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A Remark on a Paper by D. E. Edmunds, A. Kufner and J. Rákosník "Embeddings of Sobolev Spaces with Weights of Power Type"[1])

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Die Arbeit ist Approximationseigenschaften für gewichtete Sobolev-Slobodeckij-Räume gewidmet.

Работа посвящена свойствам аппроксимации в пространствах типа Соболева-Слободецкого с весами.

The paper deals with approximation properties in weighted spaces of Sobolev-Slobodeckij type.

1. Definitions and Results

Let M be a non-empty closed subset of the Euclidean n-space $\mathbf{R}_n$. Let Ω be a domain in $\mathbf{R}_n$ with $\Omega \subset \mathbf{R}_n - M$ (we may assume, without restriction of generality, that M is a subset of the boundary of Ω, i.e. $M \subset \partial\Omega = \bar{\Omega} - \Omega$; however this is not important in the sequel). If $x \in \mathbf{R}_n$ then $\operatorname{dist}(x, M) = \inf |x - y|$ is the usual distance, where the infimum is taken over all $y \in M$. We mollify $\operatorname{dist}(x, M)$: There exists a positive function $d_M(x) \in C^\infty(\mathbf{R}_n - M)$ and two positive numbers c_1 and c_2 with

$$c_1 d_M(x) \leq \operatorname{dist}(x, M) \leq c_2 d_M(x), \qquad x \in \mathbf{R}_n - M,$$

cf. the construction in [2: Remark 3.2.3/1], where $d_M(x)$ coincides essentially with $\varrho^{-1}(x)$ (in contrast to the construction given there we do not care about the behaviour of $d_M(x)$ if $|x|$ tends to infinity, but this does not affect our arguments). Let $1 < p < \infty$ and $-\infty < \varepsilon < \infty$. If s is a non-negative integer then $W_p^s(\Omega, \varepsilon)$ is the collection of all complex distributions $f \in D'(\Omega)$ such that

$$\|f \mid W_p^s(\Omega, \varepsilon)\| = \sum_{|\alpha| \leq s} \left(\int_\Omega |(D^\alpha f)(x)|^p \, d_M^{\varepsilon - (s - |\alpha|)p}(x) \, dx \right)^{1/p} < \infty \tag{1}$$

holds. If $0 < s = [s] + \{s\}$, where $[s]$ is an integer and $0 < \{s\} < 1$ then $W_p^s(\Omega, \varepsilon)$ is the collection of all complex distributions $f \in D'(\Omega)$ such that

$$\|f \mid W_p^s(\Omega, \varepsilon)\| = \sum_{|\alpha| \leq [s]} \left(\int_{\Omega \times \Omega} \frac{|d_M^{\varepsilon/p - [s] + |\alpha|}(x) \, D^\alpha f(x) - d_M^{\varepsilon/p - [s] + |\alpha|}(y) \, D^\alpha f(y)|^p}{|x - y|^{n + \{s\} p}} \, dx \, dy \right)^{1/p} + \|f \mid W_p^{[s]}(\Omega, \varepsilon - \{s\} p)\| < \infty \tag{2}$$

holds. These weighted Sobolev-Slobodeckij spaces are near to the spaces $W_p^s(\Omega, \varrho^\mu, \varrho^\nu)$ in [2], where $\varrho^{-1}(x)$ is essentially $d_M(x)$, $\mu = -\varepsilon$ and $\nu = \mu + sp$: Formula (3.2.3/11)

[1]) The preceding paper of this issue.

(where the misprint $B^s_{p,q}$ must be corrected by $B^s_{p,p}$), Theorems 1, 2, and 3 in 3.2.4. The differences between the above spaces $W_p{}^s(\Omega, \varepsilon)$ and the spaces $W_p{}^s(\Omega, \varrho^\mu, \varrho^\nu)$ are characterized by the different behaviour of $\varrho(x)$ and $d_M{}^{-1}(x)$ at infinity and that we have $\partial\Omega$ instead of M in [2]. However in what follows these differences are immaterial. We give a full proof of our main result and take over only some technicalities from [2: 3.2.3 and 3.2.4]. Obviously, $W_p{}^s(\Omega, \varepsilon)$ is a Banach space.

Theorem: *Let* $1 < p < \infty$, $s > 0$ *and* $-\infty < \varepsilon < \infty$. *Then*

$$\{f \mid f \in W_p{}^s(\Omega, \varepsilon), \operatorname{supp} f \cap M = \emptyset\} \tag{3}$$

is dense in $W_p{}^s(\Omega, \varepsilon)$.

Remark 1: We recall that we may assume that M is a subset of $\partial\Omega$. If s is a natural number then $W_p{}^s(\Omega, \varepsilon)$ coincides with the spaces $H^{s,p}(\Omega; d_M, \varepsilon)$ from [1]. In [1] it has been proved that

$$C_M{}^\infty(\Omega) = \{w \mid w \in C^\infty(\bar{\Omega}), \operatorname{supp} w \cap M = \emptyset\} \tag{4}$$

is dense in $H^{s,p}(\Omega, d_M, \varepsilon)$. Our theorem (restricted to natural s) is not a new proof of this assertion, however it shows that the weighted case can be reduced to the unweighted case, and this simplifies the task considerably. On the other hand, in contrast to [1], we deal also with fractional spaces (and only for that purpose we mollified dist (x, M)).

