On the Limit of some Diffusion-Reaction System with Small Parameter

LI

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Es wird ein Reaktions-Diffusionssystem mit kleinem Parameter a betrachtet, das einen PolykondensationaprozeB beschreibt, in dem die chemische Reaktion schneller als der Massentransport verläuft. Für $\epsilon \rightarrow 0$ ergibt sich eine nichtlineare Evolutionsgleichung vom Typ $v_i = A f(v)$.

Paccматривается система с малым параметром *в* с реакциями и диффузией и описывающая процесс поликонденсации, в котором химическая реакция протекает быстрее транспорта веществ. Для $\varepsilon \to 0$ получается нелинейное эволюционное уравнение типа Es wird ein Reaktions-Diffu

kondensationsprozeß beschr

transport verläuft. Für ε –
 $v_t = \Delta f(v)$.

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вающая процесс поликон;

транспорта веществ. Для
 $v_t^* = \Delta f(v)$.

A diffusion-reaction system

A diffusion-reaction system with small parameter ε is considered describing some process of polycondensation in which the chemical reactions are faster than the mass transport. For $\varepsilon \to 0$ results a nonlinear evolution equation like $v_i = \Delta f(v)$.

Let $G \subset \mathbb{R}^n$ be a bounded domain, $\partial G = D = \bigcup D_k$ its smooth boundary with the components $D_k: D_k \cap D_l = \emptyset$ ($k + l$). In this note we want to study the following Let $G \subseteq \mathbb{R}^n$ be a bounded domain, $\partial G = D = \bigcup_{i=1}^r D_k$ its a components $D_k: D_k \cap D_l = \emptyset$ $(k + l)$. In this note we was diffusion-reaction system for $x \in G$, $t \in (0, T] = S$, $\varepsilon > 0$: *e* and *e* and *e* and *e* and *examples in* ϵ c peakly and ϵ in ϵ in ϵ in ϵ ϵ is considered des
 n-reaction system with small parameter ε is co **extra 3** *evaluated domain,* $\cos^2 2x^2 + \cos^2 2x$ *evaluated domain,* $\cos^2 2x^2 + \cos^2 2x$ *extra 5.00.*
extra = f(v) - *u*, *Bu* = 0 on *D* × *S*,
etu = *u* - *f(v)* - *u*, *Bu* = 0 on *D* × *S*,
ev_t = *u* - *f(v)*

$$
E A u = f(v) - u, \qquad Bu = 0 \quad \text{on} \quad D \times S,
$$

\n
$$
v_t = u - f(v), \qquad v(x, 0) = v_0 = \text{const} \ge 0 \quad \text{on} \quad G
$$
 (1)

where *A* is an elliptic differential operator with a suitable boundary operator *B*.

A problem of this kind occured when we tried to reduce some model of a polycondensation process in the so-called transport-limited case (see *[7:* eq. (5.3)]). The main question in that case was the convergence of the spatial *L'-norm* of *v* at each time t because that implied the convergence of some other measurable quantity the average degree of polymerization. Let $G \subseteq \mathbb{R}^n$ be a bounded domain, $\partial G = D = \bigcup P_k$ its smoto
components $D_k : D_k \cap D_l = \emptyset$ $(k + l)$. In this note we want diffusion-reaction system for $x \in G$, $t \in (0, T] = S$, $\varepsilon > 0$:
 $\varepsilon A u = f(v) - u$, $Bu = 0$ on $D \times S$,
 $\varepsilon v_i =$ $\epsilon A u = f(v) - u$, $Bu = 0$ on $D \times S$,
 $\epsilon v_t = u - f(v)$, $v(x, 0) = v_0 = \text{const} \ge 0$ on G

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on process in the so-called transport-limited case (see [7: eq. (5.3)]).

Now we will show that under certain assumptions the solutions of (1) converge

$$
v_t + A f(v) = 0, \qquad v(x, 0) = v_0, \qquad u = f(v) \tag{2}
$$

in $L^2(G\times S)$. A corollary will answer the question mentioned above. But (1) may be interesting even from a broader point of view. The second equation can be transformed to

$$
v_t + A_t f(v) := v_t + A(I + \varepsilon A)^{-1} f(v) = 0.
$$
\n
$$
(3)
$$

Here, A_t is of course the Yosida approximation of A . Thus, our convergence problem is a special case of the general question of the convergence of this approximation with inclosed nonlinearity. **be** i
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Investigations of a similar kind in the case of accretive operators in $L¹$ can be found in [2] and [3]. Other interesting diffusion-reaction equations with small parameters, even with mixed concentration terms on the right side of the equations, and their relationship to some nonlinear limit equation were studied by L. *C. EVANS in[4]..*

We assume the following conditions to be satisfied:

