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Coefficient Control in a Linear Second Order Ordinary Differential Equation

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Wir betrachten ein Steuerproblem, bei dem der Zustand durch die Lösung einer linearen Randwertaufgabe für eine gewöhnliche Differentialgleichung zweiter Ordnung beschrieben wird. Der Koeffizient der ersten Ableitung in der Zustandsgleichung ist dabei die Steuerfunktion. Es wird die Existenz optimaler Steuerungen bewiesen und deren Eindeutigkeit diskutiert. Des weiteren werden sowohl notwendige als auch hinreichende Optimalitätsbedingungen formuliert.

Рассматривается задача оптимального управления, при которой состояние является решением линейной граничной задачи для обыкновенного дифференциального уравнения второго порядка. Управление — коэффициент первой производной уравнения. Доказывается существование оптимального управления и изучается его единственность. Кроме того формулируются и необходимые и достаточные условия оптимальности.

In this paper we deal with a control problem whose behaviour is described by the solution of a linear boundary value problem of a second order ordinary differential equation. The coefficient of the first derivative in the state equation is assumed to be the control. We prove the existence of optimal control and discuss its uniqueness. Moreover, we formulate both necessary and sufficient optimality conditions.

1. Introduction

The present paper is devoted to the study of an optimal control problem for a second order ordinary differential equation. The characteristic feature of the problem is the occurence of the control as the coefficient of the first derivative in the state equation. Both existence of optimal controls is proved and a necessary optimality condition is given without any additional requirements. Uniqueness of optimal control is shown and a sufficient optimality condition is formulated under assumption that the given data are in a certain relation to each other. It is worthwhile to hint at the fact that this assumption can be easily fulfilled by appropriate choice of the parameters ε and δ in the cost functional (1). We remark that our problem is related to an inverse problem in Geophysics about which we shall report in a later paper.

We start with the statement of the problem, and then we consider the questions mentioned above in separate paragraphs. In each of them we begin with some remarks concerning the method applied in it and relevant literature.

2. Problem Statement, Preliminaries

Throughout the whole paper we use the following notations. $\mathbf{R}$ is the set of all real numbers, (a, b) is a bounded interval. $L^2(a, b)$ with the scalar product $(\cdot, \cdot)$ and the norm $\|\cdot\|$, and $L^\infty(a, b)$ with the norm $\|\cdot\|_\infty$ are the usual Lebesgue spaces of real-valued functions defined on (a, b). $H^m(a, b)$ is the known Sobolev space of order m. Its scalar product and norm will be denoted by $(\cdot, \cdot)_m$ and $\|\cdot\|_m$, respectively. $H_0^1(a, b)$ is the subspace of $H^1(a, b)$ whose elements vanish at the ends of the interval (a, b). $H^{-1}(a, b)$ is the dual space of $H_0^1(a, b)$; $\langle \cdot, \cdot \rangle$ will be used for the generalized scalar product between these both spaces. $\mathfrak{L}(X, Y)$ stands for the Banach space of all linear bounded mappings defined on the whole Banach space X with values in another Banach space Y.

The optimal control problem in question is the following:

$$\inf \mathsf{J}(y, u), \qquad \mathsf{J}(y, u) = \varepsilon \|y - g\|^2 + \delta \|u - h\|^2, \tag{1}$$

subject to

$$\left.\begin{aligned} -y''(x) + u(x)\, y'(x) + y(x) &= f(x), \qquad x \in (a, b) \\ y(a) = y(b) &= 0, \end{aligned}\right\} \tag{2}$$

and

$$u \in U_{\text{ad}} = \{u \in L^2(a, b) \mid \xi_1(x) \leqq u(x) \leqq \xi_2(x) \text{ a.e. } x \in (a, b)\}, \tag{3}$$

where the functions $f, g, h \in L^2(a, b)$ and the constants $\varepsilon > 0, \delta \geqq 0$ are given as well as the functions $\xi_{1,2} \in L^\infty(a, b)$ which are supposed to fulfil the inequality

$$-1 \leqq \xi_1(x) \leqq \xi_2(x) \leqq 1 \qquad \text{a.e. } x \in (a, b). \tag{4}$$

We start the consideration of the control problem (1—3) with rewriting the boundary value problem (2) in form of an operator equation. To this end we recall that for any fixed $u \in U_{\text{ad}}$ an element $y \in H_0^1(a, b)$ is said to be weak solution to (2) if

$$\Pi(y, z) + \Pi_u(y, z) = \langle \mathsf{F}, z \rangle \ \forall z \in H_0^1(a, b), \tag{5}$$

where

$$\left.\begin{aligned} \Pi(y, z) &= \int_a^b \{y'(x)\, z'(x) + y(x)\, z(x)\}\, dx = (y, z)_1, \\ \Pi_u(y, z) &= \int_a^b u(x)\, y'(x)\, z(x)\, dx, \\ \langle \mathsf{F}, z \rangle &= \int_a^b f(x)\, z(x)\, dx = (f, z). \end{aligned}\right\} \tag{6}$$

