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Embeddings of Sobolev Spaces with Weights of Power Type

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Die Arbeit befaßt sich mit einigen Eigenschaften gewichteter Sobolevscher Räume $W_0^{k,p}$, $W_M^{k,p}$ und $H^{k,p}$, in denen als Gewichte Potenzen der Entfernung von einem Teil M des Randes des Definitionsgebietes auftreten. Es werden Bedingungen angegeben, unter denen diese Räume gegenseitig eingebettet und gewisse Normen äquivalent sind. Dabei werden einige Ergebnisse aus [1] verallgemeinert und eine in [2] formulierte Vermutung wird bewiesen.

В работе исследованы некоторые свойства весовых пространств С. Л. Соболева $W_0^{k,p}$, $W_M^{k,p}$ и $H^{k,p}$, весовые функции которых являются степенями расстояния от части M границы области определения. Указаны условия, при которых эти пространства вкладываются друг в друга и при которых некоторые нормы эквивалентны. Обобщаются некоторые результаты из [1] и доказана гипотеза, сформулированная в [2].

The paper deals with some properties of the weighted Sobolev spaces $W_0^{k,p}$, $W_M^{k,p}$ and $H^{k,p}$, with weights which are powers of the distance from a part M of the boundary of the domain of definition. Conditions are given which guarantee the mutual embeddings of these spaces and the equivalence of certain norms. The paper generalizes some results of [1] and verifies a conjecture formulated in [2].

0. Introduction

Let Ω be a domain in the Euclidean space $\mathbf{R}^N$ and let M be a non-empty subset of the boundary $\partial\Omega$ of Ω; given any $x \in \mathbf{R}^N$ write

$$d_M(x) = \operatorname{dist}(x, M). \tag{0.1}$$

Let $\varepsilon, p \in \mathbf{R}$, with $p \geqq 1$, $k \in \mathbf{N}$, and let $W^{k,p}(\Omega; d_M, \varepsilon)$ be the weighted Sobolev space of (equivalence classes of) functions $u: \Omega \to \mathbf{R}$ such that for all $\alpha := (\alpha_i) \in \mathbf{N}_0^N$ with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N \leqq k$, the distributional derivative $D^\alpha u$ satisfies

$$\int_\Omega |D^\alpha u(x)|^p \, d_M^\varepsilon(x) \, dx < \infty.$$

The space $W^{k,p}(\Omega; d_M, \varepsilon)$ is a Banach space when equipped with the norm defined by

$$\|u\|_W := \Big(\sum_{|\alpha| \leqq k} \int_\Omega |D^\alpha u(x)|^p \, d_M^\varepsilon(x) \, dx \Big)^{1/p}. \tag{0.2}$$

We also introduce the spaces $W_M^{k,p}(\Omega; d_M, \varepsilon)$ and $W_0^{k,p}(\Omega; d_M, \varepsilon)$, which are the closures in the Banach space $W^{k,p}(\Omega; d_M, \varepsilon)$ of the sets

$$C_M^\infty(\Omega) := \{w \in C^\infty(\bar{\Omega}) : \operatorname{supp} w \cap \overline{M} = \emptyset\} \tag{0.3}$$

and $C_0^\infty(\Omega)$ respectively; and the space $H^{k,p}(\Omega; d_M, \varepsilon)$, which is the set of all functions $u: \Omega \to \mathbf{R}$ such that for all $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leqq k$,

$$\int_\Omega |D^\alpha u(x)|^p \, d_M^{\varepsilon - (k - |\alpha|)p}(x) \, dx < \infty.$$

This last space is also a Banach space when provided with the norm given by

$$\|u\|_H := \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p \, d_M{}^{\varepsilon-(k-|\alpha|)p}(x) \, dx \Big)^{1/p}. \tag{0.4}$$

If $\varepsilon = 0$, $W^{k,p}(\Omega; d_M, \varepsilon)$ and $W_0{}^{k,p}(\Omega; d_M, \varepsilon)$ are simply the classical unweighted Sobolev spaces $W^{k,p}(\Omega)$ and $W_0{}^{k,p}(\Omega)$ respectively; if $k = 0$, then $W^{k,p}(\Omega; d_M, \varepsilon)$ is nothing more than the weighted Lebesgue space $L^p(\Omega; d_M, \varepsilon)$.

