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On an Evolution Equation for a Non-Hypoelliptic Linear Partial Differential Operator from Stochastics

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Neulich hat E. B. Dynkin [2] einen nichthypoelliptischen linearen partiellen Differentialoperator von gerader Ordnung (mit konstanten Koeffizienten) eingeführt und untersucht, der aus der Theorie der mehrparametrigen stochastischen Prozesse entstanden ist. Von diesen Betrachtungen von Dynkin angeregt, haben die Verfasser in der Abhandlung [1] schon ein verallgemeinertes Dirichlet-Problem für diesen Differentialoperator gelöst. Unser Ziel in der vorliegenden Arbeit ist, das Cauchy-Problem für die entsprechende Evolutionsgleichung (in der Zeitveränderlichen von erster Ordnung) zu untersuchen; ein solches Cauchy-Problem könnte Anwendungen auf Fragen der Stochastik haben.

Недавно Е. Б. Дынкин [2] ввел и исследовал негиперболический линейный дифференциальный оператор четного порядка с постоянными коэффициентами, возникшийся в теории многопараметрических стохастических процессов. Возбуждены рассуждениями Е. Б. Дынкина авторы ранее решили в [1] для этого оператора обобщенную задачу Дирихле. Цель настоящей работы — исследовать задачу Коши для соответствующего эволюционного уравнения, первого порядка относительно временной переменной. Такая задача Коши мог бы иметь применения к вопросам стохастики.

Recently E. B. Dynkin [2] introduced and studied a non-hypoelliptic linear partial differential operator of even order (with constant coefficients) which originates from the theory of multiparametric stochastic processes. Motivated by the considerations of Dynkin the authors have solved a generalized Dirichlet problem for this differential operator in their work [1]. Our aim in the present paper is to investigate the Cauchy problem for the corresponding evolution equation (in the time variable of first order); such a Cauchy problem could have applications to some questions from the stochastics.

1. Introduction

1.1 We consider in a bounded open set $G \subset \mathbf{R}^n$ (which satisfies some conditions, cf. Property P in 1.2) the non-hypoelliptic linear partial differential operator $L(D)$ of order $2k$ ($k \in \mathrm{N}$, $k \leqq n$) introduced by E. B. Dynkin [2]; the exact definition of $L(D)$ will be given below in 1.3. In this paper we want to investigate an initial-boundary value problem, which we call the Cauchy problem, for the "abstract" evolution equation

$$\frac{d}{dt}\, u(t) + L^\sim u(t) = f(t)$$

where $L^\sim$ is the closure of the differential operator $L(D)$ in $L^2(G)$, f a given function defined on the positive time axis with values in $L^2(G)$ and the solutions $u(t)$ are searched among functions defined on the positive time axis with values in a certain subspace of $L^2(G)$ which coincides with the domain $D(L^\sim)$ of $L^\sim$.

1.2 Let us first recall some notions, notations and results from our earlier paper [1]. Let G be a set in $\mathbf{R}^n$ with

Property P: *The bounded open set $G \subset \mathbf{R}^n$ is the Cartesian product*

$$G = G_1 \times \cdots \times G_k$$

of bounded open sets $G_j \subset \mathbf{R}^{m_j}$ $(1 \leqq j \leqq k)$, $m_1 + \cdots + m_k = n$, $\mathbf{R}^n = \mathbf{R}^{m_1} \times \cdots \times \mathbf{R}^{m_k}$, with sufficiently smooth boundaries ∂G_j (if the boundaries ∂G_j are of class C^∞ all our considerations surely will be valid).

We write

$$\partial_j G := G_1 \times \cdots \times G_{j-1} \times \partial G_j \times G_{j+1} \times \cdots \times G_k .$$

By $\mathbf{N}_0{}^n$ we denote, as usual, the set of all ordered systems of n nonnegative integers (multi-indices). For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0{}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}_0{}^n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha \leqq \beta$ if $\alpha_l \leqq \beta_l$ for $1 \leqq l \leqq n$.

