Embeddings of Sobolev Spaces with Weights of Power Type

D. E. EDMUNDS, A. KUFNER and J. RAKOSNIK-

Die Arbeit befaßt sich mit einigen Eigenschaften gewichteter Sobolevscher Räume $W_0^{k,p}$ *,* $W_M^{k,p}$ *und* $H^{k,p}$ *, in denen als Gewichte Potenzen der Entfernung von einem Teil M des Ran*des des Definitionsgebietes auftreten. Es werden Bedingungen angegeben, unter denen diese Räume gegenseitig eingebettet und gewisse Normen äquivalent sind. Dabei werden einige Ergebuisse aus [1] verallgemeinert und eine in [2] formilierte Vermutung wird bewiesen.

В работе исследованы некоторые свойства весовых пространств С. Л. Соболева $W_0{}^{k,p},$ $W_M^{k,p}$ и $H^{k,p}$, весовые функции которых являются степенями расстояния от части М границы области определения. Указаны условия, при которых эти пространства вкладываются друг'в друга и при которых некоторые нормы эквивалентны. Обобщаются некоторые результаты из [1] и доказана гипотеза, сформулированная в [2].

The paper deals with some properties of the weighted Sobolev spaces $W_0^k P$, $W_M^k P$ and domain of definition. Conditions are given which guarantee the mutual embeddings of these $H^{k,p}$, with weights which are powers of the distance from a part \tilde{M} of the boundary of the spaces and the equivalence of certain norms. The paper generalizes some results of [1) and verifies a conjecture formulated in [2]. B работе исследованы некоторые свойства весовых пространств $W_M^{k,p}$ и $H^{k,p}$, весовые функции которых являются степенями границы области определения. Указаны условия, при которых этак важивали сопроше результаты из [1] pyr'n *n*pyra *u npu korophix* нек
pesyntraria *us* [1] *u n*o*kasaha* r*u*
r deals with some properties of
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d the equivalence of certain norms.

0. Introduction

Let Ω be a domain in the Euclidean space \mathbb{R}^N and let M be a non-empty subset of the boundary $\partial\Omega$ of Ω ; given any $x \in \mathbb{R}^N$ write

$$
d_M(x) = \text{dist}(x, M).
$$

(0.1)

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the boundary $\partial\Omega$ of Ω ; given any $x \in \mathbb{R}^N$ write
 $d_M(x) = \text{dist}(x, M)$. (0.1)
Let $\varepsilon, p \in \mathbb{R}$, with $p \ge 1$, $k \in \mathbb{N}$ of (equivalence classes of) functions $u: \Omega \to \mathbf{R}$ such that for all $\alpha = (\alpha_i) \in N_0^N$ with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N \leq k$, the distributional derivative D^*u satisfies **the boundary** *ost* of *st*; given any $x \in \mathbb{R}^N$ write
 $d_M(x) = \text{dist}(x, M)$.

Let $\varepsilon, p \in \mathbb{R}$, with $p \ge 1$, $k \in \mathbb{N}$, and let $W^{k,p}(\Omega; d_M, \varepsilon)$ be the weighted Sot

of (equivalence classes of) functions $u: \Omega \to$ *f*^{*I*}. the equivalence of certain norms. The paper generalizes some results of [1] and veri-
 f^{*a*}*du(x)* do f*i*^{*D*}*n*^{*x*} (*R^{<i>N*}</sup> and let *M*^{*b*} de a non-empty subset of
 dary 20 of *Ω*; given any $x \in$ 0. Introductio

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Let $\varepsilon, p \in \mathbb{R}$, v

of (equivalence
 $|\alpha| := \alpha_1 + \alpha_2$
 $\int_{\Omega} |D^{\alpha}|$

The space W^k

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 $||u||_W$

We also intro a domain in the Euclidean space \mathbb{R}^N and let M be
 $d_M(x) = \text{dist}(x, M)$.
 R, with $p \ge 1$, $k \in \mathbb{N}$, and let $W^{k,p}(\Omega; d_M, \varepsilon)$ be the

elence classes of) functions $u: \Omega \to \mathbb{R}$ such that for
 $+\alpha_2 + \cdots + \alpha_N \le k$ Let $\varepsilon, p \in \mathbb{R}$, with $p \ge 1$, $k \in \mathbb{N}$, and let $W^{k,p}(\Omega; d_M, \varepsilon)$ be therefore classes of the interiors $u: \Omega \to \mathbb{R}$ such that for $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N \le k$, the distributional derivative $\int |D^s u(x)|^p d_M^s(x) dx < \$

$$
|D^{\alpha}u(x)|^p d_M^{\alpha}(x) dx < \infty.
$$

The space $W^{k,p}(\Omega; d_M, \varepsilon)$ is a Banach space when equipped with the norm defined

$$
\iota \|_{W} := \left(\sum_{|a| \leq k} \int_{Q} |D^a u(x)|^p \ d_M^c(x) \ dx \right)^{1/p} . \tag{0.2}
$$

