent to the relations (12) with and the constant $k = u(1)$ has to be determined in fulfilling the integral condition (5).
The harmonic function $u(z)$ itself is then given by the Poisson integral of the
boundary values $u(e^{i\theta})$. Thus, Problem P is equiva with (11), and additionally (5) has to be fulfilled. $k_1 = \frac{1}{2\pi} \int \int \varphi(e^{i\sigma}) d\sigma$

and the constant $k = u(1)$ has to be determined in fulfilling

The harmonic function $u(z)$ itself is then given by th

boundary values $u(e^{i\theta})$. Thus, Problem P is equivalent

arbitrary k (1)

g the integral condition (i.e.

le Toisson integral of the relations (12) with $X(z)$ the solution of (
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 θ)
 θ) $\in L_{e}(\Gamma), \rho > 1$, the Rinds $X(z)$ with $\varphi(e^{is}) = \text{Re } X$

(1) *= a(s) h(s) + fl(s) e (s),* • • (15) and the constant $k = u(1)$ has to be determined in fulfilling the integral condition (5).

The harmonic function $u(z)$ itself is then given by the Poisson integral of the

boundary values $u(e^{i\theta})$. Thus, Problem P is equi

§ 2. **Reduction to'a fixed point problem**

For prescribed continuous function $A(s)$ and function $g(s) \in L_{\rho}(F)$, $\rho > 1$, the Riemann-Hilbert problem (8) with (11) has a unique solution $X(z)$ with $\varphi(e^{iz}) = \text{Re } X(t)$ with (11), and additionally (5) has to be fulfilled.

§ 2. Reduction to a fixed point problem

For prescribed continuous function $A(s)$ and function $g(s) \in L_e(T)$, $\varrho > 1$, the Riemann-Hilbert problem (8) with (11) has a u *u*ching the discontinuous function $A(s)$ and function $g(s) \in L_{\epsilon}$
 u(b) ϵ arc tan $\lambda(s) + \beta(s) e^{-H(\mu)(s)} H(e^{H(\mu)}h)(s)$,
 $\alpha(s) = A(s)/\sqrt{1 + A^2(s)}$, $\beta(s) = 1/\sqrt{1 + A^2(s)}$,
 $\mu(s) = \arctan A(s)$, $h(s) = g(s)/\sqrt{1 + A^2(s)}$,

motes the Hilbert § 2. Reduction to a fixed point problem

For prescribed continuous function $A(s)$ and function $g(s) \in L_{e}(T)$,

mann-Hilbert problem (8) with (11) has a unique solution $X(z)$ with

on *T* given by (cf. [5: § 29])
 $\varphi(e^{is})$

$$
\varphi(e^{is}) = \alpha(s) h(s) + \beta(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\}(s), \qquad (15)
$$

$$
\alpha(s) = A(s)/\sqrt{1 + A^2(s)}, \qquad \beta(s) = 1/\sqrt{1 + A^2(s)},
$$
\n
$$
\mu(s) = \arctan A(s), \qquad h(s) = g(s)/\sqrt{1 + A^2(s)},
$$
\n(17)

$$
\alpha(s) = A(s)/\gamma 1 + A^2(s), \qquad \beta(s) = 1/\gamma 1 + A^2(s), \tag{16}
$$

$$
\mu(s) = \arctan A(s), \qquad h(s) = g(s)/\gamma 1 + A^2(s), \tag{17}
$$

and H denotes the Hilbert transform

5.

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For prescribed continuous function
$$
A(s)
$$
 and function $g(s) \in L_{\rho}(T)$, $\rho > 1$, the Riemann-Hilbert problem (8) with (11) has a unique solution $X(z)$ with $\varphi(e^{is}) = \text{Re } X(t)$
on Γ given by (cf. [5: § 29])
 $\varphi(e^{is}) = \alpha(s) h(s) + \beta(s) e^{-H(\mu)(s)} H(e^{H(\mu)}h)$ (s), (15)
where
 $\alpha(s) = A(s)/\sqrt{1 + A^2(s)}$, $\beta(s) = 1/\sqrt{1 + A^2(s)}$, (16)
 $\mu(s) = \text{arc tan } A(s)$, $h(s) = g(s)/\sqrt{1 + A^2(s)}$, (17)
and H denotes the Hilbert transform
 $(\underline{H}\nu)$ (s) = $\frac{1}{2\pi} \int_{-\pi}^{\pi} \nu(\sigma) \cot \frac{\sigma - s}{2} d\sigma$. (18)
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- 0
0
0
0 This solution exists if and only if *g(s)* satisfies the orthogonality condition

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\nThis solution exists if and only if
$$
g(s)
$$
 satisfies the orthogonality condition

\n
$$
\int_{-\pi}^{\pi} \gamma(s) g(s) ds = 0
$$
\nwith the nonnegative function

\n
$$
\gamma(s) = e^{H(\mu)(s)} / \sqrt{1 + A^2(s)}.
$$
\nIn virtue of a known theorem of A. ZvOMUND (cf. [22]) the function $e^{H(\mu)(s)}$ and there.

with the nonnegative function
\n
$$
\gamma(s) = e^{H(\mu)(s)} / \sqrt{1 + A^2(s)}.
$$
\n
$$
(20)
$$
\nIn virtue of a known theorem of A. ZYGMUND (cf. [22]) the function $e^{H(\mu)(s)}$ and there-

• fore $y(s)$ is summable to any power.

For given continuous functions $\xi(s)$, $\eta(s)$ on Γ with $\int \eta(s) ds = 0$, i.e. $\xi \in C(\Gamma)$,

P(*s*) = $e^{H(\mu)(s)} / \sqrt{1 + A^2(s)}$. (20)
 μ virtue of a known theorem of A. ZYGMUND (cf. [22]) the function $e^{H(\mu)(s)}$ and there-

re $\gamma(s)$ is summable to any power.

For given continuous functions $\xi(s)$, $\eta(s)$ on Γ condition $\int_{-\pi}^{\pi} \gamma(s) g(s) ds$
 $\int_{-\pi}^{\pi} \gamma(s) g(s) ds$

with the nonnegativ
 $\gamma(s) = e^{H(\mu)}$

In virtue of a knowr

fore $\gamma(s)$ is summabl

For given continu
 $\gamma \in C_0(\Gamma)$, we introved

condition
 $A(s, \xi, \eta) \varphi$

and the additional c
 fluce the *auxiliary problem*
 $(t) + \psi(t) = g_0(s, \xi, \eta)$ $y(s) = e^{2\pi i t \kappa/2} / \gamma + A$ (so

In virtue of a known theorem of A

fore $y(s)$ is summable to any pown

For given continuous function
 $\eta \in C_0(\Gamma)$, we introduce the *auxi*

condition
 $A(s, \xi, \eta) \varphi(t) + \psi(t) = g$

and the additi

$$
A(s, \xi, \eta) \varphi(t) + \psi(t) = g_0(s, \xi, \eta) \quad \text{a.e. on } \Gamma
$$
 (21)

$$
A(s, \xi, \eta) \varphi(t) + \psi(t) = g_0(s, \xi, \eta) \quad \text{a.e. on } \Gamma
$$
\n
$$
\text{and the additional condition (11) for } X(z) = \varphi(z) + i\psi(z), \text{ where}
$$
\n
$$
A(s, \xi, \eta) = \Phi_\omega(s, \xi, \eta),
$$
\n
$$
g_0(s, \xi, \eta) = g(s, \xi, \eta) - m(s, \xi, \eta) \tag{23}
$$

$$
g(s, \xi, \eta) = \Phi_s(s, \xi, \eta) + \Phi_u(s, \xi, \eta) \eta - f'(s),
$$

\n
$$
m(s, \xi, \eta) = m_0[\xi, \eta] \cdot [\gamma(s, \xi, \eta)]^{q-1}, \qquad 1/\varrho + 1/\sigma = 1,
$$
\n(25)

$$
A(s, \xi, \eta) \varphi(t) + \psi(t) = g_0(s, \xi, \eta) \quad \text{a.e. on } \Gamma
$$
\n(21)

\nadditional condition (11) for $X(z) = \varphi(z) + i\psi(z)$, where

\n
$$
A(s, \xi, \eta) = \Phi_\omega(s, \xi, \eta),
$$
\n
$$
g_0(s, \xi, \eta) = g(s, \xi, \eta) - m(s, \xi, \eta)
$$
\n
$$
g(s, \xi, \eta) = \Phi_s(s, \xi, \eta) + \Phi_u(s, \xi, \eta) \eta - f'(s),
$$
\n
$$
m(s, \xi, \eta) = m_0[\xi, \eta] \cdot [\gamma(s, \xi, \eta)]^{s-1}, \qquad 1/\varrho + 1/\sigma = 1,
$$
\n
$$
(25)
$$

$$
g_0(s, \xi, \eta) = g(s, \xi, \eta) - m(s, \xi, \eta)
$$
(23)

$$
g(s, \xi, \eta) = \Phi_s(s, \xi, \eta) + \Phi_u(s, \xi, \eta) \eta - f'(s),
$$
(24)

$$
m(s, \xi, \eta) = m_0[\xi, \eta] \cdot [\gamma(s, \xi, \eta)]^{q-1}, \qquad 1/\varrho + 1/\sigma = 1,
$$
(25)

$$
m_0[\xi, \eta] = \int_0^{\pi} g(s, \xi, \eta) \gamma(s, \xi, \eta) ds / \int_{-\pi}^{\pi} \gamma^o(s, \xi, \eta) ds,
$$
(26)

$$
\gamma(s, \xi, \eta) = e^{H[\mu(\cdot, \xi, \eta)](s)} / \sqrt{1 + A^2(s, \xi, \eta)},
$$
(27)

$$
u(s, \xi, \eta) = \arctan A(s, \xi, \eta).
$$
(28)

and

$$
\gamma(s,\xi,\eta) = e^{H(\mu(\cdot,\xi,\eta)\{s\}}/\sqrt{1+A^2(s,\xi,\eta)},
$$
\n
$$
\mu(s,\xi,\eta) = \arctan A(s,\xi,\eta).
$$
\n(27)

 $\begin{aligned} &\beta_u(s,\xi,\eta)\\ &\delta_u(s,\xi,\eta) \eta - f'(s),\\ &\xi,\eta)\rfloor^{\sigma-1},\quad 1/\varrho +\\ &\dot\xi,\eta)\,ds\Bigg/\int\limits_{-\pi}^{\pi}\gamma^\sigma(s,\xi),\\ &\overline{+A^2(s,\xi,\eta)},\\ &\eta),\\ &\hat u\text{ unique solution}\\ &\text{as }u(s)=u(\mathrm{e}^{is})\text{ of }u\end{aligned}$ The auxiliary problem $P_{\xi,\eta}$ has a unique solution $X(z)$. According to (12), (13), the corresponding boundary values $u(s) = u(e^{is})$ of $u(z)$ and $u_1(s)$ of $u_s(z)$ are given with
 $g(s, \xi, \eta) = \Phi_s(s, \xi, \eta) + \Phi_u(s, \xi, \eta)$
 $m(s, \xi, \eta) = m_0[\xi, \eta] \cdot [\gamma(s, \xi, \eta)]$
 $m_0[\xi, \eta] = \int_{-\pi}^{\pi} g(s, \xi, \eta) \gamma(s, \xi, \eta)$

and
 $\gamma(s, \xi, \eta) = e^{H[\mu(\cdot, \xi, \eta)](s)} / \sqrt{1 + A!}$
 $\mu(s, \xi, \eta) = \arctan A(s, \xi, \eta)$.

The auxiliary problem $P_{$ • $m_0[s, \eta] = \int_{-\pi}^{\pi} g(s, \xi, \eta) \gamma(s, \xi, \eta) ds \Big/ \int_{-\pi}^{\pi} \gamma^*(s, \xi, \eta) ds,$

and
 $\gamma(s, \xi, \eta) = e^{H(\mu(\cdot, \xi, \eta)l(s)} / \sqrt{1 + A^2(s, \xi, \eta)},$
 $\mu(s, \xi, \eta) = \arctan A(s, \xi, \eta).$

The auxiliary problem $P_{\xi, \eta}$ has a unique solution $X(z)$. Accordi and
 $\gamma(s, \xi, \eta) = e^{H[\mu(\cdot, \xi, \eta)](s)} / \sqrt{1 + A^2(s, \xi, \eta)},$
 $\mu(s, \xi, \eta) = \arctan A(s, \xi, \eta).$

The auxiliary problem $P_{\xi, \eta}$ has a unique solution $X(z)$. The corresponding boundary values $u(s) = u(e^{is})$ of $u(z)$ and

by the expression

$$
u(s) = N_1[\eta](s) = \int_0^s \eta(\sigma) d\sigma + k, \qquad (29)
$$

$$
u_1(s) = N_2[\xi, \eta] (s) = - \int_0^s \varphi(\sigma, \xi, \eta) d\sigma + k_1,
$$
 (30)

the corresponding boundary values
$$
u(s) = u(e^{-s})
$$
 of $u(z)$ and $u_1(s)$ of $u_3(z)$ are given
\nby the expressions
\n
$$
u(s) = N_1[\eta](s) = \int_0^s \eta(\sigma) d\sigma + k,
$$
\n(29)
\n
$$
u_1(s) = N_2[\xi, \eta](s) = -\int_0^s \varphi(\sigma, \xi, \eta) d\sigma + k_1,
$$
\nwhere
\n
$$
k_1 = \frac{1}{2\pi} \int_0^{\pi} \int_0^s \varphi(\sigma, \xi, \eta) d\sigma ds
$$
\n(31)
\nand $k = k[\eta]$ has to be determined in fulfilling the relation
\n
$$
\int_0^{\pi} \varphi(s, \int_0^s \eta(\sigma) d\sigma + k, \eta(s)) ds = \int_0^{\pi} f(s) ds.
$$
\n(32)

and $k = k[\eta]$ has to be determined in fulfilling the relation

$$
\int_{-\pi}^{\pi} \Phi\left(s, \int_{0}^{s} \eta(\sigma) \, d\sigma + k, \, \eta(s)\right) ds = \int_{-\pi}^{\pi} f(s) \, ds. \tag{32}
$$

 $\mathbf{r}^{\prime}=\mathbf{r}^{\prime}$

The function $\varphi(s, \xi, \eta) = \varphi(e^{is}, \xi, \eta)$ is defined by (15) with (16), (17), where *A* is replaced by $A(s, \xi, \eta)$ from (22) and g by $g_0(s, \xi, \eta)$ from (23). By Assumption A the function $\tilde{A}(s, \xi, \eta)$ is continuous on $\tilde{\Gamma}$ and the function $g_0(s, \xi, \eta) \in L_{\rho}(\Gamma)$ for any pair of continuous functions $\xi(s), \eta(s).$ Moreover, g_0 fulfils the orthogonality condition (19) by construction.