With $n, k, m_j \in \mathbf{N}$ as in Property P we put

$$l_1 = 0, \quad l_j := \sum_{l=1}^{j-1} m_l \qquad (2 \leqq j \leqq k).$$

For each j $(1 \leqq j \leqq k)$ we define m_j multi-indices $\varepsilon_{t_j} \in \mathbf{N}_0{}^n$ with $|\varepsilon_{t_j}| = 1$ each having its only nonvanishing coordinate in the t_j-th position, $l_j + 1 \leqq t_j \leqq l_j + m_j$. We introduce the set

$$\Gamma := \left\{ \alpha \in \mathbf{N}_0{}^n \mid \alpha = \sum_{j=1}^{k} \varepsilon_{t_j} \right\}.$$

Note that Γ has $m_1 \ldots m_k$ elements. Further, we write

$$\Gamma^j := \left\{ \gamma \in \mathbf{N}_0{}^n \mid \gamma = \sum_{\substack{l=1 \\ l \neq j}}^{k} \varepsilon_{t_l} \right\} \qquad (1 \leqq j \leqq k).$$

1.3 Now we can introduce the differential operator

$$L(D) := \sum_{\alpha \in \Gamma} D^{2\alpha}$$

where we use the abbreviation

$$D^\alpha = D_1{}^{\alpha_1} \ldots D_n{}^{\alpha_n} \quad \text{with} \quad D_j := -\sqrt{-1}\, \frac{\partial}{\partial x_j}.$$

This operator has also the expression

$$L(D) = \Delta_1 \ldots \Delta_k$$

where

$$\Delta_j = \sum_{s=1}^{m_j} D_{l_j+s}^2 \qquad (1 \leqq j \leqq k)$$

denotes the Laplace operator which acts on functions defined on subsets of $\mathbf{R}^{m_j} (\subset \mathbf{R}^n)$. As we have shown in [1] the operator $L(D)$ is not hypoelliptic.

1.4 Let $C^k(G)$, $k \in \mathbf{N}_0$, be the linear space of all complex valued functions u which are k times continuously differentiable in G. By $C_0{}^k(G)$ we denote the space of all functions $u \in C^k(G)$ each having a compact support in G. We write also $C_0{}^\infty(G) = \bigcap_{k \in \mathbf{N}_0} C_0{}^k(G)$. Further let be

$$C_*{}^k(G) := \{u \in C^k(G) \mid D^\alpha u \in L^2(G),\ \alpha \in \mathbf{N}_0{}^n,\ |\alpha| \leqq k\}.$$

In $C_*^k(G)$ (with K from 1.2) we associate with the operator $L(D)$ the sesquilinear form

$$B(u, v) := \sum_{\alpha \in \Gamma} \int_G \overline{D^\alpha u(x)}\, D^\alpha v(x)\, dx. \tag{1.1}$$

We put

$$(u, v)_\Gamma := B(u, v) + (u, v)_0 \tag{1.2}$$

where $(.,.)_0$ denotes the scalar product in $L^2(G)$, $(u, v)_0 = \int_G \overline{u(x)}\, v(x)\, dx$. Thus, we have a scalar product $(.,.)_\Gamma$ on $C_*^k(G)$ with the corresponding norm $\|.\|_\Gamma$. We denote the completion of $C_*^k(G)$ with respect to the scalar product (1.2) by $H^\Gamma(G)$. For the closure of $C_0^\infty(G)$ in $H^\Gamma(G)$ we write $H_0^\Gamma(G)$.

On the space $H_0^\Gamma(G)$ the sesquilinear form (1.1) defines a scalar product equivalent to (1.2), in particular one has the inequalities

$$B(u, u) \leq \|u\|_\Gamma^2 \leq c^* B(u, u) \quad \text{for all } u \in H_0^\Gamma(G) \tag{1.3}$$

with a constant c^* (cf. [1: Lemma 2]). The norm induced by $B(.,.)$ on $H_0^\Gamma(G)$ will be denoted by $|||\cdot|||_\Gamma$.