Both $\Pi(\cdot, \cdot)$ and $\Pi_u(\cdot, \cdot)$ are bilinear forms defined on $H_0^1(a, b) \times H_0^1(a, b)$. From the generalized Lax-Milgram Theorem and from

$$\Pi(y, y) = \|y\|_1^2, \qquad \Pi(y, z) \leqq \|y\|_1 \|z\|_1 \ \forall y, z \in H_0^1(a, b)$$

it follows that the linear operator $\mathsf{A}: H_0^1(a, b) \to H^{-1}(a, b)$ induced by $\Pi(\cdot, \cdot)$ via

$$\langle \mathsf{A}y, z \rangle = \Pi(y, z) \ \forall y, z \in H_0^1(a, b) \tag{7}$$

is bounded and invertible; we have

$$\|\mathsf{A}\|_{\mathfrak{L}(H_0^1(a,b), H^{-1}(a,b))}, \|\mathsf{A}^{-1}\|_{\mathfrak{L}(H^{-1}(a,b), H_0^1(a,b))} \leqq 1. \tag{8}$$

(In (8) we could write the equality sign, because A is the duality operator of $H_0^1(a, b)$.) But the bilinear form $\Pi_u(\cdot, \cdot)$ is only bounded because

$$\Pi_u(y, z) \leqq \|u\|_\infty \|y'\| \|z\| \leqq \|y\|_1 \|z\|_1 \ \forall u \in U_{\text{ad}}, \forall y, z \in H_0^1(a, b)$$

and not coercive on $H_0^1(a, b)$. The latter can be seen in case of constant control $u(x) = u_0$ from the identity

$$2\Pi_u(y, y) = 2u_0 \int_a^b y'(x)\, y(x)\, dx = u_0\big(y^2(b) - y^2(a)\big) = 0$$

which is obviously valid for any $y \in H_0^1(a, b)$. This means concerning the linear operator $\mathsf{B}(u): H_0^1(a, b) \to H^{-1}(a, b)$ induced by

$$\langle \mathsf{B}(u)\, y, z\rangle = \Pi_u(y, z) \ \forall y, z \in H_0^1(a, b) \tag{9}$$

we may state only its boundedness and the estimate

$$\|\mathsf{B}(u)\|_{\mathfrak{L}(H_0^1(a,b),H^{-1}(a,b))} \leqq 1. \tag{10}$$

Here u is an arbitrary element belonging to U_{ad}. Despite the last fact the operator

$$\mathsf{T}(u) = \mathsf{A} + \mathsf{B}(u)$$

being an element of $\mathfrak{L}\big(H_0^1(a, b), H^{-1}(a, b)\big)$ with

$$\|\mathsf{T}(u)\|_{\mathfrak{L}(H_0^1(a,b),H^{-1}(a,b))} \leqq 2 \tag{11}$$

is invertible and it yields

$$\|\mathsf{T}(u)^{-1}\|_{\mathfrak{L}(H^{-1}(a,b),H_0^1(a,b))} \leqq 2$$

for any fixed $u \in U_{\text{ad}}$. This is a consequence of the estimate

$$\begin{aligned}\Pi(y, y) + \Pi_u(y, y) &= \|y\|_1^2 + \int_a^b u(x)\, y'(x)\, y(x)\, dx \\ &\geqq \|y\|_1^2 - \|y'\|\,\|y\| \geqq \|y\|_1^2 - \frac{1}{2}\left(\|y'\|^2 + \|y\|^2\right) \\ &= \frac{1}{2}\|y\|_1^2 \ \forall u \in U_{\text{ad}},\ \forall y \in H_0^1(a, b)\end{aligned} \tag{12}$$

showing the coercitivity of the bilinear form $\Pi(\cdot,\cdot) + \Pi_u(\cdot,\cdot)$ for any fixed control $u \in U_{\text{ad}}$.

Finally, taking into account that F defined in the last equation of (6) lies in $H^{-1}(a, b)$ we conclude the following: For fixed $u \in U_{\text{ad}}$ an element $y \in H_0^1(a, b)$ is a solution to the boundary problem (2) if and only if it solves the operator equation

$$\mathsf{T}(u)\, y = \mathsf{F}, \qquad \mathsf{T}(u) = \mathsf{A} + \mathsf{B}(u), \tag{13}$$

where A, B and F are defined by (7, 9), and (6), respectively. The above considerations lead to the known

Lemma 1: (i) *For any admissible control $u \in U_{\text{ad}}$ there exists an unique weak solution $y \in H_0^1(a, b)$ to* (2) *related to that control.*

(ii) *It yields $\|y\|_1 \leqq 2\,\|f\|$ for any admissible control and any weak solution to* (2).

We have seen that our control problem (1—3) can be interpreted as a control problem in Banach spaces with $H_0^1(a, b)$ as the state space and $L^2(a, b)$ or $L^\infty(a, b)$ as the control space. The state equation is given in (13), the set of admissible controls in (3), and the cost functional in (1). In the section on uniqueness of optimal control and

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on optimality conditions, respectively, we shall work within the framework just described. However, dealing with existence of optimal control we have to consider the boundary value problem (2) more intensively. This will be done in the next section.