In [1] the case $M = \partial\Omega$ was considered, and it was shown that if for $i = 1, 2, \ldots, k$ the inequality $\varepsilon \neq ip - 1$ holds, then

$$W_0{}^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon). \tag{0.5}$$

the symbol $\hookrightarrow$ denoting continuous embedding; it was also shown that for all $\varepsilon \in \mathbf{R}$,

$$H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_0{}^{k,p}(\Omega; d_M, \varepsilon). \tag{0.6}$$

In particular, if $\varepsilon = 0$, then for any $u \in W^{k,p}(\Omega)$ such that $D^\alpha u \in L^p(\Omega; d_M, (k - |\alpha|)\, p)$ for all $\alpha \in \mathbf{N}_0{}^N$ with $|\alpha| \leq k - 1$, we have $u \in W_0{}^{k,p}(\Omega)$. The assumption that Ω had a Lipschitz boundary was important in [1], although insofar as the embedding (0.6) is concerned this condition can be weakened.

When $\varepsilon = 0$ it is possible, by following unpublished suggestions of D. J. HARRIS, to establish (0.6) under the sole assumption on Ω that it should be bounded (see Theorem 1.1). The method used relies on the properties of the maximal function, and can be extended to the case of an arbitrary set $M \subset \partial\Omega$ and an arbitrary $\varepsilon \in \mathbf{R}$: in Theorem 1.2 it is shown that

$$H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M{}^{k,p}(\Omega; d_M, \varepsilon).$$

The embedding (0.5) follows from embedding theorems in weighted spaces; in the case $M = \partial\Omega$, such theorems hold if $\partial\Omega$ is Lipschitzian. It was conjectured in [2] that

$$W_M{}^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon)$$

for certain special sets M, provided that for $i = 1, 2, \ldots, k$ the inequality $\varepsilon \neq ip - 1$ holds. This conjecture is verified in Corollary 3.1, (ii); the main tool is due to the third author (J. R.) and is a fuller version of the short communication [4].

1. The embedding $H \hookrightarrow W$

First, we shall prove a theorem, which implies (0.6) for the case $\varepsilon = 0$ and $M = \partial\Omega$. This result is contained in [1], but here, the condition on Ω is weakened and the method is completely new.

Theorem 1.1: *Let Ω be a non-empty open subset of $\mathbf{R}^N$, $\Omega \neq \mathbf{R}^N$, and for each $x \in \Omega$ put $d(x) = \operatorname{dist}(x, \partial\Omega)$; let $p \in (1, \infty)$ and $k \in \mathbf{N}$. Suppose that $u \in W^{k,p}(\Omega)$ is such that $ud^{-k} \in L^p(\Omega)$. Then $u \in W_0{}^{k,p}(\Omega)$.*

Proof: First suppose Ω is bounded. Let $0 < h < \delta$; define $u^{(\delta)}$ by $u^{(\delta)}(x) = u(x)$ if $d(x) \geq \delta$ and $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise; let $\phi \in C_0{}^\infty(\mathbf{R}^N)$ be such that $\phi(x) = 0$ if $|x| \geq 1$, $\phi(x) > 0$ if $|x| < 1$, $\int_{\mathbf{R}^N} \phi(x)\, dx = 1$; and put $\phi_h(x) = h^{-N}\phi(x/h)$. Let us agree that any function g defined on Ω will be supposed extended to the whole $\mathbf{R}^N$, if necessary, by setting $g(x) = 0$ for all $x \in \mathbf{R}^N \setminus \Omega$. Note that $\phi_h * u^{(\delta)} \in C_0{}^\infty(\Omega)$.

First suppose that $d(x) > \delta + h$, $x \in \Omega$ and $\alpha \in \mathbf{N}_0^N$, $|\alpha| \leq k$. Then since $B(x, h) := \{y \in \mathbf{R}^N : |x - y| < h\} \subset B(x, \delta + h) \subset \Omega$, and $d(y) > \delta$ if $y \in B(x, h)$,

$$D^\alpha(\phi_h * u^{(\delta)})(x) = D^\alpha \int_{B(x,h)} \phi_h(x - y)\, u^{(\delta)}(y)\, dy$$

$$= D^\alpha \int_{\mathbf{R}^N} \phi_h(x - y)\, u(y)\, dy = \int_{\mathbf{R}^N} \phi_h(x - y)\, D^\alpha u(y)\, dy;$$

that is,

$$D^\alpha(\phi_h * u^{(\delta)})(x) = (\phi_h * D^\alpha u)(x) \quad \text{if} \quad d(x) > \delta + h, \quad x \in \Omega. \tag{1.1}$$

It follows that, with $\mathcal{M}(g)$ as the maximal function defined by

$$\mathcal{M}(g)(x) = \sup_{r>0} |B(0, r)|^{-1} \int_{B(x,r)} |g(y)|\, dy$$

($|B(0, r)| = \omega_N r^N$ being the volume of the ball $B(0, r)$), we have

$$|D^\alpha(\phi_h * u^{(\delta)})(x)| \leq \mathcal{M}(D^\alpha u)(x) \cdot \omega_N \sup_{y \in B(x,h)} \phi\left(\frac{x - y}{h}\right)$$

$$= \mathcal{M}(D^\alpha u)(x) \cdot \omega_N \sup_{z \in B(0,1)} \phi(z) = \mathcal{M}(D^\alpha u)(x)\, b(\phi), \quad \text{say}.$$