The elements of $H_0^\Gamma(G)$ can be interpreted as functions with generalized homogeneous boundary data: Namely, for each $u \in H_0^\Gamma(G) \cap C^{k-1}(\overline{G})$ the relation

$$D^\beta u|_{\partial G} = 0 \tag{1.4}$$

holds for all $\beta \in \mathbb{N}_0^n$ with $\beta \leq \gamma$ for some $\gamma \in \Gamma^j$ $(1 \leq j \leq k)$ (we proved this assertion in [1: Theorem 4], cf. also [4: p. 28]).

Further for every $u \in H_0^\Gamma(G)$ the strong L^2-derivative $D^\tau u$ exists for all $\tau \in \mathbb{N}_0^n$ with $\tau \leq \sigma$ for some $\sigma \in \Gamma$ (the corresponding assertion for $u \in H^\Gamma(G)$ is not valid; cf. [1: Lemma 1], and also [5: Theorem 1]).

Because the relation (1.4) is valid for functions $u \in H_0^\Gamma(G) \cap C^{k-1}(\overline{G})$ one can apply partial integration for functions from the set $X := H_0^\Gamma(G) \cap C^\infty(G)$ and gets

$$\int_G \overline{D^\alpha u(x)}\, D^\alpha v(x)\, dx = \int_G \overline{D^{2\alpha} u(x)}\, v(x)\, dx \tag{1.5}$$

for all $u, v \in X$ and all $\alpha \in \Gamma$ (cf. [1: Lemma 5]).

1.5 In the Hilbert space $L^2(G)$ we associate to the differential operator $L(D)$ a linear operator L by

$$D(L) := X\big(\subset L^2(G)\big),$$
$$Lu := L(D)\, u \quad \text{for all } u \in X.$$

This operator is densely defined and has an adjoint operator L^*, and by partial integration one sees that $D(L^*) \supset X$ and that L^* is also densely defined. The operator L is therefore closable with the closure (smallest closed extension) $L^\sim := L^{**}$. On the other hand, we have on X a scalar product defined by

$$(u, v)_{(k)} := \big(L(D)\, u, L(D)\, v\big)_0 + (u, v)_0. \tag{1.6}$$

The completion of X with respect to the scalar product is a Hilbert space $H^{(k)}(G)$.

In [1: Theorem 6] we proved that the domain of the closure $L^\sim$ of L in $L^2(G)$ coincides with the Hilbert space $H^{(k)}(G)$, $D(L^\sim) = H^{(k)}(G)$. On $H^{(k)}(G)$ we have hence the scalar product $(u, v)_{(k)} = (L^\sim u, L^\sim v)_0 + (u, v)_0$. Partial integration in (1.5) yields

$$B(u, \phi) = \big(L(D)\, u, \phi\big)_0 \tag{1.7}$$

for all $u \in X$ and $\phi \in C_0^\infty(G)$ and since (1.7) depends continuously on u in the topology defined by (1.6) we have further

$$B(w, \phi) = (L^\sim w, \phi)_0 \tag{1.8}$$

for all $w \in H^{(k)}(G)$ and $\phi \in C_0^\infty(G)$.

In [1] we proved that the elements of $H^{(k)}(G)$ have the same boundary behaviour as the elements of $H_0^r(G)$, i.e.,

$$H^{(k)}(G) \cap H_0^r(G) = H^{(k)}(G). \tag{1.9}$$

1.6 In [1] we proved the existence and uniqueness of the solution of a generalized Dirichlet problem. We formulate the results for homogeneous boundary data:

For a given $f \in L^2(G)$ there exists a unique element $u \in H_0^r(G)$ such that

$$B(u, \phi) = (f, \phi)_0 \tag{1.10}$$

holds for all $\phi \in C_0^\infty(G)$.

From the regularity result of [1: Theorem 9] it follows that the solution u of (1.10) lies in $H^{(k)}(G)$.