We also introduce the spaces $W_M^{k,p}(\Omega; d_M, \varepsilon)$ and $W_0^{k,p}(\Omega; d_M, \varepsilon)$, which are the closures in the Banach space $W^{k,p}(\Omega; d_M, \varepsilon)$ of the sets
 $C_M^{\infty}(\Omega) := \{ w \in C^{\infty}(\overline{\Omega}) : \text{supp } w \cap \overline{M} = \emptyset \}$ (0.3) closures in the Banach space $W^{k,p}(Q; d_M, \varepsilon)$ of the sets

$$
C_M^{\infty}(\Omega) := \{ w \in C^{\infty}(\bar{\Omega}) : \text{supp } w \cap \overline{M} = \emptyset \}
$$
\n
$$
(0.3)
$$

and $C_0^{\infty}(\Omega)$ respectively; and the space $H^{k,p}(\Omega; d_M, \varepsilon)$, which is the set of all functions Coloures in the Banach space $W^{k,p}(\Omega; d_M, \varepsilon)$ of the $C_M^{\infty}(\Omega) := \{w \in C^{\infty}(\bar{\Omega}) : \text{supp } w \cap \overline{M} = \text{and } C_0^{\infty}(\Omega)$ respectively; and the space $H^{k,p}(\Omega; d_M, \varepsilon) \to \mathbf{R}$ such that for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq k$ *ID" ID" ID"*

$$
\int_{2} |D_{\cdot}^{\alpha} u(x)|^{p} d_{M}^{\epsilon-(k-|\alpha|)p}(x) dx < \infty.
$$

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D. E. EDMUNDS, A. KUFNER and J. Rárosník
\nspace is also a Banach space when provided with the norm given by
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$$
||u||_H := \left(\sum_{|a| \leq k} \int_{\Omega} |D^a u(x)|^p d_M^{e-(k-|a|)p}(x) dx\right)^{1/p}.
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$$
\n
$$
||u||_H := \left(\sum_{|a| \leq k} \int_{\Omega} |D^a u(x)|^p d_M^{e-(k-|a|)p}(x) dx\right)^{1/p}.
$$

If $\varepsilon = 0$, $W^{k,p}(Q; d_M, \varepsilon)$ and $W_0^{k,p}(Q; d_M, \varepsilon)$ are simply the classical unweighted Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ respectively; if $k=0$, then $W^{k,p}(\Omega; d_M, \varepsilon)$ is nothing more than the weighted Lebesgue space $L^p(\Omega; d_M, \varepsilon)$. D. E. EDMUNDS, A. KUFNER and J. RÁKOSNÍK

space is also a Banach space when provide
 $||u||_H := \left(\sum_{|a| \leq k} \int_D |D^a u(x)|^p d_M^{e-(k-|a|)p}(x) dx\right)^1$
 $0, W^{k,p}(\Omega; d_M, \varepsilon)$ and $W_0^{k,p}(\Omega; d_M, \varepsilon)$ are spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\$ $||u||_H := \left(\sum_{|a| \leq k} \int_{Q} |D^s u(x)|^p d_M^{e-(k-|a|)p}(x) dx\right)^{1/2}$
 $0, W^{k,p}(\Omega; d_M, \varepsilon)$ and $W_0^{k,p}(\Omega; d_M, \varepsilon)$ are singness $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ respectively; if

nore than the weighted Lebesgue space $L^p(\cdot)$

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In [1] the case $M = \partial \Omega$ was considered, and it was shown that if for $i = 1, 2, ..., k$ the inequality $\varepsilon \neq ip-1$ holds, then

$$
W_0^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon).
$$
\n
$$
(0.5)
$$

the symbol \hookrightarrow denoting continuous embedding; it was also shown that for all $\varepsilon \in \mathbb{R}$,

$$
H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_0^{k,p}(\Omega; d_M, \varepsilon). \tag{0.6}
$$

In particular, if $\varepsilon = 0$, then for any $u \in W^{k,p}(\Omega)$ such that $D^{\alpha}u \in L^p(\Omega; d_M, (k - |\alpha|) p)$ for all $\alpha \in N_0^N$ with $|\alpha| \leq k - 1$, we have $u \in W_0^{k,p}(\Omega)$. The assumption that Ω had a Lipschitz boundary was important in [1], although insofar as the embedding (0.6) is concerned this condition can be weakened.

When $\varepsilon = 0$ it is possible, by following unpublished suggestions of D. J. HARRIS, to establish (0.6) under the sole assumption on Ω that it should be bounded (see Theorem 1.1). The method used relies on the properties of the maximal function and can be extended to the case of an arbitrary set $M\subset \partial\Omega$ and an arbitrary $\epsilon\in{\bf R}$: in Theorem *1.2* it is shown that ish (0.6) under the sole assumption
1.1). The method used relies on the
be extended to the case of an arbitrary
 $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon)$.

$$
H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon).
$$

The embedding (0.5) follows from embedding theorems in weighted spaces;. in the case $M = \partial \Omega$, such theorems hold if $\partial \Omega$ is Lipschitzian. It was conjectured in [2] that

$$
W_M{}^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon)
$$

for certain special sets M, provided that for $i=1, 2, ..., k$ the inequality $\varepsilon \neq ip-1$ holds. This conjecture is verified in Corollary 3.1, (ii); the main tool is due to the third author (J. R.) and is a fuller version of the short communication [4].

1. The embedding $H \hookrightarrow W$

First, we shall prove a theorem, which implies (0.6) for the case $\varepsilon = 0$ and $M = \partial \Omega$. This result is contained in [1], but here, the condition on *Q is* weakened and the method is completely new.

Theorem 1.1: Let Ω be a non-empty open subset of \mathbb{R}^N , Ω \neq \mathbb{R}^N , and for each $x \in \Omega$ put $d(x) = \text{dist}(x, \partial \Omega)$; iet $p \in (1, \infty)$ and $k \in \mathbb{N}$. Suppose that $u \in W^{k,p}(\Omega)$ is *such that* $ud^{-k} \in L^p(\Omega)$ *. Then* $u \in W_0^{k,p}(\Omega)$.