Remark 2: Let Ω be a bounded domain and $M = \partial\Omega$. Then we have

$$W_p{}^s(\Omega, \varepsilon) = W_p{}^s(\Omega, \varrho^{-\varepsilon}, \varrho^{-\varepsilon+sp})$$

with $\varrho^{-1}(x) = d_M(x)$ if $x \in \Omega$, where $1 < p < \infty$, $s \geqq 0$ and $-\infty < \varepsilon < \infty$, cf. [2: Theorem 3.2.4/3] and Remark 3 at the end of this paper. By [2: Theorem 3.2.4/1], $C_0{}^\infty(\Omega)$ is dense in $W_p{}^s(\Omega, \varepsilon)$. This extends Theorem 1.1 in [1] to the fractional spaces, at least if Ω is bounded.

2. Proof of the Theorem

We prove the above theorem in two steps.

Step 1: Let s be a natural number. Let

$$M^{(j)} = \{x \mid x \in \mathbf{R}_n, d_M(x) > 2^{-j}\} \quad \text{if} \quad j = 1, 2, \ldots \tag{5}$$

and

$$M_1 = M^{(3)}, \quad M_k = M^{(k+2)} - \overline{M^{(k-1)}} \quad \text{if} \quad k = 2, 3, \ldots \tag{6}$$

We assume $M^{(1)} \cap \Omega \neq \emptyset$ (without restriction of generality). Let $\psi = \{\psi_j(x)\}_{j=1}^\infty$ be a system of infinitely differentiable functions on $\mathbf{R}_n$ with the following properties:

$$0 \leqq \psi_j(x) \leqq 1, \quad \operatorname{supp} \psi_j \subset M_j, \quad \sum_{j=1}^\infty \psi_j(x) = 1 \quad \text{if} \quad x \in \mathbf{R}_n - M; \tag{7}$$

or any multi-index γ there exists a positive number c_γ with

$$|D^\gamma \psi_j(x)| \leqq c_\gamma 2^{j|\gamma|} \quad \text{for all } j = 1, 2, 3, \ldots \text{ and all } x \in \mathbf{R}_n. \tag{8}$$

Systems of this type exist, we refer to [2: 3.2.3]. We claim that

$$\left(\sum_{j=1}^{\infty} \|\psi_j f \mid W_p^s(\Omega, \varepsilon)\|^p\right)^{1/p} = \|f \mid W_p^s(\Omega, \varepsilon)\|^\psi \tag{9}$$

is an equivalent norm in $W_p^s(\Omega, \varepsilon)$. It is easy to see that $\|f \mid W_p^s(\Omega, \varepsilon)\|$ can be estimated from above by $c \|f \mid W_p^s(\Omega, \varepsilon)\|^\psi$. In order to prove the reverse assertion we remark that

$$\begin{aligned} \|\psi_j f \mid W_p^s(\Omega, \varepsilon)\|^p &\leqq c \sum_{|\alpha| \leqq s} 2^{-j\varepsilon + jp(s-|\alpha|)} \int_\Omega |D^\alpha(\psi_j f)(x)|^p \, dx \\ &\leqq c' \sum_{|\alpha| \leqq s} 2^{-j\varepsilon + jp(s-|\alpha|)} \int_{\Omega \cap M_{j_h}} |D^\alpha f(x)|^p \, dx \\ &\leqq c'' \sum_{|\alpha| \leqq s} \int_{\Omega \cap M_j} d_M^{\varepsilon - (s-|\alpha|)p}(x) \, |D^\alpha f(x)|^p \, dx. \end{aligned} \tag{10}$$

Summation yields the desired result. Now it follows easily that

$$\left(\sum_{j=1}^{N} \psi_j\right) f \to f \quad \text{if } N \to \infty \quad \big(f \in W_p^s(\Omega, \varepsilon)\big). \tag{11}$$

This proves the theorem if s is a non-negative integer.

Step 2: Let $0 < s = [s] + \{s\}$, where $[s]$ is an integer and $0 < \{s\} < 1$. Again we claim that $\|f \mid W_p^s(\Omega, \varepsilon)\|^\psi$ from (9) is an equivalent norm in $W_o^s(\Omega, \varepsilon)$. It is easy to see that $\|f \mid W_p^s(\Omega, \varepsilon)\|$ can be estimated from above by $c \|f \mid W_p^s(\Omega, \varepsilon)\|^\psi$, cf. (2). In order to prove the reverse assertion we use (2) with $\psi_j f$ instead of f. The terms $\|\psi_j f \mid W_p^{[s]}(\Omega, \varepsilon - \{s\} p)\|$ can be treated in the same way as in (10). In other words, the problem is reduced to an estimate of