2. The resolvent of $L^\sim$

2.1 We first derive a priori estimates for the operator $L^\sim$. Take a finite open interval $J := \{t \in \mathbf{R} \mid -T < t < T\}$ and let G be a bounded open set in $\mathbf{R}^n$ with Property P. The set of all functions which obey the condition

$$\omega(x, .) \in C_0^\infty(J) \quad \text{for each } x \in G,$$
$$\omega(., t) \in X \quad \text{for each } t \in J$$

will be denoted by $\mathscr{C}(G \times J)$. On this set we introduce the norm

$$\|\omega\|^2_{(k,2),G\times J} := \int_{-T}^{T} \int_G \{|L(D)\,\omega(x, t)|^2 + |\partial_t^2\omega(x, t)|^2 + |\omega(x, t)|^2\}\, dt\, dx \tag{2.1}$$

where we use the notation ∂_t for the differential operator $\partial/\partial t$.

Lemma 1: *The estimate*

$$\|\{L(D) + e^{i\theta}\,\partial_t^2\}\,\omega\|^2_{0,G\times J} \geqq \|\omega\|^2_{(k,2),G\times J} - \|\omega\|^2_{0,G\times J}$$

holds for each $\theta \in \mathbf{R}$ with $\pi/2 \leqq \theta \leqq 3\pi/2$ and for all $\omega \in \mathscr{C}(G \times J)$.

Proof: For $\omega \in \mathscr{C}(G \times J)$ we have

$$\|\{L(D) + e^{i\theta}\,\partial_t^2\}\,\omega\|^2_{0,G\times J} = \int_{-T}^{T} \int_G |\{L(D) + e^{i\theta}\,\partial_t^2\}\,\omega(x, t)|^2\, dt\, dx$$

$$= \int_{-T}^{T} \int_G \{|L(D)\,\omega(x, t)|^2 + |\partial_t^2\omega(x, t)|^2\}\, dt\, dx + e^{i\theta}\varkappa + e^{-i\theta}\bar{\varkappa}$$

$$= \|\omega\|^2_{(k,2),G\times J} - \|\omega\|^2_{0,G\times J} + 2\,\mathrm{Re}\,(e^{i\theta}\varkappa) \tag{2.2}$$

with

$$\varkappa := \int_{-T}^{T} \int_G \overline{L(D)\,\omega(x, t)}\; \partial_t^2\omega(x, t)\, dt\, dx.$$

By partial integration one gets (note that the boundary terms vanish)

$$\varkappa = \int_{-T}^{T} \int_{G} \sum_{\alpha \in \Gamma} \overline{D_x^{2\alpha}\omega(x, t)}\, \partial_t^2\omega(x, t)\, dt\, dx$$

$$= \int_{-T}^{T} \int_{G} \sum_{\alpha \in \Gamma} \overline{D_x^{\alpha}\omega(x, t)}\, D_x^{\alpha}\, \partial_t^2\omega(x, t)\, dt\, dx$$

$$= -\int_{-T}^{T} \int_{G} \sum_{\alpha \in \Gamma} |D_x^{\alpha}\, \partial_t\omega(x, t)|^2\, dt\, dx.$$

Thus one has $\varkappa \in \mathbf{R}$, $\varkappa \leqq 0$. We get therefore by the assumption $2\,\mathrm{Re}\,(e^{i\theta}\varkappa) = 2\varkappa \cos\theta \geqq 0$, and (2.2) gives the desired result ∎

Lemma 2: *Let G be a bounded open set with Property P. To the operator $L^{\sim}$ there exist two real constants $c^* > 0$ and $\lambda_0 > 0$ such that*[1])

$$c^* \,\|(L^{\sim} - \lambda I)\, w\|_{0,G} \geqq (1 + |\lambda|)\, \|w\|_{0,G}$$

for all $w \in H^{(k)}(G)$ and for all $\lambda \in \mathbf{C}$ with $|\lambda| \geqq \lambda_0$ and $\mathrm{Re}\,\lambda \leqq 0$.

