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Redundancy Conditions for the Functional Equation $f(x + h(x)) = f(x) + f(h(x))$

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Es wird die Funktionalgleichung $f(x + h(x)) = f(x) + f(h(x))$ betrachtet, wobei $h: \mathbf{R} \to \mathbf{R}$ eine gegebene stetige Funktion mit $h(0) = 0$ ist. Es wird bewiesen: falls die Nullstellen von h und die Stellen, wo $h(x) = -x$ gilt, nicht „zu dicht liegen", dann ist die stetige und im Ursprung differenzierbare Lösung der obigen Funktionalgleichung $f(x) = xf'(0)$ für alle reellen x.

Рассматривается функциональное уравнение $f(x + h(x)) = f(x) + f(h(x))$, где $h: \mathbf{R} \to \mathbf{R}$ есть заданная непрерывная функция с условием $h(0) = 0$. Пусть множество всех нулей функции h и всех точек x с $h(x) = -x$ не слишком „густо". Показывается, что тогда непрерывное и в начале координат дифференцируемое решение этого уравнения имеет вид $f(x) = xf'(0)$ для всех вещественных x.

Consider the functional equation $f(x + h(x)) = f(x) + f(h(x))$, where $h: \mathbf{R} \to \mathbf{R}$ is a given continuous function, $h(0) = 0$. It is proved if the set of all zeros of h and of all points where $h(x) = -x$ is not "too much dense", then the continuous and at $x = 0$ differentiable solution $f: \mathbf{R} \to \mathbf{R}$ of the functional equation under consideration is $f(x) = xf'(0)$ for all real x.

1. This paper deals with the functional equation

$$f(x + h(x)) = f(x) + f(h(x)), \qquad x \in \mathbf{R}, \tag{1}$$

where $h: \mathbf{R} \to \mathbf{R}$ is a given function with $h(0) = 0$. Equation (1) has been studied by many authors in order to obtain conditions of redundancy, that is conditions on the given function h and on the class of solutions which ensure that

$$f(x + h(x)) = f(x) + f(h(x)) \quad \textit{iff} \quad f(x + y) = f(x) + f(y), \qquad x, y \in \mathbf{R}$$

(see [1–4]). In [2] we got redundancy in the class of the continuous functions differentiable at zero, by requiring the continuity of h and that its graph (for $x \neq 0$) lies in one of the following regions of the plane:

$$\mathcal{R}_1 = \{(x, y): xy > 0\}, \qquad \mathcal{R}_2 = \left\{(x, y): -1 < \frac{y}{x} < 0\right\},$$

$$\mathcal{R}_3 = \left\{(x, y): \frac{y}{x} < -1\right\}.$$

Herein we achieve redundancy under considerably weaker conditions, essentially by requiring that the set of all zeros of h and of all points where $h(x) = -x$ be not "too much dense". In the proofs we use the techniques developed in [2], so for the details we refer to that note.

2. From now on we assume the continuity of h on $\mathbf{R}$ and we shall use the following notations:

$$Z_+ = \{x > 0 : h(x) = 0\}, \qquad Z_- = \{x < 0 : h(x) = 0\}, \qquad Z = Z_+ \cup Z_-;$$
$$K_+ = \{x > 0 : h(x) + x = 0\}, \qquad K_- = \{x < 0 : h(x) + x = 0\},$$
$$K = K_+ \cup K_-;$$
$$B_+ = \{x > 0 : h(x) + x < 0\}, \qquad B_- = \{x < 0 : h(x) + x > 0\};$$
$$A_+ = B_+ \setminus (-B_-); \qquad A_- = B_- \setminus (-B_+);$$
$$E_+ = Z_+ \cup \{K_+ \cap [(-K_-) \cup (-Z_-)]\}, \qquad E_- = Z_- \cup \{K_- \cap [(-K_+) \cup (-Z_+)]\},$$
$$E = E_+ \cup E_-;$$

if T is a subset of $\mathbf{R}$, by $D^1(T)$ we denote the set of the limit points of T and, for every $n > 1$, $D^n(T) = D^1(D^{n-1}(T))$.