- (F) $f: \mathbf{R} \to \mathbf{R}$ is an increasing function of class C^1 , f' is locally Lipschitz continuous, $f(a) = a$ for some $a \le 0$.
- (F) $f: \mathbf{R} \to \mathbf{R}$ is an increasing function of class C^1 , f' is locally I
nuous, $f(a) = a$, for some $a \le 0$.
(A) $\cdot Au = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n a_{ij} u_{x_j} + a_i u \right) + a_0 u$ on G ,
 $Bu = \delta_k \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} u$ $Bu = \delta_k \sum_{i=1}^n \left(\sum_{i=1}^n a_{ij} u_{x_i} + a_i u \right) v_i + b_0 u$ on D_k , $1 \leq k \leq r$, *a*₁ (i) $\begin{aligned}\n\begin{aligned}\n\text{a} &= \text{a} \quad \text{b} \quad \text{c} \quad \text{c} \quad \text{d} \$ $a_{ij} = a_{ji}$, $a_i \in C^{1+\mu}(\overline{G})$, $1 \leq i$, $j \leq n$, $a_0 \in C^{\mu}(\overline{G})$, $b_0 \in C^{1+\mu}(D)$;
there exists a $\varkappa > 0$ such that $\sum_{i=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \varkappa |\xi|^2$ for all $x \in G$ and $\xi \in \mathbb{R}^n$; $a_{ij} (= a_{ji}),$ $a_i \in C^{1+\mu}(\overline{G}),$ $1 \leq i, j \leq n,$ $a_0 \in C^{\mu}(\overline{G}),$ $b_0 \in C^{1+\mu}(D);$

there exists $a \times > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \kappa |\xi|^2$ for all $x \in G$ and $\xi \in \mathbb{R}^n;$
 $a_0 \geq 0, \text{ div } \overline{a} = 0, \delta_k \overline{a} \$

$$
\delta_k \in \{0, 1\}, \, \delta_k = 0 \text{ implies } b_0 = 1 \text{ on } D_k, \, b_0 = 0.
$$

Here and in the following an index *x* or *t* to some function will mean the partial derivative of this function with respect to the named variable.

The notations C^l , C^{l+a} , H^l describe the usual spaces (see e.g. [8]). $C^{a,b}(G \times H) \ni u$ means $u(\cdot, y) \in C^{a}(G), u(x, \cdot) \in C^{b}(H)$ ($x \in G, y \in H$). By $\langle \cdot, \cdot \rangle$ we will always describe means $u(\cdot, y) \in C^o(G)$, $u(x, \cdot) \in C^o(H)$ ($x \in G, y \in H$). By $\langle \cdot, \cdot \rangle$ we will always describe

a pure L^2 - scalar product, not distinguishing - if there can be no misunderstanding

- between the scalar products in $L^2(G$ - between the scalar products in $L^2(G)$, $L^2(S, L^2(G))$ or similar spaces. $(0, 1), b_k = 0$ implies b_0
 $(0, 1), \delta_k = 0$ implies b_0
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tations C^l , C^{l+a} , H^l desc,
 \blacksquare , y) $\in C^a(G)$, $u(x, \cdot) \in C^t$
 \blacksquare , y) $\in C^a(G)$, *G* 1. $G_0 = 1$ on D_k , $G_0 = 0$.
 G 1. G^{l+a} , H^l describe the named variable.
 G, $u(x, \cdot) \in C^b(H)$ ($x \in G$, $y \in H$). By $\langle \cdot, \cdot \rangle$ we will all

product, not distinguishing $-$ if there can be no miss

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Let us denote

$$
D'=\bigcup_{1}\{D_k\colon \delta_k\neq 0\},\qquad H=\{u\in H^1(G)\colon Bu=0\text{ on }D\setminus D'\}.
$$

Now we can identify the pair of operators (A, B) with the operator $A: H \to H^*$. given by

$$
\langle Au, v \rangle = \int_{G} \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_i} dx + \int_{G} a_0 uv dx + \int_{G} \sum_{i=1}^{n} a_i uv_{x_i} dx + \int_{D'} b_0 uv dx.
$$

It is known that A has a continuous and isotone inverse (see e.g. $[1, 8, 5]$). The same is true for the "symmetric part" A_s given by

(A,u, *v)* ⁼ *^f*Z *dx '+ f a0uv dx. Gi,j=l C*

Both, A and A_s are continuous operators themselves and strongly monotone. For A_s this is quiet clear, in the case of A it follows from the monotonicity of the "rest" operator" $A_r = A - A_s$:

be of this function with respect to the named v
btations
$$
C^l
$$
, C^{l+a} , H^l describe the usual spaces
 $(\cdot, y) \in C^o(G)$, $u(x, \cdot) \in C^b(H)$ ($x \in G, y \in H$). By
 z^2 - scalar product, not distinguishing - if there
then the scalar products in $L^2(G)$, $L^2(S, L^2(G))$ on
denote

$$
D' = \bigcup_i \{D_k : \delta_k \neq 0\}, \qquad H = \{u \in H^1(G): Bu
$$

can identify the pair of operators (A, B) wi
 $\langle Au, v \rangle = \int \sum_{i=1}^n a_{ij} u_{x_i} v_{x_i} dx + \int_G a_0 uv dx + \int_{G} \sum_{i=1}^n u_{i+1} u_{x_i} v_{x_i} dx$
with that A has a continuous and isotope inverso
or the "symmetric part" A_s given by
 $\langle A_s u, v \rangle = \int \sum_{i=1}^n a_{ij} u_{x_i} v_{x_i} dx + \int_G a_0 uv dx$.
and A_s are continuous operators themselves
s quite clear, in the case of A it follows from t
" $A_r = A - A_s$:
 $\langle A_r u, u \rangle = \frac{1}{2} \int_0^r \vec{a} \cdot \text{grad}(u^2) dx + \int_0^r b_0 u^2 dx$
 $= \frac{1}{2} \int_0^r u^2 (\vec{a} \cdot \vec{n} + 2b_0) dx \ge 0$.