Next, suppose that $d(x) \leq lh$ for some $l \in \mathbf{N}$, and let $\alpha \in \mathbf{N}_0^N$, $|\alpha| \leq k$. Then

$$|D^\alpha(\phi_h * u^{(\delta)})(x)| = \left| \int_{\mathbf{R}^N} h^{-N-|\alpha|} (D^\alpha \phi)\left(\frac{x - y}{h}\right) u^{(\delta)}(y)\, dy \right|$$

$$\leq h^{-N-|\alpha|} \int_{\mathbf{R}^N} \left| (D^\alpha \phi)\left(\frac{x - y}{h}\right) u^{(\delta)}(y)\, d^{-k}(y) \right| h^k (l + 1)^k\, dy$$

$$\leq \omega_N h^{k-|\alpha|} (l + 1)^k\, \mathcal{M}(ud^{-k})(x) \sup_{z \in B(0,1)} |(D^\alpha \phi)(z)|.$$

Thus if we take $\delta = 2h = 2/j$ ($j \in \mathbf{N}$) and $l = 3$, we see that for all $x \in \Omega$, for all $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leq k$, and for all $j \in \mathbf{N}$,

$$|D^\alpha(\phi_{1/j} * u^{(2/j)}(x))|$$
$$\leq \max \{b(\phi)\, \mathcal{M}(D^\alpha u)(x),\ \ 4^k c(\alpha, \phi)\, \mathcal{M}(ud^{-k})(x)\} := G_{\alpha,\varphi}(x), \tag{1.2}$$

where $c(\alpha, \phi) = \omega_N \sup_{z \in B(0,1)} |(D^\alpha \phi)(z)|$. Since, by [5: Chap. I, Th. 1], $\mathcal{M}$ is a bounded mapping from $L^p(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N)$, it follows that $G_{\alpha,\varphi} \in L^p(\mathbf{R}^N)$. Moreover, for each $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leq k$, $\phi_{1/j} * D^\alpha u \to D^\alpha u$ in $L^p(\Omega)$ as $j \to \infty$, and hence there is a subsequence $(j(l))$ of the sequence of positive integers such that for all $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leq k$, $(\phi_{1/j(l)} * D^\alpha u)(x) \to D^\alpha u(x)$ almost everywhere in Ω. Thus by (1.1),

$$D^\alpha(\phi_{1/j(l)} * u^{(2/j(l))})(x) \to D^\alpha u(x) \tag{1.3}$$

a.e. in Ω. Together with Lebesgue's dominated convergence theorem, (1.2) and (1.3) show that $(\phi_{1/j(l)} * u^{(2/j(l))}) \to u$ in $W^{k,p}(\Omega)$; that is, $u \in W_0^{k,p}(\Omega)$.

If Ω is unbounded we simply apply the arguments above to $u\psi_m := u_m$, where $\psi_m \in C_0^\infty(\mathbf{R}^N)$, $\psi_m(x) = 1$ ($|x| < m$), $\psi_m(x) = 0$ ($|x| > 2m$), and note that $u_m d^{-k} \in L^p(\Omega)$ and $u_m \to u$ in $W^{k,p}(\Omega)$ as $m \to \infty$ ∎

Note that if Ω is bounded, the validity of (0.6), when $\varepsilon = 0$ and $M = \partial\Omega$, follows immediately from Theorem 1.1. We now set about the proof of (0.6) for general ε and M.

Theorem 1.2: *Let Ω be a bounded domain in $\mathbf{R}^N$, let $M \subset \partial\Omega$, $M \neq \emptyset$, and let $p \in (1, \infty)$, $k \in \mathbf{N}$ and $\varepsilon \in \mathbf{R}$. Then*

$$H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M{}^{k,p}(\Omega; d_M, \varepsilon).$$

Proof: Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, and for any $\delta > 0$ write $u^{(\delta)}(x) = u(x)$ if $d_M(x) > \delta$ and $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise. Let ϕ be as in the proof of Theorem 1.1 and write $\phi_h(x) = h^{-N}\phi(x/h)$ $(h > 0, x \in \mathbf{R}^N)$. Let $h > 0$ and set $v_h = \phi_h * u^{(3h)}$. We distinguish various cases.

(i) Let $x \in \Omega$ be such that

$$d_M(x) \geqq 2h. \tag{1.4}$$

Then $v_h(x) = h^{-N} \int\limits_{B(x,h)} \phi\left(\frac{x-y}{h}\right) u^{(3h)}(y)\, dy = 0$ since by (1.4) if $y \in B(x, h)$ then $d_M(y) < 3h$ and consequently $u^{(3h)}(y) = 0$. It follows that

$$\operatorname{supp} v_h \subset \Omega_h := \{x \in \Omega : d_M(x) > h\}. \tag{1.5}$$

Note that $\Omega_h \cap \overline{M} = \emptyset$.

