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Remarks on Duality Mapping and the Lax-Milgram Property

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Es werden einige Bedingungen abgeleitet, die sicherstellen, daß die Dualitäts-Abbildung ein Homeomorphismus von einem Banach-Raum X auf X^* ist, der überdies auf einer dichten Teilmenge von X oberhalbstetig ist. Es werden ferner einige weitere Eigenschaften der Dualitäts-Abbildung bewiesen, die mit der Struktur des Banachraumes X zusammenhängen, und es wird die sogenannte Lax-Milgram Eigenschaft der betrachteten Bilinearformen untersucht.

Находятся условия, при которых дуальное отображение является гомеоморфизмом из Банахова пространства X на X^* и является полунепрерывным сверху на некотором плотном в X множестве. Рассматриваются дальнейшие свойства дуального отображения в связи с геометрической структурой Банахова пространств X и свойство Лакс-Миллграма билинейных форм.

Some conditions under which the duality map is a homeomorphism from a Banach space X onto X^* and upper-semicontinuous at some dense subset of X are derived. Some further properties of the duality mapping are established in connection with the structure of the Banach space X. The so-called Lax-Milgram property of the bilinear forms is also investigated.

1. Introduction

The concept of duality mapping introduced independently by Beurling and Livingston [2] and Cudia [4] has been used in several branches in functional analysis and its applications: theory of monotone and accretive operators, fixed point theory of nonexpansive (and related) operators, theory of approximations and geometry of Banach spaces.

Cudia [4] proved that the duality mapping J is always upper-semicontinuous on X when X has the norm and the dual space has the $\sigma(X^*, X)$-topology, while Kenderov [18] extended this result to maximal monotone operators. Upper-semicontinuity of duality mapping and subdifferential maps has been studied by Giles, Gregory and Sims [13] and Gregory [14], where upper-semicontinuity is characterized in terms of slices of the closed unit ball and upper-semicontinuity properties are related to the geometric structure of the spaces and properties of convex functions.

The purpose of this note is to derive some conditions under which the duality map J is a homeomorphism of X onto X^* and upper-semicontinuous at the dense subset of the given space. Some further properties of the duality mapping are derived in connection with the structure of Banach spaces. Furthermore, so-called Lax-Milgram property of the bilinear forms is investigated.

2. Definitions and notations

Let X be a real normed linear space, X^* its dual space, $\langle\cdot,\cdot\rangle$ the pairing between X^* and X. Let $B_1(0)$, $B_1^*(0)$, $B_1^{**}(0)$ denote the closed unit balls; $S_1(0)$, $S_1^*(0)$, $S_1^{**}(0)$ their boundaries in X, X^*, X^{**}, respectively. Denote by $\sigma(X, X^*)$, $\sigma(X^*, X)$ the weak and weak* topologies on X, X^*, respectively and by $\tau: X \to X^{**}$ a canonical mapping of X into X^{**}. We use the notion of rotundity (or strict convexity) of spaces in usual sense. A normed linear space X is said to be:

(i) *smooth at* $u \in S_1(0)$ if its norm $\|\cdot\|$ is Gâteaux differentiable at u;
(ii) *smooth*, if the norm of X is Gâteaux differentiable on $S_1(0)$;
(iii) an (F)-*space*, if its norm is Fréchet differentiable on $S_1(0)$;
(iv) *weakly locally uniformly rotund* (WLUR) at $u \in S_1(0)$ if for every sequence $(u_n) \subset S_1(0)$ with $\|u_n + u\| \to 2$ there is $u_n \to u$ in the $\sigma(X, X^*)$-topology;
(v) WLUR *at the points of some subset* Q of $S_1(0)$, if X is WLUR at each point u of Q;
(vi) an (h)-*space* if for each sequence (u_n) in X converging in the $\sigma(X, X^*)$ topology of X to u_0 and $\|u_n\| \to \|u_0\|$ we have that $u_n \to u_0$ in the norm topology of X.