Lemma 1: *Assume that $A_+ = A_- = \emptyset$ and let $f: \mathbf{R} \to \mathbf{R}$ be a solution of equation* (1). *For every $x \notin E$, $x \neq 0$, there exist z and y such that $|z| < |x|$, $|y| < |x|$ and*

$$\frac{f(z)}{z} \leqq \frac{f(x)}{x} \leqq \frac{f(y)}{y}.$$

Proof: For $t \neq 0$ we put $g(t) = \dfrac{f(t)}{t}$ and $k(t) = \dfrac{h(t)}{t}$. Take $x \notin E$ and consider $x + h(x)$; we have the following possibilities:

$$\alpha)\ \frac{x + h(x)}{x} > 1, \quad \beta)\ 0 < \frac{x + h(x)}{x} < 1, \quad \gamma)\ \frac{x + h(x)}{x} < 0, \quad \delta)\ x + h(x) = 0.$$

α) There exists t_0 such that $t_0 + h(t_0) = x$, $0 < \dfrac{t_0}{x} < 1$, so it is $k(t_0) > 0$. If $k(t_0) \leqq 1$, equation (1) yield $\dfrac{f(t_0)}{t_0} = \dfrac{f(t_0 + h(t_0)) - f(h(t_0))}{t_0}$, then (see [2: Theorem 3])

$$g(t_0) \leqq g(t_0 + h(t_0)) = g(x) \leqq g(h(t_0)) \quad \text{or} \quad g(h(t_0)) \leqq g(x) \leqq g(t_0). \tag{2}$$

If $k(t_0) > 1$, from $\dfrac{f(h(t_0))}{h(t_0)} = \dfrac{f(t_0 + h(t_0)) - f(t_0)}{h(t_0)}$ again we get one of the relations (2). Thus since $|h(t_0)| < |x|$ we have the desired conclusion.

β) From equation (1) we have $\dfrac{f(x)}{x} = \dfrac{f(x + h(x)) - f(h(x))}{x}$, then (see [2: Theorem 5])

$$g(h(x)) \leqq g(x) \leqq g(x + h(x)) \quad \text{or} \quad g(x + h(x)) \leqq g(x) \leqq g(h(x)). \tag{3}$$

Since $|h(x)| < |x|$ and $|x + h(x)| < |x|$, from (3) we obtain the conclusion.

γ) Since $A_+ = A_- = \emptyset$, it is $\dfrac{-x + h(-x)}{x} > 0$, that is $\dfrac{h(-x)}{x} > 1$. By the continuity of h, there exists s such that $-1 < \dfrac{s}{x} < 0$ and $h(s) = x$. Equation (1) gives $\dfrac{f(h(s))}{h(s)} = \dfrac{f(s + h(s)) - f(s)}{h(s)}$, then (see [2: Theorem 4])

$$g(s) \leqq g(x) \leqq g(s + h(s)) \quad \text{or} \quad g(s + h(s)) \leqq g(x) \leqq g(s). \tag{4}$$

Since $|s + h(s)| < |x|$, from (4) we get the conclusion.

δ) If $x + h(x) = 0$, equation (1) gives $f(x) = -f(-x)$. By the hypothesis it is $-x + h(-x) \neq 0$ and $h(-x) \neq 0$, so, as in the previous cases, we get z and y such that $|z| < |x|$, $|y| < |x|$ and $g(z) \leqq g(-x) \leqq g(y)$. Since $g(-x) = g(x)$, the proof ends ∎

Lemma 2: *Let $h: \mathbf{R} \to \mathbf{R}$ satisfy the following conditions:*

i) $A_+ = A_- = \emptyset$,

ii) *for every $x \neq 0$ there exists $N = N(x)$ such that $\{D^{N+1}(E) \cap (-|x|, |x|)\} \setminus \{0\} = \emptyset$.*

Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous solution of equation (1), differentiable at zero. If $x \in E$ then either there exist z and y such that $|z| < |x|$, $|y| < |x|$ and $g(z) \leqq g(x) \leqq g(y)$ or $g(x) = f'(0)$.