3. Existence of Optimal Controls

There are no difficulties in proving the existence of at least one optimal control to (1—3) if the given set of admissible controls is replaced, e.g., by a convex closed bounded subset of the Sobolev space $H^1(a, b)$; cf. [2, 4, 5]. Under this assumption even the top-order coefficient in the differential equation may depend nonlinearly on the control. However, if we take U_{ad} as given in (3) it is impossible to prove a general existence theorem for such a coefficient control problem. We refer to paper [8] where some related counterexamples are given. We mention also Theorem 1.17 in [2] by means of which the existence of optimal controls can be proved provided the function h occuring in the cost functional (1) belongs to a certain dense subset of $L^2(a, b)$. However, since this subset is not described explicitly it is nearly impossible to decide in this way whether an optimal control exists for a given $h \in L^2(a, b)$. The aim of this section is to prove the existence of at least one optimal control for our model problem (1—3).

First we have to deal again with the boundary value problem (2) for fixed $u \in U_{ad}$. The following lemma is essential for proving the existence of optimal controls.

Lemma 2: (i) *For fixed $u \in U_{ad}$ the weak solution to* (2) *lies in $H^2(a, b)$.*

(ii) *There is a positive constant c not depending on y, u, and f such that $\|y\|_2 \leqq c\,\|f\|$ for any admissible control and any weak solution to* (2).

To prove this lemma we study a.e.-solution of (2) which are defined as solutions $y \in H^2_{0,0}(a, b)$, $H^2_{0,0}(a, b) = H_0{}^1(a, b) \cap H^2(a, b)$ of the operator equation

$$\mathsf{A}_1 y = f, \tag{14}$$

where A_1 is the second order differential operator given by

$$\mathsf{A}_1 y = -y''(x) + u(x)\, y'(x) + y(x), \qquad y \in H^2_{0,0}(a, b).$$

We have $\mathsf{A}_1 \in \mathfrak{L}\big(H^2_{0,0}(a, b), L^2(a, b)\big)$ and

$$\|\mathsf{A}_1 y\| \leqq \gamma\, \|y\|_2 \quad \text{with} \quad \gamma = \sqrt{3}\ \forall y \in H^2_{0,0}(a, b). \tag{15}$$

Obviously, if y is an a.e.-solution to (2), then it is also a weak solution to (2). And vice versa, if $y \in H^2(a, b)$ is a weak solution to (2), then y solves (2) almost everywhere in (a, b). Thus, because of Lemma 1 statement (i) is proved after showing that (14) is uniquely solvable for arbitrary $f \in L^2(a, b)$.

Let $\mathsf{A}_0 \in \mathfrak{L}\big(H^2_{0,0}(a, b), L^2(a, b)\big)$ be another second order differential operator defined by

$$\mathsf{A}_0 y = -y''(x), \qquad y \in H^2_{0,0}(a, b).$$

We have

$$\|\mathsf{A}_0 y\| \leqq \|y\|_2\ \forall y \in H^2_{0,0}(a, b). \tag{16}$$

In virtue of [10: p. 305, Satz 23.5] the operator A_0 is invertible, and its range is the whole $L^2(a, b)$. In other words, A_0 has all the properties which we would like to have for A_1. Now we introduce the operator family $A_t \in \mathfrak{L}(H^2_{0.0}(a, b), L^2(a, b))$, $t \in [0,1]$, by setting

$$A_t = A_0 + t(A_1 - A_0), \qquad t \in [0, 1].$$

The following two lemmas give basic properties of A_t needed to prove Lemma 2.

Lemma 3: *There is a constant $c > 1$ which depends neither on t nor on y such that*

$$\|y\|_2 \leqq c\,\|A_t y\| \ \forall t \in [0, 1], \quad \forall u \in U_{ad}, \quad \forall y \in H^2_{0.0}(a, b).$$

Proof: The proof consists of four steps. Let $y \in H^2_{0.0}(a, b)$ be arbitrary, but fixed.

1. Using Cauchy-Schwarz inequality and integrating by parts we get

$$\|A_t y\|\,\|y\| \geqq (A_t y, y) = \|y'\|^2 + t\,\|y\|^2 + t \int_a^b u(x)\, y'(x)\, y(x)\, dx.$$

To estimate below the last term we make use of (3, 4) and the elementary inequality

$$2ab \geqq -\left(\varrho a^2 + \frac{1}{\varrho} b^2\right) \forall a, b \in \mathbf{R}, \quad \forall \varrho > 0 \tag{17}$$

with $\varrho = 1$. We obtain

$$\|A_t y\|\,\|y\| \geqq \left(1 - \frac{t}{2}\right) \|y'\|^2 + \frac{t}{2}\,\|y\|^2 \geqq c_0(t)\,\|y\|^2; \tag{18}$$

with $c_0(t) = \left(1 - \frac{t}{2}\right) k^2 + \frac{t}{2}$, where the constant k is taken out from Friedrichs inequality. Since

$$c_0(t) \geqq d^{-1} = \min\left\{k^2, \frac{1}{2}\,(k^2 + 1)\right\} \forall t \in [0, 1]$$