By the *gauge function* $\mu: \mathbf{R}^+ \to \mathbf{R}^+$ we mean a real valued strictly increasing continuous function such that $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = +\infty$. A set-valued mapping $J: X \to 2^{X^*}$ is said to be a *duality mapping* of X into X^* *with the gauge function* μ if $J(0) = \{0\}$ and for each $u \in X$, $u \neq 0$,

$$J(u) = \{u^* \in X^*: \langle u^*, u\rangle = \|u^*\| \cdot \|u\|, \|u^*\| = \mu(\|u\|)\}.$$

For $u \in X$, $J(u)$ is non-empty convex and $\sigma(X^*, X)$-compact subset of X^*. X is smooth at $u \in S_1(0)$ if and only if J is single-valued at u (see [4]). The duality mapping of X^* into X^{**} is denoted by J^* and the duality map $J^*: X \to 2^{X^{**}}$ with the gauge function $\mu^* = \mu^{-1}$ is called the associated duality map with J (see [23]) and we denote it by J_a^*. By normalized duality mapping we mean the duality mapping with gauge function $\mu(s) = s$. Let t denote the $\sigma(X^*, X)$, $\sigma(X, X^*)$, or norm topology on X^*. The duality mapping $J: X \to 2^{X^*}$ is said to be *upper-semicontinuous* at $u_0 \in X$ from the norm topology of X to the t-topology of X^* if for any t-open set V in X^* with $J(u_0) \subset V$ there exists an open neighborhood U of u_0 such that $J(U) \subset V$. We shall use the following result of ŠMULIAN [26]. Let X be a Banach space, then the norm of X is Gâteaux (Fréchet) differentiable at $u_0 \in S_1(0)$ if and only of the following implication holds: when $(u_n^*) \subset S_1^*(0)$, $\langle u_n^*, u_0\rangle \to \|u_0\| = 1$, then (u_n^*) is a weak* (strong) Cauchy sequence. Similarly, necessary and sufficient condition for the fact that the norm of X^* is Gâteaux (Fréchet) differentiable at $u_0^* \in S_1^*(0)$ is that the following implication is valid: if $(u_n) \subset S_1(0)$, $\langle u_0^*, u_n\rangle \to 1$, then (u_n) is a weak (strong) Cauchy sequence in X.

3. Some properties of duality mapping

We shall use the following result, which is a special case of the more general statement proved by GILES [9] (compare also [4, 11]) on the base of the BISHOP-PHELPS [3] theorem.

Lemma 1: *If X is a Banach space and X^* is an* (F)-*space, then X is reflexive.*

Proof: Let $u_0^* \in S_1^*(0)$ be arbitrary. We show that there exists $u_0 \in S_1(0)$ such that $\langle u^*, u_0\rangle = 1$. Choose a sequence $(u_n) \subset S_1(0)$ such that $\langle u_0^*, u_n\rangle \to 1$. By the

Šmuljan theorem (u_n) is a Cauchy sequence in X. Hence $u_n \to u_0$ and $u_0 \in S_1(0)$, while $\langle u_0^*, u_0 \rangle = 1$. By the JAMES [17] characterization of reflexivity X is reflexive ∎

Lemma 2: *Let X be a Banach space. Then the following statements are valid:*

(i) (GILES [10]). *A duality mapping J is an homeomorphism of X onto X^* if and only if X and X^* are both* (F)*-space, then X is an* (F)*-spaces.*

(ii) *If X is smooth reflexive and X^* is an* (h)*-space, then X is an* (F)*-space.*

Proof: We give it for the sake of completeness.

(i) In comparison with [10] we use here a slight different argument. Assume that X, X^* are both (F)-spaces. By Lemma 1 X is reflexive. Moreover, J is one-to-one, onto and J, J^{-1} are continuous. Conversely, assume that J is a homeomorphism of X onto X^*. Then X is reflexive and X is an (F)-space. Since $J_a^* = \tau J^{-1}$ (see [23]) and J_a^* is continuous, J^* is also continuous from X^* into X^{**}. Hence X^* is an (F)-space.