Moreover, A_s defines some new scalar products and equivalent norms in H and H^* :

Diffusion Reaction System
\n
$$
A_s
$$
 defines some new scalar products and eq
\n
$$
(u, v)_H = \langle A_s' u, v \rangle, \qquad (g, h)_{H^*} = \langle A_s^{-1}g, h \rangle.
$$
\nuse these new norms throughout this paper.

We will use these new norms throughout this paper and note that thus A_s becomes the duality mapping between H and H^* . Further, we should remark that $A: L^2(G) \to H^*$ is a continuous operator.

To prove convergence of the solutions to (1) we have first to make sure that there is anything to speak about at all. The solvability of (1) was already stated in [7] in a somewhat less general form. Nevertheless, the idea of the proof remains unchanged.
Theorem 1: *Problem* (1) has for fixed ε in [7] in a somewhat less general form. Nevertheless, the idea of the proof remains Diffusion-Reaction System with Small Paramet

Moreover, A_s defines some new scalar products and equivalent norms in $(u, v)_H = \langle A_s u, v \rangle$, $(g, h)_{H^*} = \langle A_s^{-1}g, h \rangle$.

We will use these new norms throughout this paper and not

Theorem 1: *Problem* (1) has for fixed $\varepsilon > 0$ exactly one solution

$$
u \in C^{2+\mu,\mu/2}(\overline{G}\times \overline{S}), \qquad v \in C^{\mu,1+\mu/2}(\overline{G}\times \overline{S})
$$

and it holds $a \leq u \leq f(v_0), a \leq v \leq v_0$ *.*

Proof: (1) is equivalent to

olds
$$
a \le u \le f(v_0), a \le v \le v_0
$$
.
f: (1) is equivalent to

$$
u = (I + \varepsilon A)^{-1} f(v),
$$

$$
v_t = -\frac{1}{\varepsilon} (I - (I + \varepsilon A)^{-1}) f(v), \qquad v(0) = v_0.
$$

 $(I + \varepsilon A)^{-1}$ is continuous in $C^{\mu,1+\mu/2}(\overline{G}\times\overline{S})$ (see [1, 8, 5]), the same is true for f because of (F). So there exists a unique solution of the ordinary differential equation in *v,* local in time. To prove the global existence of this solution it suffices to show the maintained inequalities. The iterations $u \in C^{2+\mu,\mu/2}(\overline{G}\times \overline{S}), \quad v \in C^{\mu,1+\mu/2}(\overline{G}\times \overline{S})$
 $lds \ a \leq u \leq f(v_0), \ a \leq v \leq v_0.$
 $\colon (1) \text{ is equivalent to}$
 $u = (I + \varepsilon A)^{-1} f(v),$
 $v_t = -\frac{1}{\varepsilon} (I - (I + \varepsilon A)^{-1}) f(v), \quad v(0) = v_0.$
 \vdots
 $v_1 = \frac{1}{\varepsilon}$ is continuous in C^{μ $u = (I + \varepsilon A)^{-1} f(v),$
 $v_i = -\frac{1}{\varepsilon} (I - (I + \varepsilon A)^{-1}) f(v),$ $v(0) =$
 $(I + \varepsilon A)^{-1}$ is continuous in $C^{\mu,1+\mu/2}(\overline{G}\times \overline{S})$ (see [1, because of (F). So there exists a unique solution of the in v, local in time. To prove th

of (F). So there exists a unique solution of the ordinary diffal in time. To prove the global existence of this solution it
to be the dualities. The iterations

$$
u_j = (I + \epsilon A)^{-1} f(v_{j-1}),
$$

$$
v_{jl} = u_j - f(v_j), \qquad v_j(0) = v_0 \qquad (j \ge 1)
$$
with the maximum principle and the monotonicity of f in
the sequences

$$
a \leq \cdots \leq u_{j+1} \leq u_j \leq \cdots \leq f(v_0),
$$

$$
a' = \cdots = v_{j+1} = v_j = \cdots = v_0.
$$

together with the maximum principle and the monotonicity of f imply the monotonicity of the sequences

$$
a \leq \cdots \leq u_{j+1} \leq u_j \leq \cdots \leq f(v_0),
$$

$$
a = \cdots = v_{j+1} = v_j = \cdots = v_0.
$$

(To prove the validity of the bounds it suffices to note that $A/(v_0) \ge 0$ and $Aa \le 0$, $a = f(a)$ and $a \leq v_0$.) Dini's Theorem and the uniqueness of the solution now guarantee the convergence of these iterations to the solution of (1) and thus the asserted inequalities \blacksquare . micity of the sequences

micity of the sequences
 $a \leq \cdots \leq u_{j+1}$
 $a' = \cdots = v_{j+1}$

(To prove the validity
 $a = f(a)$ and $a \leq v_0$.) lantee the convergence

inequalities

Corollary: The sole

Corollary: The solutions \acute{u} , \dot{v} of (1) belong to $C^{2,1}(G \times S)$.