Any fixed point $\{u, u_1\} \in C(\Gamma) \times C_0(\Gamma)$ of the operator $N = \{N_1, N_2\}$ yields a solution $\tilde{u}(z)$ of Problem P with boundary values $u(s)$. For, $u = \xi$ and $u_1 = \eta$ in (29), (30) at first implies $u_1 = u'(s)$ and further leads to the equation (4) with an additional term $m_0[u, u_s] \cdot [\gamma(s, u, u_s)]^{s-1}$ in the right-hand side which is seen to be zero by integrating the equation over *f*. Also, from (30) follows that $u_1 = u_s|_r \in C_0(T)$ indeed is a Hölder continuous function such that also the boundary values of the normal derivative u_r , of u are (Hölder-) continuous and $u(z) \in C^1(\overline{G})$. Thus, Problem P is reduced to the determination of fixed points for the integral operators (29), (30) with (31), (32). (30) at first implies $u_1 = u'(s)$ is

(30) at first implies $u_1 = u'(s)$ is

tional term $m_0[u, u_s] \cdot [\gamma(s, u, u_s]$

by integrating the equation ov

indeed is a Hölder continuous

normal derivative u_r of u are (H

is reduced tional term $m_0[u, u_s] \cdot [\gamma(s, u, u_s)]^{q-1}$ in
by integrating the equation over Γ . A
indeed is a Hölder continuous functior
normal derivative u_r of u are (Hölder-) is
reduced to the determination of fixe
with (31), (32) de which is seen to be zero

lows that $u_1 = u_s|_r \in C_0(\Gamma)$

the boundary values of the
 $z) \in C^1(\overline{G})$. Thus, Problem P

integral operators (29), (30)

integral operators (29), (30)

ompact subset $\Re = \Re_1 \times \Re_2$
 $|s_1 -$

We consider the operator $N = \{N_1, N_2\}$ on the convex compact subset $\mathfrak{X} = \mathfrak{K_1} \times \mathfrak{K_2}$ $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$ the operator $N = \{N : C(\Gamma) \times C_0(\Gamma) \text{ with }$
= $\{ \xi \in C(\Gamma) : |\xi(s)| \le$ *F*₁, *N*₂) on the convex compact subset $\hat{\mathbf{R}} = \Re_1 \times \Re_2$
 P, $|\xi(s_1) - \xi(s_2)| \le P_0 |s_1 - s_2|$, (33)
 R, $|\eta(s_1) - \eta(s_2)| \le R_0 |s_1 - s_2|^2$ (34) •
•
•
•
•
•
• **²**= (7⁷ € *C^O (T) :* I()l 5 *R,* I01) - *⁷*7(*^S*2)1

$$
\Re_1 = \{ \xi \in C(\Gamma) : |\xi(s)| \le P, \, |\xi(s_1) - \xi(s_2)| \le P_0 \, |s_1 - s_2| \},\tag{33}
$$

$$
\Re_2 = \{ \eta \in C_0(\Gamma) : |\eta(s)| \leq R, \quad |\eta(s_1) - \eta(s_2)| \leq R_0 \, |s_1 - s_2|^2 \} \tag{34}
$$

for any $s, s_1, s_2 \in [-\pi, \pi]$, where $\lambda = 1/q$, q the conjugate exponent to $p, 1 < p < q$, and P, P₀, R, R₀ are fixed positive real numbers to be specified later.
Further, we make the *Assumption* B that for all $\eta \in \mathbb{R}_2$ the equation (32) for

We consider the operator $N = \{N_1, N_2\}$ on the convex compact subse
of the space $C(\Gamma) \times C_0(\Gamma)$ with
 $\hat{\mathbb{R}}_1 = \{ \xi \in C(\Gamma) : |\xi(s)| \le P, |\xi(s_1) - \xi(s_2)| \le P_0 |s_1 - s_2| \},$
 $\hat{\mathbb{R}}_2 = \{ \eta \in C_0(\Gamma) : |\eta(s)| \le R, |\eta(s_1) - \eta(s_2)| \le R_0 |s_1 - s$ $k \in {\bf R}$ has a root $k=k[\eta]$ which depends continuously on η and is uniformly bounded with respect to $\eta \in \mathbb{R}_2$: If the operator $N = \{N_1, N_2\}$ on the convex compact subs
 $\operatorname{ace} C(\Gamma) \times C_0(\Gamma)$ with
 $\Re_1 = \{\xi \in C(\Gamma) : |\xi(s)| \leq P, |\xi(s_1) - \xi(s_2)| \leq P_0 |s_1 - s_2|\},$
 $\Re_2 = \{\eta \in C_0(\Gamma) : |\eta(s)| \leq R, |\eta(s_1) - \eta(s_2)| \leq R_0 |s_1 - s_2|^2\}$
 $s_1, s_2 \in \{-\pi, \$ For any *s*, s_1 , $s_2 \in \{-\pi, \pi\}$, where $x = 1/q$, q the conjugate originate and P , P_0 , R , R_0 are fixed positive real numbers to be specified la
Further, we make the *Assumption* B that for all $\eta \in \mathbb{R}_2$ l₀, *K*, *K*₀ are fixed positive real numbers to be specified later.

r, we make the *Assumption* B that for all $\eta \in \Re_2$ the equation (32) for s a root $k = k[\eta]$ which depends continuously on η and is uniformly

$$
|k[n]| \leq K \text{ for any } \eta \in \mathfrak{X}_2 \text{ with } K = K(R, R_0). \tag{35}
$$

This is especially fulfilled if the (continuous) function $\Phi(s, u, \omega)$ is strictly monotone with respect to u for any $s \in [-\pi, \pi]$, $\omega \in \mathbb{R}$, there exist the limits

$$
\lim_{u \to \pm \infty} \Phi(s, u, \omega) = \Phi_{\pm}(s) \in C(\Gamma) \text{ uniformly in } s \in [-\pi, \pi], \quad \omega \in \mathbb{R}, \qquad (36)
$$

 $\mathfrak{R}_2 = \{ \eta \in C_0(\Gamma) : |\eta(s)| \leq R, \quad |\eta(s_1) - \eta(s_2)| \leq R_0 |s_1 - s \}$
for any $s, s_1, s_2 \in [-\pi, \pi]$, where $\lambda = 1/q$, q the conjugate exponen
and P, P_0, R, R_0 are fixed positive real numbers to be specified $\frac{1}{n}$.
Further, we Let us now estimate $u = N_1[\eta]$ and $u_1 = N_2[\xi, \eta]$ for $\xi \in \mathbb{R}_1$, $\eta \in \mathbb{R}_2$. Obviously, $\int ds$, lies 1
 $|u(s)| \leq$
 $\int_0^1 s^2 \, ds$ **27***Z 27z* *****27 27 27 27 27 27 27 28 <i>27* *****27 27 27* *****27 28 27 27* *****27 27* *****27 27 27* *****27 27 27 27* *****27 27 27 27 27* **** $\frac{1}{2}$ (37) Example of the continuous) function φ , it, w , it is expected to *u* for any $s \in [-\pi, \pi]$, $\omega \in \mathbf{R}$, there exist the limits $\lim_{n \to \infty} \Phi(s, u, \omega) = \Phi_{\pm}(s) \in C(\Gamma)$ uniformly in $s \in [-\pi, \pi]$, $\omega \in \mathbf{R}$, \Rightarrow (36) *Vpj* = SU ^P *^I ⁸ (s,u, w)l ⁰* < oo,, and $\int f(s) ds$ lies between the values $\Phi_{\pm} = \int \Phi_{\pm}(s) ds$.

Let us now estimate $u = N_1[\eta]$ and $u_1 = N_2[\xi, \eta]$ for $\xi \in \Re_1, \eta \in \Re_2$. Obviously,
 $|u(s)| \leq 2\pi R + K(R, R_0)$, $|u(s_1) - u(s_2)| \leq R |s_1 - s_2|$ (37)

for any $s, s_1,$

$$
|u(s)| \leq 2\pi R + K(R, R_0), \qquad |u(s_1) - u(s_2)| \leq R |s_1 - s_2| \tag{37}
$$

for any $s, s_1, s_2 \in [-\pi, \pi]$. Further, for the L_{ϱ} norm of the function g_0 there holds

$$
||g_0||_{\rho} \le 2 ||g||_{\rho} \le 2\{M + v_{P,R} + R\beta_{P,R}\},\tag{38}
$$

where $M = ||f||_{\rho}$,

$$
P_{\mathbf{p},\mathbf{r}} = \sup ||\boldsymbol{\Phi}_s(s, u, \omega)||_{\rho} < \infty, \tag{39}
$$

$$
\beta_{P,R} = \sup \|\Phi_u(s, u, \omega)\|_p < \infty, \tag{40}
$$

 $\begin{aligned} \n\mathcal{P}_{P,R} &= \sup \|\Phi_s(s, u, \omega)\|_e < \infty, \\
\beta_{P,R} &= \sup \|\Phi_u(s, u, \omega)\|_e < \infty, \\
\text{the suprema are taken over } s \in [-\pi, \pi], |u| \leq P, |\omega| \leq \text{triangle inequality}, \text{ the elementary inequalities } |\beta(s)| \leq \text{triangle inequality}. \n\end{aligned}$ *R.* Finally, applying the $n_{P,R} = \sup ||\Phi_s(s, u, \omega)||_e < \infty$,
 $\beta_{P,R} = \sup ||\Phi_u(s, u, \omega)||_e < \infty$,

the suprema are taken over $s \in [-\pi, \pi]$, $|u| \le P$, $|\omega| \le R$

triangel inequality, the elementary inequalities $|\beta(s)| \le$ triangel inequality, the elementary inequalities $|\beta(s)| \leq 1$ and $|\alpha(s)\beta(s)| \leq 1/2$,

Hölder's inequality and the boundedness of the Hilbert transform in Lebesgue spaces, for the L_p norm of the function φ we obtain the estimate

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\nHölder's inequality and the boundedness of the Hilbert transform in Lebesgue
\nspaces, for the
$$
L_p
$$
 norm of the function φ we obtain the estimate
\n
$$
||\varphi||_p \leq (1/2) ||g_0||_p + A_r ||e^{-H(\mu)}||_x \cdot ||e^{H(\mu)}g_0||_r
$$
\n
$$
\leq ||g_0||_e \{(1/2) (2\pi)^{2/\alpha} + A_r ||e^{-H(\mu)}||_x \cdot ||e^{H(\mu)}||_x\}, \qquad (*)
$$
\nwhere $\alpha = 2pq/[\varrho - p]$, $r = \alpha[\varrho - p]/[\varrho + p]$, and A_r is the M. Riesz constant, the
\nnorm of the Hilbert transform in $L_r(\Gamma)$.
\nWe make the Assumption C that
\n
$$
2\gamma_{P,R} = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa
$$
\n(41)
\nfor the oscillation of the function $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$, the supremum
\nand infimum are again taken over $s \in [-\pi, \pi]$, $|u| \leq P$, $|\omega| \leq R$. Then
\n
$$
||e^{H(\mu)}||_x, ||e^{-H(\mu)}||_x \leq \left(\frac{2\pi}{\cos \pi \gamma_{P,R}}\right)^{1/\kappa},
$$

where $x = 2pq/[\rho - p]$, $r = x[\rho - p]/[\rho + p]$, and A_r is the M. Riesz constant, the norm of the Hilbert transform in $L_r(\Gamma)$.