(ii) It is sufficient to show that J is continuous on $S_1(0)$. Let $u_0 \in S_1(0)$, $(u_n) \subset X$, $u_n \to u_0$. Then $\mu(\|u_n\|) \to \mu(\|u_0\|)$ and hence $\|J(u_n)\| \to \|J(u_0)\|$ as $n \to \infty$. Since X is smooth and reflexive, J is single-valued and continuous from the norm topology of X to the $\sigma(X, X^*)$-topology of X^*. As X^* is an (h)-space we conclude that $J(u_n) \to J(u_0)$ in the norm topology of X^* ∎

Theorem 1: *Let X be a reflexive smooth* (h)*-Banach space. Then the following statements are valid:*

(i) *J is a proper map (i.e. for each compact set G of X^* the set $J^{-1}(G)$ is compact in X). In addition, if X is rotund, then J^{-1} is continuous on X^*.*

(ii) *If J is a local homeomorphism of X into X^*, then J is a homeomorphism of X onto X^*.*

(iii) *If X^* is smooth and* (h)*-space, then J is a homeomorphism of X onto X^*.*

Proof: It is sufficient to prove our assertions for normalized duality mapping.

(i) Since X is smooth, J is singlevalued on X. Reflexivity of X implies that J is onto. Indeed, each $u^* \in X^*$ is $\sigma(X, X^*)$-continuous and $B_1(0)$ is $\sigma(X, X^*)$-compact. By the Weierstrass Theorem each $u^* \in X^*$ attains its supremum on $S_1(0)$. Hence $J(X) = X^*$. Let G be a compact subset of X^*. Take $(u_n) \subset J^{-1}(G)$ and set $u_n^* = J(u_n)$. Then $(u_n^*) \subset G$ and in view of the compactness of G in X^* there exist a point $u^* \in G$ and a subsequence of (u_n^*), say (u_n^*), such that $u_n^* \to u^*$. From $\|J(u_n)\| = \|u_n^*\|$ we conclude that (u_n) is bounded in X. Hence there exist $u_0 \in X$ and a subsequence of (u_n), say (u_{n_j}) such that $u_{n_j} \to u_0$ in the $\sigma(X, X^*)$-topology of X. As $\|u_n^*\| \to \|u^*\|$ we have that $\lim_j \|u_{n_j}\| = \|u^*\|$. Furthermore,

$$\|u_0\| \leqq \varliminf_j \|u_{n_j}\| = \lim_j \|u_{n_j}\| = \lim_j \|J(u_{n_j})\| = \|u^*\|,$$

$$\lim_j \|u_{n_j}\|^2 = \lim_j \langle J(u_{n_j}), u_{n_j} \rangle = \langle u^*, u_0 \rangle .$$

Hence $\|u^*\|^2 = \langle u^*, u_0 \rangle \leqq \|u^*\| \cdot \|u_0\|$, which implies together with the first inequality that $\|u^*\| = \|u_0\|$ and $\langle u^*, u_0 \rangle = \|u_0\| \cdot \|u^*\|$. Since J is single-valued, $u^* = J(u_0)$ and therefore $u_0 \in J^{-1}(G)$. Moreover, $u_{n_j} \to u_0$ in the $\sigma(X, X^*)$-topology of X and $\|u_{n_j}\| \to \|u^*\| = \|u_0\|$. As X is an (h)-space, we get that $u_{n_j} \to u_0$, which proves that $J^{-1}(G)$ is compact in X.

In addition, assume that X is rotund. Then J is onto and one-to-one by reflexivity and rotundity of X. Let $(u_n^*) \subset X^*$, $u_0^* \in X^*$, $u_n^* \to u_0^*$ in X^*. From the previous considerations it follows that each subsequence (u_{n_j}) of (u_n), where $u_n = J^{-1}(u_n^*)$, contains a subsequence converging to $u_0 = J^{-1}(u_0^*)$ in the $\sigma(X, X^*)$-topology of X.

Hence $u_n \to u_0$ in the $\sigma(X, X^*)$-topology of X and $\|u_n\| \to \|u_0\|$. Therefore $u_n \to u_0$, which proves the continuity of J^{-1}.

(ii) J is proper by (i). Since J is a local homeomorphism of X into X^*, by the Banach-Mazur theorem (see [24]) J is a global homeomorphism of X onto X^*.

(iii) This assertion follows at once from Lemma 2 ∎

Note that under stronger assumptions on X, X^* it was proved in [6] that J is a proper map.

Among others, Yorke [27] proved the following assertions:

(i) If a Banach space X is WLUR at $x \in S_1(0)$, then X^* is smooth at the points of $J(x) \subset S_1(0)$.

(ii) If X is WLUR at some $x^* \in J(x)$, then X is very smooth at x (i.e. every support mapping on X is norm to $\sigma(X^*, X^{**})$ continuous at x). In fact, these results are given in Theorem 2(i) where a different proof method is used.

