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On the Existence of Solutions for a General Form of Variational and Quasi-Variational Inequalities

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In der Arbeit werden Aussagen über die Existenz von Lösungen für eine allgemeine Form von Variations- und Quasivariationsungleichungen gemacht. Dazu findet eine Modifikation der klassischen Koerzivitätsbedingung und eine Monotonie der Ungleichung Verwendung.

В данной работе изучается существование решения вариационных и квазивариационных неравенств некоторого общего вида. Для этого модифицируется классическое условие коэрцитивности и используется некоторая монотонность неравенств.

The paper studies the existence of solutions for a general form of variational and quasi-variational inequalities. For this there is used a modification of the classical coerciveness condition and a monotonicity of inequalities.

1. Introduction

This paper studies the existence of solutions for a general form of variational and quasi-variational inequalities by using a modification of the classical coerciveness condition and a monotonicity of inequalities. Specifically, let X be a real Hausdorff locally convex space, C a closed convex subset of X, f a function from $C \times C$ into $\mathbf{R}$ and Q a multivalued mapping from C into C. The monotonicity used in the paper is that of the function f (see the definition in Section 2). We shall deal with the following general form of variational inequalities:

$$\textit{Find} \quad x \in C \quad \textit{such that} \quad f(x, y) \leqq 0 \quad \textit{for all} \quad y \in C.$$

The coerciveness condition used in our investigation of this problem is the following. There exists a convex compact set $B \subset C$ such that for each $x \in C \setminus B$ there is $y \in B$ satisfying $f(x, y) > 0$. In this condition, the existence of $y \in B$ may depend on $x \in C \setminus B$. Here it is important to note that in the classical coerciveness condition the existence of such an $y \in B$ is required independently of $x \in C \setminus B$ (see Theorem of Mosco [9] given in Section 2).

Well known results for variational inequalities, obtained by using the classical coerciveness condition, can be found, for instance, in BROWDER [4], HARTMAN and STAMPACCHIA [5], LIONS [8] and MOSCO [9]. The coerciveness condition in the above form was first used by ALLEN [1] for an extension to non-compact sets of the inequality of KY FAN [7]. Another extension of this inequality is the result of BREZIS, NIRENBERG and STAMPACCHIA [3] obtained by using the classical coerciveness condition.

The general form of quasi-variational inequalities considered in this paper is the following:

$$\textit{Find} \quad x \in C \quad \textit{such that} \quad x \in Q(x) \quad \textit{and} \quad f(x, y) \leqq 0 \quad \textit{for all} \quad y \in Q(x).$$

Our coerciveness condition for this problem is the following: There exists a convex compact set $B \subset C$ such that for each $x \in B$ and each $y \in Q(x) \setminus B$ there is $z \in Q(x) \cap B$ satisfying $f(y, z) > 0$. As we can see, the existence of z in this condition may depend on the pair (x, y). This is in contrast to the classical coerciveness condition used for instance in Mosco [9] where the point z is required to exist independently of the pair (x, y) (see Remark 3.4).

The results obtained in this paper are extensions of some classical results. We shall see their efficiency in some cases to which classical results cannot be applied (see the example given in Section 2). The results obtained for variational inequalities are presented in Section 2 and for quasi-variational inequalities in Section 3.

2. Variational inequalities

Let X be a real Hausdorff locally convex space, $C \subset X$ a closed convex subset and f a function from $C \times C$ into $\mathbf{R}$. We consider the following problem:

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(x, y) \leqq 0 \quad \text{for all} \quad y \in C. \tag{2.1}$$

The function f is said to be *monotone* if $f(x, y) + f(y, x) \geqq 0$ for all $x, y \in C$, and f is said to be *hemi-continuous* if the function $f\big(x + t(y - x), y\big)$ of the real variable $t \in [0, 1]$ is lower-semicontinuous for arbitrary given $x, y \in C$.

The existence of solutions to problem (2.1) is established in the following theorem.