we have

$$d^2\,\|A_t y\|^2 \geqq \|y\|^2 \ \forall t \in [0, 1]. \tag{19}$$

2. Combining (18) and (19) it follows

$$d\,\|A_t y\|^2 \geqq \left(1 - \frac{t}{2}\right) \|y'\|^2 + \frac{t}{2}\,\|y\|^2 \geqq \left(1 - \frac{t}{2}\right) \|y'\|^2 \geqq \frac{1}{2}\,\|y'\|^2,$$

i.e.,

$$2d\,\|A_t y\|^2 \geqq \|y'\|^2 \ \forall t \in [0, 1]. \tag{20}$$

3. We have $\|A_t y\|^2 = \|y''\|^2 + t^2\,\|uy' + y\|^2 - 2t(y'', uy' + y)$. Applying again Cauchy-Schwarz inequality and (17) with $\varrho = 1/2$ we find $\|A_t y\|^2 \geqq \left(1 - \frac{t}{2}\right) \|y''\|^2 + t(t - 2)\,\|uy' + y\|^2$ and, consequently, $\|A_t y\|^2 + 2t\,\|uy' + y\|^2 \geqq \left(1 - \frac{t}{2}\right) \|y''\|^2$. Keeping in mind $t \in [0, 1]$ we get $\|A_t y\|^2 + 2\,\|uy' + y\|^2 \geqq \frac{1}{2}\,\|y''\|^2$. We apply (19, 20) to $\|uy' + y\|$ and obtain

$$\|uy' + y\|^2 \leqq (\|y'\| + \|y\|)^2 \leqq 2(\|y'\|^2 + \|y\|^2) \leqq 2d(2 + d)\,\|A_t y\|^2,$$

i.e.,

$$2\big(1 + 4d(2 + d)\big)\,\|A_t y\|^2 \geqq \|y''\|^2 \ \forall t \in [0, 1]. \tag{21}$$

4. Finally, adding (19—21) and extracting the root on both sides we obtain the desired inequality with the constant $c = \sqrt{2 + 18d + 9d^2} > 1$ ∎

Lemma 4: *If for a chosen parameter $t_0 \in [0, 1)$ the operator equation*

$$\mathsf{A}_t y = f \tag{22}$$

is uniquely solvable in $H^2_{0,0}(a, b)$ for any $f \in L^2(a, b)$, then it has the same property for any parameter $t \in [t_0, t_0 + t_1] \cap [0, 1]$, where $t_1 = 1/2c(1 + \gamma)$ with the constants γ and c introduced in (15) and Lemma 2, respectively.

Proof: We write A_t in the form $\mathsf{A}_t = \mathsf{A}_{t_0} + (t - t_0)(\mathsf{A}_1 - \mathsf{A}_0)$ so that (22) is equivalent to

$$\mathsf{A}_{t_0} y + (t - t_0)(\mathsf{A}_1 - \mathsf{A}_0)\, y = f.$$

Since by assumption the inverse of A_t for $t = t_0$ exists and its range coincides with all of $L^2(a, b)$ the last equation in its turn is equivalent to the fixed point equation

$$\mathsf{B}_t y = y$$

with

$$\mathsf{B}_t y = \mathsf{A}_{t_0}^{-1} f - (t - t_0)\, \mathsf{A}_{t_0}^{-1-1}(\mathsf{A}_1 - \mathsf{A}_0)\, y, \qquad y \in H^2_{0,0}(a, b).$$

B_t is for any $t \in [t_0, t_0 + t_1] \cap [0, 1]$ a contracting mapping in $H^2_{0,0}(a, b)$ because of

$$\begin{aligned} \|\mathsf{B}_t y - \mathsf{B}_t z\|_2 &\leqq (t - t_0)\, \|\mathsf{A}_{t_0}^{-1}\|_{\mathfrak{L}(L^2(a,b), H^2_{0,0}(a,b))} \left(\|\mathsf{A}_1(y - z)\| + \|\mathsf{A}_0(y - z)\|\right) \\ &\leqq t_1 c(1 + \gamma)\, \|y - z\|_2 \ \forall y, z \in H^2_{0,0}(a, b), \end{aligned}$$

where we have used (15, 16), and the inequality for $t = t_0$ proved in Lemma 3. By Banach Fixed Point Theorem the statement follows ∎

We may give now the proof of Lemma 2: For $t = 0$ the operator equation (22) reduces to $\mathsf{A}_0 y = f$. As we have stated above this equation is uniquely solvable in $H^2_{0,0}(a, b)$ for each $f \in L^2(a, b)$. Hence, by Lemma 4 the operator equation (22) has the same property for any $t \in [0, t_1]$. Putting now $t_0 = t_1$, we see this is also true for any $t \in [t_1, 2t_1]$. Continueing this procedure we come after a finite number of steps to an interval which contains $t = 1$. This means, the operator equation $\mathsf{A}_1 y = f$ is uniquely solvable in $H^2_{0,0}(a, b)$ for arbitrary $f \in L^2(a, b)$, and the first assertion (i) of Lemma 2 is proved. The second statement (ii) follows immediately from Lemma 3 for $t = 1$ ∎

After this preparations we are ready to prove the wanted existence theorem for our control problem.