$$\sum_{j=1}^{\infty} \int_{\Omega \times \Omega} \frac{|d_M^{(\varepsilon/p) - [s] + |\alpha|}(x) \, D^\alpha(\psi_j f)(x) - d^{(\varepsilon/p) - [s] + |\alpha|}(y) \, D^\alpha(\psi_j f)(y)|^p}{|x - y|^{n + \{s\}p}} \, dx \, dy \tag{12}$$

with $|\alpha| \leqq [s]$. Without restriction of generality we may assume that

$$\operatorname{dist}(\operatorname{supp} \psi_j, \mathbf{R}_n - M_j) \geqq c 2^{-j}, \tag{13}$$

where c is an appropriate positive number which is independent of j. Let $\Omega_j = \Omega \cap M_j$. Then the sum in (12) can be estimated from above by

$$\begin{aligned} 2 \sum_{j=1}^{\infty} \Bigg[&\int_{\Omega_j \times \Omega_j} \frac{|d_M^{(\varepsilon/p) - [s] + |\alpha|}(x) \, D^\alpha(\psi_j f)(x) - d_M^{(\varepsilon/p) - [s] + |\alpha|}(y) \, D^\alpha(\psi_j f)(y)|^p}{|x - y|^{n + \{s\}p}} \, dx \, dy \\ &+ \int_{\Omega_j} d_M^{\varepsilon - p([s] - |\alpha|)}(x) \, |D^\alpha(\psi_j f)(x)|^p \int_{\Omega - \Omega_j} \frac{dy}{|x - y|^{n + \{s\}p}} \Bigg]. \end{aligned} \tag{14}$$

Under the hypothesis (13) the second terms in (14) can be estimated from above by

$$\begin{aligned} &c \sum_{j=1}^{\infty} \int_{\Omega_j} 2^{-j\varepsilon + jp\{s\} + jp([s] - |\alpha|)} |D^\alpha(\psi_j f)(x)|^p \, dx \\ &\leqq c \|f \mid W_p^{[s]}(\Omega, \varepsilon - \{s\} p)\|^p \leqq c' \|f \mid W_p^s(\Omega, \varepsilon)\|^p, \end{aligned} \tag{15}$$

where we used inequalities of type (10). The first terms in (14) can be estimated from above by a sum of terms of the form

$$\sum_{j=1}^{\infty}\left[\int_{\Omega_j}|(D^\beta f)(x)|^p\int_{M_j}\frac{|d_M{}^{(\varepsilon/p)-[s]+|\alpha|}(x)\,D^\gamma\psi_j(x)-d_M{}^{(\varepsilon/p)-[s]+|\alpha|}(y)\,D^\gamma\psi_j(y)|^p}{|x-y|^{n+\{s\}p}}\,dy\,dx\right.$$
$$\left.+\int_{\Omega_j\times\Omega_j}d_M{}^{\varepsilon-p([s]-|\alpha|)}(y)\,|D^\gamma\psi_j(y)|^p\,\frac{|(D^\beta f)(x)-(D^\beta f)(y)|^p}{|x-y|^{n+\{s\}p}}\,dx\,dy\right] \tag{16}$$

with $|\beta|+|\gamma|=|\alpha|$. The integral over M_j in (16) can be estimated from above by $c2^{-j\varepsilon+j\{s\}p+jp([s]-|\beta|)}$. This follows from [2: Lemma 3.2.4/2] (here we need that $d_M(x)$ is a mollified distance). Consequently, the first terms in (16) can be estimated from above by $c\,\|f\mid W_p{}^{[s]}(\Omega,\varepsilon-\{s\}\,p)\|^p$. The second terms in (16) can be estimated from above by

$$\sum_{j=1}^{\infty}2^{-j\varepsilon+jp([s]-|\beta|)}\int_{\Omega_j\times\Omega_j}\frac{|(D^\beta f)(x)-(D^\beta f)(y)|^p}{|x-y|^{n+\{s\}p}}\,dx\,dy$$
$$\leqq c\int_{\Omega\times\Omega}\frac{|d_M{}^{(\varepsilon/p)-[s]+|\beta|}(x)\,D^\beta f(x)-d_M{}^{(\varepsilon/p)-[s]+|\beta|}(y)\,D^\beta f(y)|^p}{|x-y|^{n+\{s\}p}}\,dx\,dy$$
$$\leqq c\,\|f\mid W_p{}^s(\Omega,\varepsilon)\|^p. \tag{17}$$

This completes the proof that (9) is an equivalent norm in $W_p{}^s(\Omega,\varepsilon)$. Then we have again (11), and this proves the theorem if s is a fractional number ∎

Remark 3: At the first glance we would try to replace $\sum\limits_{|\alpha|\leq[s]}$ in (2) by $\sum\limits_{|\alpha|=[s]}$. This seems to be the better definition of the fractional spaces $W_p{}^s(\Omega,\varepsilon)$ and it agrees also completely with our theory in [2: 3.2.3 and 3.2.4]. If $M=\partial\Omega$ (this is the case treated in [2]) then such a replacement is possible. However in the general case $M\neq\partial\Omega$ this is not clear: The difficulties come from the estimate (17) of the terms with $|\beta|<[s]$.

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