Theorem 2.1: *Let X be a real Hausdorff locally convex space and C a closed convex subset of X. Let $f: C \times C \to \mathbf{R}$ be a monotone and hemi-continuous function with $f(x, x) \leqq 0$ for all $x \in C$ and for which the function $f(x, \cdot)$ is concave and upper semi-continuous for each $x \in C$. Assume that there exists a convex compact subset $B \subset C$ such that for each $x \in C \setminus B$ there is an element $y \in B$ satisfying $f(x, y) > 0$ (the coerciveness condition).*

Then the set of all solutions to Problem (2.1) is non-empty, convex and compact.

In order to prove Theorem 2.1 we need some results from Mosco [9].

Lemma 2.2 [9: Cor. 1 of Th. 6.1]: *Let C be a non-empty convex compact subset of a real Hausdorff locally convex space X and $f: C \times C \to \mathbf{R}$ a monotone hemi-continuous function such that $f(x, x) \leqq 0$ and $f(x, \cdot)$ is a concave and upper semi-continuous function for each $x \in C$.*

Then Problem (2.1) admits a solution.

Lemma 2.3 [9: Lemma 3.1]: *Let C be a convex closed subset of a real Hausdorff locally convex space X and $f: C \times C \to \mathbf{R}$ a monotone and hemi-continuous function such that $f(x, x) \leqq 0$ and $f(x, \cdot)$ is a concave and upper semi-continuous function for each $x \in C$. For each $y \in C$ set*

$$G(y) = \{x \in C: f(x, y) \leqq 0\}, \qquad H(y) = \{x \in C: f(y, x) \geqq 0\}$$

and $F(y)$ the closure of $G(y)$ in the space X.

Then

$$\bigcap_{y \in C} G(y) = \bigcap_{y \in C} F(y) = \bigcap_{y \in C} H(y)$$

and each of these intersections is a closed convex subset of C.

Proof of Theorem 2.1: For each $y \in C$ we define

$$K(y) = \{x \in B : f(x, y) \leqq 0\}$$

and let $Q(y)$ denote the closure of $K(y)$. It is easy to check that $K(y)$ is non-empty. In fact, if we define D to be the convex hull of the set $B \cup \{y\}$, then D is convex and compact. By Lemma 2.2, the Problem

$$\begin{cases} x \in D \\ f(x, y) \leqq 0 \quad \text{for all} \quad y \in D \end{cases}$$

admits a solution. Let $\bar{x}$ be such a solution. Then, by the coerciveness condition, we obtain $\bar{x} \in B$. That means $K(y) \neq \emptyset$.

We shall now show that

$$\bigcap_{y \in C} K(y) = \bigcap_{y \in C} Q(y). \tag{2.2}$$

It is obvious that $\bigcap_{y \in C} K(y) \subseteq \bigcap_{y \in C} Q(y)$. Therefore, we have only to show the converse inclusion. Let $G(y) = \{x \in C : f(x, y) \leqq 0\}$ and let $F(y)$ be the closure of $G(y)$. It is clear that $K(y) = G(y) \cap B$, hence $\bigcap_{y \in C} K(y) = \bigcap_{y \in C} G(y) \cap B$. But, from the coerciveness condition it follows that the set of all solutions to Problem (2.1) is contained in B, that means $\bigcap_{y \in C} G(y) \subset B$. Therefore

$$\bigcap_{y \in C} K(y) = \bigcap_{y \in C} G(y). \tag{2.3}$$

From $K(y) \subseteq G(y)$ we get $Q(y) \subseteq F(y)$, hence $\bigcap_{y \in C} Q(y) \subseteq \bigcap_{y \in C} F(y)$. By Lemma 2.3 we have $\bigcap_{y \in C} F(y) = \bigcap_{y \in C} G(y)$. Therefore, together with (2.3) we obtain $\bigcap_{y \in C} Q(y) \subseteq \bigcap_{y \in C} K(y)$, which means (2.2) holds.