We make the *Assumption* Cthat

$$
2\gamma_{P,R} = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa \qquad (41)
$$

for the oscillation of the function $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$, the supremum We make the Assumption C that
 $2\gamma_{P,R} = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa$

for the oscillation of the function $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$, then

and infimum are again taken over $s \in [-\pi, \pi]$, $|u| \leq P$, $|\omega| \leq R$. Then and infimum are again taken over $s \in [-\pi, \pi]$, $|u| \le P$, $|\omega| \le R$. Then $2\gamma_{P,R} = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa$ (41)

socillation of the function $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$, the supremum

num are again taken over $s \in [-\pi, \pi], |u| \leq P, |\omega| \leq R$. Then
 $||e^{H(u)}||_{\kappa}, ||e^{-H(u)}||_{\kappa} \leq \left(\frac{2\pi}{\cos \frac{x$ $2\gamma_{P,K} = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa$ (41)

iscillation of the function $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$, the supremum

num are again taken over $s \in [-\pi, \pi]$, $|u| \leq P$, $|\omega| \leq R$. Then
 $||e^{H(\mu)}||_{\kappa}$, $||e^{-H(\mu)}||_{\kappa} \leq \$

\n The following inequality is:\n
$$
\mu(s, u, \omega)
$$
\n with the function $\mu(s, u, \omega)$.\n

according to the well-known Zygmund lemma (cf. [22, 18]). Therefore,

$$
\|\varphi\|_p \leq (2\pi)^{2/\kappa} \|g_0\|_p \left\{ (1/2) + A_r(\cos \kappa \gamma_{P,R})^{-2/\kappa} \right\},\
$$

and because of

$$
|k_1| \leq \int |\varphi(s,\xi,\eta)| \, ds \leq (2\pi)^{1/q} \, ||\varphi||_p \tag{43}
$$

for the oscillation of the function
$$
\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)
$$
, the
\nand infimum are again taken over $s \in [-\pi, \pi]$, $|u| \leq P$, $|\omega| \leq R$. Then
\n
$$
||e^{H(\mu)}||_{\kappa}, ||e^{-H(\mu)}||_{\kappa} \leq \left(\frac{2\pi}{\cos \kappa \gamma_{PR}}\right)^{1/\kappa},
$$
\naccording to the well-known Zygmund lemma (cf. [22, 18]). Therefore,
\n
$$
||\varphi||_p \leq (2\pi)^{2/\kappa} ||g_0||_e \{(1/2) + A_r(\cos \kappa \gamma_{PR})^{-2/\kappa}\},
$$
\nand because of
\n
$$
|k_1| \leq \int_{-\pi}^{\pi} |\varphi(s, \xi, \eta)| ds \leq (2\pi)^{1/q} ||\varphi||_p
$$
\nwe have the estimation
\n
$$
|u_1(s)| \leq 2(2\pi)^{1/q} ||\varphi||_p
$$
\n
$$
\leq (2\pi)^{1/q} ||g_0||_e \{1 + 2A_r(\cos \kappa \gamma_{PR})^{-2/\kappa}\},
$$
\ntaking into account that $1/\sigma = 1/q + 2/\kappa$. Moreover.

$$
||\varphi||_p \leq (2\pi)^{2/8} ||g_0||_e ((1/2) + A_r(\cos \times \gamma_{P,R})^{-2/\kappa}),
$$
\n(42)
\nand because of
\n
$$
|k_1| \leq \int_{-\pi}^{\pi} |\varphi(s, \xi, \eta)| ds \leq (2\pi)^{1/q} ||\varphi||_p
$$
\n(43)
\nwe have the estimation
\n
$$
|u_1(s)| \leq 2(2\pi)^{1/q} ||\varphi||_p
$$
\n
$$
\leq (2\pi)^{1/q} ||\varphi||_p
$$
\n
$$
\leq (2\pi)^{1/q} ||\varphi||_p
$$
\n(44)
\ntaking into account that $1/\sigma = 1/q + 2/\kappa$. Moreover,
\n
$$
|u_1(s_1) - u_1(s_2)| \leq |s_1 - s_2|^{1/q} ||\varphi||_p
$$
\n
$$
\leq |s_1 - s_2|^2 (2\pi)^{2/\kappa} ||g_0||_e ((1/2) + A_r(\cos \times \gamma_{P,R})^{-2/\kappa}).
$$
\n(45)
\nFinally, we put $P_0 = R$ and make the Assumption D that there exist $P, R > 0$
\nwith $R = 2(2\pi)^{1/q} R_0, P = 2\pi R + K(R, R_0)$ such that
\n
$$
R_0 \geq C_{P,R} = (2\pi)^{2/\kappa} [M + \gamma_{P,R} + R\beta_{P,R}] \{1 + 2A_r(\cos \times \gamma_{P,R})^{-2/\kappa}\},
$$
\n(46)
\nwhere as above $1 < p < \varrho, \kappa = 2p\varrho/[\varrho - p], r = \kappa[\varrho - p]/[\varrho + p], A_r$ the M. Riesz
\nconstant, and $\gamma_{P,R} = \beta_{P,R} \approx \gamma_{P,R} \approx \gamma_{P,R} \approx 2p\varrho/[\varrho - p], r = \kappa[\varrho - p]/[\varrho + p], A_r$ the M. Riesz

Finally, we put $P_0 = R$ and make the *Assumption* D that there exist $P, R > 0$ with $R = 2(2\pi)^{1/q} R_0$, $P = 2\pi R + K(R, R_0)$ such that

$$
R_0 \geq C_{P,R} = (2\pi)^{2/\kappa} \left[M + v_{P,R} + R\beta_{P,R} \right] \{ 1 + 2A_r (\cos \kappa \gamma_{P,R})^{-2/\kappa} \}, \tag{46}
$$

constant, and $v_{P,R}$, $\beta_{P,R}$, $\gamma_{P,R}$ are defined by (39), (40), (41), respectively. Then, on account of the estimations (37) and (44), (45) with (38), the operator N maps the convex compact subset \Re of $C(\Gamma) \times C_0(\Gamma)$ into itself. Besides, the operator $N: \Re \to \Re$ is continuous in the maximum norm of uniform convergence. This is obvious for N_1 and can be shown for N_2 like for the corresponding operator in [18].

The Schauder fixed point theorem applied to the equations (29) , (30) in \Re now yields the existence of a solution $u(z)$ to Problem P. The boundary values $u(s)$ of the solution u and $u_1(s)$ of $\partial u/\partial s$ are lying in \mathbb{R}_1 and \mathbb{R}_2 , respectively. Therefore, $u(z)$ has Hölder continuous partial derivatives in \overline{G} , viz. $u(z) \in \overline{C}^{1/4}(\overline{G})$

Theorem 1: *Under Assumptions A*-D *Problem P has a solution* $u(z) \in C^{1,1}(\overline{G})$ *with the Hölder exponent* $\lambda = 1 - (1/p), 1 < p < \varrho$. α and
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$$
\sim
$$

Corollary: Assumption D is fulfilled for arbitrary M with a sufficiently large $R>0$ if Poincaré Boundary Value Problems 391

lary: Assumption D is fulfilled for arbitrary M with a sufficiently large
 $v = \sup ||\Phi_s(s, u, \omega)||_e < \infty, \qquad \beta = \sup ||\Phi_u(s, u, \omega)||_e < \infty,$ (47)
 $2\gamma = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/x;$ (48) Foincaré Boundary Value Problems 391

y: Assumption D is fulfilled for arbitrary M with a sufficiently large

sup $\|\Phi_e(s, u, \omega)\|_e < \infty$, $\beta = \sup \|\Phi_a(s, u, \omega)\|_e < \infty$, (47)

= $\sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/x$; (48)

im and infimu Poincaré Boundary Value Problems 391

lary: Assumption D is fulfilled for arbitrary M with a sufficiently large
 $\nu = \sup ||\Phi_a(s, u, \omega)||_e < \infty$, $\beta = \sup ||\Phi_u(s, u, \omega)||_e < \infty$, (47)
 $2\gamma = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\varkappa$; (48)

mum an

$$
\nu = \sup ||\varPhi_{\mathbf{s}}(s, u, \omega)||_{\varrho} < \infty, \qquad \beta = \sup ||\varPhi_{\mathbf{s}}(s, u, \omega)||_{\varrho} < \infty,
$$
 (47)

$$
2\gamma = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/\kappa,
$$
\n(48)

the supremum and infimum are taken over $s \in [-\pi, \pi]$, $u \in \mathbf{R}$, $\omega \in \mathbf{R}$, and

$$
2(2\pi)^{1/\sigma}\beta\{1+2A_r(\cos x\gamma)^{-2/\kappa}\}<1.
$$
\n(49)

Especially, for $\rho = \infty$ with $\sigma = 1$, $\varkappa = r = 2p$ and p sufficiently near to 1 the condition e supre
peciall
ion
sures t

$$
4\pi\beta(1+2(\cos 2\gamma)^{-1})<1
$$
\n(50)

Follows Boundary Value Problems 391

lary: Assumption D is fulfilled for arbitrary M with a sufficiently large
 $\nu = \sup ||\Phi_s(s, u, \omega)||_e < \infty$, $\beta = \sup ||\Phi_u(s, u, \omega)||_e < \infty$, (47)
 $2\gamma = \sup \mu(s, u, \omega) - \inf \mu(s, u, \omega) < \pi/x$; (48)

mum and in insures the existence of a solution to Problem P if the inequalities (2) , (3) are-fulfilled uniformly in $u, \omega \in \mathbb{R}$, the oscillation 2γ of $\mu(s, u, \omega) = \arctan \Phi_{\omega}(s, u, \omega)$ taken over in $[-\pi, \pi] \times \mathbf{R} \times \mathbf{R}$ is smaller than $\pi/2$ and Assumption B is satisfied for every $\eta \in C_0(\Gamma)$. pecially, for $\varrho = \infty$ with $\sigma = 1$, $\varkappa = r = 2p$ and p

ion
 $4\pi\beta(1 + 2(\cos 2\gamma)^{-1}) < 1$

sures the existence of a solution to Problem P if t

led uniformly in $u, \omega \in \mathbb{R}$, the oscillation 2γ of $\mu(s, u)$

er in $[-\$ *a*_{*a*} *a*_{*a*</sup> *a*_{*a*} *a*_{*a*} *a*_{*a*</sup> *a*_{*a*} *a*_{*a*} *a*_{*a*} *a*_{*a*} *a*_{*a*</sup> *a*_{*a*} *a*_{*a*} *a*_{*a*} *a*_{*a*</sup> *a*_{*a*} *a*_*}}}} R* $x = r = 2p$ and *p* surfies that P is the inet of $\mu(s, u, \omega) = r$ than $\pi/2$ and Assumption $\frac{m}{2} + b^2$ and $\frac{m}{2} + b^2$ and filled uniformly in $u, \omega \in \mathbb{R}$, the oscillation 2γ of $\mu(s, u, \omega) = \arctan \Phi_u(s, u)$

over in $[-\pi, \pi] \times \mathbb{R} \times \mathbb{R}$ is smaller than $\pi/2$ and Assumption B is satisfied
 $\eta \in C_0(\Gamma)$.

Remark: Using the elementary ine

$$
|\alpha(s)\cdot a + \beta(s)\cdot b| \leq \sqrt{a^2 + b^2}
$$

together with Minkowski's inequality in the estimation $(*)$ of $||\varphi||_p$ we obtain the slightly smaller expression. obtain the estimation (*) of $\|\varphi\|_p$ we obtain the
 $R\beta_{P,R}$] $\{1 + A_r^2(\cos x \gamma_{P,R})^{-4/s}\}^{1/2}$ (46')
 ∞ so that the criterion (50) applies.
 $cu + \Psi(\omega)$, i.e.

(3) on Γ , (51) *u*(*s*) $\cdot a + \beta(s) \cdot b \le \sqrt{a^2 + b^2}$
 *u*_{*P*,*R* = 2(2*x*)^{2/*x*} [*M* + *v*_{*P*},*R* + *Rβ*_{*P*,*R*}] (1 + *A*²(cos *x*_{*YP*,*R*})^{-4/*x*}]^{1/2} (46^{*c*})
 c,*R* = 2(2*x*)^{2/*x*} [*M* + *v*_{*P*},*R* + *Rβ*_{*P*,}}

$$
C_{P,R} = 2(2\pi)^{2/\kappa} \left[M + v_{P,R}^2 + R\beta_{P,R} \right] \left\{ 1 + A_r^2 (\cos \kappa \gamma_{P,R})^{-4/\kappa} \right\}^{1/2}
$$
(46')

in Assumption D for $p \geq 2$.

•

§ 4. Special cases
We give some examples taking $\varrho = \infty$ so that the criterion (50) applies.

Example 1: Let be $\Phi(s, u, \omega) = cu + \Psi(\omega)$, i.e.

$$
\frac{\partial u}{\partial r} + cu + \Psi(\frac{\partial u}{\partial s}) = f(s) \quad \text{on } \Gamma,
$$

with a constant c and a continuously differentiable function $\Psi(\omega)$. Then the condition (50) reads

$$
4\pi |c| (1 + 2(\cos 2\nu)^{-1}) < 1
$$

 $C_{P,R} = 2(2\pi)^{2/\kappa} \left[M + \nu_{P,R} + R\beta_{P,R}\right] \left(1 + A_r^2(\cos \times \nu_{P,R})^{-4/\kappa}\right)^{1/2}$ (46')

in Assumption D for $p \ge 2$.

§ 4. Special cases

We give some examples taking $\varrho = \infty$ so that the criterion (50) applies.

Example 1: Let with $2\gamma = \sup_{\omega \in \mathbb{R}} [\arctan \Psi'(\omega)] - \inf_{\omega \in \mathbb{R}} [\arctan \Psi'(\omega)] < \pi/2$ which is fulfilled for are examples taking $\varrho = c$
 $\partial^2 \varrho^2 + c u + \varrho^2(\partial u/\partial s) = f(c)$
 $\partial \varrho^2 + c u + \varrho^2(\partial u/\partial s) = f(c)$

tant c and a continuous

eads
 $|c| \{1 + 2(cos 2\gamma)^{-1}\} < 1$

sup [are tan $\varrho^2(\omega)$] — inf

small $|c|$ if $0 < l_1 \leq \varrho^2(l_2)$

we sufficiently small $|c|$ if $0 < l_1 \leq \Psi'(\omega) < \infty$ or $0 \leq \Psi'(\omega) \leq l_2 < \infty$, for instance. Further, c must be different from zero to insure the solvability of the corresponding equation (32) for the parameter *k.* $4\pi |c| (1 + 2(\cos 2\gamma)^{-1}) < 1$
 $2\gamma = \sup_{\omega \in \mathbf{R}} \left[\arctan \Psi'(\omega) \right] - \inf_{\omega \in \mathbf{R}} \left[\arctan \Psi''(\omega) \right]$

ciently small $|c|$ if $0 < l_1 \leq \Psi'(\omega) < \infty$ or

her, c must be different from zero to insure

tion (32) for the parameter k.
 up [arc tan $\Psi'(\omega)$] – inf [arc tan $\Psi'(\omega)$] $< \pi$
up [arc tan $\Psi'(\omega)$] – inf [arc tan $\Psi'(\omega)$] $< \pi$
mall |c| if $0 < l_1 \leq \Psi'(\omega) < \infty$ or $0 \leq \Psi'(\omega)$
) for the parameter k.
ition (52) is very restrictive for the cons

The condition (52) is very restrictive for the-constant *c.* It always demands that $|c| < (12\pi)^{-1}$, whereas in the limit case $\Psi = 0$ existence of a solution is present for all $c \neq 0, -1, -2, ...$ We therefore apply another method for positive values *c*

introducing the new unknown function
 $\therefore U_0(z) = ru_r(z) + cu(z)$. (53) • introducing the new unknown function

$$
U_0(z) = ru_r(z) + cu(z).
$$

With u also U_0 is a regular harmonic function in G which is continuous in \bar{G} if $u \in C^{1}(\overline{G})$. Moreover, $u = P[U_0]$ is uniquely determined by U_0 and the conjugate

(53)

I

harmonic function V_0 to U_0 is given by $V_0(z) = -u_s(z) + cv(z)$, where *v* is the conjugate harmonic function to u . The conjugate functions v , V_0 are normalized by $V_0(0) = v(0) = 0$ in the origin such that $v = -H(u) = -H(P[U_0])$ and $V_0 = -H(U_0)$ with *H* the Hilbert operator (18). 392
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jugate
 $V_0(0) =$

with *H*

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which *b*

function *L.* **v.** WOLFERSDORF

function V_0 to U_0 is given by $V_0(z) = -u_s(z) + cv(z)$, where v is the con-

principle function to u. The conjugate functions v, V_0 are normalized by
 $(0) = 0$ in the origin such that $v = \frac{-H(u)}{-H(v$ 392 L. v. WOLFERSDORF

harmonic function V_0 to U_0 is give

jugate harmonic function to u. Tl
 $V_0(0) = v(0) = 0$ in the origin such t

with H the Hilbert operator (18).