Proof: First suppose *Q* is bounded. Let $0 < h' < \delta$; define $u^{(\delta)}$ by $u^{(\delta)}(x) = u(x)$ if $d(x) \geq \delta$ and $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise; let $\phi \in C_0^{\infty}(\mathbb{R}^N)$ be such that $\phi(x) = 0$ if $|x| \ge 1$, $\phi(x) > 0$ if $|x| < 1$, $\int \phi(x) dx = 1$; and put $\phi_h(x) = h^{-N} \phi(x/h)$. Let us

agree that any function g defined on Ω will be supposed extended to the whole \mathbb{R}^N , if necessary, by setting $g(x) = 0$ for all $x \in \mathbb{R}^N \setminus \Omega$. Note that $\phi_h * u^{(\delta)} \in C_0^{\infty}(\Omega)$.

Embeddings of Sobolev Spaces

First suppose that $d(x) > \delta + h$, $x \in \Omega$ and $\alpha \in N_0^N$, $|\alpha| \leq k$. Then since $B(x, h)$
 $\Rightarrow {y \in R^N : |x - y| < h} \subset B(x, \delta + h) \subset \Omega$, and $d(y) > \delta$ if $y \in B(x, h)$, uppose that $d(x) > \delta +$
 $R^N : |x - y| < h$ $\subset B(x,$
 $D^{\alpha}(\phi_h * u^{(\delta)}) (x) = D^{\alpha} \int_{B(x, h)}$
 $= D^{\alpha} \int_{R^N} \phi_h(x - y) u(y) dy$ Embeddings of Sobolev
 $> \delta + h$, $x \in \Omega$ and $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq k$. Then
 $\subset B(x, \delta + h) \subset \Omega$, and $d(y) > \delta$ if $y \in B$
 $\subset D^{\alpha} \int \phi_h(x - y) u^{(\delta)}(y) dy$
 $\sim u(y) dy = \int \phi_h(x - y) D^{\alpha}u(y) dy$; **Embeddings of Sobolev Spaces**
 D(c) $\delta + h, x \in \Omega$ and $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq k$. Then since $B(x, h)$
 $\mathbb{R}^N : |x - y| < h$, $\mathbb{D}(x, \delta + h) \subset \Omega$, and $d(y) > \delta$ if $y \in B(x, h)$,
 $D^3(\phi_h * u^{(\delta)}) (x) = D^2 \int \phi_h(x - y) u^{(\delta)}(y) dy$
 $= D^2 \$

$$
D^{x}(\phi_h * u^{(\delta)}) (x) = D^{a} \int_{B(x,h)} \phi_h(x-y) u^{(\delta)}(y) dy
$$

$$
= D^a \int_{\mathbf{R}^N} \phi_h(x - y) u(y) dy = \int_{\mathbf{R}^N} \phi_h(x - y) D^2 u(y) dy;
$$

that is,

$$
D^a(\phi_h * u^{(\delta)}) (x) = (\phi_h * D^2 u) (x) \text{ if } d(x) > \delta + h, x
$$

It follows that, with $\mathcal{M}(g)$ as the maximal function defined by
 $\mathcal{M}(g) (x) = \sup_{r>0} |B(0, r)|^{-1} \int_{B(x, r)} |g(y)| dy$
 $(|B(0, r)| = \omega_N r^N \text{ being the volume of the ball } B(0, r)), \text{ we have}$

that is,

$$
D^{\circ}(\phi_h * u^{(\delta)}) (x) = (\phi_h * D^{\circ} u) (x) \quad \text{if} \quad d(x) > \delta + h, \quad x \in \Omega.
$$

It follows that, with $\mathcal{M}(g)$ as the maximal function defined by

$$
\mathcal{M}(g) (x) = \sup_{r>0} |B(0, r)|^{-1} \int_{B(x,r)} |g(y)| dy
$$

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$$

\n
$$
|B(0, r)| = \omega_N r^N \text{ being the volume of the ball } B(0, r)), \text{ we have}
$$

\n
$$
|D^{\alpha}(\phi_h * u^{(\delta)}) (x)| \leq \mathcal{M}(D^{\alpha}u) (x) \cdot \omega_N \sup_{y \in B(x,h)} \phi \left(\frac{x-y}{h}\right)
$$

\n
$$
= \mathcal{M}(D^{\alpha}u) (x) \cdot \omega_N \sup_{z \in B(0, 1)} \phi(z) = \mathcal{M}(D^{\alpha}u) (x) b(\phi), \text{ say.}
$$

\nNext, suppose that $d(x) \leq lh$ for some $l \in \mathbb{N}$, and let $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq k$. Then

$$
= \mathcal{M}(D^{\alpha}u) (x) \cdot \omega_{N} \sup_{z \in B(0,1)} \phi(z) = \mathcal{M}(D^{\alpha}u) (x) b(\phi), \text{ say.}
$$

\n
$$
\text{suppose that } d(x) \leq lh \text{ for some } l \in \mathbb{N}, \text{ and let } \alpha \in \mathbb{N}_{0}^{N}, |\alpha| \leq k. \text{ Then}
$$

\n
$$
|D^{\alpha}(\phi_{h} * u^{(\delta)}) (x)| = \left| \int_{\mathbb{R}^{N}} h^{-N-|\alpha|} (D^{\alpha} \phi) \left(\frac{x-y}{h} \right) u^{(\delta)} (y) dy \right|
$$