Existence Theorem: *For any parameters $\varepsilon > 0$ and $\delta \geqq 0$ there is at least one optimal pair $\{y_0, u_0\} \in H_0^1(a, b) \times L^2(a, b)$ to the control problem (1—3).*

Proof: We may follow the scheme given by Weierstrass Theorem. Let $\{(y_n, u_n)\} \subset H_0^1(a, b) \times L^2(a, b)$ be a minimizing sequence of J, i.e.,

$$\lim_{n \to \infty} \mathsf{J}(y_n, u_n) = \inf \mathsf{J}(y, u), \qquad u_n \in U_{\text{ad}} \quad \text{and} \quad y_n \in H_0^1(a, b)$$

with

$$\Pi(y_n, z) + \Pi_{u_n}(y_n, z) = \langle \mathsf{F}, z \rangle \ \forall z \in H_0^1(a, b), \quad n = 1, 2, \ldots \tag{23}$$

We notice that U_{ad} is bounded and, closed and convex in $L^2(a, b)$, that $\{y_n\}$ is a bounded sequence in $H^2(a, b)$ by Lemma 2, and that the imbedding of $H^2(a, b)$

into $H^1(a, b)$ is completely continuous. Thus, without loss of generality, we may assume

$$u_n \to u_0 \text{ weak* in } L^2(a, b),\ y_n \to y_0 \text{ strong in } H_0^1(a, b).$$

As $n \to \infty$ in (23) we see, $y_0 \in H_0^1(a, b)$ is the weak solution of (2) corresponding to control u_0. Since u_0 belongs to U_{ad} the pair $\{y_0, u_0\}$ is admissible. The simple structure of cost functional J implies

$$\underline{\lim}\ \mathsf{J}(y_n, u_n) \geqq \mathsf{J}(y_0, u_0)$$

from which follows that the constructed pair $\{y_0, u_0\}$ solves the control problem (1—3) ∎

4. Uniqueness of Optimal Control

In this section we are concerned with the problem of uniqueness of optimal control to (1—3). This important question has been tackled with success, e.g., in [1, 4, 5, 7] for different types of operator equations as state equation. However, none of the results found in these papers are just applicable to our special control problem (1—3), or say (1, 13, 3). In particular, we cannot use the uniqueness theorem due to Bruckner [1] since the operator B defined in (9, 6) is not coercive. Nevertheless, as the following shows we may proceed along the lines of Bruckner's paper. We remark that on the basis of [1, 4, 5] one can obtain an uniqueness theorem if the control occurs in all coefficients of the differential operator, i.e., also in the top-order coefficient. This is very remarkable.

First we prove the uniqueness of optimal control to (1—3) under the assumption that a certain relation between the given data is satisfied. Afterwards we give in the general case an upper bound for the diameter of the set of optimal controls.

Uniqueness Theorem: *If the inequality*

$$\delta > 16(b - a)\,(2\,\|f\| + \|g\|)\,\|f\|\,\varepsilon \tag{24}$$

holds, then the control problem (1—3) *has not more than one solution.*

Proof: It suffices to show the strong convexity of the cost functional (1) if we regard it as a functional defined on U_{ad}. (This makes sense since y depends on u.) For this we have to prove various auxiliary inequalities.

1. Because of

$$|z(x)| = \left|\int_a^x z'(x)\,dx\right| \leqq \sqrt{b-a}\,\|z'\| \leqq \sqrt{b-a}\,\|z_1\|\ \forall x \in [a,b],\ \forall z \in H_0^1(a,b)$$

and, hence,

$$|\Pi_u(y, z)| \leqq \sqrt{b-a}\,\|z\|_1 \int_a^b |u(x)\,y'(x)|\,dx \leqq \sqrt{b-a}\,\|u\|\,\|y\|_1\,\|z\|_1$$

we get

$$\langle \mathsf{B}(u)\,y, z\rangle \leqq \sqrt{b-a}\,\|u\|\,\|y\|_1\,\|z\|_1\ \ \forall u \in L^2(a,b),\ \forall y, z \in H_0^1(a,b). \tag{25}$$

2. Let $y_0, y_1 \in H_0^1(a, b)$ be the unique weak solutions to (2) corresponding to $u_0, u_1 \in U_{ad}$, respectively. In virtue of (12) we have

$$\|y_0 - y_1\|_1^2 \leqq 2\langle \mathsf{T}(u_0)\,(y_0 - y_1), y_0 - y_1\rangle.$$

Using $\mathsf{T}(u_0)\, y_0 = \mathsf{T}(u_1)\, y_1 (= \mathsf{F})$, (13), and (25) one obtains

$$\|y_0 - y_1\|_1^2 \leqq 2\langle \mathsf{B}(u_1 - u_0)\, y_1, y_0 - y_1\rangle \leqq 2\sqrt{b-a}\, \|u_0 - u_1\|\, \|y_1\|_1\, \|y_0 - y_1\|_1 .$$

Together with the inequality from Lemma 1 we get

$$\|y_0 - y_1\|_1 \leqq 4\sqrt{b-a}\, \|f\|\, \|u_0 - u_1\| \quad \forall u_{0,1} \in U_{\mathrm{ad}}. \tag{26}$$