The boundary condition (51) nov
 $U_0 + \Psi(cv - V_0) =$

The boundary condition **(51)** now writes

$$
U_0 + \Psi(cv - V_0) = f(s) \quad \text{on } \Gamma,\tag{54}
$$

which has the form of a nonlinear Riemann-Hilbert condition for the holomorphic. function $W_0(z) = U_0(z) + iV_0(z)$. Like in our paper [18] we replace (54) by the ilbert con
paper [18 $\mathcal{L} \mathcal{L}(cv - V_0) = f(s)$ on *I*, (54)

the form of a nonlinear Riemann-Hilbert condition for the holomorphic,
 $\mathcal{L}(z) = U_0(z) + iV_0(z)$. Like in our paper [18] we replace (54) by the condition
 $-\Psi'(cv - V_0) \frac{\partial U_0}{\partial r} + c\Psi'(cv$ The boundary condition (51) now writes
 $U_0 + \Psi(cv - V_0) = f(s)$ on Γ , (54)

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unction $W_0(z) = U_0(z) + iV_0(z)$. Like in our paper [18] we replace (5 The boundary condition (51) now writes
 $U_0 + \Psi(cv - V_0) = f(s)$ on Γ ,

iich has the form of a nonlinear Riemann-Hilbert condition for

notion $W_0(z) = U_0(z) + iV_0(z)$. Like in our paper [18] we rep

ifterentiated condition
 $\$ *a(s)* = **i + ë,** (59)

$$
\frac{\partial U_0}{\partial s} - \Psi'(cv - V_0) \frac{\partial U_0}{\partial r} + c\Psi'(cv - V_0) [U_0 - cu] = f'(s) \text{ on } \Gamma \qquad (55)
$$

together with the integral condition

function
$$
W_0(z) = U_0(z) + iV_0(z)
$$
. Like in our paper [18] we replace (54) by the
\ndifferentiated condition
\n
$$
\frac{\partial U_0}{\partial s} - \Psi'(cv - V_0) \frac{\partial U_0}{\partial r} + c\Psi'(cv - V_0) [U_0 - cu] = f'(s) \text{ on } \Gamma
$$
\n(55)
\ntogether with the integral condition
\n
$$
\int_0^{\pi} U_0 ds + \int_0^{\pi} \Psi(cv - V_0) ds = \int_0^{\pi} f(s) ds.
$$
\n(56)
\nFrom (55) it follows that
\n
$$
U_0(e^{is}) = k + \int_0^s \varphi(\sigma) d\sigma
$$
\nwith
\n
$$
\varphi(s) = \beta(s) h(s) - \alpha(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\},
$$
\nwhere
\n
$$
\alpha(s) = \Psi'/\sqrt{1 + \Psi'^2}, \quad \beta(s) = 1/\sqrt{1 + \Psi'^2},
$$
\n(58)
\n
$$
\mu(s) = \arctan \Psi', \quad h(s) = g(s)/\sqrt{1 + \Psi'^2},
$$
\n(60)

From (55) it follows that

$$
U_0(e^{is}) = k + \int_0^s \varphi(\sigma) d\sigma
$$
 (57)
\n
$$
\varphi(s) = \beta(s) h(s) - \alpha(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\},
$$
 (58)
\n
$$
\alpha(s) = \frac{\Psi'}{\sqrt{1 + \Psi'^2}}, \quad \beta(s) = \frac{1}{\sqrt{1 + \Psi'^2}},
$$
 (59)
\n
$$
\mu(s) = \arctan \Psi', \quad h(s) = \frac{g(s)}{\sqrt{1 + \Psi'^2}},
$$
 (60)
\n
$$
g(s) = f'(s) - c\Psi'(cv - V_0) [U_0 - cu]
$$
 - (61)
\nIfil the orthogonality condition

with

$$
\varphi(s) = \beta(s) h(s) - \alpha(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\},\tag{58}
$$

$$
\varphi(s) = \beta(s) h(s) - \alpha(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\},\tag{58}
$$
\n
$$
\alpha(s) = \frac{\Psi'}{\sqrt{1 + \Psi'^2}}, \qquad \beta(s) = \frac{1}{\sqrt{1 + \Psi'^2}},\tag{59}
$$

$$
\mu(s) = \arctan \Psi', \qquad h(s) = g(s)/\sqrt{1 + \Psi'^2}, \tag{60}
$$

and

$$
g(s) = f'(s) - c\Psi'(cv - V_0) [U_0 - cu]
$$
 (61)

has to fulfil the orthogonality condition

with
\n
$$
\varphi(s) = \beta(s) h(s) - \alpha(s) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\},
$$
\nwhere
\n
$$
\alpha(s) = \frac{\psi'}{\sqrt{1 + \psi'^2}}, \quad \beta(s) = 1/\sqrt{1 + \psi'^2},
$$
\n(59)
\n
$$
\mu(s) = \arctan \Psi', \quad h(s) = g(s)/\sqrt{1 + \psi'^2},
$$
\n(60)
\nand
\n
$$
g(s) = f'(s) - c\Psi'(cv - V_0) [U_0 - cu]
$$
\nhas to fulfill the orthogonality condition
\n
$$
\int_{-\pi}^{\pi} \gamma(s) g(s) ds = 0
$$
\n(62)
\nwith the nonnegative function
\n
$$
\gamma(s) = e^{H(\mu)(s)}/\sqrt{1 + \Psi'^2}.
$$
\n(63)
\nThe constant k in (57) is determined by substituting U_0 into (56).
\nThe investigation of the integral equation (57) for U_0 can be performed like for

with the nonnegative function

$$
\nu(s) = e^{H(\mu)(s)} / \sqrt{1 + \Psi'^2}.
$$
\n(63)

The constant k in (57) is determined by substituting U_0 into (56).

The investigation of the integral equation (57) for U_0 can be performed like for the corresponding integral equation in [18], i.e. like in the above proof of Theorem 1, where now only the elementary inequalities $|\alpha(s)|$, $|\beta(s)| \leq 1$ in the estimation for $\gamma(s) = e^{H(\mu)(s)} / \sqrt{1 + \Psi'^2}$. (63)
The constant k in (57) is determined by substituting U_0 into (56).
The investigation of the integral equation (57) for U_0 can be performed like for
the corresponding integral equatio $\|\varphi\|_p$ are used. Restricting ourselves to the case $\varrho = \infty$ again, we have the estimation $y(s) = e$
tant k in
vestigation
sponding
w only t
sed. Res *f*(μ)(s) / $\sqrt{1 + \Psi'^2}$. (63).
 f(μ)(s) / $\sqrt{1 + \Psi'^2}$. (63).

(57) is determined by substituting U_0 into (56).

on of the integral equation (57) for U_0 can be performed like for integral equation in [18] $\int_{-\pi}^{\pi} \gamma(s) g(s) ds = 0$

with the nonnegative function
 $\gamma(s) = e^{H(\mu)(s)} / \sqrt{1 + \Psi'^2}$.

The constant k in (57) is determined by substituting U_0 into (56).

The investigation of the integral equation (57) for U_0 can be s) = $e^{H(\mu)(s)}/\sqrt{1 + \Psi'^2}$.

s) = $e^{H(\mu)(s)}/\sqrt{1 + \Psi'^2}$.

nt k in (57) is determined by substituting U_0 into (56).

stigation of the integral equation (57) for U_0 can be onding integral equation in [18], i.e. like i (62)
(62)
 $\left(63\right)$
 $\left(63\right)$
 $\left(64\right)$
 $\left(64\right)$
 $\left(65\right)$ the correspondin
where now only
 $\|\varphi\|_p$ are used. Re
 $\|\varphi\|_{\infty} \leq$
with
 $\delta = \sup_{\omega \in \mathbb{R}}$

$$
\|g\|_{\infty} \leq \|f'\|_{\infty} + 2c\delta \, \|U_0\|_{\infty}
$$

•

/

$$
\delta = \sup_{\omega \in \mathbf{R}} |\Psi'(\omega)| < \infty \quad \text{(by assumption)}
$$

Poincaré Boundary Value Problems
\nfor the function (61) because
\n
$$
\max_{\Gamma} |cu| \leq \max_{\Gamma} |cu + u_r| = \max_{\Gamma} |U_0|
$$
\nin write of the maximum-minimum property for harmonic functions. In the analogue

Poincaré Boundary Value

inction (61) because
 $\max |cu| \leq \max_{f} |cu + u_{r}| = \max_{f} |U_{0}|$

of the maximum-minimum property for harmonic function

indition (50) the constant β can therefore put equal to $2c\delta$ in virtue of the maximum-minimum property for. harmonic functions. In the analogon of the condition (50) the constant β can therefore put equal to $2c\delta$ and we obtain the condition Point relation (61) because
 $\max_{\Gamma} |cu| \leq \max_{\Gamma} |cu + u_r| = \max_{\Gamma} |U_{\{r\}}|$

of the maximum-minimum property for
 $16\pi c\delta\{1 + (\cos 2\gamma)^{-1}\} < 1$

We constant c. In the limit case Ψ
 $\sum_{n=1}^{\infty} \frac{1}{n!} c_n^{\frac{1}{n}} \log \Phi(c_n; c_n) = \$ *au* $|e(u)| \leq \max |cu| + u_r| = \max |U_0|$
 au $|cu| \leq \frac{1}{r}$ *f* $|u - u_r| = \max |U_0|$
 f the maximum-minimum property for harmonic
 dition (50) the constant β can therefore put equal
 $16\pi c\delta\{1 + (\cos 2\gamma)^{-1}\} < 1$
 we constan $\max_{r} |cu| \leq \max_{r} |cu + u_r| = \max_{r} |U_0|$

in virtue of the maximum-minimum property for harmonic function

condition

(50) the constant β can therefore put equal to 2cδ

condition

16πcδ{1 + (cos 2γ)⁻¹} < 1

for positive c

$$
16\pi c\delta \{1 + (\cos 2\gamma)^{-1}\} < 1\tag{66}
$$

for positive constant *c*. In the limit case $\Psi \equiv 0$ this condition is fulfilled for all $c > 0$. $c > 0$. d for all
 $-$ (67)

Example 2: Let be $\Phi(s, \hat{u}, \omega) = X(u) \Psi(\omega)$, i.e.

$$
\frac{\partial u}{\partial r} + X(u) \Psi(\frac{\partial u}{\partial s}) = f(s) \quad \text{on } \Gamma,
$$
 (67)

with continuously differentiable functions $\mathcal{Y}(\omega)$ and $X(u)$. Further shall be

\n
$$
\text{ple } 2 \colon \text{Let } \mathbf{b} \in \Phi(s, u, \omega) = X(u) \, \Psi(\omega), \text{ i.e.}
$$
\n

\n\n $\frac{\partial u}{\partial r} + X(u) \, \Psi(\partial u/\partial s) = f(s) \quad \text{on } \Gamma,$ \n

\n\n In a complex interval, $\text{F}(u) = \int_{u \in \mathbf{R}} |\Psi(u)| \, du$, where $\text{F}(u) = \int_{u \in \mathbf{R}} |\Psi(u)| \, du$, where $\text{F}(s) = \int_{u \in \mathbf{R}} |\Psi(u)| \, du$, where $\text{F}(s) = \int_{u \in \mathbf{R}} |\Psi(u)| \, du$ is the following property:\n

\n\n $\frac{\partial u}{\partial r} = \sup_{u \in \mathbf{R}} |W(u)| - \inf_{u \in \mathbf{R}} \left[\int_{u \in \mathbf{R}} |\Psi(u)| \, du \right] \, du$ \n

\n\n $\frac{\partial u}{\partial r} = \sup_{u \in \mathbf{R}} |W(u)| - \inf_{u \in \mathbf{R}} \left[\int_{u \in \mathbf{R}} |\Psi(u)| \, du \right] \, du$ \n

\n\n $\frac{\partial u}{\partial r} = \sup_{u \in \mathbf{R}} |W(u)| - \inf_{u \in \mathbf{R}} |W(u)| - \$

and

$$
2\gamma = \sup\left[\arctan\left(X(u)\Psi'(\omega)\right)\right] - \inf\left[\arctan\left(X(u)\Psi'(\omega)\right)\right] < \pi/2, \qquad (69)
$$

where supremum and infimum are taken over $u \in \mathbb{R}$, $\omega \in \mathbb{R}$. Then the condition (50) reads ive constant c. In the limit case $\Psi = \Psi$

ple 2: Let be $\Phi(s, u, \omega) = X(u) \Psi(\omega)$, i.
 $\partial u/\partial r + X(u) \Psi(\partial u/\partial s) = f(s)$ on Γ ,

iinuously differentiable functions $\Psi(\omega)$ a
 $\beta_1 = \sup_{\omega \in \mathbf{R}} |\Psi(\omega)| < \infty$, $\beta_2 = \sup_{u \in \mathbf{R}} |X'(u)|$
 c > 0.