This fact follows from the regularity assertion of Theorem 1 and the theorem about solutions of ordinary differential equations depending on parameters \blacksquare

The following theorem will be the main result of this paper. It gives an answer to the question what happens to (1) if $\varepsilon \to 0$. To distinguish between solutions for different parameters we will keep *e* as an index to these solutions. Further, set $Q=G\times S.$ *u*, $u \leq v_0$, $D \text{ini}^s$ Theorem and the uniqueness of the solution not convergence of these iterations to the solution of (1) and thus the iss \blacksquare .

Lary: The solutions \dot{u} , v of (1) belong to $C^{2,1}(G \times S)$.

Theorem 2: For $\varepsilon \to 0$ the solutions $(u_{\varepsilon}, v_{\varepsilon})$ of (1) converge in the following sense:

$$
u_{\epsilon} \to f(v), \qquad v_{\epsilon} \to v \quad in \; L^2(Q),
$$

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where v is a function satisfying the following conditions

(1)
 (1) (1) (1) (1) (1) (1) (1) (1) (1) (1) (1) (2) (1) (484 (*ii***)** *(iii) (iii) v* \in *L*[∞](*Q*), $a \leq v$;

(*ii*) $v \in L^{\infty}(Q)$, $a \leq v$;

(*iii*) *v i H*¹(*S*, *H*^{*}), *f*(*v*) \in (*iii*) *v is the unique solution of*
 $v_t + A f(v) = 0$, $v(x, y)$ $f(v) \in L^2(S, H)$, *vg + A/(v)* α *i A/(v)* α *i i i i i y i i y i i y i y i y i i j i y i y i y i y i y i y i y i y i y i y i y i y i y i y*

with the properties (i), (ii) mentioned above.

Proof: The proof of this theorem will be given in steps $(a) - (t)$. (a) Theorem 1 and its corollary say that for every $\varepsilon > 0$ we have $u_{\varepsilon}, v_{\varepsilon} \in C^{2,1}(Q)$,

 $a \le u_i \le f(v_0), a \le v_i \le v_0$ in Q.
(b) For every $\varepsilon > 0$ it holds $f(v_i) \ge u_i$. Froof: The proof of this theorem will

(a) Theorem 1 and its corollary say the
 $\le u_t \le f(v_0), a \le v_t \le v_0$ in Q.

(b) For every $\varepsilon > 0$ it holds $f(v_\varepsilon) \ge u_t$:

tt $w = u_{tt}$, $z = v_{tt}$. (1) implies

Put $w = u_{\epsilon t}$, $z = v_{\epsilon t}$. (1) implies $Aw=f'(v_1)z-w,$

$$
v \in L^{\infty}(Q), \quad a \leq v \leq v_{0},
$$

\n
$$
v \in H^{1}(S, H^{*}), \quad f(v) \in L^{2}(S, H),
$$

\n
$$
v \text{ is the unique solution of}
$$

\n
$$
v_{t} + A f(v) = 0, \quad v(x, 0) = v_{0} \quad a.e. \text{ in } G
$$

\nwith the properties (i), (ii) mentioned above.
\n
$$
\therefore
$$
 The proof of this theorem will be given in steps (a) – (t).
\neorem 1 and its corollary say that for every $\varepsilon > 0$ we have $u_{\varepsilon}, v_{\varepsilon} \in C^{2,1}(Q),$
\n
$$
f(v_{0}), a \leq v_{\varepsilon} \leq v_{0} \text{ in } Q.
$$

\n
$$
v \text{ every } \varepsilon > 0 \text{ it holds } f(v_{\varepsilon}) \geq u_{\varepsilon}:
$$

\n
$$
u_{\varepsilon t}, z = v_{\varepsilon t}. \quad (1) \text{ implies}
$$

\n
$$
A w = f'(v_{\varepsilon}) z - w,
$$

\n
$$
z_{t} = w - f'(v_{\varepsilon}) z, \quad z(0) = \frac{1}{\varepsilon} \left(u_{\varepsilon}(0) - f(v_{0}) \right) \leq 0.
$$