\n
$$
\leq h^{-N-|\alpha|} \int_{\mathbb{R}^{N}} \left| (D^{\alpha} \phi) \left(\frac{x-y}{h} \right) u^{(\delta)} (y) d^{-k} (y) \right| h^{k} (l+1)^{k} dy
$$

\n
$$
\leq \omega_{N} h^{k-|\alpha|} (l+1)^{k} \mathcal{M}(ud^{-k}) (x) \sup_{z \in B(0,1)} | (D^{\alpha} \phi) (z)|.
$$

\nwe take $\delta = 2h = 2/j \ (j \in \mathbb{N})$ and $l = 3$, we see that for all $x \in \Omega$, for all
\nwith $|\alpha| \leq k$, and for all $j \in \mathbb{N}$,
\n
$$
|D^{\alpha}(\phi_{1/j} * u^{(2/j)} (x))|
$$

\n
$$
\leq \max \{b(\phi) \mathcal{M}(D^{\alpha}u) (x), \quad 4^{k}c(\alpha, \phi) \mathcal{M}(ud^{-k}) (x) \} := G_{\alpha, \varphi}(x), \qquad (1.2)
$$

\n
$$
x, \phi) = \omega_{N} \sup_{z \in B(0,1)} | (D^{\alpha} \phi) (z)|. \text{ Since, by } [5: \text{Chap. I, Th. 1}], \mathcal{M} \text{ is a bounded}
$$

Thus if we take $\delta = 2h = 2/j$ $(j \in \mathbb{N})$ and $l = 3$, we see that for all $x \in \Omega$, for all therefore the *s* of the take *δ* = K_0^N with $|α|$ ≤ $|D^2/d|$ + \ge $|D^2/d|$ $\alpha \in N_0^N$ with $|\alpha| \leq k$, and for all $j \in N$,

$$
|D^{2}(\phi_{1/j} * u^{(2/j)}(x))|
$$

\n
$$
\leq \max \{b(\phi) \mathcal{M}(D^{2}u)(x), \quad 4^{k}c(\alpha, \phi) \mathcal{M}(ud^{-k})(x)\} := G_{\alpha,\alpha}(x).
$$
 (1.2)

where $c(\alpha, \phi) = \omega_N$ sup $|(D^3\phi)(z)|$. Since, by [5: Chap. I, Th. 1], *M* is a bounded mapping from $L^p(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N)$, it follows that $G_{a,p} \in L^p(\mathbf{R}^N)$. Moreover, for each $\alpha \in N_0^N$ with $|\alpha| \leq k$, $\phi_{1/j} * D^2 u \to D^2 u$ in $L^p(\Omega)$ as $j \to \infty$, and hence there is a subsequence $(j(l))$ of the sequence of positive integers such that for all $\alpha \in N_0$ ^{*N*} with $|\alpha| \leq k$, $(\phi_{1/i(1)} * D^{\alpha}u)(x) \rightarrow D^{\alpha}u(x)$ almost everywhere in Ω . Thus by (1.1), $\delta = 2h = 2/j \ (j \in \mathbb{N})$ and $l = 3$,
 $\leq k$, and for all $j \in \mathbb{N}$,
 $\langle u^{(2j)}(x) \rangle$
 $b(\phi) \mathcal{M}(D^a u)(x)$, $4^k c(\alpha, \phi) \mathcal{M}(u, \theta)$
 $\mathcal{M} \leq \mathcal{M}(D^a u)(x)$, $4^k c(\alpha, \phi) \mathcal{M}(u, \theta)$
 $\mathcal{M} \leq \mathcal{M}(D^a u)(x)$, $4^k c(\alpha, \phi$ *(0,1)*

hat for all $x \in \Omega$, for all
 $= G_{\alpha,\varphi}(x)$, (1.2)

Th. 1], *M* is a bounded
 (\mathbb{R}^N) . Moreover, for each
 ∞ , and hence there is a

that for all $\alpha \in \mathbb{N}_0^N$ with

Thus by (1.1), (1.3)
 \mapsto theorem, Example $\alpha \in N_0^N$ with $|\alpha| \leq k$, $\phi_{1/j} * D^3 u \to D^3 u$ in $L^p(\Omega)$ as subsequence $(j(l))$ of the sequence of positive integer $|\alpha| \leq k$, $(\phi_{1/j(l)} * D^3 u)(x) \to D^3 u(x)$ almost everywhere $D^3(\phi_{1/j(l)} * u^{(2/j(l))})(x) \to D^3 u(x)$
a.e. in

$$
D^{2}(\phi_{1/j(t)} * u^{(2/j(t))})(x) \to D^{2}u(x)
$$
\n(1.3)

a.e. in Ω . Together with Lebesgue's dominated convergence theorem, (1.2) and (1.3) show that $(\phi_{1/2}(t) * u^{(2/2)})) \rightarrow u$ in $W^{k,p}(\Omega)$; that is, $u \in W_0^{k,p}(\Omega)$.

If Ω is unbounded we simply apply the arguments above to $u\psi_m := u_m$, where **a.e. in** Ω **. Together with Lebesgue's dominated convergence theorem, (1.2) and (1.3)**
show that $(\phi_{1/j(t)} * u^{(2/j(t))}) \rightarrow u$ in $W^{k,p}(\Omega)$; that is, $u \in W_0^{k,p}(\Omega)$.
If Ω is unbounded we simply apply the arguments above to

Note that if Ω is bounded, the validity of (0.6), when $\varepsilon = 0$ and $M = \partial \Omega$, follows immediately from Theorem 1.1. We now set about the proof of (0.6) for general ε and M.