Proof: We choose a real valued function $\varrho \in C_0^{\infty}(\mathbf{R})$ satisfying the conditions

$$\varrho(t) = \begin{cases} 1 & \text{for } |t| \leqq 1 \\ 0 & \text{for } |t| \geqq 2 \end{cases}$$

and $0 \leqq \varrho(t) \leqq 1$ elsewhere. For an arbitrary function $u \in X$ and an arbitrary $\mu \in \mathbf{R}$ we define the function ω by

$$\omega(x, t) := u(x)\, \varrho(t)\, e^{i\mu t}.$$

Now let us take $T > 2$. Then one has $\omega \in \mathscr{C}(G \times J)$, and we get by Lemma 1 for $\theta \in \mathbf{R}$, $\pi/2 \leqq \theta \leqq 3\pi/2$,

$$\|\omega\|^2_{(k,2),G\times J} \leqq \|\{L(D) + e^{i\theta}\, \partial_t^2\}\, \omega\|^2_{0,G\times J} + \|\omega\|^2_{0,G\times J}. \tag{2.3}$$

We estimate now the first term on the right hand side of (2.3):

$$\|\{L(D) + e^{i\theta}\, \partial_t^2\}\, \omega\|^2_{0,G\times J}$$

$$= \int_{-T}^{T} \int_{G} |L(D)\, \omega(x, t) + e^{i\theta}u(x)\, e^{i\mu t}\{\varrho''(t) + 2i\mu\varrho'(t) - \mu^2\varrho(t)\}|^2\, dt\, dx$$

$$\leqq 2 \int_{-T}^{T} \int_{G} \{|\varrho(t)|^2\, |L(D)\, u(x) - \mu^2\, e^{i\theta}u(x)|^2 + |u(x)|^2\, |\varrho''(t) + 2i\mu\varrho'(t)|^2\, dt\, dx$$

$$\leqq C_1 \int_{G} |L(D)\, u(x) - \mu^2\, e^{i\theta}u(x)|^2\, dx + C_2 \int_{G} |u(x)|^2\, dx$$

$$= C_1\, \|\{L(D) - \mu^2\, e^{i\theta}\}\, u\|^2_{0,G} + C_2\, \|u\|^2_{0,G} \tag{2.4}$$

with two positive constants C_1 and C_2 (independent of u and T).

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[1]) By I we denote the identity operator on $L^2(G)$.

The second term on the right hand side of (2.3) satisfies the estimate

$$\|\omega\|_{0,G\times J}^2 = \int_{-T}^{T} \int_G |u(x) \varrho(t) e^{i\mu t}|^2 \, dt \, dx \leq C_3 \|u\|_{0,G}^2 \tag{2.5}$$

with a third positive constant C_3.

By (2.1) we receive for the left hand term of (2.3)

$$\begin{aligned} \|\omega\|_{(k,2),G\times J}^2 &\geq \int_{-1}^{1} \int_G \{|L(D)\,\omega(x,t)|^2 + |\partial_t^2 \omega(x,t)|^2 + |\omega(x,t)|^2\} \, dt \, dx \\ &= 2\,\|L(D)\,u\|_{0,G}^2 + 2\mu^4 \|u\|_{0,G}^2 + 2\,\|u\|_{0,G}^2 \\ &\geq (2\mu^4 + 2)\,\|u\|_{0,G}^2. \end{aligned} \tag{2.6}$$

From (2.3) we get by the estimates (2.4)—(2.6)

$$(2\mu^4 + 2 - C_2 - C_3)\,\|u\|_{0,G}^2 \leq C_1' \,\|\{L(D) - \mu^4 e^{i\theta}\}\,u\|_{0,G}^2.$$

We choose $\lambda_0 = 1 + (C_2 + C_3)^{1/2}$. Then for all $\lambda = \mu^2 e^{i\theta}$ with $\mu \in \mathbf{R}$, $\mu^2 \geq \lambda_0$ and $\pi/2 \leq \theta \leq 3\pi/2$ the inequality

$$c^* \,\|(L(D) - \lambda)\,u\|_{0,G} \geq (1 + |\lambda|)\,\|u\|_{0,G}$$

with $c^* := +C_1^{1/2}$ is valid for all $u \in X$. By continuous extension to $H^{(k)}(G)$ we get the assertion of the lemma ∎

2.2 Analogously to the elliptic case we prove now

Lemma 3: *Let G be a bounded open set with Property P. Then the range of the operator $L^\sim - \lambda I: H^{(k)}(G) \to L^2(G)$ coincides with the whole space $L^2(G)$ for all $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda \leq 0$, and the operator $L^\sim - \lambda I$ is bijective.*

Proof: Let be $\lambda \in \mathbf{C}$, $\operatorname{Re} \lambda \leq 0$, and take an arbitrary element $f \in L^2(G)$. We have to prove that there exists a uniquely determined element $u \in H^{(k)}(G)$ with $(L^\sim - \lambda I)\,u = f$.