Example 2: Let be $\Phi(s, u, \omega) = X(u) \Psi(\omega)$, i.e.
 $\partial u/\partial r + X(u) \Psi(\partial u/\partial s) = f(s)$ on Γ ,

with continuously differentiable functions $\Psi(\omega)$ and $X(u)$. Further shall
 $\beta_1 = \sup_{\omega \in \mathbf{R}} |\Psi(\omega)| < \infty$, $\beta_2 = \sup_{u \in \mathbf{R}} |X'(u)|$ $\partial u/\partial r + X(u) \ \Psi(\partial u/\partial s) = f(s) \quad \text{on } \Gamma,$

with continuously differentiable functions $\Psi(\omega)$ and $X(u)$. Further
 $\beta_1 = \sup_{\omega \in \mathbb{R}} |\Psi(\omega)| < \infty, \qquad \beta_2 = \sup_{u \in \mathbb{R}} |X'(u)| < \infty$

and
 $2\gamma = \sup_{\omega \in \mathbb{R}} \left[\arctan \left(X(u) \ \Psi'(\omega) \right) \right] - \$ ntinuously differentiable functions $\Psi(\omega)$ and $X(u)$. Further shall be
 $\beta_1 = \sup_{u \in \mathbf{R}} |\Psi(\omega)| < \infty$, $\beta_2 = \sup_{u \in \mathbf{R}} |X'(u)| < \infty$ (68)
 $2\gamma = \sup_{u \in \mathbf{R}} |\text{arc tan } (X(u) \Psi'(\omega))] - \text{inf } [\text{arc tan } (X(u) \Psi'(\omega))] < \pi/2$, (69)

uupremum and with continuously differ
 $\beta_1 = \sup_{\omega \in \mathbf{R}} |\Psi(\omega)$

and
 $2\gamma = \sup_{\omega \in \mathbf{R}} |\text{arc } t|$

where supremum and ir

reads
 $4\pi\beta_1\beta_2\{1 + 2(c)$

Moreover, Assumption

In particular, for bot

nondecreasing function
 $0 \le l_1 = \lim_{\$ $\beta_1 = \sup_{\omega \in \mathbf{R}} |\mathcal{W}(\omega)| < \infty,$ $\beta_2 = \sup_{u \in \mathbf{R}} |X'(u)| < \infty$ (68)

and
 $2\gamma = \sup_{\omega \in \mathbf{R}} |\arctan (X(u) \cdot \mathcal{Y}'(\omega))| - \inf \left[\arctan (X(u) \cdot \mathcal{Y}'(\omega)) \right] < \pi/2,$ (69)

where supremum and infimum are taken over $u \in \mathbf{R}, \omega \in \mathbf{R}$. T and $2\gamma = \sup \left[\arctan \left(X\right)\right]$

where supremum and infimum

reads $4\pi\beta_1\beta_2(1 + 2(\cos 2\gamma))$

Moreover, Assumption B has

In particular, for bounded

nondecreasing function $\Psi(\omega)$
 $0 \leq l_1 = \lim_{u \to -\infty} X(u)$,
 $0 < m_1 = \lim_{\omega \to -\in$ where supremum and infimum are taken over $u \in \mathbb{R}$, $\omega \in \mathbb{R}$. Then the condition (50)
reads
 $4\pi\beta_1\beta_2[1 + 2(\cos 2\gamma)^{-1}] < 1$. (70)

Moreover, Assumption B has to be fulfilled.

In particular, for bounded monotonica

$$
4\pi\beta_1\beta_2(1+2(\cos 2\gamma)^{-1})<1.
$$
 (70)

Moreover, Assumption B has to be fulfilled.
In particular, for bounded monotonically increasing function $X(u)$ and bounded

$$
0 \leq l_1 = \lim_{u \to -\infty} X(u), \quad \lim_{u \to +\infty} X(u) = L_1 < \infty,
$$
\n
$$
(71)
$$

$$
0 < m_1 = \lim_{\omega \to -\infty} \Psi(\omega), \quad \lim_{\omega \to +\infty} \Psi(\omega) = M_1 < \infty, \tag{72}
$$

$$
0 \leq l_2 \leq X'(u) \leq L_2 < \infty, \qquad 0 \leq m_2 \leq \Psi'(\omega) \leq M_2 < \infty,\tag{73}
$$

where supremum and infimum are taken over
$$
u \in \mathbb{R}
$$
, $\omega \in \mathbb{R}$. Then the condition (50) reads\n\n
$$
4\pi\beta_1\beta_2(1 + 2(\cos 2\gamma)^{-1}) < 1.
$$
\n\nMoreover, Assumption B has to be fulfilled.\n\nIn particular, for bounded monotonically increasing function $X(u)$ and bounded nondecreasing function $\Psi(\omega)$ with\n\n
$$
0 \leq l_1 = \lim_{u \to -\infty} X(u), \quad \lim_{u \to +\infty} X(u) = L_1 < \infty,
$$
\n(71)\n\nand\n\n
$$
0 < m_1 = \lim_{\omega \to -\infty} \Psi(\omega), \quad \lim_{\omega \to +\infty} \Psi(\omega) = M_1 < \infty,
$$
\n(72)\n\nand\n\n
$$
0 \leq l_2 \leq X'(u) \leq L_2 < \infty, \quad 0 \leq m_2 \leq \Psi'(\omega) \leq M_2 < \infty,
$$
\n(73)\n\nthe condition (70) takes the form\n\n
$$
4\pi M_1 L_2 \left\{ 1 + 2 \right\} \left[1 + \left[\frac{L_1 M_2 - l_1 m_2}{1 + l_1 L_1 m_2 M_2} \right]^2 \right\} < 1.
$$
\n\nAnd Assumption B is fulfilled if $l_1 M_1 < L_1 m_1$ and the right-hand side $f(s)$ satisfies the Landesman-Lazer condition

And Assumption B is fulfilled if $l_1M_1 < L_1m_1$ and the right-hand side $f(s)$ satisfies the Landesman-Lazer condition

$$
0 < m_1 = \lim_{\omega \to -\infty} \psi(\omega), \quad \lim_{\omega \to +\infty} \psi(\omega) = M_1 < \infty, \tag{72}
$$
\n
$$
0 \le l_2 \le X'(u) \le L_2 < \infty, \quad 0 \le m_2 \le \Psi'(\omega) \le M_2 < \infty, \tag{73}
$$
\n
$$
4\pi M_1 L_2 \left\{ 1 + 2 \left| \int_1 + \left[\frac{L_1 M_2 - l_1 m_2}{1 + l_1 L_1 m_2 M_2} \right]^2 \right| < 1. \tag{74}
$$
\n
$$
\text{sumption B is fulfilled if } l_1 M_1 < L_1 m_1 \text{ and the right-hand side } f(s) \text{ satisfies}
$$
\ndesman-Lazer condition\n
$$
l_1 M_1 < \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds < L_1 m_1. \tag{75}
$$
\n
$$
\text{where } \alpha \in \mathbb{R} \text{ such that } \alpha \in \mathbb{R} \text{
$$

Example 3: If $\Phi = \Phi(s, \omega)$ does not depend on *u*, only the equation (30) with the operator N_2 for the unknown function η has to be considered. A solution *u* of Problem P exists only if the solution u_1 of this equation satisfies the condition he equat
lered. A
fies the complete
constant

$$
\int_{\pi}^{\pi} \Phi(s, u_1(s)) ds = \int_{-\pi}^{\pi} f(s) ds,
$$
\n(76)

the solution u is then determined up to an arbitrary additive constant k in (29).

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Since u_1 is determined by $f'(s)$ only, the boundary condition (1) should be modified in the following way *au/Or* + *b(s, t9u/as)* = *i(s)* + *2* on *I', (77)*

/

$$
\frac{\partial u}{\partial r} + \Phi(s, \frac{\partial u}{\partial s}) = f(s) + \lambda \quad \text{on } \Gamma,
$$
\n(77)

where λ is an arbitrarily variable constant which will be determined by

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\n
$$
u_1
$$
 is determined by $f'(s)$ only, the boundary condition (1) should be modi-
\nthe following way
\n $\frac{\partial u}{\partial r} + \Phi(s, \frac{\partial u}{\partial s}) = f(s) + \lambda$ on Γ , (77)
\nis an arbitrarily variable constant which will be determined by
\n $2\pi\lambda = \int_{-\pi}^{\pi} \Phi(s, u_1(s)) ds - \int_{-\pi}^{\pi} f(s) ds$ (78)
\nving the equation (30). Besides, u may be fixed through u_1 preserving the
\n $k = u(1)$ or

after solving the equation (30). Besides, u may be fixed through u_1 prescribing the value of $k = u(1)$ or.

i is an arbitrarily variable
\n
$$
2\pi\lambda = \int_{-\pi}^{\pi} \Phi(s, u_1(s)) ds -
$$
\n
$$
2\pi\lambda = \int_{-\pi}^{\pi} \Phi(s, u_1(s)) ds -
$$
\n
$$
i k = u(1) \text{ or}
$$
\n
$$
u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{is}) ds.
$$

The existence of a solution u_1 to the equation (30) is insured if the corresponding Assumptions A and C'bove and the following *modified Assumption* .D is fulfilled' where λ is an arbitrarily variable constant
 $2\pi\lambda = \int_{-\pi}^{\pi} \Phi(s, u_1(s)) ds - \int_{-\pi}^{\pi} f(s) ds$

after solving the equation (30). Besides,

value of $k = u(1)$ or
 $u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{is}) ds$.

The existence of a solution (There exists $R > 0$ with $R = 2(2\pi)^{1/q} R_0$ such that istence of a

ions A and
 $[18]$)':
 $R > 0$
 $R_0 \geq C_R =$ (*xietia) as*

(2x)^{*x*} *k y c* above and the equation (30) is insured if the corresponding C above and the following modified Assumption D is fulfilled

(2x)^{2/x} [*M* + ν_R] $\{1 + 2A_r(\cos x \gamma_R)^{-2/\kappa}\}$, (79)
 after solving the equation (30). Besides, *u* may be fixed through *u*₁ prescribing the

value of $k = u(1)$ or
 $u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) ds$.

The existence of a solution *u*₁ to the equation (30) is insured if t to the equ $2(2\pi)^{1/q}R_0 \stackrel{\cdot}{\scriptstyle \infty} \ + \ \nu_R]\ \{1\ +\ 2 \ \infty \$
, $\left(\begin{matrix} 0 \ 0 \end{matrix} \right) - \inf$ value of $k = u(1)$ or
 $u(0) = \frac{1}{2\pi} \int_0^{\pi} u(e^{i\theta}) ds$.

The existence of a solution u_1 to the equation (30) is insured if the Assumptions A and C above and the following modified Assumpt

(cf. also [18]):

There exist $u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) ds.$

e existence of a solution u_1 to the equation (30) is insured if the corresponding

mptions A and C above and the following modified Assumption D is fulfilled

lso [18]):

ere exists

$$
R_0 \geq C_R = (2\pi)^{2/\kappa} \left[M + \nu_R \right] \{ 1 + 2A_r(\cos \kappa \gamma_R)^{-2/\kappa} \},\tag{79}
$$

 \cdot \cdot

where
\n
$$
v_R = \sup ||\Phi_s(s, \omega)||_e < \infty
$$
\nand
\n
$$
2y_R = \sup [\text{arc tan } \Phi_{\omega}(s, \omega)] - \inf [\text{arc tan } \Phi_{\omega}(s, \omega)] < \pi/\varkappa,
$$
\n(81)
\nthe supremum and infimum are taken over $s \in [-\pi, \pi], |\omega| \leq R$.
\n§ 5. The quasilinear case
\nIn the quasilinear case $\Phi(s, u, \omega) = \Psi(s, u) + \omega X(u)$, i.e.
\n
$$
\frac{\partial u}{\partial r} + X(u) \frac{\partial u}{\partial s} + \Psi(s, u) = f(s) \quad \text{on } \Gamma,
$$
\n(82)

$$
2\gamma_R = \sup \left[\arctan \varPhi_{\omega}(s, \omega) \right] - \inf \left[\arctan \varPhi_{\omega}(s, \omega) \right] < \pi/\varkappa, \tag{81}
$$

§ 5. The quasilinear case

$$
2y_R = \sup \left[\arctan \Phi_{\omega}(s, \omega) \right] - \inf \left[\arctan \Phi_{\omega}(s, \omega) \right] < \pi/\nu, \tag{8}
$$
\nthe supremum and infimum are taken over $s \in [-\pi, \pi], |\omega| \leq R$.

\n§ 5. The quasilinear case

\n
$$
\lim_{\delta \to 0} \frac{\sin(\delta x + \mu)}{\delta x} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x
$$

$$
\frac{\partial u}{\partial r} + X(u) \frac{\partial u}{\partial s} + \Psi(s, u) = f(s) \quad \text{on } \Gamma, \tag{82}
$$

 $2\gamma_R = \sup \left[\arctan \Phi_\omega(s, \omega)\right] - \inf \left[\arctan \Phi_\omega(s, \omega)\right] < \pi/\nu,$

premum and infimum are taken over $s \in [-\pi, \pi], |\omega| \leq R.$
 \therefore
 \therefore
 $\text{The quasilinear case } \Phi(s, u, \omega) = \Psi(s, u) + \omega X(u), \text{ i.e.}$
 $\frac{\partial u}{\partial r} + X(u) \frac{\partial u}{\partial s} + \Psi(s, u) = f(s) \quad \text{on } \Gamma,$
 $\text{[older$ with Hölder continuous functions $X(u)$, $\Psi(s, u)$, and $f(s)$ we can the above methodapply directly to the boundary condition (82) without differentiate it before. As In the quasilinear case $\Phi(s, u, \omega) = \Psi(s, u) + \omega X(u)$, i.e.
 $\partial u/\partial r + X(u) \partial u/\partial s + \Psi(s, u) = f(s)$ on Γ , (82)

with Hölder continuous functions $X(u)$, $\Psi(s, u)$, and $f(s)$ we can the above method

apply directly to the boundary cond $2\gamma_R = \sup \left[\arctan \Phi_{\omega}(s, \omega) \right] - \inf \left[\arctan \Phi_{\omega}(s, \omega) \right] < \pi/s$

the supremum and infimum are taken over $s \in [-\pi, \pi], |\omega| \leq R$.