By using the finite intersection property for the family of the closed sets $Q(y)$ in the compact set B, we now show the existence of a solution to Problem (2.1)[1]. Let $y_1, \ldots, y_n$ be arbitrary points of C. Let C_0 denote the convex hull of the set $B \cup \{y_1, \ldots, y_n\}$. Then C_0 is a convex compact set. Hence, from Lemma 2.2 the Problem

$$\begin{cases} x \in C_0 \\ f(x, y) \leqq 0 \quad \text{for all} \quad y \in C_0 \end{cases}$$

admits a solution. Let x_0 be such a solution. So we have $x_0 \in B$ (by virtue of the coerciveness condition) and $f(x_0, y_i) \leqq 0$ for all $i = 1, \ldots, n$. That means

$$x_0 \in \bigcap_{i=1}^{n} K(y_i) \subseteq \bigcap_{i=1}^{n} Q(y_i).$$

Therefore, the family $\{Q(y) : y \in C\}$ has the finite intersection property. Hence, there exists $x \in \bigcap_{y \in C} Q(y)$. Together with (2.2) and (2.3) we obtain $x \in \bigcap_{y \in C} G(y)$. That means x is a solution of Problem (2.1).

The convexity and the compactness of the set M of all solutions of (2.1) follow from the coerciveness condition, the relation (2.3) (shown above) and Lemma 2.3. M is then a closed convex subset of the compact set B ∎

[1]) The technique of using the finite intersection property here is similar to that used in ALLEN [1].

We consider now a special case of Problem (2.1). Let X be a reflexive Banach space, X^* the dual space of X, $C \subset X$ a closed convex subset, A an operator from C into X^* and φ a function from C into $\mathbf{R}$. For the following variational inequality

$$\begin{cases} x \in C \\ \langle Ax - v', x - y\rangle + \varphi(x) - \varphi(y) \leqq 0 \quad \text{for all} \quad y \in C, \end{cases} \tag{2.4}$$

where v' is an arbitrary given element of X^*, Theorem 2.1 has an immediate corollary.

Corollary 2.4: *Let the space X and the set C be given as above. Let A be a monotone and hemi-continuous operator and φ a convex and lower-continuous function. Suppose that there exists a bounded set $B_0 \subset C$ such that* $\sup\{\varphi(x): x \in B_0\} < +\infty$ *and for each $x \in C$ there is $y(x) \in B_0$ satisfying*

$$\frac{\langle Ax, x - y(x)\rangle + \varphi(x)}{\|x\|} \to +\infty \quad as \quad \|x\| \to +\infty. \tag{2.5}$$

Then the set of all solutions to Problem (2.4) *is non-empty, convex and compact.*

Proof: Apply Theorem 2.1 with the weak topology on X, $f(x, y) = \langle Ax - v', x - y\rangle + \varphi(x) - \varphi(y)$ and the convex compact set $B = \{x \in X: \|x\| \leqq r\} \cap C$ with $r > 0$ sufficiently large. The coerciveness condition is then satisfied, since the set $B' := \{x \in C: f(x, y) \leqq 0 \ \forall y \in B_0\}$ is now bounded (by virtue of (2.5)) and thus we can take $B \supset B_0 \cup B'$ for sufficiently large $r > 0$. The set B so defined has the desired property of the convex compact set in the coerciveness condition ∎

We would like here to note that by using the subdifferential of the function φ and the indicatorfunction I_C for C we can write (2.4) in the form $v' \in Ax + \partial(I_C + \varphi)(x)$. The assertion of Corollary 2.4 then follows also from the surjectivity criterion of ROCKAFELLAR [10].

Remark 2.5: From BROWDER [4] and HARTMAN and STAMPACCHIA [5] we have the following result: Let X, C, A and v' be given as in Corollary 2.4. Suppose that there is a point $y_0 \in C$ such that

$$\frac{\langle Ax, x - y_0\rangle}{\|x\|} \to +\infty \quad \text{as} \quad \|x\| \to \infty, \qquad x \in C.$$

Then the variational inequality

$$\begin{cases} x \in C \\ \langle Ax, x - y\rangle \leqq \langle v', x - y\rangle \quad \text{for all} \quad y \in C \end{cases}$$

admits a solution. We obtain this result by applying Corollary 2.4 with $\varphi \equiv 0$ and the bounded set $B_0 := \{y_0\}$.