3. Let $y_\lambda \in H_0^1(a, b)$ be the unique weak solution of (2) corresponding to $u_\lambda = (1 - \lambda)\, u_0 + \lambda u_1 \in U_{\mathrm{ad}}, \forall \lambda \in [0, 1]$. ($y_\lambda$ does not equal $(1 - \lambda)\, y_0 + \lambda y_1$!) Taking into account $\mathsf{T}(u_\lambda)\, y_\lambda = \mathsf{T}(u_0)\, y_0 = \mathsf{T}(u_1)\, y_1$, (13), and $u_0 - u_\lambda = \lambda(u_0 - u_1)$, $u_1 - u_\lambda = -(1 - \lambda) \times (u_0 - u_1)$ we find the identity

$$\mathsf{T}(u_\lambda)\big(y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\big) = \lambda(1 - \lambda)\, \mathsf{B}(u_0 - u_1)\, (y_0 - y_1).$$

In virtue of (12, 25) and (26) it follows from this

$$\begin{aligned} &\|y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\|_1^2 \\ &\leqq 2\lambda(1 - \lambda)\, \langle \mathsf{B}(u_0 - u_1)\, (y_0 - y_1), y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\rangle \\ &\leqq 2\lambda(1 - \lambda)\sqrt{b-a}\, \|u_0 - u_1\|\, \|y_0 - y_1\|_1\, \|y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\|_1 \\ &\leqq 8\lambda(1 - \lambda)\, (b - a)\, \|f\|\, \|u_0 - u_1\|^2\, \|y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\|_1 \end{aligned}$$

or

$$\begin{aligned} &\|y_\lambda - (1 - \lambda)\, y_0 - \lambda y_1\| \leqq 8\lambda(1 - \lambda)\, (b - a)\, \|f\|\, \|u_0 - u_1\|^2 \quad \forall u_{0,1} \in U_{\mathrm{ad}}, \\ &\forall \lambda \in [0, 1], \end{aligned} \tag{27}$$

because $\|y\| \leqq \|y\|_1\ \forall y \in H_0^1(a, b)$.

4. We consider now the cost functional. Obviously, it holds

$$\begin{aligned} &(1 - \lambda)\, \|u_0 - h\|^2 + \lambda\, \|u_1 - h\|^2 - \|u_\lambda - h\|^2 = \lambda(1 - \lambda)\, \|u_0 - u_1\|^2 \geqq 0 \\ &\forall u_{0,1}, h \in L^2(a, b), \quad \forall \lambda \in [0, 1]. \end{aligned} \tag{28}$$

So we have

$$\begin{aligned} (1 - \lambda)\, \|y_0 - g\|^2 + \lambda\, \|y_1 - g\|^2 - \|y_\lambda - g\|^2 \geqq \|(1 - \lambda)\, y_0 + \lambda y_1 - g\|^2 \\ - \|y_\lambda - g\|^2. \end{aligned}$$

The right-hand side may be written as $\big((1 - \lambda)\, y_0 + \lambda y_1 + y_\lambda - 2g, (1 - \lambda)\, y_0 + \lambda y_1 - y_\lambda\big)$ and by Cauchy-Schwarz inequality, triangel inequality, and (27) the inequality becomes

$$\begin{aligned} &(1 - \lambda)\, \|y_0 - g\|^2 + \lambda\, \|y_1 - g\|^2 - \|y_\lambda - g\|^2 \\ &\geqq \big((1 - \lambda)\, y_0 + \lambda y_1 + y_\lambda - 2g, (1 - \lambda)\, y_0 + \lambda y_1 - y_\lambda\big) \\ &\geqq -2\big(y_\lambda - g, (1 - \lambda)\, y_0 + \lambda y_1 - y_\lambda\big) \qquad (29) \\ &\geqq -16\lambda(1 - \lambda)\, (b - a)\, (2\, \|f\| + \|g\|)\, \|f\|\, \|u_0 - u_1\|^2 \qquad (30) \\ &\forall u_{0,1} \in U_{\mathrm{ad}}, \forall \lambda \in [0, 1]. \end{aligned}$$

Considering (28, 30) we see that

$$\begin{aligned} &(1 - \lambda)\, \mathsf{J}(y_0, u_0) + \lambda \mathsf{J}(y_1, u_1) - \mathsf{J}(y_\lambda, u_\lambda) \\ &\geqq \lambda(1 - \lambda)\big(\delta - 16(b - a)\, (2\, \|f\| + \|g\|)\, \|f\|\, \varepsilon\big)\, \|u_0 - u_1\|^2. \end{aligned} \tag{31}$$

In other words, J regarded as a functional on $U_{ad} \subset L^2(a, b)$ is strongly convex provided (24) is fulfilled. Thus, the theorem is proved ∎

Theorem: *The inequality*

$$\|u_0 - u_1\| \leqq 16\delta^{-1}\varepsilon \sqrt{b-a}\,(2\,\|f\| + \|g\|)\,\|f\|$$

holds for any two optimal controls $u_{0,1} \in U_{ad}$ *to* (1—3).