§ 5. The quasilinear case

In the quasilinear case

In the quasilinear case
 $\frac{\partial u}{\partial r} + X(u) \frac{\partial u}{\$ mum and infimum are taken over $s \in [-\pi, \pi]$
 quasilinear case
 uasilinear case
 i $\phi(s, u, \omega) = \Psi(s, u) + \omega X(u)$,
 $\partial u/\partial r + X(u) \partial u/\partial s + \Psi(s, u) = f(s)$ on Γ ,

der continuous functions $X(u)$, $\Psi(s, u)$, and

rectly to the bound ³

with Höld

apply dire

above this

[18])

u

with
 $\frac{1}{2}$ uasilinear case $\Phi(s, u, \omega) = \Psi(s, u) + \omega X(u)$, i.e.
 $u/\partial r + X(u) \partial u/\partial s + \Psi(s, u) = f(s)$ on Γ ,

er continuous functions $X(u)$, $\Psi(s, u)$, and $f(s)$ we

cely to the boundary condition (82) without dileads to the fixed point problem f *vasumear* case $\varphi(s, u, \omega) = \varphi(s, u) + \omega X(u)$, i.e.
 $\partial u/\partial r + X(u) \partial u/\partial s + \varphi(s, u) = f(s)$ on Γ , (82)

ilder continuous functions $X(u)$, $\varphi(s, u)$, and $f(s)$ we can the above method

irectly to the boundary condition (82) withou $\frac{1}{2}$ above

[18])

with

where der continuous functions $X(u)$, $\Psi(s, u) = f(s)$ on I ,
der continuous functions $X(u)$, $\Psi(s, u)$, and $f(s)$ we can the above
rectly to the boundary condition (82) without differentiate it
is leads to the fixed point problem

$$
u(s) = (N\xi)(s) = k + \int_{0}^{s} \varphi(\sigma, \xi) d\sigma
$$
\n(83)

 $\begin{aligned} \text{with} \quad \text{with}$

$$
\varphi(s,\xi)=\beta(\xi)\,h(s,\xi)+\alpha(\xi)\,\mathrm{e}^{-H(\mu)(s)}H\{\mathrm{e}^{H(\mu)}h\}\,(s),\qquad \qquad (84)
$$

where

$$
u(s) = (N\xi) (s) = k + \int_{0}^{s} \varphi(\sigma, \xi) d\sigma
$$
\n(83)
\n
$$
\varphi(s, \xi) = \beta(\xi) h(s, \xi) + \alpha(\xi) e^{-H(\mu)(s)} H\{e^{H(\mu)}h\} (s),
$$
\n(84)
\n
$$
\alpha(\xi) = 1/\sqrt{1 + X^{2}(\xi)}, \quad \beta(\xi) = X(\xi)/\sqrt{1 + X^{2}(\xi)},
$$
\n(85)
\n
$$
\mu(\xi) = \arctan X(\xi), \quad h(s; \xi) = g_{0}(s; \xi)/\sqrt{1 + X^{2}(\xi)}
$$
\n(86)

$$
\mu(\xi) = \arctan X(\xi), \qquad h(s; \xi) = g_0(s; \xi) / \sqrt{1 + X^2(\xi)} \tag{86}
$$

0

O

$$
Poincaré Boundary Value Problemsand
$$
g_0(s, \xi) = g(s, \xi) - m_0[\xi],
$$

$$
g(s, \xi) = f(s) - \Psi(s, \xi),
$$
(88)
$$

$$
g(\dot{s},\xi) = f(s) - \Psi(s,\xi),\tag{88}
$$

$$
y_{0}(s, \xi) = g(s, \xi) - m_{0}[\xi],
$$
\n
$$
g(s, \xi) = f(s) - \Psi(s, \xi),
$$
\n
$$
m_{0}[\xi] = \int_{-\pi}^{\pi} g(s, \xi) \gamma(s, \xi) ds \bigg\langle \int_{-\pi}^{\pi} \gamma(s, \xi) ds, \qquad (89)
$$
\n
$$
\gamma(s, \xi) = e^{H(\mu)(s)} / \sqrt{1 + X^{2}(\xi)}.
$$
\n(90)
\nHere the constant k in (83) is to be determined in fulfilling the integral condition

$$
\gamma(s,\xi) = e^{H(\mu)(s)} \big/ \sqrt{1 + X^2(\xi)}.
$$
\n(90)

d
-
-
ere the Here the constant *k* in (83) is to be determined in fulfilling the integral condition

and
\n
$$
g_0(s, \xi) = g(s, \xi) - m_0[\xi],
$$
\n
$$
g(s, \xi) = f(s) - \Psi(s, \xi),
$$
\n(88)
\n
$$
m_0[\xi] = \int_0^s g(s, \xi) \gamma(s, \xi) ds \Big/ \int_{-\pi}^{\pi} \gamma(s, \xi) ds,
$$
\n(89)
\n
$$
\gamma(s, \xi) = e^{H(\mu)(s)} / \sqrt{1 + X^2(\xi)}.
$$
\n(90)
\nHere the constant k in (83) is to be determined in fulfilling the integral condition
\n
$$
\int_0^{\pi} \Psi \left(s, k + \int_0^s \varphi(\sigma, \xi) d\sigma \right) ds = \int_0^{\pi} f(s) ds.
$$
\n(91)
\nWe assume that
\n
$$
2\gamma = \sup \{ \text{arc tan } X(u) \} - \inf \{ \text{arc tan } X(u) \} < \pi/2,
$$
\n(92)
\nthe supremum and infimum are taken over $u \in \mathbb{R}$, and
\n
$$
|\Psi(s, u)| \leq c_1 + c_2 |u|^s, \quad 0 \leq \delta < 1,
$$
\n(93)
\nfor all $u \in \mathbb{R}$ with positive constants c_1, c_2 . Furthermore, the equation (91) for
\n $k \in \mathbb{R}$ shall have a root $k = k[\xi]$ for any $\xi \in C(\Gamma)$ which depends continuously upon ξ

$$
2\gamma = \sup_{u \in \mathbb{R}} [\arctan X(u)] - \inf [\arctan X(u)] < \pi/2,
$$
 (92)
imum and infimum are taken over $u \in \mathbb{R}$, and

$$
|\Psi(s,u)| \leq c_1 + c_2 |u|^{s}, \qquad 0 \leq \delta < 1,
$$
 (93)

for all $u \in \mathbb{R}$ with positive constants c_1, c_2 . Furthermore, the equation (91) for $k \in \mathbf{R}$ shall have a root $k = k[\xi]$ for any $\xi \in C(F)$ which depends continuously upon ξ and satisfies the estimation $|\Psi(s, u)| \leq c_1 + c_2 |u|^\delta$, $0 \leq \delta < 1$,
 $\in \mathbb{R}$ with positive constants c_1, c_2 . Furthermore,
 jll have a root $k = k[\xi]$ for any $\xi \in C(\Gamma)$ which deplies the estimation
 $|k[\xi]| \leq K_1 + K_2 R^* \equiv K(R)$, $0 \leq v < 1$,
 $C(\Gamma)$ 2,
e, the equa
eends continu
ve constants
unction $\mathcal{Y}(s)$ *c*
 (91)
 l
 l
 l
 k
 *K*₁, *K*₂, *l*
 l

$$
|k[\xi]| \leq K_1 + K_2 R' \equiv K(R), \qquad 0 \leq \nu < 1, \tag{94}
$$

 $2\gamma = \sup$ [arc tan $X(u)$] – inf [arc tan $X(u)$] < $\pi/2$, (92)

the supremum and infimum are taken over $u \in \mathbb{R}$, and
 $|\Psi(s, u)| \leq c_1 + c_2 |u|^s$, $0 \leq \delta < 1$, (93)

for all $u \in \mathbb{R}$ with positive constants c_1, c_2 . Fu is especially fulfilled with $\delta < v < 1$ if the (continuous) function $\Psi(s, u)$ is strictly monotone with respect to u for any $s \in [-\pi, \pi]$, there exist the (finite or identically infinite) limit functions monotone with respect to u for any $s \in [-\pi, \pi]$, there exist the (finite or identically We assume that
 $2\gamma = \sup$ [arc tan $X(u)$] - inf [

the supremum and infimum are taken of
 $|\Psi(s, u)| \leq c_1 + c_2 |u|^{s}$, $0 \leq$

for all $u \in \mathbb{R}$ with positive constants
 $k \in \mathbb{R}$ shall have a root $k = k[\xi]$ for any

and s ntir
, th
. . $X(u)$ $\leq \pi/2$, (92)
 R, and (93)

'urthermore, the equation (91) for which depends continuously upon ξ
 ≤ 1 , (94)

orm positive constants K_1 , K_2 . This

tinuous) function $\Psi(s, u)$ is strictly

there exist \mathbf{d} $k \in \mathbb{R}$ shall have a root $k = k[\xi]$ for any $\xi \in C(\Gamma)$ which depends continuously u

and satisfies the estimation
 $\langle k[\xi] | \leq K_1 + K_2 R = K(R)$, $0 \leq r < 1$,

for all $\xi \in C(\Gamma)$ with $|\xi(s)| \leq R$, $R > 0$, and uniform positive *IRI*(*II*) $\le K_1 + K_2 R$ = $K(R)$, $0 \le v < 1$, (94)
 IRI(I) with $|\xi(s)| \le R$, $R > 0$, and uniform positive constants K_1 , K_2 . This
 IIII fulfilled with $\delta < v < 1$ if the (continuous) function $\Psi(s, u)$ is strictly

imi is especially fulfilled with $\delta < v < 1$ if the (continuous) function Ψ
monotone with respect to u for any $s \in [-\pi, \pi]$, there exist the (finit
infinite) limit functions
 $\lim_{u \to \pm \infty} \Psi(s, u) = \Psi_{\pm}(s)$ uniformly in $s \in [-$

$$
\lim \Psi(s, u) = \Psi_{\pm}(s) \quad \text{uniformly in } s \in [-\pi, \pi]
$$

and, f *f(s) ds* lies between the values $\Psi_{\pm} = \int \Psi_{\pm}(s) ds$. Namely, because of the and $\int f(s)$
 $\begin{array}{c} -\pi \\ -\pi \\ -\pi \end{array}$

for all $\xi \in$

shown like

Now we
 $\begin{array}{c} \text{if } \\ \text{if }$

$$
\int_{-\pi}^{\pi} |\varphi(s,\xi)| ds \leq L_1 + L_2 R^3
$$
\nfor all $\xi \in C(\Gamma)$ with $|\xi(s)| \leq R$ and uniform positive constants L_1 , L_2 as can be shown like above.

\nNow we consider the operator N on the convex compact set \Re of $C(\Gamma)$ defined by

\n
$$
\Re = \{\xi \in C(\Gamma) : |\xi(s)| \leq R, |\xi(s_1) - \xi(s_2)| \leq R_0 |s_1 - s_2|^2\}
$$
\nwith $\lambda = 1 - (1/p), 1 < p < \pi/(4\gamma)$, and sufficiently large positive constant R such

Now we consider the operator N on the convex compact set \mathfrak{K} of $C(\Gamma)$ defined by

$$
\widehat{\mathfrak{R}} = \{ \xi \in C(\Gamma) : |\xi(s)| \leq R, \ |\xi(s_1) - \xi(s_2)| \leq R_0 \ |s_1 - s_2|^2 \} \tag{97}
$$

with $\lambda = 1 - (1/p), 1 < p < \pi/(4\gamma)$, and sufficiently large positive constant *R* such *that* $(2\pi)^{1} R_{0} = \hat{R} - K(R) = R - K_{1} - K_{2}R^{r} > 0$ and

$$
R_0 \geq C_R = (2\pi)^{1/p} \left[M + c_1 + c_2 R^3 \right] \{ 1 + 2A_{2p} (\cos 2p\gamma)^{-1/p} \}
$$
(98)

and *f* $f(s) ds$ lies between the values $\Psi_{\pm} = \int \Psi_{\pm}(s) ds$. Namely, because of the

assumptions (92) and (93), there holds an estimation of the form
 $\int_{s}^{s} |\varphi(s, \xi)| ds \leq L_1 + L_2 R^s$ (96)

for all $\xi \in C(\Gamma)$ with $|\xi(s)| \le$ with $M = \max |f(s)|$ and A_r , the M. Riesz constant. Then like in [18] Schauder's (s, 5) $ds \leq L_1 + L_2 R^3$
 Γ) with $|\xi(s)| \leq R$ and uniform positive constants L_1 , L_i

nove.

sider the operator N on the convex compact set \Re of $C(\Gamma)$
 $|\xi \in C(\Gamma) : |\xi(s)| \leq R$, $|\xi(s_1) - \xi(s_2)| \leq R_0 |s_1 - s_2|^2$
 $\cdot ($ fixed point theorem yields the existence of a fixed point $u \in \mathbb{R}$ for the operator *N*.

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Moreover, because of $\partial u/\partial s = \varphi(s, u)$ the corresponding solution $u(z)$ to Problem P has Hölder continuous partial derivatives in \bar{G} . Using the Lipschitz continuity of *u* in (84) the corresponding Hölder exponent is seen to be the minimum of the Hölder 396 L. v. WOLFERSDORF

Moreover, because of $\partial u/\partial s = \varphi(s, u)$ the corresponding s

has Hölder continuous partial derivatives in \overline{G} . Using the

in (84) the corresponding Hölder exponent is seen to be the

exponents of boundary condition

In the boundary Castillan Boundary of the Hilder Hilder

In the corresponding Hilder exponent is sequently to the corresponding Hilder exponent is sequently of $\{f, \Psi, X\}$.