Let X be a Banach space, G a subset of $S_1(0)$ of all functionals of $S_1^*(0)$ which attains its norm on $S_1(0)$. By the Bishop-Phelps theorem G is norm-dense in $S_1^*(0)$. For each $u^* \in G$ there is some $u \in S_1(0)$ such that $\langle u^*, u\rangle = 1$. Denote by Q the set of such points $u \in S_1(0)$ having the above property, where in Q is included just one point u for each $u^* \in G$. In next we can assume without loss of generality that J, J^* are normalized duality mappings on X, X^*, respectively.

Theorem 2: *Let X be a Banach space. Then the following conclusions are valid:*

(i) *If X is WLUR at the points of $Q \subset S_1(0)$, then J^* is single-valued at the points of dense set G of $S_1^*(0)$. In addition, if X^{**} is WLUR at the points of $J^*(G) \subset S_1^{**}(0)$, then J^* is upper-semicontinuous at the points of G from the norm topology of X^* to the $\sigma(X^{**}, X^{***})$-topology of X^{**}.*

(ii) *If J^* is single-valued and upper-semicontinuous at the points of G when X^*, X^{**} have the norm topologies, then the strong and weak convergence of sequences of $S_1(0)$ coincide at the points of $S_1(0)$. Conversely, if the last condition is satisfied and X is sequentially $\sigma(X, X^*)$-complete and J^* is single-valued at the points of G, then J^* is upper-semicontinuous at the points of G when X^* and X^{**} have the norm topologies.*

Proof: (i) It is sufficient to show that the norm of X^* is smooth at the points of dense set G in $S_1^*(0)$. Let $u_0^* \in G$ be arbitrary. Then there exists a point $u_0 \in Q \subset S_1(0)$ such that $\langle u_0^*, u_0\rangle = \|u_0^*\| = 1$. There exists a sequence $(u_n) \subset S_1(0)$ such that $\langle u_0^*, u_n\rangle \to \langle u_0^*, u_0\rangle = 1$. Then $2 \geqq \|u_n + u_0\| = \langle u_0^*, u_0 + u_n\rangle$. Since $\langle u_0^*, u_0 + u_n\rangle \to 2$ as $n \to \infty$, we have that $\|u_0 + u_n\| \to 2$. As X is WLUR at the points of Q, we conclude that $u_n \to u_0$ in the $\sigma(X, X^*)$-topology of X. By the Šmuljan theorem X^* is smooth at u_0^*.

Suppose, in addition, that X^{**} is WLUR at the points of the set $J^*(G) \subset S_1^{**}(0)$. Let $u_0^* \in G$, $(u_n^*) \subset X^*$, $u_n^* \to u_0^*$. Without loss of generality one may assume that $u_n^* \in S_1^*(0)$. Assume that $v_n^{**} \in J^*(u_n^*)$, $n = 1, 2, \ldots$ By (i) J^* is single-valued at u_0^*. Put $v_0^{**} = J^*(u_0^*)$, it is sufficient to prove that $v_n^{**} \to v_0^{**}$ in the $\sigma(X^{**}, X^{***})$-topology of X^{**}. The properties of J^* imply that

$$2(\|u_n^*\|^2 + \|u_0^*\|^2) - \langle v_n^{**} - v_0^{**}, u_n^* - u_0^*\rangle = \langle v_n^{**} + v_0^{**}, u_0^* + u_n^*\rangle.$$

Hence

$$4 - \|v_n^{**} - v_0^{**}\| \, \|u_n^* - u_0^*\| \leqq 2\,\|v_n^{**} + v_0^{**}\| \leqq 2(\|v_n^{**}\| + \|v_0^{**}\|) = 4.$$

Therefore $\|v_n^{**} + v_0^{**}\| \to 2$ as $n \to \infty$. Since X^{**} is WLUR at the points of $J^*(G)$, $v_n^{**} \to v_0^{**}$ in the $\sigma(X^{**}, X^{***})$-topology of X^{**}, which proves that J^* is upper-semicontinuous at $u_0^* \in G$ from the norm topology of X^* in the $\sigma(X^{**}, X^{***})$-topology of X^{**}.