\n
$$
v \geq v \text{ in every equation } v \geq 0 \quad (4)
$$

\n
$$
v \geq v_{\varepsilon} \quad (5)
$$

\n
$$
v \geq v_{\varepsilon} \quad (6)
$$

\n
$$
v \geq v_{\varepsilon} \quad (7)
$$

\n
$$
v \geq v_{\varepsilon} \quad (8)
$$

\n
$$
v \geq v_{\varepsilon} \quad (9)
$$

\n
$$
v \geq v \quad (1)
$$

\n
$$
v \ge
$$

This problem has (in the sense of "=" replaced by " \geq " in every equation) the supersolution $W = Z = 0$. For the iterated system

oblem has (in the sense of "=" replaced by "
$$
\ge
$$
" in every
ution $W = Z = 0$. For the iterated system

$$
Aw_n = f'(v_\epsilon) z_{n-1} - w_n,
$$

$$
z_{nt} = w_n - f'(v_\epsilon) z_n, \qquad z_n(0) = z(0), \qquad z_0 = 0 \qquad (n \ge 1)
$$
was with the usual mechanism $(R := (I + \epsilon A)^{-1}$ is an iso)

one shows with the usual mechanism $(R_i)=(I+\varepsilon A)^{-1}$ is an isotone operator!) that

$$
A w_n = f(v_\epsilon) z_{n-1} - w_n,
$$

\n
$$
z_{nt} = w_n - f'(v_\epsilon) z_n, \qquad z_n(0) = z(0),
$$

\none shows with the usual mechanism $(R_\epsilon :=$
\nthat
\n
$$
0 = W = w_1 \geq \cdots \geq w_n \geq w_{n+1} \geq \cdots
$$

\n
$$
0 = Z = z_0 \geq \cdots \geq z_n \geq z_{n+1} \geq \cdots
$$

\nOn the other hand, we have
\n
$$
w_n = R_\epsilon(f'(v_\epsilon) z_{n-1}), \qquad z_n = F(z_{n-1}),
$$

\nwhere F is a continuous homomorphism on $C(\epsilon)$

On the other hand, we have

$$
w_n = R_{\iota}(f'(v_{\iota}) z_{n-1}), \qquad z_n = F(z_{n-1}),
$$

where F is a continuous homomorphism on $C(G)$ given by

$$
z_{nt} = w_n - f'(v_{\epsilon}) z_n, \t z_n(0) = z(0), \t z_0 = 0 \t (n \ge 1)
$$

as with the usual mechanism $(R_{\epsilon} := (I + \epsilon A)^{-1}$ is an isot-
 $0 = W = w_1 \ge \dots \ge w_n \ge w_{n+1} \ge \dots$,
 $0 = Z = z_0 \ge \dots \ge z_n \ge z_{n+1} \ge \dots$
then hand, we have
 $w_n = R_{\epsilon}(f'(v_{\epsilon}) z_{n-1}), \t z_n = F(z_{n-1}),$
is a continuous homomorphism on $C(G)$ given by

$$
\frac{1}{\epsilon} \int_{0}^{t} f'(v_{\epsilon}) ds \left(\frac{1}{\epsilon} \int_{0}^{t} \frac{1}{\epsilon} \int_{0}^{t} f'(v_{\epsilon}) ds \cdot R_{\epsilon}(f'(v_{\epsilon}) y) dr - z(0) \right).
$$

to prove that F is a contraction in $C(G)$ for some norm eq.
This would give us the convergence of (w, z) to the solution

We want to prove that F is a contraction in $C(G)$ for some norm equivalent to the original. This would give us the convergence of (w_n, z_n) to the solution of (4) and thus the inequality $z \leq 0$, which in connection with (1) and the definition of *z* proves our statement. *o* a contraction in $C(G)$ for the convergence of (w_n)
thich in connection with (1)
the Lipschitz continuity of
 $\frac{c}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_{0}^{t} r(v_{\varepsilon})ds} \int_{0}^{t} e^{\frac{1}{\varepsilon}\int_{0}^{t} r(v_{\varepsilon})ds}$ ove that *F* is a
would give us tility $z \leq 0$, which λ), (F) and the I
 $-F(d)$ $||c \leq \frac{c}{\varepsilon}$ e e of (w_n, z_n) to the s

in with (1) and the det

inuity of R_{ϵ} we have
 $\frac{1}{e} \int_{0}^{t} f'(v_{\epsilon}) ds$
 $||b - d||_{c} dr$.

Because of (a), (F) and the Lipschitz continuity of R_{ϵ} we have

$$
||F(b) - F(d)||_C \leq \frac{c}{\varepsilon} e^{-\frac{1}{\varepsilon} \int_0^t f'(v_\varepsilon) ds} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^t f'(v_\varepsilon) ds} ||b - d||_C dr.
$$

Introducing the Bielecki norm

$$
||F(b) - F(d)||_C \leq \frac{c}{\varepsilon} e^{-\varepsilon} \int_0^{\varepsilon} \int_0^{\varepsilon} e^{\varepsilon} e^{-\varepsilon} \int_0^{\varepsilon} e^{-\varepsilon} e^{-\varepsilon} dx
$$

Introducing the Bielecki norm

$$
||y||_r = \sup \{e^{-\gamma t} ||y(t)||_C : t \in S\}
$$

we get
$$
||F(b) - F(d)||_r \leq \tilde{c} ||b - d||_r, \tilde{c} < 1 \text{ if } \gamma > \frac{c}{\varepsilon}
$$