Theorem 2: If the assumpti

• Theorem 2: *If the assumptions (92),* (93) *are fulfilled and there exists a continunus* $solution k[ξ] of (91) with (94) the boundary value problem (82) with Hölder continuous$ *functions* $X(u)$, $\Psi(s, u)$, and $f(s)$ has a solution $u(z) \in C^{1,\epsilon}(\overline{G})$, where ε is the minimum *of the Hölder exponents of* f, Ψ, X *.*

If the function $\Psi(s, u)$ in (82) is strictly increasing in *u*, existence of solutions to this problem can also be proved for functions Ψ with linear and superlinear growth in *u* using results of SCHLETF [16] for the corresponding semilinear boundary condition. in (84) the exponent
 P_1 Theor
 P_2
 P_3
 P_4
 P_5
 P_5
 P_6
 P_7
 P_8
 P_7
 P_8
 P_9
 P_9
 P_8
 P_9
 P_9
 P_1
 P_2
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 P_4
 P_5
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 P_9
 P_9
 P_9
 P_9 *au/arian au/as partial derivatives in* \overline{G} . Using the Lipschitz continuity of u are corresponding Hölder exponent is seen to be the minimum of the Hölder of f, Ψ, X .
 $\overline{G} = 2$: *If the assumptions* (92), (93 Theorem 2: If the assumptions (92), (93)
solution $k[\xi]$ of (91) with (94) the boundary v
functions $X(u)$, $\mathcal{Y}(s, u)$, and $f(s)$ has a soluti
of the Hölder exponents of f , \mathcal{Y} , X .
If the function $\mathcal{Y}(s, u)$ i *If* $g(f) = f(x)$ *value (see formalism and the boundary value problem (s2) with Holder continuous*
 If $g(x)$, $\psi(s, u)$, and $f(s)$ has a solution $u(z) \in C^{1,2}(\overline{G})$, where ε is the minimum

der exponents of f , ψ ,

$$
\frac{\partial u}{\partial r} + X(s, u) \frac{\partial u}{\partial s} + \Psi(s, u) = f(s) \quad \text{on } \Gamma
$$

the functions $X(s, u)$, $\psi(s, u)$, $f(s)$ shall be Hölder continuous and moreover, $\psi(s, u)$ shall possess a Hölder continuous derivative $\Psi_u(s, u) > 0$ for all $s \in [-\pi, \pi], u \in \mathbb{R}$. + $X(s, u)$ $\partial u/\partial s$ + $\Psi(s, u) = f(s)$ on Γ
 $\mathbf{K}(s, u)$, $\Psi(s, u)$, $f(s)$ shall be Hölder continuous and moreover, Ψ

Hölder continuous derivative $\Psi_u(s, u) > 0$ for all $s \in [-\pi, \pi]$, u

ume that
 $\langle f(s) \langle \Psi_+(s) \rangle$ for *u* vising results of SCHLETFF [16] for the corres
 tion.

In the boundary condition
 $\partial u/\partial r + X(s, u) \partial u/\partial s + \Psi(s, u) = f(s)$

are functions $X(s, u), \Psi(s, u), f(s)$ shall be Hölder

and possess a Hölder continuous derivative $\Psi_u(s)$
 au/ar π *au/as* + *W(s, u)* $\partial u/\partial s$ + *W(s, u)* = *f(s)* on *P* (99)
 au/ar + *X(s, u), W(s, u), f(s)* shall be Hölder continuous and moreover, *W(s, u)*

ions *X(s, u), W(s, u), f(s)* shall be Hölder continuous a

$$
\Psi_{-}(s) < f(s) < \Psi_{+}(s) \quad \text{for } s \in [-\pi, \pi], \tag{100}
$$

where $\Psi_{\pm}(s) = \lim_{n \to \pm \infty} \Psi(s, u).$

Then, according to Theorem 2 of [16: Part 2] for any Hölder continuous function $\xi = \xi(s)$ the auxiliary problem (100)
 z (100)
 z derivatives in \overline{G} . More-
 z $\in \overline{G}$, (102)
 z to the equation F(8) < f(8) < F₊(8) for $s \in [-\pi, \pi]$,
 $\lim_{u \to \pm \infty} \Psi(s, u)$.

according to Theorem 2 of [16: Part 2]

he auxiliary problem
 $\partial u/\partial r + X(s, \xi) \partial u/\partial s + \Psi(s, u) = f(s)$

ique solution $u(z)$ with Hölder continus

satisfies the estim

$$
\frac{\partial u}{\partial r} + X(s, \xi) \frac{\partial u}{\partial s} + \Psi(s, u) = f(s) \quad \text{on } \Gamma \tag{101}
$$

 $\partial u/\partial r + X(s, \xi) \partial u/\partial s + \Psi(s, u) = f(s)$ on Γ (101
has a unique solution $u(z)$ with Hölder continuous partial derivatives in \bar{G} . More
over, $u(z)$ satisfies the estimation
 $\eta_1 \equiv \min_{\Gamma} \eta(s) \le u(z) \le \max_{\Gamma} \eta(s) = \eta_2, \qquad z \in \bar{G$ over, *u(z)* satisfies the estimation *Stellary* $\alpha_1 \in \mathbb{R}^n$ and $\alpha_2 \in \mathbb{R}^n$ is a statistic the estimation $\eta_1 \equiv \min_{\Gamma}(s) \leq u(z) \leq \max_{\Gamma} \eta(s) \equiv \eta_2, \qquad z \in \overline{G},$

continuous function $\eta(s)$ is the solution u to the equation $\Psi(s, u) = f(s), \qquad s \in [-\pi, \pi$

$$
\eta_1 \equiv \min_{\Gamma} \eta(s) \le u(z) \le \max_{\Gamma} \eta(s) \equiv \eta_2, \qquad z \in \overline{G}, \tag{102}
$$

where the continuous function $\eta(s)$ is the solution *u* to the equation

$$
\Psi(s, u) = f(s), \qquad s \in [-\pi, \pi]. \tag{103}
$$

We consider the problem (101) for $\xi \in \mathbb{R}$ with \mathbb{R} defined by

$$
\mathfrak{F} = \{ \xi \in C(T) \colon \eta_1 \leq \xi(s) \leq \eta_2, \quad |\xi(s_1) - \xi(s_2)| \leq R_0 \, |s_1 - s_2|^2 \},\tag{104}
$$

where $\lambda = 1 - (1/p), 1 < p < \pi/(4\gamma)$. The solution u of (101) depends continuously on $\xi \in \mathbb{R}$ in the maximum norm topology such that the operator $u = N\xi$ of the boundary values $u(s)$ of *u* is a continuous operator from $\hat{\mathfrak{X}}$ into $C(\Gamma)$.

Namely, let $\xi_n \in \mathbb{R}$ converge uniformly to $\xi_0 \in \mathbb{R}$ and let u_n, u_0 be the corresponding solutions of (101), respectively. The difference function $U_n = u_n - u_0$ satisfies the boundary condition $\hat{\mathbf{R}} = \{\xi \in C(\Gamma) : \eta_1 \leq \xi(s) \leq \eta_2, \quad |\xi(s_1) - \xi(s_2)| \leq R_0 |s_1 - s_2|^2\},$ (104)
 $= 1 - (1/p), 1 < p < \pi/4$. The solution *u* of (101) depends continuously

in the maximum norm topology such that the operator $u = N\xi$ of the
 v g, let $\xi_n \in \mathbb{R}$ converge uniformly to $\xi_0 \in \mathbb{R}$ and let u_n, u
of (101), respectively. The difference function $U_n =$
 $\frac{\partial U_n}{\partial r} + X(s, \xi_n) \frac{\partial U_n}{\partial s} + \Psi(s, U_n + u_0) = G_n(s)$
Hölder continuous function
 $G_n(s) = \Psi(s, u_0) + [X$

$$
\partial U_n/\partial r + X(s, \xi_n) \partial U_n/\partial s + \Psi(s, U_n + u_0) = G_n(s) \quad \text{on } \Gamma \tag{105}
$$

with the Hölder continuous function

$$
G_n(s) = \Psi(s, u_0) + [X(s, \xi_0) - X(s, \xi_n)] \partial u_0/\partial s.
$$

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(113)

• Again by Theorem 2 of [16: Part 2] there holds the estimation

\n Poincaré Boundary Value Problems 397
\n Again by Theorem 2 of [16: Part 2] there holds the estimation
\n
$$
\min \eta_n(s) \leq U_n(s) \leq \max \eta_n(s), \quad s \in [-\pi, \pi],
$$
\n (106)
\n for the boundary values of U_n on Γ , where $\eta_n(s)$ is the unique solution of the equation
\n $\Psi(s, \eta_n(s) + u_0(s)) = G_n(s), \quad s \in [-\pi, \pi].$ \n (107)
\n But $G(s)$ converges uniformly to $\Psi(s, u_0(s))$ as $n \to \infty$ such that $\eta_n(s)$ converges\n

$$
\Psi(s,\,\eta_n(s) \,+\, u_0(s)) = G_n(s), \qquad s \in [-\pi,\,\pi]. \tag{107}
$$

But $G_n(s)$ converges uniformly to $\Psi(s, u_0(s))$ as $n \to \infty$ such that $\eta_n(s)$ converges uniformly to zero. Due to (106) then also $U_n(s)$ converges uniformly to zero.

Further, analogously to (83) one obtains the following expression for $u = N\xi$:

$$
u(s) = k + \int\limits_{0}^{s} \varphi(\sigma, \xi, u) d\sigma \qquad (108)
$$

Poincaré Boundary Va

Theorem 2 of [16: Part 2] there holds the estimation
 $\min \eta_n(s) \leq U_n(s) \leq \max \eta_n(s), \quad s \in [-\pi, \pi],$
 uudary values of U_n on Γ , where $\eta_n(s)$ is the unique sol
 $\Psi(s, \eta_n(s) + u_0(s)) = G_n(s), \quad s \in [-\pi, \pi].$
 ρ with $k = k[\xi]$ the value of $u(z)$ for $t = 1$ and $\varphi(s, \xi, u)$ is given by (84), where in the with $k = k[\xi]$ the value of $u(z)$ for $t = 1$ and $\varphi(s, \xi, u)$ is given by (84), where in the formulas (84)—(90) $X(s, \xi)$ instead of $X(\xi)$, $\Psi'(s, u)$ instead of $\Psi(s, \xi)$, and $g(s, u)$ instead of $g_0(s, \xi)$ is to be written $u(s) = k + \int_{0}^{\infty} \varphi(\sigma, \xi, u) d\sigma$ (108)
with $k = k[\xi]$ the value of $u(z)$ for $t = 1$ and $\varphi(s, \xi, u)$ is given by (84), where in the
formulas (84)-(90) $X(s, \xi)$ instead of $X(\xi)$, $\varphi'(s, u)$ instead of $\varphi(s, \xi)$, and $g(s, u$ instead of $g_0(s, \xi)$ is to be written. As just proved, $u(z)$ depends continuously upon ξ , therefore also $k = k[\xi]$.
Finally, we assume that The boundary values of U_n on Γ , where $\eta_n(s)$ is the unique so $\Psi(s, \eta_n(s) + u_0(s)) = G_n(s)$, $s \in [-\pi, \pi]$.

It $G_n(s)$ converges uniformly to $\Psi(s, u_0(s))$ as $n \to \infty$ such iformly to zero. Due to (106) then also $U_n(s)$ conve $\Psi(s, \eta_n(s) + u_0(s)) = G_n(s), \quad s \in [-\pi, \pi].$ (107)

But $G_n(s)$ converges uniformly to $\Psi(s, u_0(s))$ as $n \to \infty$ such that $\eta_n(s)$ converges

uniformly to zero. Due to (106) then also $U_n(s)$ converges uniformly to zero.

Further, anal But $\sigma_n(s)$ converges uniformly to $\alpha_1(s)$ then also $U_n(s)$ converges uniformly

multionary to zero. Due to (106) then also $U_n(s)$ converges uniformly

Further, analogously to (83) one obtains the following expression

$$
2\gamma = \sup\left[\arctan X(s, u)\right] - \inf\left[\arctan X(s, u)\right] < \pi/2, \tag{109}
$$

therefore also $k = k[\xi]$.
Finally, we assume that
 $2\gamma' = \sup \left[\arctan X(s, u)\right] - \inf \left[\arctan X(s, u)\right] < \pi/2$, (109)
the supremum and infimum are taken over $s \in [-\pi, \pi]$, $\eta_1 \leq u \leq \eta_2$. Then there
holds the estimation $2\gamma = \sup \left[\arctan X(s, u)\right] - \inf \left[\arctan X(s, u)\right]$

Emum and infimum are taken over $s \in [-\pi, \pi]$
 $\|\varphi\|_p \leq (2\pi)^{1/p} \{M + M_0\} \{(1/2) + A_{2p}(\cos 2p\gamma) \in \mathbb{R}, \text{ where } 1 < p < \pi/(4\gamma), M = \max_{\Gamma} |f(s)|,$
 $M_0 = \max_{\Gamma} \max_{\Gamma} [\varPsi(s, \eta_2), -\varPsi(s, \eta_1)],$
 $\lim_{$

$$
\|\varphi\|_n \le (2\pi)^{1/p} \left\{ M + M_0 \right\} \left\{ (1/2) + A_{2p} (\cos 2p\gamma)^{-1/p} \right\} = C_0 \tag{110}
$$

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for any $\xi \in \mathfrak{N}$, where $1 < p < \pi/(4\gamma)$, $M = \max |f(s)|$,

$$
M_0=\max_{\Gamma}\max_{\Gamma}\left[\Psi(s,\eta_2),-\Psi(s,\eta_1)\right],
$$

and A_r , the'M. Riesz constant.

Taking the constant R_0 in (104) equal to C_0 and applying the Schauder fixed point theorem to the operator *N* on \Re , we obtain the existence of a fixed point of \overline{N} , i.e., the existence of \tilde{a} solution u to the boundary value problem (99).

Theorem **3:** *Under the assumptions* (100) *and* (109) *the boundary value problem* (99) with Hölder continuous functions $X(s, u)$, $\mathcal{W}(s, u)$, $f(s)$, where $\Psi(s, u)$ possesses a *Holder continuous derivative* $\Psi_u(s, u) > 0$ *, has a solution* $u(z) \in C^{1, \epsilon}(\overline{G})$ *, where* ϵ *is the minimum of the Hölder exponents of* f, Ψ, X *.