(ii) If J^* is single-valued and upper-semicontinuous at the points of $G \subset S_1^*(0)$ when X^*, X^{**} have the norm topologies, then the norm of X^* is Fréchet-differentiable at the points of G (see [11, 13]). Let $u_0 \in S_1(0)$, $(u_u) \subset S_1(0)$, $u_n \to u_0$ in the $\sigma(X, X^*)$-topology of X. Then there exists $u_0^* \in S_1^*(0)$ such that $\langle u_0^*, u_0\rangle = \|u_0\| = 1$. Hence $u_0^* \in G$. Since $\langle u_0^*, u_n\rangle \to \langle u_0^*, u_0\rangle = 1$ and the norm of X^* is Fréchet-differentiable at the points of G, according to the Šmuljan theorem (u_n) converges to u_0 in the norm topology of X. Conversely, assume that X is sequentially $\sigma(X, X^*)$-complete and that J^* is single-valued at the points of G. Then X^* is smooth at these points. Let $u_0^* \in G$ be arbitrary, $(u_n) \subset S_1(0)$ be such that $\langle u_0^*, u_n\rangle \to 1$. By the Šmuljan theorem (u_n) is a weak Cauchy sequence in X. Hence $u_n \to u_0$ in the $\sigma(X, X^*)$-topology of X for some $u_0 \in X$. Clearly, $u_0 \in S_1(0)$. According to our hypothesis $u_n \to u_0$ in the norm of X. Again, in view of the Šmuljan theorem, the norm of X^* is Fréchet-differentiable at u_0^*. Hence J is upper-semicontinuous at u_0^* when X^* and X^{**} have the norm topologies (see [11—13]) ▮

We shall use the following

Lemma 3 [23]: *Let X be a real normed linear space, $J: X \to 2^{X^*}$, $J^*: X^* \to 2^{X^{**}}$ normalized duality mappings. Then an element $u^* \in X^*$ lies in $J(u)$ for some $u \in X$ if and only if $\tau(u) \in J^*(u^*)$.*

Proposition 1: *Let X be a non-reflexive normed linear space such that X^* is smooth, $J: X \to 2^{X^*}$, $J^*: X^* \to X^{**}$ duality mappings. If $R(J) = X^*$, then there exists a separable closed linear subspace W of X^* such that J^* is not onto W^*.*

Proof: It depends on the arguments of [23] and [5] and it is given here for the sake of completeness. First of all, we show that if X^* is smooth and $R(J) = X^*$, then $R(J^*) = \tau(X)$. Clearly, $\tau(X) \subset R(J^*)$. Indeed, if $u_0^{**} \in \tau(X)$, then u_0^{**} is $\sigma(X^*, X)$-continuous on X^*. Since $B_1^*(0)$ is $\sigma(X^*, X)$-compact, by the Weierstrass theorem u_0^{**} attains its supremum on $S_1^*(0)$. Therefore $u_0^{**} \in R(J^*)$. We assert that $R(J^*) \subset \tau(X)$. Assume that $u_0^{**} \in R(J^*)$, $u_0^{**} \neq 0$. Then there is $u_0^* \in X^*$ such that $u_0^{**} = J^*(u_0^*)$. As $R(J) = X^*$ there exists $u_0^* \in X^*$ such that $u_0^* \in J(u_0)$. By Lemma 3 $\tau(u_0) = J^*(u_0^*)$. Since X^* is smooth we get that $\tau(u_0) = u_0^{**}$ and hence $\tau(X) \supset R(J^*)$. According to our hypothesis X is not reflexive. The ball $B_1^*(0)$ of X^* is $\sigma(X^*, X)$-compact but it is not $\sigma(X^*, X^{**})$-countably compact. Indeed, if $B_1^*(0)$ would be $\sigma(X^*, X^{**})$-countably compact, then $B_1^*(0)$ would be also $\sigma(X^*, X^{**})$-compact by the Eberlein-Šmuljan theorem, which is impossible, because the $\sigma(X^*, X^{**})$- and the $\sigma(X^*, X)$-topologies agree on X^* if and only if X is reflexive. As $B_1^*(0)$ is not $\sigma(X^*, X^{**})$-countably compact there is a sequence $(u_n^*) \subset B_1^*(0)$, having no $\sigma(X^*, X^{**})$-convergent subnet. Since, $B_1^*(0)$ is $\sigma(X^*, X)$-compact, there is a subnet $(u_\alpha^*) \subset (u_n^*)$ and a point $u^* \in B_1^*(0)$ such that $u_\alpha^* \to u^*$ in the $\sigma(X^*, X)$-topology. Put $W = \overline{\text{span}}\,\{(u_n^*) \cup (u^*)\}$. Then W is closed separable subspace of X^*. For each fixed $u \in X$ $\tau(u)$ is a $\sigma(X^*, X)$-continuous linear functional on X^* and therefore $\langle \tau(u), u_\alpha^*\rangle \to \langle \tau(u), u^*\rangle$ for each (fixed) $u \in X$. Since (u_α^*) is a subnet of (u_n^*) and (u_n^*) contains no $\sigma(X^*, X^{**})$-convergent subnet, we conclude that $u_\alpha^* \nrightarrow u^*$ in the $\sigma(X^*, X^{**})$-topology of X^*. Each $z^{**} \in W^*$ is a restriction of some u^{**} of X^{**} to W and conversely, each linear continuous functional z^{**} defined on W can be continuously extended on the whole space X^*. Hence there is $z^{**} \in W^*$ such that $\langle z^{**}, u_\alpha^*\rangle \nrightarrow \langle z^{**}, u^*\rangle$. Thus z^{**} is not $\sigma(X^*, X)$-continuous, i.e. $z^{**} \notin \tau(X) = R(J^*)$, which proves the assertion ▮