(ii) Suppose that $x \in \Omega$ is such that

$$2h < d_M(x) \leqq 4h. \tag{1.6}$$

Then if $y \in B(x, h)$ we see that

$$h < d_M(y) < 5h; \tag{1.7}$$

and (1.6) and (1.7) imply the equivalence of h, $d_M(x)$ and $d_M(y)$:

$$d_M(x) \approx h, \qquad d_M(y) \approx h, \tag{1.8}$$

with the obvious meaning of the symbol $\approx$. For all $\alpha \in \mathbf{N}_0{}^N$, $|\alpha| \leqq k$, we have

$$\begin{aligned}
|D^\alpha v_h(x)| &= \left| D^\alpha \int\limits_{\mathbf{R}^N} \phi_h(x-y) u^{(3h)}(y)\, dy \right| = h^{-N} \left| \int\limits_{\mathbf{R}^N} D^\alpha \phi\left(\frac{x-y}{h}\right) u^{(3h)}(y)\, dy \right| \\
&\leqq h^{-N-|\alpha|} \int\limits_{B(x,h)} \left| (D^\alpha \phi)\left(\frac{x-y}{h}\right)\right| |u(y)|\, dy \\
&\leqq \Big(\sup_{z \in \mathbf{R}^N} |D^\alpha \phi(z)|\Big) |B(0, 1)|\, |B(x, h)|^{-1} h^{-|\alpha|} \int\limits_{B(x,h)} |u(y)|\, dy \\
&\leqq c_0 h^{-|\alpha|} |B(x, h)|^{-1} \int\limits_{B(x,h)} |u(y)|\, d_M{}^{(\varepsilon/p)-k}(y)\, dy \cdot h^{k-(\varepsilon/p)} \\
&\leqq c_1 h^{k-|\alpha|} |B(x, h)|^{-1} \int\limits_{B(x,h)} |u(y)|\, d_M{}^{(\varepsilon/p)-k}(y)\, dy \cdot [d_M(x)]^{-\varepsilon/p},
\end{aligned}$$

the last two inequalities following from the equivalence of $d_M(y)$ and h, and that of $d_M(x)$ and h. Thus we have

$$|D^\alpha v_h(x)|\, d_M{}^{\varepsilon/p}(x) \leqq c_1 h^{k-|\alpha|} \mathcal{M}(u d_M{}^{(\varepsilon/p)-k})(x), \tag{1.9}$$

where, as before, $\mathcal{M}$ stands for the maximal function and any function on Ω is extended by zero in $\mathbf{R}^N \setminus \Omega$.

(iii) Let $x \in \Omega$ be such that for some $l \geqq 4$,

$$lh < d_M(x) < (l+1)\, h. \tag{1.10}$$

Let $y \in B(x, h)$. Then

$$(l-1)\,h < d_M(y) < (l+2)\,h, \tag{1.11}$$

and (1.10) together with (1.11) imply

$$d_M(x) \approx lh, \qquad d_M(y) \approx lh. \tag{1.12}$$

From (1.11) it follows that $u^{(3h)}(y) = u(y)$, since $d_M(y) > (l-1)\,h \geqq 3h$. For $\alpha \in \mathbf{N}_0^N$, $|\alpha| \leqq k$, we find, as in (ii), that

$$\begin{aligned}
|D^\alpha v_h(x)| &= \left|D^\alpha \int_{\mathbf{R}^N} \phi_h(x-y)\, u^{(3h)}(y)\, dy\right| = \left|D^\alpha \int_{\mathbf{R}^N} \phi_h(x-y)\, u(y)\, dy\right| \\
&= \left|\int_{\mathbf{R}^N} \phi_h(x-y)\, D^\alpha u(y)\, dy\right| = h^{-N} \left|\int_{B(x,h)} \phi\left(\frac{x-y}{h}\right) D^\alpha u(y)\, dy\right| \\
&\leqq c_2\left(\sup_{z\in\mathbf{R}^N} \phi(z)\right) |B(0,1)|\, |B(x,h)|^{-1} \\
&\quad \times \int_{B(x,h)} |D^\alpha u(y)|\, d_M^{(\varepsilon/p)-(k-|\alpha|)}(y)\, dy \cdot (lh)^{-(\varepsilon/p)+k-|\alpha|} \\
&\leqq c_3 \mathcal{M}(D^\alpha u \cdot d_M^{(\varepsilon/p)-(k-|\alpha|)})\,(x) \cdot [d_M(x)]^{-(\varepsilon/p)+k-|\alpha|}.
\end{aligned}$$