The following existence theorem is proved by Mosco [9: Theorem 3.1]. Let X be a real Hausdorff topological linear space and C a closed convex subset of X. Let $f: C \times C \to \mathbf{R}$ be a monotone and hemi-continuous function with $f(x, x) \leqq 0$ for all $x \in C$ and for which the function $f(x, \cdot)$ is concave and upper semi-continuous for each $x \in C$. Suppose that there exists a compact subset $B \subset C$ and a point $y_0 \in B$ such that $f(x, y_0) > 0$ for each $x \in C \setminus B$ (the coerciveness condition). Then the set of all solutions to Problem (2.1) is non-empty, convex and compact. A connection between this result and Theorem 2.1 above is given in the following remark.

Remark 2.6: In the case where X is a Banach space, Theorem 2.1 is a generalization of Theorem 3.1 in [9]. Since in a Banach space the closed convex hull of a compact set (weakly compact set) is a compact set (weakly compact set, respectively), it is easy to see that from the coerciveness condition in Theorem 3.1 [9] we get the coerciveness condition in Theorem 2.1. Theorem 3.1 [9] follows then from Theorem 2.1.

We give now an example from stationary point problems, to which Theorem 3.1 [9] cannot be applied, but Theorem 2.1 above implies the existence of a solution. Let $C \subset \mathbf{R}^n$ be a convex set and A an operator from C into $\mathbf{R}^n$. A point $x \in C$ is said to be a *stationary point* of the variational problem defined by the pair (A, C) if the inequality

$$Ax(x - y)^T \leqq 0 \quad \text{for all} \quad y \in C$$

is satisfied (see [6]). Here we also say that x is a solution of the stationary point Problem (A, C) (the superscript T denotes the transposition). Our example is the stationary point problem (A, C) where

$$\begin{cases} C = \mathbf{R}^n \\ Ax = (x_2, -x_1, x_4, -x_3, \ldots, x_n, -x_{n-1}), \; n \quad \text{even} \\ x = (x_1, \ldots, x_n). \end{cases}$$

For the above example we define f as $f(x, y) = Ax(x - y)^T$.

In this case, it is easy to check that the set $\{x \in C: f(x, y) \leqq 0\}$ is unbounded for each $y \in C$. Hence, it is impossible to find a compact set B and a point $y_0 \in B$ satisfying the coerciveness condition in Theorem 3.1 [9]. Conversely, by taking $B = \{x = (x_1, \ldots, x_n) \in C: x_i \in [-1, +1] \text{ for } i = 1, \ldots, n\}$ we can verify that the coerciveness condition in Theorem 2.1 is satisfied. The other assumptions in Theorem 2.1 are also satisfied. Hence, in the above example Problem (2.1) admits a solution, that means, the above stationary point problem admits a solution.

3. Quasi-variational inequalities

We assume again that X is a real Hausdorff locally convex space and $C \subset X$ a closed convex subset. Let Q be a multivalued mapping from C into C and f a function from $C \times C$ into $\mathbf{R}$. We consider the following problem:

$$\textit{Find} \quad x \in C \quad \textit{such that} \quad x \in Q(x) \quad \textit{and} \quad f(x, y) \leqq 0 \quad \textit{for all} \quad y \in Q(x). \tag{3.1}$$

The mapping Q is said to be *closed* if for every generalized sequence $\{(x_\alpha, y_\alpha)\}$ converging to (x, y) in $C \times C$ and satisfying $y_\alpha \in Q(x_\alpha)$ we have in the limit $y \in Q(x)$.

Q is said to be *lower semi-continuous with respect to* f if for every generalized sequence $\{(x_\alpha, y_\alpha)\}$ converging to (x, y) in $C \times C$ and satisfying

$$\begin{cases} y_\alpha \in Q(x_\alpha) \\ f(y_\alpha, z) \leqq 0 \quad \text{for all} \quad z \in Q(x_\alpha) \end{cases}$$

and for every $z \in Q(x)$ there is $z_\alpha \in Q(x_\alpha)$ such that

$$\underline{\lim} \, [f(x_\alpha, z) - f(x_\alpha, z_\alpha)] \leqq 0.$$

For Problem (3.1) we shall prove the following existence theorem.