Proof: Let $u_0, u_1 \in U_{ad}$ be two distinct optimal controls to (1—3); and let $u_\lambda, y_0, y_1, y_\lambda$ have the same meaning as in the proof of the previous theorem. Then from (26) it follows

$$\begin{aligned} \|y_\lambda - y_0\|_1 &\leqq 4\sqrt{b-a}\,\|f\|\,\|u_\lambda - u_0\| = 4\lambda\sqrt{b-a}\,\|f\|\,\|u_0 - u_1\|, \\ \|y_\lambda - y_1\|_1 &\leqq 4\sqrt{b-a}\,\|f\|\,\|u_\lambda - u_1\| = 4(1-\lambda)\sqrt{b-a}\,\|f\|\,\|u_0 - u_1\| \end{aligned} \tag{32}$$

and, hence, via $\|y\| \leqq \|y\|_1$ valid for every $y \in H_0^1(a, b)$ and via triangel inequality

$$\begin{aligned} \|y_\lambda - (1-\lambda)\,y_0 - \lambda y_1\| &\leqq (1-\lambda)\,\|y_\lambda - y_0\|_1 + \lambda\,\|y - y_1\|_1 \\ &\leqq 8\lambda(1-\lambda)\sqrt{b-a}\,\|f\|\,\|u_0 - u_1\|. \end{aligned}$$

Applying this inequality, (29), and $J(y_0, u_0) = J(y_1, u_1) = \inf J(y, u)$ we get

$$J(y_0, u_0) - J(y_\lambda, u_\lambda) \geqq \lambda(1-\lambda)\,d\,\|u_0 - u_1\| \;\forall \lambda \in [0, 1]$$

with $d = \delta\,\|u_0 - u_1\| - 16\varepsilon\sqrt{b-a}\,(2\,\|f\| + \|g\|)\,\|f\|$. Suppose d to be strictly greater than zero one gets

$$J(y_0, u_0) > J(y_\lambda, u_\lambda) \;\forall \lambda \in (0, 1)$$

which contradicts the optimality of u_0 and thus $d \leqq 0$ ∎

5. Optimality Conditions

In the last section we want to formulate optimality conditions for our control problem. We work within in the framework given by its reformulation (1, 13, 3). There is a lot of papers dealing with optimality conditions for such control problems in Banach spaces. Here we rely on paper [9] which is an offspring of the work done by v. Wolfersdorf and Goebel [2, 3]; we refer to [11] where in the introductory section one may find a short summary of the basic results. After *considerable* adequate simplifications the needed result can be described as follows.

Let Y, Z, U be real Banach spaces, $U_{ad} \subset U$ be a non-emty subset, T be an operator acting on the whole space $Y \times U$ with values in Z, and J be a real-valued functional which is also defined on the whole $Y \times U$. Consider the control problem

$$\inf J(y, u) \quad \text{subject to} \quad u \in U_{ad}, \qquad T(y, u) = 0. \tag{33}$$

Let $\{y_0, u_0\}$ be optimal for (33). We assume:

H1: U_{ad} is convex.

H2: T *is* of the form

$$T(y, u) = Ay + B(u)\,y - F,$$

where $A \in \mathfrak{L}(Y, Z)$, $B(u) \in \mathfrak{L}(Y, Z)\ \forall u \in U$ and $B(\cdot)\,y \in \mathfrak{L}(U, Z)\ \forall y \in Y$, and $F \in Z$ is fixed. $A + B(u) \in \mathfrak{L}(Y, Z)$ is invertible for any $u \in U_{ad}$.

H3: J is Fréchet-differentiable at $\{y_0, u_0\}$.

The latter means the existence of elements $J_y^0 \in Y^*$ and $J_u^0 \in U^*$, called partial derivatives of J with respect to y and u, respectively, at $\{y_0, u_0\}$, such that

$$J(y_0 + y, u_0 + u) - J(y_0, u_0) = \langle J_y^0, y \rangle + \langle J_u^0, u \rangle + R(y, u) \tag{34}$$

with $R(y, u) = o(\|y\| + \|u\|)$ as $\|y\| + \|u\| \to 0$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing between the corresponding spaces.

We quote the necessary optimality condition for problem (33) as

Lemma 5 (cf. [9, 2] or [11]): *Let $\{y_0, u_0\}$ be optimal to* (33) *and let* **H1–H3** *be fulfilled. Then*

$$\begin{aligned} &\text{(i)}\ \ u_0 \in U_{\mathrm{ad}}, A y_0 + B(u_0)\, y_0 = F; \\ &\text{(ii)}\ \langle J_u^0, u - u_0 \rangle + \langle z_0, B(u - u_0)\, y_0 \rangle \geqq 0 \ \forall u \in U_{\mathrm{ad}}, \end{aligned} \tag{35}$$

where $z_0 \in Z^*$, called adjoint state, is the unique solution of

$$A^* z + B(u_0)^* z = -J_y^0. \tag{36}$$

On the basis of this lemma we show the following

Necessary Optimality Condition: *Let $\{y_0, u_0\} \in H_0^1(a, b) \times L^2(a, b)$ be an optimal pair to* (1–3). *Then it yields:*