Hence

$$|D^\alpha v_h(x)|\, d_M^{(\varepsilon/p)}(x) \leqq c_3 \mathcal{M}(D^\alpha u \cdot d_M^{(\varepsilon/p)-(k-|\alpha|)})\,(x) \cdot d_M^{k-|\alpha|}(x). \tag{1.13}$$

However, since Ω is bounded and $|\alpha| \leqq k$, $d_M^{k-|\alpha|}(x)$ is bounded. Thus there is a constant c_4 such that

$$|D^\alpha v_h(x)|\, d_M^{(\varepsilon/p)}(x) \leqq c_4 \mathcal{M}(D^\alpha u \cdot d_M^{(\varepsilon/p)-(k-|\alpha|)})\,(x).$$

It follows from (i), (ii) and (iii) that if $0 < h \leqq 1$, there is a constant c_5, independent of h, such that for all $x \in \Omega$ and all $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leqq k$,

$$|D^\alpha v_h(x)|\, d_M^{\varepsilon/p}(x) \leqq c_5 G(x), \tag{1.14}$$

where $G(x) := \max\,\{\mathcal{M}(u d_M^{(\varepsilon/p)-k})\,(x), \mathcal{M}(D^\alpha u \cdot d^{(\varepsilon/p)-k+|\alpha|})\,(x)\}$. Since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, $D^\alpha u \cdot d_M^{(\varepsilon/p)-k+|\alpha|} \in L^p(\Omega)$ and hence $D^\alpha u \cdot d_M^{(\varepsilon/p)-k+|\alpha|} \in L^p(\mathbf{R}^N)$; in particular, $u d_M^{(\varepsilon/p)-k} \in L^p(\mathbf{R}^N)$. The properties of the maximal function now imply that

$$G \in L^p(\mathbf{R}^N). \tag{1.15}$$

By the Lebesgue dominated convergence theorem, the assumption that as $h \to 0$,

$$|D^\alpha v_h(x)|\, d_M^{\varepsilon/p}(x) \to |D^\alpha u(x)|\, d_M^{\varepsilon/p}(x) \tag{1.16}$$

for almost all $x \in \Omega$, implies (in view of (1.14) and (1.15)) that as $h \to 0$, $d_M^{\varepsilon/p} D^\alpha v_h \to d_M^{\varepsilon/p} D^\alpha u$ in L^p; that is,

$$v_h \to u \quad \text{in} \quad W^{k,p}(\Omega; d_M, \varepsilon), \tag{1.17}$$

which is what we need.

All that remains to complete the proof of the Theorem is to establish (1.16). To do this, let $\delta > 0$, $\Omega_\delta = \{x \in \Omega : d_M(x) > \delta\}$, and note that since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, $u \in W^{k,p}(\Omega_\delta)$. For $h \in (0, \delta/4)$ and $x \in \Omega_\delta$, we have $u^{(3h)}(x) = u(x)$ and, for $|\alpha| \leqq k$, α fixed,

$$D^\alpha v_h(x) = D^\alpha(\phi_h * u^{(3h)})\,(x) = D^\alpha(\phi_h * u)\,(x) = (\phi_h * D^\alpha u)\,(x),$$

and as $h \to 0$,

$$D^\alpha v_h \to D^\alpha u \quad \text{in} \quad L^p(\Omega_\delta). \tag{1.18}$$

since $\phi_h * D^\alpha u \to D^\alpha u$ in $L^p(\Omega_\delta)$. From (1.18) it follows that there is a sequence (h_n) of positive real numbers, converging to 0, such that $D^\alpha v_{h_n} \to D^\alpha u$ almost everywhere in Ω_δ. Now take $\delta = 1/k_0$, $k_0 \in \mathbf{N}$, and let $(v_n^{(1)})$ be the sequence (v_{h_n}) for this particular δ. Let $(v_n^{(2)})$ be a subsequence of $(v_n^{(1)})$ such that

$$D^\alpha v_n^{(2)} \to D^\alpha u \quad \text{a.e. in} \quad \Omega_{1/(k_0+1)};$$

more generally let $(v_n^{(l)})$ be a subsequence of $(v_n^{(l-1)})$ such that

$$D^\alpha v_n^{(l)} \to D^\alpha u \quad \text{a.e. in} \quad \Omega_{1/(k_0+l)}.$$