Proof: Let $x \in E_+$ and suppose that do not exist z and y with the required property. To make not uselessy intricate the proof, we assume $N(x) = 2$. We have to prove that $g(x) = f'(0)$. Since by ii) the set E has not interior points, there exists an increasing sequence $\{t_n\}$ such that $t_n \to x$ and $t_n \notin E$. If $x \in Z_+$ we can assume that $|h(t_n)| < t_n$ and we can split $\{t_n\}$ in two subsequences (one possibly empty or finite) $\{t_{n_k}\}$ and $\{t_{n_s}\}$ such that $h(t_{n_k}) > 0$, $h(t_{n_s}) < 0$. By Lemma 1 we have

$$g(t_{n_k}) \leqq g\big(t_{n_k} + h(t_{n_k})\big) \leqq g\big(h(t_{n_k})\big) \quad \text{or} \tag{5_1}$$

$$g\big(h(t_{n_k})\big) \leqq g\big(t_{n_k} + h(t_{n_k})\big) \leqq g(t_{n_k}) \tag{5_2}$$

and

$$g\big(h(t_{n_s})\big) \leqq g(t_{n_s}) \leqq g\big(t_{n_s} + h(t_{n_s})\big) \quad \text{or} \tag{6_1}$$

$$g\big(t_{n_s} + h(t_{n_s})\big) \leqq g(t_{n_s}) \leqq g\big(h(t_{n_s})\big). \tag{6_2}$$

If $x \in K_+ \cap [(-K_-) \cup (-Z_-)]$, we choose $\{t_n\}$ such that $h(t_n) < 0$ and $\{t_{n_k}\}$, $\{t_{n_s}\}$ such that $h(t_{n_k}) > -t_{n_k}$ and $h(t_{n_s}) < -t_{n_s}$. By Lemma 1 we have

$$g\big(h(t_{n_k})\big) \leqq g(t_{n_k}) \leqq g\big(t_{n_k} + h(t_{n_k})\big) \quad \text{or} \tag{7_1}$$

$$g\big(t_{n_k} + h(t_{n_k})\big) \leqq g(t_{n_k}) \leqq g\big(h(t_{n_k})\big) \tag{7_2}$$

and

$$g(u_{n_s}) \leqq g(t_{n_s}) \leqq g\big(u_{n_s} + h(u_{n_s})\big) \quad \text{or} \tag{8_1}$$

$$g\big(u_{n_s} + h(u_{n_s})\big) \leqq g(t_{n_s}) \leqq g(u_{n_s}), \tag{8_2}$$

where $h(u_{n_s}) = t_{n_s}$ and $-1 < \dfrac{u_{n_s}}{t_{n_s}} < 0$.

Assume $g(x) < f'(0)$. Then if $x \in Z_+$, for infinite indexes k or s either (5_1) or (6_2) holds; if $x \in K_+ \cap [(-K_-) \cup (-Z_-)]$, for infinite indexes k or s either (7_1) or (8_1) holds. Since in (5_1) (or (6_2), or (7_1), or (8_1)) we cannot have for each index the equalities (otherwise $g(x) = f'(0)$) and since we have either two equalities or two strict inequalities, we can infer the existence of y_0 and u with $|y_0| < x$, $|u| < |y_0|$ and $g(x) < g(u) < g(y_0) < f'(0)$. Moreover for any t with $|t| \leq x$ we have $g(x) < g(t)$.

Let now $\{a_n\}$ be a strictly increasing sequence such that $g(u) < a_n < g(y_0)$ and $a_n \to g(y_0)$ as $n \to +\infty$. For every n, let $\{a_{n,k}\}$ be a strictly increasing sequence such that $g(u) < a_{1,k} < a_1$, $a_{n-1} < a_{n,k} < a_{n+1}$, and $a_{n,k} \to a_n$ as $k \to +\infty$.

Let $u_{1,1}$ be the element (or one of the elements) of minimum modulus for which $g(u_{1,1}) \leqq g(u)$; obviously $|u_{1,1}| < |y_0|$. If $u_{1,1} = 0$ we have a contradiction; so $u_{1,1} \neq 0$ and (by Lemma 1) $u_{1,1} \in E$. By the continuity of f, we can take $y_{1,1}$ with the following properties:

$$\frac{u_{1,1}}{y_{1,1}} > 1, \qquad y_{1,1} \notin E, \qquad g(y_{1,1}) < a_{1,1}.$$

By Lemma 1 there is a point z, $|z| < |y_{1,1}|$, such that $g(z) \leqq g(y_{1,1})$. Let $u_{1,2}$ be the element of minimum modulus such that $g(u_{1,2}) \leqq g(y_{1,1})$. As above it is $u_{1,2} \neq 0$, then $u_{1,2} \in E$ and there exists $y_{1,2} \notin E$, such that $\frac{u_{1,2}}{y_{1,2}} > 1$ and $g(y_{1,2}) < a_{1,1}$. Continuing the construction we obtain a sequence $\{y_{1,m}\}$ such that

$$|y_{1,m}| < |u_{1,m}| < |y_{1,m-1}|; \qquad \frac{u_{1,m}}{y_{1,m}} > 1, \qquad g(y_{1,m}) < a_{1,1}.$$