Finally, we briefly deal with the boundary condition of the form \partial u/\partial r + \partial/* the minimum of the Hölder exponents of f, Ψ, X . *au/ar + a/3s[b(s, u)]* = */(s)* on *1' . f*. A. Riesz constant.

the constant R_0 in (104)
 f. or the operator N on $\hat{\mathbb{R}}$,
 f. or $\hat{\mathbb{R}}$ and $\hat{\mathbb{R}}$ an

Finally, we briefly deal with the boundary condition of the form

$$
\partial u/\partial r + \partial/\partial s[\Phi(s, u)] = f(s) \quad \text{on } \Gamma \tag{111}
$$

with Holder continuous function */(s)* satisfying the necessary solvability condition

$$
\mathbf{y}, \text{ we briefly deal with the boundary condition of the form}
$$
\n
$$
\frac{\partial u}{\partial r} + \frac{\partial}{\partial s}[\Phi(s, u)] = f(s) \quad \text{on } \Gamma
$$
\n
$$
\text{Ider continuous function } f(s) \text{ satisfying the necessary solvability condition}
$$
\n
$$
\int_{-\pi}^{\pi} f(s) \, ds = 0 \tag{112}
$$

and Hölder continuously differentiable function $\Phi(s, u)$ with respect to *s* and *u*.

Denoting by $v(z)$ the conjugate harmonic function to $u(z)$ normalized by $v(1) = 0$ or $v(0) = 0$, respectively, the condition (111) is equivalent to the Riemann-Hilbert condition- > 0 , has a

of f, Ψ , X.

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on (111) is
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$$
v + \varPhi(s, u) = \bar{k} + F(s) \quad \text{on } \Gamma
$$

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398 L. v. WOLFERSDORF

398 L. v. WOLFERSDORF
with $F(s) = \int_{0}^{s} f(\sigma) d\sigma$ and an arbitrary constant $\bar{k} \in \mathbb{R}$ together with the additional
condition 0 398 L. v. WOLFERSDORF
with $F(s) = \int_{0}^{s} f(\sigma) d\sigma$ and an arbitrary constant $\bar{k} \in \mathbb{R}$ rogether
condition $v(1) = 0$ L. V. WOLFERSDORF
 $\begin{aligned}\n\psi(1) &= 0 \\
\psi(0) &= 0\n\end{aligned}$
 $\begin{aligned}\n\psi(1) &= 0 \\
\psi(0) &= 0\n\end{aligned}$ the additional
 (114)
 (115) L. V. WOLFERSDORF
 $\psi(1) = 0$
 $\psi(0) = 0$,
 $\psi(0) = 0$,
 $\psi(1) = 0$
 $\psi(0) = 0$,
 $\psi(0) = 0$,
 $\psi(0) = 0$,
 $\psi(1) = 0$,
 $\psi(1) = 0$,
 $\psi(0) = 0$

$$
v(1) = 0
$$
 (114)

or

$$
v(0) = 0
$$

respectively.

The Riemann-Hilbert problems (113) with (114) or (115) have' been considered in our papers [18, 19]. In case of the additional condition (114)-let $k \in \mathbb{R}$ be an arbitrary constant and put $\bar{k} = \Phi(0, k)$. Then (114) can be replaced by the additional condition 398 L. v. Wolfer

with $F(s) = \int f(\sigma) d\sigma$

condition $v(1) = 0$

or
 $v(0) = 0$,

respectively.

The Riemann-Hill

in our papers [18, 1

arbitrary constant are

condition
 $u(1) = k$

to the boundary con- $F(s) = \int_{0}^{s} f(\sigma) d\sigma$ and an

ion $v(1) = 0$
 $v(0) = 0$,

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ion
 $u(1) = k$

boundary condition
 $v + \Phi(s, u) = \Phi(0, k)$ *v*(1) = 0
 v(0) = 0, (114)
 v(0) = 0, (115)

ely. (115)

(interpretent problems (113) with (114) or (115) have been considered

apers [18, 19]. In case of the additional condition (114) let $k \in \mathbb{R}$ be an

constan

$$
k \tag{116}
$$

(115)

to the boundary, condition

$$
v + \Phi(s, u) = \Phi(0, k) + F(s) \quad \text{on } T.
$$

To this problem Theorem 2 of [18] with $\rho = \infty$ can be applied.

For strictly monotone function $\Phi(s, u)$ in u also Theorem 4 of [18] with $\rho = \infty$ may be applied directly to the problem (113) with (115), where the Landesman-Lazer type condition in this theorem can be fulfilled by a suitable choice of the parameter \vec{k} . at other y constant and put $k = \Psi(0, k)$. Then (114) can be replaced by the

condition
 $u(1) = k$

to the boundary condition
 $v + \Phi(s, u) = \Phi(0, k) + F(s)$ on T.

To this problem Theorem 2 of [18] with $\varrho = \infty$ can be applied.

F

 \cdot .

- [1] AMANN, H.: Nonlinear operators in ordered Banach spaces and some applications to nonlinear boundary value problems. In: Nonlinear Operators and the Calculus of Variations, Bruxelles 1975 (Lecture Notes Math. 543). Springer-Verlag: Berlin-Heidelberg-New York 1976, pp. $1-55$. his problem Theorem 2 of [18] with $\varrho = \infty$ can be applied directly monotone function $\Phi(s, u)$ in u also Theo
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COMEX AMANN, H.: Nonlinear operators in ordered

nonlinear boundary value problems. In: Nonl

ations, Bruxelles 1975 (Le AMANN, H.: Nonlinear operators in order
nonlinear boundary value problems. In: N.
ations, Bruxelles 1975 (Lecture Notes Math.
New York 1976, pp. 1–55.
AMANN, H.: Nonlinear elliptic equations w
Developments in Differential FERENCES

AMANN, H.: Nonlinear operators in ordered Banach spaces and some

nonlinear boundary value problems. In: Nonlinear Operators and the C

ations, Bruxelles 1975 (Lecture Notes Math. 543). Springer-Verlag: Berlin
 [1] AMANN, H.: Nonlinear operators in ordered Banach spaces and some applications to nonlinear boundary value problems. In: Nonlinear Operators and the Calculus of Variations, Bruxelles 1975 (Lecture Notes Math. 543). Spri
- [2] AMANN, H.: Nonlinear elliptic equations with nonlinear boundary conditions. In: New Developments in Differential Equations (ed. by W. Eckhaus). North-Holland Publ. the Calcu

Berlin – H

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North-Ho

y Gleichu Developments in Differential Equations (ed. by W. Eckhaus). North-Holland Comp., Amsterdam 1976, p. 43–63.
CARLEMAN, T.: Uber eine nichtlineare Randwertaufgabe bei der Gleichung Δu Math. Zeitschrift 98 (1921), 35–43.
CUS
- [3] CARLEMAN, T.: Uber eine nichtlineare Randwertaufgabe bei der Gleichung- $\Delta u = 0$.
Math. Zeitschrift 93 (1921), 35-43.
- [4] CUSHINO, J. M.: Nonlinear Steklov problems on the unit circle. J. Math. Anal. Appl. 38
-
- [5] Γ AXOB, Φ . Π .: Краевые задачи. Гос. изд-во физ.-мат. лит.: Москва 1963.
[6] INKMANN, F.: Existence and multiplicity theorems for semi-linear elliptic equations with nonlinear boundary conditions. Indiana Univ.
- [7] KLINGELHÖFER, K.: Uber nichtlineare Randwertaufgaben der Potentialtheorie I, II. Mitteil. math. Sem. Gießen 76 (1967), 1-70; 79 (1968), 1-27.
- [8] KLINGELHOFER, K.: Nonlinear harmonic boundary value problems I. Arch. Rat. Mcch. Anal. **31.** (1968), 364-371. mercent integral equations). J. Math. Anal. Appl. 25 (1969), 592-606.

Anal. 31 (1968), 364-371.

[9] KLINGELHOFER, K.: Nonlinear harmonic boundary value problems II. (modified Hammerstein integral equations). J. Math. An
- [9] KLINGELHOFER, K.: Nonlinear harmonic boundary value problems II. (modified Hammerstein integral equations). J. Math. Anal. Appl. 25 (1969), 592 606.
- boundary value problems. J. Math. Anal. AppI. 28 (1969), 77-87.
- [11] KLINGELHÖFER, K.: Nonlinear boundary value problems with simple eigenvalue of the linear part. Arch. Rat. Mech. Anal. 37 (1970), 381 —398.
- [12] MARUHN, K.: Über einige Klassen nichtlincarer Randwertaufgaben der Potentialtheorie.
Math. Z. 51 (1949), 36-60.
- INKMANN, F.: Existence and multiplicity theorems for semi-linear
nonlinear boundary conditions. Indiana Univ. Math. J. 31 (1982),
KLINGELHÖFER, K.: Uber nichtlineare Randwertaufgaben der
Mitteil. math. Sem. Gießen 76 (1967 [13] Muschellschwill, N. I.: Singuläre Integralgleichungen. Randwertprobleme der Funktionentheorie und cinige Anwendungen auf die mathematische Physik (Math. Lehr-
bücher u. Monographien: Abt. 2, Bd. 20). Akademie-Verlag: Berlin 1965. • butten: math. Sem. Gielen *A6* (1967), 1—70; 79 (1968), 1—27.

[8] KLINGELHÖFER, K.: Nonlinear harmonic boundary value problems I. Arch. Rat.

[9] KLINGELHÖFER, K.: Nonlinear harmonic boundary value problems II. (modifie Anal. 31. (1968), 364—371.

KLINGELHÖFER, K.: Nonlinear harmonic boundary value problem

merstein integral equations). J. Math. Anal. Appl. 25 (1969), 592—

KLINGELHÖFER, K.: Modified Hammerstein integral equations an

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- [141 POGORZELSKI, W.: Integral Equations and their Applications, Vol. I (International Series of Monographs in' Pure and Applied Mathematics: Vol. 88). Pergamon Press: Oxford, Poi

Pocor ELSKI, W.: Integral Equations and the

of Monographs in Pure and Applied Mathen

and PWN: Warszawa 1966.

ROTHE, F.: Solutions for systems of nonlinear

conditions. Math. Methods Appl. Sci. 1 (1979)

SCHLEIFF, M Poincaré Boundary Value Problems

Poconzels. W.: Integral Equations and their Applications, Vol. I (International Sof Monographs in Pure and Applied Mathematics: Vol. 88). Pergamon Press: Ox

and PWN: Warszawa 1966.

ROTHE [14] Pooonzer.ski, W.: Integral Equations and their Applications, Vol. I (International Section Monographs in Pure and Applied Mathematics: Vol. 88). Pergamon Press: Oxf. and PWN: Warszawa 1966.

[15] Rorne, F.: Solutions
	- [15] ROTHE, F.: Solutions for systems of nonlinear elliptic equations with nonlinear boundary conditions. Math. Methods Appl. Sci. 1 (1979), 545-553.
	- [16] SCITLEIFF, M.: Uber einige niehtlineare Vera ilgemeinerungen des Rand wertproblems von Poincaré, 1. und 2. Teil. Wiss. Z. Univ. Halle 19 (1970), 87-93 und 95-100.
	- [17] \%TIIDENAUER, P.: Existence of a minimal solution and a maximal solution of nonlinear elliptic boundary value problems. Indiana Univ. Math. J. 29 (1980), $455-462$.
	- [18] WOLFERSDORF, L. $v: A$ class of nonlinear Riemann-Hilbert problems for holomorphic • JUNCTURE 1, 1981. ISLEE TO A MILDENAUER, P.: Existence of a minimal solution
elliptic boundary value problems. Indiana Univ.
[1] WOLFERSDORF, L. v.: A class of nonlinear Riem
• Tunctions. Math. Nachr. 116 (1984), 89–107.
	- [19] WOLFERSDORF, L. v;: Landesman-Lazer's type boundary value problems for holomorphic
	- [20] WOLSKA-BOCHENEK, J.: Un problème aux limites à dérivée tangentielle pour l'equation. du type elliptique. Ann. Polon. Math. 4 (1958), 257-287.
- [21] WOLSKA-BOCHENEK, J.: Problème non-linéaire à dérivée oblique. Ann. Polon. Math. 9 KOTHE, F.: Solutions Ior systems of nonlinear elliptic equations with nonlinear bound
conditions. Math. Methods Appl. Sci. 1 (1979), 545-553.
ScintLier, M.: Uber einige nichtlineare Verallgemeinerungen des Randwertproblems SCIILEIFF, M.: Uber emge mentlineare Verallgemeinerungen d

Poincaré, 1. und 2. Teil. Wiss. Z. Univ. Halle 19 (1970), 87–9

WLDENAUER, P.: Existence of a minimal solution and a max

elliptic boundary value problems. Indian For boundary value problems. Indian Onteraction, the proteins. Indian Onlinear Riemann-Hilbert problem

Tunctions. Math. Nachr. 116 (1984), 89-107.

Tunctions. Math. Nachr. 116 (1984), 89-107.

[20] WOLFERSDORF, L. v.: La • Prof. Dr. LOTHAR VON VOLFERSDORF • - • • Secuelar Mathematic Mathematic der Bergakademie Freiberg

Peelliptique. Ann. Polon. Math. 4 (1958), 257-287.

FRA-BOCHENEK, J.: Problème non-linéaire à dérivée oblique. Ann. Polon. Math. 9

(1955), 253-264.

1080, A.: Trig
	- [22] ZYGMUND, A.: Trigonometrical Series (Monografie mathematyczne: T. 5). Univ.: War-

/

VERFASSER:

pe emperature. Ann. Foron. math. 4 (1938), 257-281.

EXA-BOCHENEK, J.: Problème non-linéaire à dérivée oblique.

0.253-264.

UND, A.: Trigonométrical Series (Monografie mathematyczne

and Lwów: 1935.

Manuskripteingang: 07