It follows that the diagonal sequence (w_n), $w_n = v_n^{(n)}$, has the property that

$$D^\alpha w_n \to D^\alpha u \quad \text{a.e. in } \Omega. \tag{1.19}$$

Of course, $\{w_n : n \in \mathbf{N}\} \subset \{v_h : h > 0\}$. Moreover, while the multi-index α was fixed, it is easy to see that matters can be arranged so that (1.19) holds for *every* $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \leq k$. Thus

$$d_M^{\varepsilon/p} D^\alpha w_n \to d_M^{\varepsilon/p} D^\alpha u \quad \text{a.e. in } \Omega,$$

and the arguments used to pass from (1.16) to (1.17) show that $w_n \to u$ in $W^{k,p}(\Omega; d_M, \varepsilon)$. Our discussion in (i) shows that $\operatorname{supp} w_n \cap \overline{M} = \emptyset$ (see (1.5)); moreover, $w_n \in C^\infty(\mathbf{R}^N)$. Hence $u \in W_M{}^{k,p}(\Omega; d_M, \varepsilon)$, and the proof is complete, since the continuity of the embedding of H in W is clear ∎

2. The embedding $W \hookrightarrow H$

Now we shall deal with embeddings inverse to those of § 1. For this case, we need more special domains.

Definition 2.1: Put $Q = (0, 1)^N$ and $Q(m) = \{x \in \overline{Q} : x_{m+1} = \cdots = x_N = 0\}$, $m = 0, 1, \ldots, N-1$. A closed subset M of $\partial\Omega$ is said to be a *manifold of dimension m on* $\partial\Omega$ if there is an open covering $\{U_i\}_{i=0}^{\omega}$ (ω finite or $\omega = \infty$) of Ω with the following properties:

(i) $M \subset \bigcup_{i=1}^{\omega} U_i$, and there exists $s \in \mathbf{N}$ such that every system of $(s+1)$ sets U_i is disjoint;

(ii) there exists $\delta > 0$ such that

$$d_M(x) \geq \delta, \quad \text{for all} \quad x \in U_0; \tag{2.1}$$

(iii) there are numbers c_1, c_2, with $c_2 \geq c_1 > 0$, and a system of one-to-one mappings $T_i : \overline{Q} \to \overline{\Omega \cap U_i}$ ($i = 1, 2, \ldots, \omega$) such that $T_i(Q(m)) = M \cap \overline{U}_i$ and

$$c_1 |x - y| \leq |T_i(x) - T_i(y)| \leq c_2 |x - y|$$

for all $x, y \in \overline{Q}$ and $i = 1, \ldots, \omega$. (2.2)

Remark 2.2: In what follows we shall see that this notion of an m-dimensional manifold on $\partial\Omega$ is suitable for dealing with embeddings of Sobolev spaces with weights which are powers of d_M. We use simple idea to avoid technical complications, but theorems of the type of the following Theorem 2.3 hold for more general sets M, and in particular for the set

$$M^* := \bigcup_{i=1}^{N} \{x \in \bar{Q} : x_j = 0 \text{ for } j \neq i\} \subset \partial\Omega,$$

which is not a one-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

Theorem 2.3: *Let $\Omega \subset \mathbf{R}^N$ be a domain, let $M \in \partial\Omega$ be an m-dimensional manifold on $\partial\Omega$, let $m \in \{0, 1, \ldots, N-1\}$ and suppose that $p \in (1, \infty)$. Then:*

$$V \hookrightarrow L^p(\Omega; d_M, \varepsilon - p) \tag{2.3}$$

where

(i) $V = W^{1,p}(\Omega; d_M, \varepsilon)$ *if*

$$\varepsilon > p + m - N \tag{2.4}$$

or

$$\varepsilon \leqq m - N; \tag{2.5}$$

(ii) $V = W_M^{1,p}(\Omega; d_M, \varepsilon)$ *if*

$$\varepsilon \neq p + m - N; \tag{2.6}$$

(iii) $V = W_0^{1,p}(\Omega; d_M, \varepsilon)$ *if $m < N - 1$ and ε is an arbitrary real number.*

Proof: Let $(\phi_i)_{i=0}^{\omega}$ be a partition of unity subordinate to the covering $\{U_i\}_{i=0}^{\omega}$ mentioned in Definition 2.1. Let $u \in W^{1,p}(\Omega; d_M, \varepsilon)$ and put $u_i = u\phi_i$, $i = 0, 1, \ldots, \omega$. It is enough to prove that there is a constant c, independent of u, such that for $i = 0, 1, \ldots, \omega$,

$$\int_{U_i \cap \Omega} |u_i(x)|^p \, d_M^{\varepsilon - p}(x) \, dx \leqq c \, \|u_i\|_W. \tag{2.7}$$

For $i = 0$, (2.7) holds trivially in view of (2.1). Now suppose that $i > 0$, and for the sake of simplicity omit the subscript i on u_i and U_i. By (2.2)

$$c_1 d_{Q(m)}(y) \leqq d_M\big(T(y)\big) \leqq c_2 d_{Q(m)}(y) \tag{2.8}$$

for all $y \in \bar{Q}$; and from [3: Chap. 2, Lemma 3.1] it follows that there are positive constants c_3, c_4, depending only on c_1, c_2, p and N, such that

$$c_3 \int_{\Omega \cap U} |w(x)|^p \, dx \leqq \int_Q |w(T(y))|^p \, dy \leqq c_4 \int_{\Omega \cap U} |w(x)|^p \, dx \tag{2.9}$$

or all $w \in L^p(\Omega \cap U)$.