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Theorem 1.2: Let Ω be a bounded domain in \mathbb{R}^N , let $M \subset \partial \Omega$, $M + \emptyset$, and let $\in (1, \infty)$, $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Then **28 p. E.** EDMUNDS, A. KUFNER and J. RAKOSNIK
 PE(1, ∞ **),** $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Then
 $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon)$.

Proof: Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, and for any $\delta > 0$ write $u^{(\delta)}(x) = v$ *H*_e Ω *Let* Ω *be a bounded dom*
 H^k, $P(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon)$.

$$
H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon).
$$

Proof: Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, and for any $\delta > 0$ write $u^{(\delta)}(x) = u(x)$ if $d_M(x) > \delta$ and $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise. Let ϕ be as in the proof of Theorem 1.1 and write $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon).$

Proof: Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, and for any $\delta > 0$ write $u^{(\delta)}$

and $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise. Let ϕ be as in the proof of Ω
 $\phi_h(x) = h^{-N}\phi(x/h)$ $(h > 0, x \in \mathbb{R}^N)$. ***** u(3h). We distinguish various cases. $\epsilon \cdot (1, \infty)$, $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Then
 $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega)$

Proof: Let $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, ε

d $x \in \Omega$, $u^{(\delta)}(x) = 0$ otherwise. I
 $(x) = h^{-N}\phi(x/h)$ ($h > 0, x \in \mathbb{R}^N$)

rious cases.

(i) Le **28 D. E. ÉDMUNDS, A.** *KOFNER* and **J. RAKOSNIK**
 Theorem 1.2: Let Ω be a bounded domain in \mathbb{R}^N , let $M \subset \partial \Omega$, $M \neq \emptyset$, and let
 $p \in (1, \infty)$, $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$. Then
 $H^{k,p}(\Omega; d_M, \vare$ $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon).$
 $H^{k,p}(\Omega; d_M, \varepsilon) \hookrightarrow W_M^{k,p}(\Omega; d_M, \varepsilon).$
 $H^{k,p}(\Omega; d_M, \varepsilon)$, and for any $\delta > 0$ write $u^{(\delta)}(x) = u(x)$ if $d_M(x) > \delta$,
 $v^{(\delta)}(x) = 0$ otherwise. Let ϕ be as in the proof of Theorem 1

$$
d_M(x) \geq 2h. \tag{1.4}
$$

 $d_M(x) \geq 2h$.
 $d_M(x) \leq 4h$.
 $d_M(x) \geq 4h$. Then $v_A(x) = h^{-N} \int \phi \left(\frac{x-y}{h} \right) u^{(3h)}(y) dy = 0$ since by (1.4) if $y \in B(x, h)$ then
 $d_M(y) < 3h$ and consequently $u^{(3h)}(y) = 0$. It follows that
 \therefore supp $v_h \subset \Omega_h := \{x \in \Omega : d_M(x) > h\}$.

Note that $\Omega_h \cap \overline{M} = \emptyset$.

(ii) Sup $\begin{align*} \mathcal{L}_\mathcal{D}(x) &= h^{-N} \int \phi \left(\frac{dy}{dx} \right) u^{(3h)}(y) \, dy = 0, \ \mathcal{L}_\mathcal{B}(x, h) \ \mathcal{L}_\mathcal{B}(x, h) &= \mathcal{L}_\mathcal{B}(x) u^{(3h)}(y) = 0. \ \mathcal{L}_\mathcal{B}(x, h) &= \mathcal{L}_\mathcal{B}(x, h) \mathcal{L}_\mathcal{B}(x) &= \mathcal{L}_\mathcal{B}(x, h) \mathcal{L}_\mathcal{B}(x, h) &= \mathcal$

 $d_M(y) < 3h$ and consequently $u^{(3h)}(y) = 0$. It follows that

$$
\operatorname{supp} v_h \subset \Omega_h := \{x \in \Omega : d_M(x) > h\} \,. \tag{1.5}
$$

Note that $\Omega_h \cap \overline{M} = \emptyset$.

(ii) Suppose that $x \in \Omega$ is such that

$$
2h < d_M(x) \leq 4h.
$$

Then if $y \in B(x, h)$ we see that

$$
h < d_M(y) < 5h \tag{1.7}
$$

$$
d_M(x) \approx h, \qquad d_M(y) \approx h,\tag{1.8}
$$

with the obvious meaning of the symbol ∞ . For all $\alpha \in N_0^N$, $|\alpha| \leq k$, we have