Theorem 3.1: *Let X be a real Hausdorff locally convex space and C a closed convex subset of X. Let $f: C\times C \to \mathbf{R}$ be a monotone and hemi-continuous function such that $f(x,x) \leqq 0$ and $f(x,\cdot)$ is a concave, upper semi-continuous function for each $x \in C$. Let the mapping $Q: C \to 2^C$ be closed and lower semi-continuous with respect to f. Assume that there exists a convex compact subset $B \subset C$ such that for each $x \in B$ and each $y \in Q(x) \setminus B$ there is $z \in B \cap Q(x)$ satisfying $f(y,z) > 0$ (the coerciveness condition).*

Then Problem (3.1) *admits a solution.*

In order to prove Theorem 3.1 we use Theorem 2.1 and the following Lemma.

Lemma 3.2: *Under the assumptions of Theorem* 3.1, *for each $x \in B$ the problem*

$$\begin{cases} y \in C, \quad y \in Q(x) \\ f(y,z) \leqq 0 \quad \text{for all} \quad z \in Q(x) \end{cases} \tag{3.2}$$

admits a solution. Moreover, if we define $S(x)$ to be the set of all solutions of this problem, then the multivalued mapping $S: x \mapsto S(x)$ is closed.

Proof: The existence of a solution to Problem (3.2) follows from Theorem 2.1 above. We now show that the mapping S is closed. Let G be the graph of S, that means $G = \{(x,y) \in C\times C: x \in B, y \in S(x)\}$. Let $\{(x_\alpha, y_\alpha)\}$ be a given sequence in G converging to (x,y). We have to show that $(x,y) \in G$. By $(x_\alpha, y_\alpha) \in G$ we have

$$\begin{cases} y_\alpha \in Q(x_\alpha) \\ f(y_\alpha, z) \leqq 0 \quad \text{for all} \quad z \in Q(x_\alpha) \end{cases} \tag{3.3}$$

Since Q is lower semi-continuous with respect to f, for every $z \in Q(x)$ there is $z_\alpha \in Q(x_\alpha)$ such that $\underline{\lim}\,[f(y_\alpha, z) - f(y_\alpha, z_\alpha)] \leqq 0$. Hence, if follows from (3.3) that $\underline{\lim} f(y_\alpha, z) \leqq 0$. Therefore, by taking account of the monotonicity of f we get $\overline{\lim} f(z, y_\alpha) \geqq 0$. By this, it follows from the upper semi-continuity of $f(z,\cdot)$ that

$$f(z,y) \geqq 0. \tag{3.4}$$

On the other hand, the mapping Q is closed and as above $\{(x_\alpha, y_\alpha)\}$ converges to (x,y), $y_\alpha \in Q(x_\alpha)$. Therefore $y \in Q(x)$. In (3.4) $z \in Q(x)$ is arbitrary. Thus y is exactly a solution of the problem

$$\begin{cases} y \in Q(x) \\ f(z,y) \geqq 0 \quad \text{for all} \quad z \in Q(x). \end{cases}$$

By Lemma 2.3, y is then a solution of the problem

$$\begin{cases} y \in Q(x) \\ f(y,z) \leqq 0 \quad \text{for all} \quad z \in Q(x). \end{cases}$$

That means $y \in S(x)$ or $(x,y) \in G$. The proof of Lemma 3.2 is complete ∎

Proof of Theorem 3.1: For each $x \in B$, by Lemma 3.2 the problem

$$\begin{cases} y \in Q(x) \\ f(y,z) \leqq 0 \quad \text{for all} \quad z \in Q(x) \end{cases}$$

admits a solution and by the coerciveness condition the set $S(x)$ of all solutions of this problem is contained in B. Thus, the closed mapping $S: B \to 2^B$ has the

compact image B. Therefore S is upper semi-continuous. By Kakutani's fixed point Theorem (see BERGE [2]) there exists a point $x \in B$ such that $x \in S(x)$. That means x is a solution of Problem (3.1) ∎

We now consider a special case. Let X be a reflexive Banach space, $C \subset X$ a closed convex subset, X^* the dual space of X, A an operator from C into X^*, Q a multivalued mapping from C into C and φ a function from C into $\mathbf{R}$. Let v' be a given point of X^*.