Diffusion-Reaction System with Small Parameter *⁴⁸⁵*

(c) $u_{\epsilon} \rightarrow u$, $v_{\epsilon} \rightarrow v$, $f(v_{\epsilon}) \rightarrow z$ at least for some subsequences in the reflexive Banach space $L^2(Q)$ because of (a). (Here and in the following we will replace the whole (generalized) sequences by their converging parts not changing our notations.) Diffusion Reaction System with

(c) $u_t \rightharpoonup u, v_t \rightharpoonup v, f(v_t) \rightharpoonup z$ at least for some subsequences

ace $L^2(Q)$ because of (a). (Here and in the following we

eneralized) sequences by their converging parts not chang

(

 (d) $||f(v_{\epsilon}) - u_{\epsilon}||_{L^{1}(\mathcal{Q})} \rightarrow 0$: By (a), (b) and the second equation of (1) this sequence is bounded by ε const and this expression tends to zero if $\varepsilon \to 0$.

(e) $||f(v_{s}) - u_{s}||_{L^{1}(\Omega)} \to 0$:

This is true because $f(v_{\epsilon})$ and u_{ϵ} belong to $L^{\infty}(Q)$ ((a)) and (d) holds.

(f) From the above statements we can now deduce $u = z$ a.e. in Q .

(g) $\{u_{\ell}\}\$ is bounded in $L^2(S, H)$:
Comparing the two equations of (1) we get $Au_{\ell} = -v_{\ell\ell} \geq 0$ ((b)) and because of (a) $\{Au_{\ell}\}\$ is bounded in $L^1(Q)$. Thus, shows statements we can now deduce $u = 3$
unded in $L^2(S, H)$:
two equations of (1) we get $Au_t = -v_{tt} \ge$
ed in $L^1(Q)$. Thus,
 $||Au_t||_{L^1(Q)} ||u_t||_{L^{\infty}(Q)} \ge \langle Au_t, u_t \rangle \ge c_2 ||u_t||_{L^1(S, H)}^2$.
use of the monotonicity of A in the new

Here we made use of the monotonicity of *A* in the new norm.

(h) $u_{\epsilon} \rightarrow u$ in $L^2(S, H)$ at least for some subsequence:

Because of (g) $\{u_i\}$ contains a weakly converging subsequence in this space. Because of (c) the limit of this subsequence must be u .

(j) $v_{\text{et}} \rightarrow v_t$ in $L^2(S, H^*)$ at least for some subsequence:

 $\{u_i\}$ is bounded in $L^2(S, H)$ and *A* is continuous. So $v_{ii} = -Au_i$ is bounded in $L^2(S, H^*)$. Now we can repeat the argument used in (h) and recall the definition This is true because $f(v_\epsilon)$ and u_ϵ belong to $L^\infty(Q)$

(f) From the above statements we can now dec

(g) $\{u_\epsilon\}$ is bounded in $L^2(S, H)$:

Comparing the two equations of (1) we get $Au_\epsilon =$
 $\{Au_\epsilon\}$ is bounded in L *(k)* ${u_{\ell}}$ is bounded in $L^2(S, H^*)$: *(i)* $v_{tt} \rightarrow v_t$ in $L^2(S, H^*)$ at least for some subsequence:
 $\{u_\epsilon\}$ is bounded in $L^2(S, H)$ and *A* is continuous. So $v_{tt} = -Au_t$, is bounded $L^2(S, H^*)$. Now we can repeat the argument used in (h) and recall the def

As before we have for $w_{\epsilon} = u_{\epsilon t}$: $Aw_{\epsilon} + w_{\epsilon} = f'(v_{\epsilon}) v_{\epsilon t} = i t_{\epsilon}$. At first, we will show that $\{f_i\}$ is bounded in $L^2(S, H^*)$. It follows from (j) that $\{v_{\epsilon i}\}\$ is bounded in this space. For an arbitrary $\varphi \in L^2(S, H)$ we have $|\varphi| \in L^2(S, H)$ and because of (b) is bounded in $L^2(S, H^*)$:

we have for $w_{\epsilon} = u_{\epsilon t}$: $Aw_{\epsilon} + w_{\epsilon} = f'(v_{\epsilon}) v_{\epsilon t} =: f_{\epsilon}$. At first we will show

s bounded in $L^2(S, H^*)$. It follows from (j) that $\{v_{\epsilon t}\}$ is bounded in this

r an arbitrary $\varphi \in L^2(S$

Here we used once more condition (F) . Now we have

 $\varepsilon ||w_{\varepsilon}||_{L^1(S,H)}^2 \leq c_4 ||f_{\varepsilon}||_{L^1(S,H^*)} ||w_{\varepsilon}||_{L^1(S,H)}$

because of the strong monotonicity of *A.* The continuity of *A* on the other hand gives $\|w_{\epsilon}\|_{L^{q}(S,H^{s})} \leq c_{4} \|f_{\epsilon}\|_{L^{q}(S,H^{s})} \|w_{\epsilon}\|_{L^{q}(S,H)}$
 f the strong monotonicity of *A*. The continuity of *A* on the other
 $w_{\epsilon}\|_{L^{q}(S,H^{s})} \leq \epsilon \|Av_{\epsilon}\|_{L^{q}(S,H^{s})} + \|f_{\epsilon}\|_{L^{q}(S,H^{s})} \leq \epsilon c_{5} \|w_{\epsilon}\|_{L^{q}(S,H)} + \|$

and these two estimates taken together give the claimed result.