ote that
$$
\Omega_h \cap \overline{M} = \emptyset
$$
.
\n(ii) Suppose that $x \in \Omega$ is such that
\n $2h < d_M(x) \le 4h$.
\nthen if $y \in B(x, h)$ we see that
\n $h < d_M(y) < 5h$;
\nand (1.6) and (1.7) imply the equivalence of h, $d_M(x)$ and $d_M(y)$:
\n $d_M(x) \approx h$, $d_M(y) \approx h$,
\n(iii) the obvious meaning of the symbol \approx . For all $\alpha \in N_0^N$, $|\alpha| \le k$, we have
\n $|D^*v_h(x)| = |D^* \int_{\mathbb{R}^N} \phi_h(x - y)u^{(3h)}(y) dy| = h^{-N} \left| \int_{\mathbb{R}^N} D^* \phi \left(\frac{x - y}{h} \right) u^{(3h)}(y) dy \right|$
\n $\le h^{\alpha, N-|\alpha|} \int_{\mathbb{R}^N} |D^* \phi(x)| |D^* \phi(x)| |D^* \phi(x)| |D^* \phi(x)| dy$
\n $\le \int_{\mathbb{R}^N} |D^* \phi(x)| |D^* \phi(x)| |D(x, h)|^{-1} |D^* \phi(x)| dy$
\n $\le c_0 h^{-|\alpha|} |B(x, h)|^{-1} \int_{B(x, h)} |u(y)| dy^{(t/p)-k}(y) dy \cdot h^{k-(t/p)}$
\n $\le c_1 h^{k-|\alpha|} |B(x, h)|^{-1} \int_{B(x, h)} |u(y)| dy^{(t/p)-k}(y) dy \cdot [d_M(x)]^{-t/p}$,
\nthe last two inequalities following from the equivalence of $d_M(y)$ and h, and that of
\n $u(x)$ and h. Thus we have
\n $|D^* v_h(x)| d_M^{t/p}(x) \le c_1 h^{k-|\alpha|} M (u d_M^{t(p)-k}) (x)$,
\nthere, as before, At stands for the maximal function and any function on Ω is extended
\ny zero in $\mathbb{R}^N \setminus \Omega$.
\n(iii) Let $x \in \Omega$ be such that for some $l \ge 4$,
\n $lh < \frac{d_M(x)}{d$

the last two inequalities following from the equivalence of $d_M(y)$ and h , and that of $d_M(x)$ and *h*. Thus we have *Bia*

wo inequalities following from
 lh. Thus we have
 $|D^{\alpha}v_h(x)| d_M^{s/p}(x) \leq c_1 h^{k-|\alpha|} M(u)$

before, *M* stands for the maxim

n $\mathbb{R}^N \setminus \Omega$.
 $t x \in \Omega$ be such that for some *l*
 $lh < d_M(x) < (l + 1) h$.

$$
|D^{\alpha}v_n(x)| d_M^{t/p}(x) \leq c_1 h^{k-|\alpha|} \mathcal{M}(ud_M^{(t/p)-k})(x), \qquad (1.9)
$$

(1.10)

where, as before, ${\mathscr M}$ stands for the maximal function and any function on Ω is extended by zero in $\mathbb{R}^N\setminus\Omega$. $\frac{1}{2}$ on Ω is external to $\frac{1}{2}$

(iii) Let $x \in \Omega$ be such that for some $l \geq 4$,

$$
lh < d_M(x) < (l+1) \, h \, .
$$

Embeddings of Sobolev Spaces 29

Let $y \in B(x, h)$. Then

| Embeddings of Sobolev Spaces | 29 | |
|--------------------------------|------------------------------------|--------|
| $B(x, h)$. Then | (l - 1) $h < d_M(y) < (l + 2) h$, | (1.11) |
| 10) together with (1.11) imply | $d_M(x) < l h$ | (1.12) |

$$
d_M(x) \approx lh, \qquad d_M(y) \approx lh. \tag{1.12}
$$

Embed

Let $y \in B(x, h)$. Then
 $(l-1) h < d_M(y) < (l+2) h$,

and (1.10) together with (1.11) imply
 $d_M(x) \approx lh$, $d_M(y) \approx lh$.

From (1.11) it follows that $u^{(3h)}(y) = u(y)$, since $d_M(y) >$ Embeddings of Sobolev Spaces
 d(x, h). Then

(*l* - 1) $h < d_M(y) < (l + 2) h$,

(1.11) imply

(*d_M(x)* $\approx lh$, $d_M(y) \approx lh$.

(1.12)

(1.12)

(1) it follows that $u^{(3h)}(y) = u(y)$, since $d_M(y) > (l - 1) h \ge 3h$. For $\alpha \in N_0^N$,

we and (1.10) together with (1.11) imply
 $d_M(x) \approx lh$, $d_M(y) \approx lh$. (1.12)

From (1.11) it follows that $u^{(3h)}(y) = u(y)$, since $d_M(y) > (l - 1)$ $h \ge 3h$. For $\alpha \in N_0^N$,
 $|\alpha| \le k$, we find, as in (ii), that

Let
$$
y \in B(x, h)
$$
. Then
\n
$$
(l-1) h < d_M(y) < (l+2) h,
$$
\n
$$
(l-1) h < d_M(y) \approx lh.
$$
\n(1.12)
\nFrom (1.11): it follows that $u^{(3h)}(y) = u(y)$, since $d_M(y) > (l-1) h \ge 3h$. For $\alpha \in N_0^N$,
\n $|\alpha| \le k$, we find, as in (ii), that
\n
$$
|D^{\alpha}v_h(x)| = |D^{\alpha} \int \phi_h(x-y) u^{(3h)}(y) dy| = |D^{\alpha} \int \phi_h(x-y) u(y) dy|
$$
\n
$$
= \left| \int_{R^N} \phi_h(x-y) D^{\alpha}u(y) dy \right| = |D^{\alpha} \int \phi_h(x-y) u(y) dy|
$$
\n
$$
= \left| \int_{R^N} \phi_h(x-y) D^{\alpha}u(y) dy \right| = h^{-N} \left| \int_{B(x,h)} \phi \left(\frac{x-y}{h} \right) D^{\alpha}u(y) dy \right|
$$
\n
$$
\le c_2 \left(\sup_{z \in R^N} \phi(z) \right) |B(0, 1)| |B(x,h)|^{-1}
$$
\n
$$
\times \int |D^{\alpha}u(y)| d_M^{(\alpha/p) - (k-|\alpha|)}(y) dy \cdot (lh)^{-(\alpha/p) + k - |\alpha|}
$$
\nHence
\n
$$
|D^{\alpha}v_h(x)| d_M^{(\alpha/p)}(x) \le c_3 M(D^{\alpha}u \cdot d_M^{(\alpha/p) - (k-|\alpha|)})(x) \cdot (d_M(x))^{-(\alpha/p) + k - |\alpha|}
$$
\nHence
\n
$$
|D^{\alpha}v_h(x)| d_M^{(\alpha/p)}(x) \le c_3 M(D^{\alpha}u \cdot d_M^{(\alpha/p) - (k-|\alpha|)})(x) \cdot d_M^{k-|\alpha|}(x).
$$
\n(1.13)
\nHowever, since Q is bounded and $|\alpha| \le k$, $d_M^{k-|\alpha|}(x)$ is bounded. Thus there is a constant c_4 such that
\n
$$
|D^{\alpha}v_h(x)| d_M^{(\alpha/p)}(x) \le c_4 M(D^{\alpha}u \cdot d_M^{(\alpha/p)
$$