The mapping Q is said to be *lower semi-continuous with respect to* (A, φ, v') if for every sequence $\{(x_n, y_n)\}$ converging to (x, y) in $C \times C$ and satisfying

$$\begin{cases} y_n \in Q(x_n) \\ \langle Ay_n - v', y_n - z\rangle + \varphi(y_n) - \varphi(z) \leqq 0 \quad \text{for all} \quad z \in Q(x_n) \end{cases}$$

and for every $z \in Q(x)$ there is $z_n \in Q(x_n)$ such that

$$\underline{\lim}\, [\langle Ax_n - v', z_n - z\rangle + \varphi(z_n) - \varphi(z)] \leqq 0.$$

For the quasi-variational inequality

$$\begin{cases} x \in C, \quad x \in Q(x) \\ \langle Ax - v', x - y\rangle + \varphi(x) - \varphi(y) \leqq 0 \quad \text{for all} \quad y \in Q(x) \end{cases} \tag{3.5}$$

we have the following corollary.

Corollary 3.3: *Let C be a closed convex subset of a reflexive Banach space X, A a monotone and hemi-continuous operator, Q a closed mapping and lower semi-continuous with respect to (A, φ, v') and φ a convex and lower semi-continuous function. Suppose that there exists a bounded set $B_0 \subset C$ such that* $\sup\{\varphi(x): x \in B_0\} < +\infty$ *and for each $y \in Q(C)$ there is $z(y) \in B_0 \cap Q(x)$ satisfying*

$$\frac{\langle Ay, y - z(y)\rangle + \varphi(y)}{\|y\|} \to +\infty \quad \text{as} \quad \|y\| \to \infty, \qquad y \in Q(C) \tag{3.6}$$

Then Problem (3.5) *admits a solution.*

Proof: Apply Theorem 3.1 with the weak topology on X, the set C and mapping Q as above, the function $f(x, y) = \langle Ax - v', x - y\rangle + \varphi(x) - \varphi(y)$ and the convex compact set $B = \{x \in X: \|x\| \leqq r\} \cap C$ with $r > 0$ sufficiently large. It is easy to check that the set $B' := \{y \in Q(C): f(y, z) \leqq 0 \ \forall z \in Q(C) \cap B_0\}$ is bounded (by virtue of (3.6)). Therefore, we can assume that B contains $B_0 \cup B'$ for sufficiently large $r > 0$ and then the coerciveness condition is satisfied. Thus, there is a solution of (3.5) ∎

For convenience we shall say that Q is *lower semi-continuous with respect to* (A, v') *if Q is lower semi-continuous* with respect to (A, φ, v') (according to the definition above) with $\varphi \equiv 0$.

Remark 3.4: From Mosco [9: Theorem 8.1 and Remark 8.1] we have the following result: Let X, C, A and v' be given as in Remark 3.3. Let Q be a closed multivalued mapping and lower semi-continuous with respect to (A, v'). Suppose that there exists a point $z_0 \in \bigcap_{x \in C} Q(x)$ such that

$$\frac{\langle Ay, y - z_0\rangle}{\|y\|} \to +\infty \quad \text{as} \quad \|y\| \to \infty, \qquad y \in Q(C).$$

Then the quasi-variational inequality

$$\begin{cases} x \in C, \quad x \in Q(x) \\ \langle Ax, x - y \rangle \leqq \langle v', x - y \rangle \quad \text{for all} \quad y \in Q(x) \end{cases}$$

admits a solution. We obtain this result by applying Corollary 3.3 with $\varphi \equiv 0$ and the set $B_0 := \{z_0\}$.

Remark 3.5: By an argument analogous to that used in the proof of Theorem 3.1, we can show that Theorem 3.1 still holds if the assumption "Q is lower semi-continuous with respect to f" is replaced by the assumption: "Q is lower semi-continuous and f is lower semi-continuous on $C \times C$". The lower semi-continuity of the mapping $Q: C \to 2^C$ here means that if $\{x_\alpha\}$ is a generalized sequence converging to x in C, then for every $y \in Q(x)$ there is $y_\alpha \in Q(x_\alpha)$ such that the sequence $\{y_\alpha\}$ converges to y in C.

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