(I) From (g) and (k) it follows that $\{u_{\epsilon}\}\$ is compact in $L^2(Q)$ (see e.g. [9: Theorem 5.1]). $||w_{\epsilon}||_{L^{1}(S,H^{*})} \leq \epsilon ||Aw_{\epsilon}||_{L^{1}(S,H^{*})} + ||f_{\epsilon}||_{L^{1}(S,H^{*})} \leq \epsilon c_{5} ||w_{\epsilon}||_{L^{1}(S,H)} + ||f_{\epsilon}||_{L^{1}(S,H^{*})}$

(d) these two estimates taken together give the claimed result.

(l) From (g) and (k) it follows that $\{u_{\epsilon}\}$ i

sequences).

(n) $u, v \in L^{\infty}(Q), a \leq u \leq f(v_0), a \leq v \leq v_0$ a.e. in Q :

We know already that *u* and *v* are weak limits in $L^2(Q)$ of some sequences which satisfy the given inequalities. Then there exist some subsequences and some convex combinations of them which converge in the strong sense (see e.g. [10: Theorem I, 1.1.8]), but this implies the convergence of some new subsequences a.e. in Q . Of course, *u* and *v* as their limits will satisfy the same inequalities. $||w_i||_{L^1(S,H^*)} \leq \varepsilon ||\varepsilon$
and these two estimates t
(1) From (g) and (k) it
5.1]).
(m) This means the c
sequences).
(n) $u, v \in L^{\infty}(Q), a \leq u$
We know already that u
satisfy the given inequali
combinations of them wh
1. $\frac{d}{dx}$ and v are weak littles. Then there exist
ties. Then there exists the converge in the
the convergence of
mits will satisfy the
 $\langle Q \rangle$ ($\xi > 0$) and b
 $-\frac{d}{dx}$
 $-\frac{1}{x}$
 $\frac{d}{dx}$

(o) $u = f(v)$ a.e. in Q: $x=v-\xi(f(v)-u) \in L^{\infty}(Q)$ ($\xi>0$) and because of the monotonicity of $f((F))$

$$
\xi \langle u - f(x), f(v) - u \rangle = \lim_{\epsilon \to 0} \langle f(v_{\epsilon}) - f(x), v_{\epsilon} - x \rangle \geq 0.
$$

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If $\xi \to 0$, then this can be true only for $u = f(v)$.

(p) $v_{\epsilon} \rightarrow v = f^{-1}(u)$ in $L^2(Q)$ at least for some subsequence:

(p) $v_e \rightarrow v = f^{-1}(u)$ in $L^2(Q)$ at least for some subsequence:

Because of (F) there exists f^{-1} and is a continuous function. Our hypothesis now

follows from (m) and (o) by Lebesgue's Theorem.
 $\lq q$, $v_t + Af(v) = 0$ (as a follows from (m) and (o) by Lebesgue's Theorem. **H. GAJEWSKI and H.-D. SPARING**
 $\ddot{x} \rightarrow 0$, then this can be true only for $u = f(v)$.

p) $v_e \rightarrow v = f^{-1}(u)$ in $L^2(Q)$ at least for some subseq

cause of (F) there exists f^{-1} and is a continuous f

lows from (m) and (o) b

(q) $v_t + Af(v) = 0$ (as an equation in $L^2(S, H^*))$:

The statements (j) and (h) together with (o) ensure that the term on the left is well defined. Further, for an arbitrary $\varphi \in L^2(S, H)$ we have

$$
\langle v_t + A f(v), \varphi \rangle = \lim_{\epsilon \to 0} \langle v_{\epsilon t} + A u_{\epsilon}, \varphi \rangle = 0.
$$

The proof of this fact amounts to a repetition of the arguments of (c) and (n) applied to $v(0, \cdot)$ instead of v .