We define

$$B_\lambda(u,v) := B(u,v) - \lambda(u,v)_{0,G} \quad \text{for all } u, v \in X.$$

Now one has by (1.3) (as $\operatorname{Re} \lambda \leq 0$) the estimate

$$\operatorname{Re} B_\lambda(u,u) = B(u,u) - \operatorname{Re} \lambda(u,u)_{0,G} \geq |||u|||_r^2. \tag{2.7}$$

On the other hand, the sesquilinear form $B_\lambda(\cdot,\cdot)$ is bounded on X, $|B_\lambda(u,v)| \leq (1 + |\lambda|)\,\|u\|_r\,\|v\|_r$. As $B_\lambda(\cdot,\cdot)$ can be continuously extended to a bounded sesquilinear form with property (2.7) to $H_0^r(G)$, we get by the theorem of Lax-Milgram (cf. [3: pp. 41—46]) the existence of a unique element $u \in H_0^r(G)$ such that $(f,\phi)_{0,G} = B_\lambda(u,\phi)$ is valid for all $\phi \in C_0^\infty(G)$. By the regularity result for the solution $u \in H_0^r(G)$ of the homogeneous Dirichlet problem mentioned in 1.6 we have $u \in H^{(k)}(G)$.

By partial integration (see (1.5)) we get further

$$(f,\phi)_{0,G} = B_\lambda(u,\phi) = \big((L^\sim - \lambda I)\,u, \phi\big)_0$$

for all $\phi \in C_0^\infty(G)$. This proves that the operator $L^\sim - \lambda I: H^{(k)}(G) \to L^2(G)$ is bijective ∎

2.3 By Lemma 3 the domain of the resolvent $(\lambda I - L^\sim)^{-1}$ of the operator $L^\sim$ coincides with the whole space $L^2(G)$ for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$, hence for every such λ the operator $(\lambda I - L^\sim)^{-1}$ is bounded. Furthermore we prove the following estimate for the resolvent.

Lemma 4: *Let G be a bounded open set with Property P. Then the resolvent $(\lambda I - L^\sim)^{-1}$ of the operator $L^\sim$ satisfies the estimate*

$$\|(\lambda I - L^\sim)^{-1}\| \leqq \frac{c}{1 + |\lambda|} \tag{2.8}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$ where c is a positive constant.

Proof: The operator $(\lambda I - L^\sim)^{-1}$ is bounded for every $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$, i.e., for every such λ there exists a positive constant c_λ such that $\|(\lambda I - L^\sim)^{-1}\| \leqq c_\lambda$. Then every $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$ has a neighborhood U_λ in $\mathbf{C}$ for which

$$\|(\lambda' I - L^\sim)^{-1}\| \leqq 2c_\lambda \quad \text{for all } \lambda' \in U_\lambda.$$

Thus, the resolvent $(\lambda I - L^\sim)^{-1}$ is uniformly bounded on each compact subset of the half plane $\operatorname{Re}\lambda \leqq 0$, and we have with a constant $c_0 > 0$ (which is independent of λ)

$$\|(\lambda I - L^\sim)^{-1}\| \leqq c_0 \tag{2.9}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$ and $|\lambda| \leqq \lambda_0$ (for λ_0 see Lemma 2). On the other hand, by Lemma 2 we have

$$\|(\lambda I - L^\sim)^{-1}\| \leqq \frac{c^*}{1 + |\lambda|} \tag{2.10}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$ and $|\lambda| \geqq \lambda_0$. Hence with $c := \max\{c^*, c_0(1 + \lambda_0)\}$ it follows from (2.9) and (2.10) that the relation

$$\|(\lambda I - L^\sim)^{-1}\| \leqq \frac{c}{1 + |\lambda|}$$

holds for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \leqq 0$ ∎

3. The Cauchy Problem

3.1 We will now investigate the generalized Cauchy problem mentioned in the Introduction. Because of (1.9) this problem is an initial-boundary value problem with generalized homogeneous boundary values on ∂G and with non-homogeneous initial values on G.