For all $y \in Q$ put $v(y) = u\big(T(y)\big)$, and introduce the "cylindrical" coordinates $y = (y', y'') \mapsto (y', \theta, r)$, $y' = (y_1, \ldots, y_m)$, $\theta \in \Xi := \left(0, \frac{\pi}{2}\right)^{N-m-1}$, $r \in \big(0, R(\theta)\big)$, $r = d_{Q(m)}(y) = \left(\sum_{i=m+1}^{N} y_i^2\right)^{1/2}$. The corresponding Jacobian is $\left|\frac{D(y)}{D(y', \theta, r)}\right| = r^{N-m-1} \times \Phi(y'; \theta)$, and

$$\int_Q |v(y)|^p \, d_{Q(m)}^{\varepsilon - p}(y) \, dy$$

$$= \int_{Q(m)} \int_{\Xi} \Phi(y', \theta) \int_0^{R(\theta)} |v(y', \theta, r)|^p \, r^{\varepsilon - p + N - m - 1} \, dr \, d\theta \, dy'. \tag{2.10}$$

Since $v(y', \theta, r) = 0$ for almost all $y' \in Q(m)$, $\theta \in \Xi$ and for all r in a neighbourhood of $R(\theta)$, we can extend the function v by zero for $r \geqq R(\theta)$ with preservation of all differentiability properties. Application of the Hardy inequality (see, for example, [2: Chap. 5]) to the inner integral on the right-hand side of (2.10) now shows that for almost all $y' \in Q(m)$, $\theta \in \Xi$,

$$\int_0^\infty |v(y', \theta, r)|^p \, r^{\varepsilon-p+N-m-1} \, dr \leqq c_H \int_0^\infty \left| \frac{\partial y}{\partial r} (y', \theta, r) \right|^p r^{\varepsilon+N-m-1} \, dr \tag{2.11}$$

if

$$\varepsilon + N - m - 1 > p - 1 \tag{2.12}$$

and $\lim_{r\to\infty} v(y', \theta, r) = 0$ (this condition is satisfied trivially), or if

$$\varepsilon + N - m - 1 < p - 1 \tag{2.13}$$

and

$$\lim_{r\to 0+} v(y', \theta, r) := v_0(y', \theta) = 0. \tag{2.14}$$

The constant c_H in (2.11) is given by $c_H = p/|\varepsilon - p + N - m|$. Integrating the inequality (2.11) over $Q(m) \times \Xi$ and passing back to Cartesian coordinates we obtain

$$\int_Q |v(y)|^p \, d_{Q(m)}^{\varepsilon-p}(y) \, dy \leqq c_H \int_Q \left| \frac{\partial v}{\partial r} (y) \right|^p d_{Q(m)}^{\varepsilon}(y) \, dy$$

$$\leqq c_5 \sum_{j=m+1}^{N} \int_Q \left| \frac{\partial v}{\partial y_j} (y) \right|^p d_{Q(m)}^{\varepsilon}(y) \, dy.$$

Use of (2.9) with $w = u d_M^{(\varepsilon/p)-1}$ and with $w = \frac{\partial u}{\partial x_j} d_M^{\varepsilon/p}$, the estimates (2.8) and the inequality $\left| \frac{\partial v}{\partial y_j} (y) \right| \leqq c_2 \sum_{l=1}^{N} \left| \frac{\partial u}{\partial y_l} \big(T(y)\big) \right|$ which follows from (2.2) now gives (2.7) with a constant c which depends only on p, ε, N, c_1 and c_2.

All that remains is to discuss the validity of the assumptions (2.12)—(2.14). The inequalities (2.4) and (2.12) are equivalent. If (2.5) holds, then (2.13) is satisfied. Further, for almost all $y' \in Q(m)$, $\theta \in \Xi$ and for $r, h > 0$ we have, by Hölder's inequality,

$$|v(y', \theta, r + h) - v(y', \theta, r)| = \left| \int_r^{r+h} \frac{\partial v}{\partial r} (y', \theta, \varrho) \, d\varrho \right|$$

$$\leqq \left(\int_r^{r+h} \left| \frac{\partial v}{\partial r} (y', \theta, \varrho) \right|^p \varrho^{\varepsilon+N-m-1} \, d\varrho \right)^{1/p}$$

$$\times \left(\int_r^{r+h} \varrho^{(\varepsilon+N-m-1)/(1-p)} \, d\varrho \right)^{(p-1)/p} = o(1)$$

as $h \to 0$, since the former of the last two integrals is finite and the exponent in the latter one is positive. Hence the function $v(y', \theta, \cdot)$ is uniformly continuous in a neighbourhood of the origin and the limit $v_0(y', \theta)$ in (2.14) exists. Since

$$\int_0^{R(\theta)} |v(y', \theta, r)| \, r^{\varepsilon+N-m-1} \, dr < \infty,$$

the assumption (2.5) yields (2.14). Thus assertion (i) holds.