We can choose a subsequence $\{y_{1,m_l}\}$ of constant sign and if $y_{1,m_l} \to y_1$ as $l \to +\infty$, it is $|y_1| = \min |y_{1,m}|$, $g(y_1) \leqq a_{1,1}$. If $y_1 = 0$ we have a contradiction; suppose $y_1 > 0$; then $u_{1,m_1} > y_{1,m_1} > u_{1,m_2} > y_{1,m_2} > \cdots > y_1$, hence $y_1 \in D^1(E) \cap (-|x|, |x|)$. Using y_1 and $a_{1,2}$ instead of u and $a_{1,1}$, we construct y_2 with $|y_2| < y_1$, $g(y_2) \leqq a_{1,2}$ and $y_2 \in D^1(E) \cap (-|x|, |x|)$. Finally we get a sequence $\{y_k\} \subset D^1(E) \cap (-|x|, |x|)$, such that $|y_{k+1}| < |y_k|$ and $g(y_k) \leqq a_{1,k} < a_1$.

There is a subsequence $\{y_{k_s}\}$ convergent to z_1 and $|z_1| = \min\limits_s |y_{k_s}| = \min\limits_k |y_k|$. It is $g(z_1) \leqq a_1$ and $z_1 \in D^2(E) \cap (-|x|, |x|)$. Iterating the described procedure, we obtain a sequence $\{z_n\} \subset D^2(E) \cap (-|x|, |x|)$, such that $|z_{n+1}| < |z_n|$ and $g(z_n) \leqq a_n < g(y_0)$. Thus there exists a subsequence $\{z_{n_l}\}$ convergent to a point z such that $g(x) < g(z) \leqq g(y_0)$; but, by construction, $z \in D^3(E) \cap (-|x|, |x|)$ so hypothesis ii) implies $z = 0$. By contradiction we have $g(x) = f'(0)$ ∎

Theorem 1: *Suppose that $h\colon \mathbf{R} \to \mathbf{R}$ satisfies the hypotheses of Lemma 2. If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous solution of equation* (1), *differentiable at zero, then $f(x) = xf'(0)$ for every $x \in \mathbf{R}$.*

Proof: Let $x \in \mathbf{R}$, $x \neq 0$. If $g(x) \neq f'(0)$, by Lemmas 1 and 2 there exist z, y such that $|z| < |x|$, $|y| < |x|$ and $g(z) \leqq g(x) \leqq g(y)$. Hence the set $U = \{z\colon |z| < |x|, g(z) \leqq g(x)\}$ is not empty; denote by z^* the point (or one of the points) of U of minimum modulus. Then either $z^* = 0$ or $g(z^*) = f'(0)$. Indeed if $g(z^*) \neq f'(0)$, then by Lemmas 1 and 2 there exists $\bar{z}$ with $|\bar{z}| < |z^*|$ and $g(\bar{z}) \leqq g(z)$, a contradiction. So $f'(0) \leqq g(x)$. Analogously we conclude that $g(x) \leqq f'(0)$ ∎

The following examples show that if Z contains an interval or if $K_+ = -K_-$ and K_+ contains an interval, then there are non linear solutions of equation (1), continuous on $\mathbf{R}$ and differentiable at zero.

Example 1: Let $h\colon \mathbf{R} \to \mathbf{R}$ be continuous, $h(0) = 0$ and

$$\begin{array}{lll} h(x) < 0 & \text{for} & x < 0, \\ 0 < h(x) \leqq a - x & \text{for} & 0 < x < a, \\ h(x) = 0 & \text{for} & a \leqq x \leqq b, \\ 0 < h(x) \leqq a & \text{for} & x > b, \end{array}$$

where $0 < a < b$. The function

$$f(x) = \begin{cases} x & \text{for} \quad x < a \quad \text{and} \quad x > b, \\ \varphi(x) & \text{for} \quad a \leqq x \leqq b, \end{cases}$$

where φ is any continuous function such that $\varphi(a) = a$, $\varphi(b) = b$, is a solution of equation (1).