Proposition 2: *Let X be a normed linear space such that X^* is smooth, M a bounded $\sigma(X, X^*)$-closed subset of X. If $R(J) = X^*$ and J^* is continuous from the $\sigma(X^*, X)$-topology of X^* to the $\sigma(X^{**}, X^*)$-topology of X^{**}, then $J(M)$ is $\sigma(X^*, X)$-compact.*

Proof: Since M is bounded we have that $\sup \{\|J(u)\|: u \in M\} < +\infty$. Hence $J(M)$ is relatively $\sigma(X^*, X)$-compact in X^*. Assume that $u_0^* \in \overline{J(M)}^{\sigma(X^*, X)}$. Then there exists a net $(u_\alpha^*)_{\alpha \in I}$ in $J(M)$ such that $u_\alpha^* \to u_0^*$ in the $\sigma(X^*, X)$-topology of X^*. Hence there are $u_\alpha \in M$ such that $u_\alpha^* \in J(u_\alpha)$ for each $\alpha \in I$. By our hypotheses J is single-valued and $R(J^*) = \tau(X)$ (see the first part of the proof of Proposition 2). In virtue of Lemma 3 $\tau(u_\alpha) = J^*(u_\alpha^*)$ for each $\alpha \in I$ and $J^*(u_\alpha^*) \to J^*(u_0^*)$ in the $\sigma(X^{**}, X^*)$-topology of X^{**}. Setting $u_0^{**} = J^*(u_0^*)$ we have that $u_0^{**} \in \tau(X)$ and hence u_0^{**} is $\sigma(X^*, X)$-continuous on X^*. There exists a point $u_0 \in X$ such that $u_0^{**} = \tau(u_0)$. Now we have that $\tau(u_\alpha) \to \tau(u_0)$ in the $\sigma(X^{**}, X^*)$-topology of X^{**}. Since τ is a linear homeomorphism from the space $(X, \sigma(X, X^*))$ into the space $(X^{**}, \sigma(X^{**}, X^*))$, we conclude that $u_\alpha \to u_0$ in the $\sigma(X, X^*)$-topology of X. Since M is $\sigma(X, X^*)$-closed, $u_0 \in M$. Because $\tau(u_0) = J^*(u_0^*)$, by Lemma 3 we get that $u_0^* \in J(u_0) \subset J(M)$, which concludes the proof ∎

4. The Lax-Milgram property of bilinear forms

In [20—22] we established some characterizations of reflexivity of Banach spaces by means of the duality mapping. Now we derive once more characterization of reflexivity of Banach spaces based on the so-called Lax-Milgram property of bilinear forms. This result is inspired by [15, 25].