Problem C: *Let G be a bounded open set in $\mathbf{R}^n$ with Property P. Further let*[2] *$f : \mathbf{R}_0^+ \to L^2(G)$ be a given uniformly Hölder continuous (with an exponent β, $0 < \beta \leqq 1$) function on $\mathbf{R}_0^+$ with values in $L^2(G)$, $f \in C^{0,\beta}(\mathbf{R}_0^+, L^2(G))$, and u_0 a given element of $H^{(k)}(G)$. We want to find all functions $u : \mathbf{R}_0^+ \to H^{(k)}(G)$ from the class $C^0(\mathbf{R}_0^+, L^2(G)) \cap C^1(\mathbf{R}^+, L^2(G))$ which solve the generalized evolution equation*

$$\frac{d}{dt} u(t) + L^\sim u(t) = f(t) \quad \textit{for } t > 0$$

and satisfy the initial condition $u(0) = u_0$.

[2] We use the notations $\mathbf{R}^+ := \{r \in \mathbf{R} \mid r > 0\}$ and $\mathbf{R}_0^+ := \mathbf{R}^+ \cup \{0\}$.

3.2 We are now able to use for this problem the theory presented by A. Friedman in [3: Part 2, especially 2.1—2.13, pp. 101—158] (cf. also [6: pp. 85—109]). For this theory it is not necessary that $L^{\sim}$ is the closure of an elliptic differential operator but that

$$D(L^{\sim}) = H^{(k)}(G)$$

is dense in $L^2(G)$ and that with a constant $c > 0$ the estimate

$$\|(\lambda I - L^{\sim})^{-1}\| \leqq \frac{c}{1 + |\lambda|}$$

is valid for all $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda \leqq 0$. These conditions guarantee that the operator $-L^{\sim}$ is an infinitesimal generator of an analytic semigroup of bounded linear operators in $L^2(G)$, with the help of this fact one proves the existence and uniqueness of the fundamental solution $V(\cdot, \tau)$ for the operator $\frac{d}{dt} + L^{\sim}$.

By a *fundamental solution* we mean a function[3])

$$V(\cdot, \cdot) : \{t \in \mathbf{R} \mid \tau \leqq t < \infty\} \times \{\tau \in \mathbf{R} \mid \tau \geqq 0\} \to \mathscr{B}(L^2(G))$$

with the properties:

I. The operator $V(t, \tau)$ $(\in \mathscr{B}(L^2(G)))$ is strongly continuous in t, τ for $0 \leqq \tau \leqq t < \infty$.

II. The derivative $\frac{\partial}{\partial t} V(t, \tau)$ exists in the strong topology of $\mathscr{B}(L^2(G))$ and belongs to $\mathscr{B}(L^2(G))$ for $0 \leqq \tau < t < \infty$ and is also strongly continuous in t for $\tau < t < \infty$.

III. The range of $V(t, \tau)$ lies in $D(L^{\sim})$ $(= H^{(k)}(G))$ for all t, τ with $0 \leqq \tau < t < \infty$.

IV. The function $V(\cdot, \tau)$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t} V(t, \tau) + L^{\sim} V(t, \tau) = 0 \quad \text{for} \quad \tau < t < \infty$$

and $V(\tau, \tau) = I$.

Finally we get from the considerations of A. Friedman [3: p. 109].

Theorem 5: *Problem C has a unique solution u. This solution has the expression*

$$u(t) := V(t, 0)\, u_0 + \int_0^t V(t, s)\, f(s)\, ds.$$

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