If $V = W_M^{1,p}(\Omega; d_M, \varepsilon)$, it is enough to deal with functions u in $C_M^\infty(\Omega)$, so that $\operatorname{supp} v \subset Q$; and condition (2.14) is then trivially satisfied. The condition (2.6) means that (2.12) or (2.13) holds.

Finally, suppose that $m < N - 1$. If $\varepsilon \neq p + m - N$, assertion (iii) follows from (ii), since $W_0^{1,p}(\Omega; d_M, \varepsilon) \subset W_M^{1,p}(\Omega; d_M, \varepsilon)$. Thus suppose that $\varepsilon = p + m - N$ and let $u \in C_0^\infty(\Omega)$. Since $m < N - 1$, we can write

$$2^{-1/2}(y_l + \varrho_l) \leqq d_{Q(m)}(y) \leqq y_l + \varrho_l, \quad \varrho_l^2 = \sum_{j=m+1, j \neq l}^{N} y_j^2$$
$$(l = m + 1, \ldots, N; y \in Q)$$

and

$$\int_Q |v(y)|^p \, d_{Q(m)}^{\varepsilon-p}(y) \, dy \leqq c_6 \int_{(0,1)^{N-1}} \int_0^1 |v(y)|^p \, (y_l + \varrho_l)^{\varepsilon-p} \, dy_l \, dy^{(l)},$$

where $y^{(l)} = (y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_N)$. Taking into account the fact that $\operatorname{supp} v \subset Q$ and $\varepsilon = p + m - N < p - 1$, we apply the generalized Hardy inequality (see [5: Lemma 5.3]) to the inner integral on the right-hand side and obtain

$$\int_Q |v(y)|^p \, d_{Q(m)}^{\varepsilon-p}(y) \, dy \leqq c_7 \int_Q \left| \frac{\partial v}{\partial y_l}(y) \right|^p d_{Q(m)}^{\varepsilon}(y) \, dy.$$

It is now sufficient to use (2.9) with $w = u \, d_M^{(\varepsilon/p)-1}$ and with $w = \dfrac{\partial u}{\partial x_l} d_M^{\varepsilon/p}$; the assertion (iii) and consequently Theorem 2.3 are proved ∎

3. Concluding remarks

We present two natural consequences of the work in § 1 and § 2.

Corollary 3.1: *Let Ω be a bounded domain in $\mathbf{R}^N$, let M be an m-dimensional manifold on $\partial\Omega$ where $m \in \{0, 1, \ldots, N - 1\}$, and let $p \in (1, \infty)$, $k \in \mathbf{N}$. Then*

$$V \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon),$$

where

(i) $V = W^{k,p}(\Omega; d_M, \varepsilon)$ *if*

$$\varepsilon > kp + m - N \tag{3.1}$$

or

$$\varepsilon \leqq m - N; \tag{3.2}$$

(ii) $V = W_M^{k,p}(\Omega; d_M, \varepsilon)$ *if*

$$\varepsilon \neq jp + m - N, \qquad j = 1, 2, \ldots, k; \tag{3.3}$$

(iii) $V = W_0^{k,p}(\Omega; d_M, \varepsilon)$ *if*

$$m < N - 1 \quad \text{and} \quad \varepsilon \in \mathbf{R}. \tag{3.4}$$

Proof: Use Theorem 2.3 successively for $D^\beta u$ with $|\beta| = k$, $|\beta| = k - 1, \ldots$ ∎

Corollary 3.2: *Let the assumptions of Corollary* 3.1 *be satisfied. If ε satisfies* (3.1) *or* (3.2), *then*

$$W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon) = W^{k,p}(\Omega; d_M, \varepsilon).$$

If ε satisfies (3.3), *then*

$$W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon). \tag{3.5}$$

If $m < N - 1$, *then* (3.5) *holds for all* $\varepsilon \in \mathbf{R}$ *with* $W_0^{k,p}(\Omega; d_M, \varepsilon)$ *instead of* $W_M^{k,p}(\Omega; d_M, \varepsilon)$.

Proof: The assertions follow immediately from Corollary 3.1 and Theorem 1.2 ∎

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