Example 2: Let $h\colon \mathbf{R} \to \mathbf{R}$ be continuous, $h(0) = 0$ and

$$\begin{aligned}
-x < h(x) < 0 \quad & \text{for} \quad 0 < x < a, \\
h(x) = -x \quad & \text{for} \quad a \leqq x \leqq b, \\
-x < h(x) < \min(-b, a - x) \quad & \text{for} \quad x > b, \\
0 < h(x) < -x \quad & \text{for} \quad -a < x < 0, \\
h(x) = -x \quad & \text{for} \quad -b \leqq x \leqq -a, \\
\max(b, x - a) < h(x) < -x \quad & \text{for} \quad x < -b,
\end{aligned}$$

where $0 < a < b$. The function

$$f(x) = \begin{cases} x & \text{for} \quad |x| < a \quad \text{and} \quad |x| > b, \\ \varphi(x) & \text{for} \quad a \leqq x \leqq b, \\ -\varphi(x) & \text{for} \quad -b \leqq x \leqq -a, \end{cases}$$

where φ is any continuous function such that $\varphi(a) = a$, $\varphi(b) = b$, is a solution of equation (1).

It remains an open problem to fill the gap between hypothesis ii) of Lemma 2 and the situation of the previous examples.

3. In Theorem 1 we assumed $A_+ = A_- = \emptyset$; in many situations we can weaken such hypothesis and get again the linearity of f. The next theorem deals with one possible situation.

Theorem 2: *Let $0 < a < b$ and suppose that $h\colon \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:*

i) *$-x < h(x) \leqq b$ for $-b < x < -a$ and $h(-a) = a$;*
ii) *$h(a) \geqq -a$, $h(x) \neq 0$ for $a \leqq x < b$, $h(x) \neq -x$ for $a < x < b$;*
iii) *if $h(a) = -a$, then there exists $\delta > 0$ such that $h(x) \geqq -a$ for $a \leqq x < a + \delta$.*

If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous solution of equation (1) and $f(x) = cx$ for $-a \leqq x \leqq a$, then $f(x) = cx$ for $-b \leqq x \leqq b$.

Proof: If $-a \leqq h(a) < 0$, then by ii) $h(x) < 0$ for $a \leqq x < b$. There exists a_1, $a < a_1 \leqq b$ such that $a < x < a_1$ implies $0 < x + h(x) < a$ and (by iii)) $-a \leqq h(x) < 0$. Whence for $a < x < a_1$ we have $f(x) = f(x + h(x)) - f(h(x)) = cx$. Let now a_1', $a_1' \leqq a_1$, be such that if $-a_1' < x \leqq -a$ then $a \leqq h(x) < a_1$; hence it is also $0 \leqq x + h(x) < a_1$. Thus for $-a_1' < x \leqq -a$ we have $f(x) = f(x + h(x)) - f(h(x)) = cx$. So, by the continuity of f, we have extended f linearly on $[-a_1', a_1]$.

Let $[-a_*', a_*]$ be the largest closed interval contained in $[-b, b]$ where $f(x) = cx$. If $a_* < b$ then $a_*' < a_*$ and, as above, we can again extend f linearly; a contradiction. So $a_* = b$ and the construction above shows that $a_*' = b$.

If $h(a) > 0$, then by ii) $h(x) > 0$ for $a \leqq x < b$: Lemma 1 and Theorem 1 give $f(x) = cx$ for $0 \leqq x \leqq b$. Hence as above we get $f(x) = cx$ on $[-b, b]$ ∎

Theorem 2 holds also when $a = 0$, by assuming the existence of $f'(0)$. The following example shows that the hypothesis iii) in Theorem 2 cannot be dispensed with.

Example 3: Take the function h of the form

$$h(x) = \begin{cases} -2x - 1 & \text{for } x < -1, \\ \varphi(x) & \text{for } |x| \leqq 1, \\ -\dfrac{x}{2} - \dfrac{1}{2} & \text{for } x > 1, \end{cases}$$

where φ is a continuous function such that $\varphi(0) = 0$, $\varphi(1) = -1$, $\varphi(-1) = 1$ and, for $x \neq 0, 1, -1$, $x\varphi(x) < 0$, $x[x + \varphi(x)] > 0$. The following function

$$f(x) = \begin{cases} 0 & \text{for } x \leqq -2, \\ -x - 2 & \text{for } -2 < x \leqq -1, \\ x & \text{for } -1 < x < 1, \\ 1 & \text{for } x \geqq 1, \end{cases}$$

is a solution of equation (1) continuous and differentiable at zero.

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