Definition 1: Let X, Y be normed linear spaces. We shall say that X has the *Lax-Milgram property* (LMP) with respect to Y if there exists a bilinear form $Q \times Y \to \mathbf{C}$ with the following property: For a given linear closed separable subspace F of X there exists a separable subspace P of Y such that Q is bounded on $F \times P$ and for each $u^* \in F^*$ there exists a unique point v_P in P such that $\langle u^*, u \rangle = Q(u, v_P)$ for each $u \in F$.

Similarly we shall say that Y has the LMP with respect to X if there is a bilinear form $Q : X \times Y - \mathbf{C}$ with the following property: For a given linear closed separable subspace V of Y there exists a separable linear subspace E of X such that Q is bounded on $E \times V$ and for each $v^* \in V^*$ there exists a unique element u_E in E such that $\langle v^*, v \rangle = Q(u_E, v)$ for each $v \in V$.

Proposition 3: *Let X, Y be normed linear spaces. If X is sequentially weakly complete and has the* LMP *with respect to Y, then X is reflexive.*

Proof: It relies on the argument of [25]. Let (u_n) be a bounded sequence in X. Put $F = \overline{\text{span}}\, \{(u_n)\}$. Then F is a closed separable subspace of X. By our hypothesis there exist a separable linear subspace P of Y and a bilinear form $Q : X \times Y \to \mathbf{C}$ such that Q is bounded on $F \times P$, i.e. $|Q(u, v)| \leqq M_{F,P} \|u\| \cdot \|v\|$ for each $u \in F$, $v \in P$ and some constant $M_{F,P} > 0$, and the representation of the elements $u^* \in F^*$ by means of Q and the unique points v_P of P is valid. Let (v_n) be a dense sequence in P. Define the linear continuous functionals (u_n^*) by $\langle u_n^*, u \rangle = Q(u, v_n)$ for each $u \in F$ and n $(n = 1, 2, \ldots)$. Clearly, $u_n^* \in F^*$ for each n. We assert that (u_n^*) is dense in F^*. Indeed, if $u^* \in F^*$ is an arbitrary point, then by our hypothesis, there exists a unique point $v_P \in P$ such that $\langle u^*, u \rangle = Q(u, v_P)$ for each $u \in F$. As (v_n) is dense in P, there exists a subsequence of (v_n), say (v_n), such that $v_n \to v_P$ as $n \to \infty$. Then

$$|\langle u^* - u_n^*, u\rangle| = |Q(u, v_P - v_n)| \leqq M_{F,P} \|u\| \cdot \|v_P - v_n\|$$

for each $u \in F$ and

$$\|u^* - u_n^*\| \leqq \sup_{\|u\|=1} |\langle u^* - u_n^*, u\rangle| \leqq M_{F,P} \|v_P - v_n\|.$$

Hence (u_n^*) is dense in F^*. Now $(\langle u_i^*, u_n\rangle)_{n=1}^{\infty}$ is a bounded sequence of reals for each fixed i $(i = 1, 2, \ldots)$. By the diagonal process one can extract a subsequence (u_k) of (u_n) such that $(\langle u_i^*, u_k\rangle)_{k=1}^{\infty}$ is convergent for each i. In view of the density of (u_n^*) in F^* we conclude that $(\langle u^*, u_k\rangle)$ is convergent for each $u^* \in F^*$. By the Hahn-Banach theorem for each $u^* \in F^*$ there exists some $v^* \in X^*$ such that $v^* \mid F = u^*$ and $\|u^*\| = \|v^*\|$. On the other hand each $v^* \in X^*$ restricted to F is an element of F^*. Hence (u_k) is the weak Cauchy sequence in X. Since X is sequentially $\sigma(X, X^*)$-complete, there exists a point $u_0 \in X$ such that $u_k \to u_0$ in the $\sigma(X, X^*)$-topology of X. Hence X is reflexive ∎

Corollary 1: *Let X, Y be sequentially weakly complete Banach spaces such that X has the* LMP *with respect to Y and Y has the* LMP *with respect to X. Then X, Y are both reflexive.*

Problems: (i) It would be interesting to describe the properties of the set Q (see Theorem 2) in connection with the geometric structure of the Banach spaces.

(ii) Definition 1 together with Proposition 4 and Theorem 1 [15] imply the following question: Let X be a Banach space and assume that each closed separable subspace F of X is isomorphic (isometric) to its dual F^*. What is the geometric structure of the space X?

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