(i) *$u_0 \in U_{\mathrm{ad}}$ and $y_0 \in H_0^1(a, b)$ is the weak solution of* (2) *with $u = u_0$.*

$$\left.\begin{aligned} &\text{(ii)}\ \left[y_0'(x)\, z_0(x) + 2\delta\big(u_0(x) - h(x)\big)\right]\big(v - u_0(x)\big) \geqq 0 \\ &\quad \text{for all real numbers } v \text{ with } v \in [\xi_1(x), \xi_2(x)] \\ &\quad \text{and almost all } x \in (a, b), \end{aligned}\right\} \tag{37}$$

where $z_0 \in H_0^1(a, b)$ is the unique weak solution of the boundary value problem

$$\left.\begin{aligned} &-(d/dx)\big(z'(x) + u_0(x)\, z(x)\big) + z(x) = -2\varepsilon\big(y_0(x) - g(x)\big), x \in a, b, \\ &z(a) = z(b) = 0. \end{aligned}\right\} \tag{38}$$

Proof: We put $Y = H_0^1(a, b)$, $Z = H^{-1}(a, b)$, $U = L^2(a, b)$ and let U_{ad}, J, A, B, and F be defined as in (3, 1, 7, 9) and (6), respectively. It is easily checked that the problem (1, 13, 3) fulfils hypotheses **H1–H3**. In particular, we have

$$\langle J_y^0, y \rangle = 2\varepsilon(y_0 - g, y) \ \forall y \in H_0^1(a, b),$$

$$\langle J_u^0, u \rangle = 2\delta(u_0 - h, u) \ \forall u \in L^2(a, b)$$

and (35) reads as

$$\int_a^b H^0(x)\, \big(u(x) - u_0(x)\big)\, dx \geqq 0 \ \forall u \in U_{\mathrm{ad}} \tag{39}$$

with the function $H^0(x) = y_0'(x)\, z_0(x) + 2\delta\big(u_0(x) - h(x)\big)$, $x \in (a, b)$, and $z_0 \in H_0^1(a, b)$ is the unique solution of (38) (cf. (36, 7, 9)). (37) follows from (39) via a lemma due to KRASNOSELSKI [6: p. 343] ∎

The next theorem shows that under certain circumstances condition (37) is not only necessary but also sufficient for an admissible pair to be optimal for (1–3).

Sufficient Optimality Condition: *Let the following both assumptions be satisfied:*

(i) $\delta \geqq 16(b-a)\,(2\,\|f\| + \|g\|)\,\|f\|\,\varepsilon$,

(ii) $\{y_0, u_0\} \in H_0^1(a, b) \times L^2(a, b)$ *is an admissible pair to* (1–3) *for which* (37) *holds.*

Then $\{y_0, u_0\}$ *is optimal to the control problem* (1–3).

Proof: Let $\{y_1, u_1\} \in H_0^1(a, b) \times L^2(a, b)$ be an arbitrary admissible pair to (1–3). We take the notations used in the proof of the Uniqueness Theorem. Assumption (i), (31), and (34) assure

$$\begin{aligned} \mathsf{J}(y_1, u_1) - \mathsf{J}(y_0, u_0) &\geqq \lambda^{-1}\big(\mathsf{J}(y_\lambda, u_\lambda) - \mathsf{J}(y_0, u_0)\big) \\ &= \lambda^{-1}\big(\mathsf{J}(y_0 + (y_\lambda - y_0), u_0 + \lambda(u_1 - u_0)) - \mathsf{J}(y_0, u_0)\big) \\ &= \lambda^{-1}\langle J_y^0, y_\lambda - y_0\rangle + \langle \mathsf{J}_u^0, u_1 - u_0\rangle + \mathsf{R}(\lambda)\ \forall \lambda \in [0, 1] \end{aligned}$$

with $\mathsf{R}(\lambda) = \lambda^{-1}\mathsf{R}\big(y_\lambda - y_0, \lambda(u_1 - u_0)\big)$. Using (36) one obtains

$$\begin{aligned} \langle \mathsf{J}_y^0, y_\lambda - y_0\rangle &= -\big\langle z_0, \big(\mathsf{A} + \mathsf{B}(u_0)\big)\,(y_\lambda - y_0)\big\rangle \\ &= -\langle z_0, \mathsf{B}(u_0 - u_\lambda)\,y_\lambda\rangle = \lambda\langle z_0, \mathsf{B}(u_1 - u_0)\,y_\lambda\rangle \end{aligned}$$

and, hence,

$$\begin{aligned} \mathsf{J}(y_1, u_1) - \mathsf{J}(y_0, u_0) \geqq{}& \langle \mathsf{J}_u^0, u_1 - u_0\rangle + \langle z_0, \mathsf{B}(u_1 - u_0)\,y_\lambda\rangle \\ &+ \mathsf{R}(\lambda)\ \forall \lambda \in [0, 1]. \end{aligned}$$

If we let $\lambda \to +0$ it follows

$$\mathsf{J}(y_1, u_1) - \mathsf{J}(y_0, u_0) \geqq \langle \mathsf{J}_u^0, u_1 - u_0\rangle + \langle z_0, \mathsf{B}(u_1 - u_0)\,y_0\rangle$$

for (32). The right-hand side of this inequality is greater than or equal to zero since (37) with $v = u_1(x)$ implies (39), i.e. (35), with u_1 instead of u. This proves the theorem ∎

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