• '• .

$$
D^{\alpha}v_h(x)| d_M^{(\epsilon/p)}(x) \leq c_3 \mathcal{M}(D^{\alpha}u \cdot d_M^{(\epsilon/p) - (\kappa - |\alpha|)}) (x) \cdot d_M^{k - |\alpha|}(x). \tag{1.13}
$$

 $|D^s v_h(x)| d_M^{(t/p)}(x) \leq c_3 \mathcal{M}(D^s u \cdot d_M^{(t/p)-(k-|\alpha|)}) (x) \cdot d_M^{k-|\alpha|}(x)$. (1.13)

However, since Ω is bounded and $|\alpha| \leq k$, $d_M^{k-|\alpha|}(x)$ is bounded. Thus there is a constant c_4 such that
 $|D^s v_h(x)| d_M^{(t/p)}(x) \leq c_4 \mathcal{M$ stant c_4 such that

$$
|D^{\mathfrak{s}} v_h(x)| d_M^{(\epsilon/p)}(x) \leq c_4 \mathcal{M}(D^{\mathfrak{s}} u \cdot d_M^{(\epsilon/p)-(\kappa-|\mathfrak{a}|)}) (x).
$$

It follows from (i), (ii) and (iii) that if $0 < h \le 1$, there is a constant c_5 , independent of *h*, such that for all $x \in \Omega$ and all $\alpha \in N_0^N$ with $|\alpha| \leq k$,

$$
|D^{\circ}v_{h}^{'}(x)| d_{M}^{\epsilon/p}(x) \leq c_{5}G(x), \qquad (1.14)
$$

where $G(x) := \max \{ \mathcal{M}(ud_M^{(\epsilon/p)-k}) (x), \mathcal{M}(D^{\alpha}u \cdot d^{(\epsilon/p)-k+|\alpha|}) (x) \}$. Since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, $D^{\alpha}u \cdot d_M^{(i/p)-k+|\alpha|} \in L^p(\Omega)$ and hence $D^{\alpha}u \cdot d_M^{(i/p)-k+|\alpha|} \in L^p(\mathbb{R}^N)$; in particular, $ud_M^{(t/p)-k} \in L^p(\mathbf{R}^N)$. The properties of the maximal function now imply that $|D^{\alpha}v_h(x)| d_M^{(\epsilon/p)}(x) \leq c_3$,
 since Ω is bounded and
 gelical i i i y n(*x*) *d_M*^{(*tp*})(*x*) $\leq c_4$,
 ws from (i), (ii) and (iii)
 s (*x*), $d_M^{(\epsilon/p)}(x) \leq c_5$
 gelical i gelical i geli It is from (i), (ii) and (iii) that if $0 < h \leq 1$, there is a constant c_5 , indepen-
 $|D^*v_h(x)| d_M^{t/p}(x) \leq c_5 G(x)$, (1.14)
 $|D^*v_h(x)| d_M^{t/p}(x) \leq c_5 G(x)$, (1.14)
 $|D^*v_h(x)| d_M^{t/p}(x) \leq c_5 G(x)$, (1.16)
 $|D^*v_h(x)| \leq c_5 G(x)$, at if $0 < h \leq 1$, there is a constant c_5 , independenting $\alpha \in N_0^N$ with $|\alpha| \leq k$,

(1.14)
 (1.14)
), (A(D²u $d^{(e/p)-k+|z|}(x)$). Since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$,

mee $D^2u \cdot d_M^{(e/p)-k+|z|} \in L^p(\mathbb{R}^N)$; in parti

 (1.15)

By the Lebesgue dominated convergence theorem, the assumption that as $h \rightarrow 0$,

$$
|D^{\mathbf{a}}v_h(x)| d_M^{e/p}(x) \to |D^{\mathbf{a}}u(x)| d_M^{e/p}(x) \tag{1.16}
$$

 $G \in L^p(\mathbf{R}^N)$. (1.15)
By the Lebesgue dominated convergence theorem, the assumption that as $h \to 0$,
 $|D^{\alpha}v_h(x)| d_M^{\epsilon/p}(x) \to |D^{\alpha}u(x)| d_M^{\epsilon/p}(x)$ (1.16)
for almost all $x \in \Omega$, implies (in view of (1.14) and (1.15)) that d_M ^{*i*} PD^u in L^p ; that is,

$$
v_h \to u \quad \text{in} \quad W^{k,p}(\Omega; d_M, \varepsilon), \tag{1.17}
$$

 $|D^{\circ}v_h(x)| d_M^{l/p}(x) \leq c_5G(x),$

where $G(x) := \max {\mathcal{M}(ud_M^{(t/p)-k})}(x), c$
 $D^{\circ}u \cdot d_M^{(t/p)-k+|\alpha|} \in L^p(\Omega)$ and hence
 $ud_M^{(t/p)-k} \in L^p(\mathbf{R}^N)$. The properties of
 $G \in L^p(\mathbf{R}^N)$.