(s) Uniqueness of the solution: (s) Uniqueness of the solution:

Let v and w be two solutions of the equation given in (q) which satisfy (i) and (ii),

set $z = v - w$. For $t \in S$ put $S_t = (0, t]$, $Q_t = G \times S_t$. Because of (F) we have
 $\langle f(v) - f(w), z \rangle_{L^1(Q_t)} \geq M ||$ set $z = v - w$. For $t \in S$ put $S_t = (0, t]$, $Q_t = G \times S_t$. Because of (F) we have *Arther, for an arbitrary* $\varphi \in L^2(S, H)$ *we have that to term on the left is well*
defined. Further, for an arbitrary $\varphi \in L^2(S, H)$ we have
 $\langle v_t + A/(v), \varphi \rangle = \lim_{\epsilon \to 0} \langle v_{\epsilon t} + A u_{\epsilon}, \varphi \rangle = 0$.
The proof of this fact amoun

$$
\langle f(v)-f(w),z\rangle_{L^1(Q_i)}\geqq M\,\|f(v)-f(w)\|_{L^1(Q_i)}^2
$$

where $1/M$ is the Lipschitz constant of *f* in the interval $[a, v_0]$. The continuity of $A_r: L^2 \to H^*$ on the other hand gives
 $||A_r(f(v) - f(w)||_{L^1(S_t, H^*)} \le c_6||f(v) - f(w)||_{L^1(Q_t)}$. set $z = v - w$. For
 $\langle f(v) - f(u) \rangle$

where $1/M$ is the I
 $A_t: L^2 \to H^*$ on the
 $||A_t(f(v)) -$

Thus, we get
 $0 = (z_t +$

$$
||A_r(f(v) - f(w))||_{L^1(S_t, H^*)} \leq c_6||f(v) - f(w)||_{L^1(Q_t)}
$$

the set

 $\frac{1}{2}$

$$
0 = (z_t + A(f(v) - f(w)), z)_{L^1(S_t, H^*)}
$$

\n
$$
= \frac{1}{2} ||z(t)||_{H^*}^2 + \langle A_s^{-1}(A_s + A_r) (f(v) - f(w)), z \rangle
$$

\n
$$
= \frac{1}{2} ||z(t)||_{H^*}^2 + \langle f(v) - f(w), z \rangle + \langle A_r(f(v) - f(w)), z \rangle_{L^1(S_t, H^*)}
$$

\n
$$
\geq \frac{1}{2} ||z(t)||_H^2 + M ||f(v) - f(w)||_{L^1(Q_t)}^2 - c(\sigma) ||z||_{L^1(S_t, H^*)}^2
$$

\n
$$
- \sigma c_6^2 ||f(v) - f(w)||_{L^1(Q_t)}^2.
$$

\ncan now choose σ sufficiently small to get $M \geq \sigma c_6^2$. This implies
\n
$$
2c(\sigma) \int_0^t ||z(r)||_{H^*}^2 dr \geq ||z(t)||_{H^*}^2,
$$

\n
$$
0
$$

\nuation which by Gronwall's Lemma leads to $z = 0$. This ends the

We can now choose σ sufficiently small to get $M \geq \sigma c_6^2$. This implies

Now choose
$$
\sigma
$$
 sufficiently small
\n
$$
2c(\sigma) \int_{0}^{t} ||z(r)||_{H^{\bullet}}^{2} dr \geq ||z(t)||_{H^{\bullet}}^{2},
$$

a situation which by Gronwall's Lemma leads to $z = 0$. This ends the proof of the uniqieness statement.

(t) The uniqueness of the solution now guarantees a posteriori the convergence of all those sequences for which we had upto now only shown the convergence of certain subsequences. This completes the proof of Theorem 2 **^I**

Cor'ollary: The convergence $v_i(t) \rightharpoonup v(t)$ in $L^2(G)$ is true for all t of S.

Proof: By (c) and (j) in the proof of the preceding theorem it holds $v_k \rightarrow v$ in $H^1(S, H^*)\subset C(S, H^*)$. Moreover, $||v(t)-v_{\epsilon}(t)||_{L^1(G)} \leq$ const ((a), (n)). Now, for an arbitrary $h \in L^2(G)$ there exists a sequence $\{h_n\} \subset H$, $h_n \to h$ in $L^2(G)$ and for fixed

 $t \in S$ we have

$$
\langle v_{\epsilon}(t) - v(t), h \rangle = \langle v_{\epsilon}(t) - v(t), h - h_n \rangle + \langle v_{\epsilon}(t) - v(t), h_n \rangle
$$

$$
\leqq c_7 \, ||h-h_n||_{L^1(G)} + \langle v_{\epsilon}(t) - v(t), h_n \rangle.
$$

We can now choose n to make the first expression on the right side smaller than a given bound and then choose ϵ to do the same with the second term **I**

Remark 1: Because of (a) and (n) the definition of f is important only in the interval $[a, v_0]$. This is true in the case of Theorem 1, too. Thus, we can weaken Diffusion-Reaction System with Small Parameter 487
 $\ell \in S$ we have
 $\langle v_t(l) - v(l), h \rangle = \langle v_t(l) - v(l), h - h_n \rangle + \langle v_t(l) - v(l), h_n \rangle$
 $\leq c_7 ||h - h_n||_{L^1(C)} + \langle v_t(l) - v(l), h_n \rangle$.

We can now choose *n* to make the first expression on the right s

Remark 2: If f is only a nondecreasing function which satisfies the other conditions of (F) , then the proof of Theorem 2 remains valid with the exception of step. (p). In that case we can only prove the weak convergence of v_{ϵ} in $L^2(Q)$.

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