By the Lebesgue dominated converges
 $|D^{\circ}v_h$ All that remains to complete the proof of the Theorem is to establish (1.16). To do this, let $\delta > 0$, $\Omega_{\delta} = \{x \in \Omega : d_M(x) > \delta\}$, and note that since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$, which is what we need.

All that remains to complete the proof of the Theorem is to establish (1.16). The this, let $\delta > 0$, $\Omega_{\delta} = \{x \in \Omega : d_M(x) > \delta\}$, and note that since $u \in H^{k,p}(\Omega; d_M, \varepsilon)$
 $u \in W^{k,p}(\Omega_{\delta})$. For $h \$ *k,* and $\int_0^{\infty} u d_M^{(x/p)-k} \in L^p(\mathbb{R}^N)$. The properties of the maximal function now imp
 $G \in L^p(\mathbb{R}^N)$.

By the Lebesgue dominated convergence theorem, the assumption the
 $|D^*v_h(x)| d_M^{(x/p)}(x) \rightarrow |D^*u(x)| d_M^{r/p}(x)$

for al

$$
D^{\alpha}v_{h}(x) = D^{\alpha}(\phi_{h} * u^{(3h)}) (x) = D^{\alpha}(\phi_{h} * u) (x) = (\phi_{h} * D^{\alpha}u) (x),
$$

and as $h\to 0$,

$$
\rightarrow 0,
$$

$$
D^*v_h \rightarrow D^*u \text{ in } L^p(\Omega_\delta).
$$

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since $\phi_h * D^{\alpha}u \to D^{\alpha}u$ in $L^p(\Omega_\delta)$. From (1.18) it follows that there is a sequence (h_n) of positive real numbers, converging to 0, such that $D^*v_{h} \to D^*u$ almost everywhere in Ω_{δ} . Now take $\delta = 1/k_0$, $k_0 \in \mathbb{N}$, and let $(v_n^{(1)})$ be the sequence (v_{h_n}) for this particular δ . Let $(v_n^{(2)})$ be a subsequence of $(v_n^{(1)})$ such that *D. E. EDMUNDS, A. KUFNER and J. RAKOSNIK*
 D'w. $\rightarrow D^*u$ in $L^p(\Omega_\delta)$. From (1.18) it follows that there is a sequence (h_n)
 e real numbers, converging to 0, such that $D^*v_{h_n} \rightarrow D^*u$ almost everywhere
 w take of positive real numbers, converging to 0, such in Ω_{δ} . Now take $\delta = 1/k_0$, $k_0 \in \mathbb{N}$, and let $(v_n$

cular δ . Let $(v_n^{(2)})$ be a subsequence of $(v_n^{(1)})$
 $D^a v_n^{(2)} \rightarrow D^a u$ a.e. in $\Omega_{1/(k_0+1)}$;

more generall

$$
D^{\alpha}v_n^{(2)} \to D^{\alpha}u \quad \text{a.e. in} \quad \Omega_{1/(k_0+1)};
$$

more generally let $(v_n^{(l)})$ be a subsequence of $(v_n^{(l-1)})$ such that

$$
D^{\alpha}v_n^{(l)} \to D^{\alpha}u \quad \text{a.e. in} \quad \Omega_{1/(k_0+l)}.
$$

It follows that the diagonal sequence (w_n) , $w_n = v_n^{(n)}$, has the property that

$$
D^*w_n \to D^*u \quad \text{a.e. in } \Omega. \tag{1.19}
$$

Of course; $\{w_n : n \in \mathbb{N}\} \subset \{v_n : h > 0\}$. Moreover, while the multi-index α was fixed, it is easy to see that matters can be arranged so that (1.19) holds for *every* $\alpha \in N_0$ ^N with

$$
d_M{}^{t/p}D^{\alpha}w_n \to d_M{}^{t/p}D^{\alpha}u \quad \text{a.e. in } \Omega,
$$

and the arguments used to pass from (1.16) to (1.17) show that $w_n \to u$ in $W^{k,p}(Q)$; d_M , ε). Our discussion in (i) shows that supp $w_n \circ \overline{M} = \emptyset$ (see (1.5)); moreover, $w_n \in C^\infty(\mathbb{R}^N)$. Hence $u \in W_M^{k,p}(\Omega; d_M, \varepsilon)$, and the proof is complete, since the continuity of the embedding of *H* in *W* is clear \blacksquare is easy to see that matters can be arranged so that (1.19) no
 $|\alpha| \leq k$. Thus
 $d_M^{t/p}D^s w_n \to d_M^{t/p}D^s u$ a.e. in Ω ,

and the arguments used to pass from (1.16) to (1.17) show
 d_M , ε). Our discussion in (i) shows

2. The embedding $W \hookrightarrow H$

Now we shall deal with embeddings inverse to those of \S 1. For this case, we need more special domains.

Definition 2.1: Put $Q=(0,1)^N$ and $Q(m)=\{x\in\overline{Q}:x_{m+1}=\cdots=x_N=0\},$ $m = 0, 1, ..., N - 1$. A closed subset *M* of $\partial\Omega$ is said to be a *manifold of dimension m* on. $\partial\Omega$ if there is an open covering ${U_i}_{i=0}^{\omega}$ (w finite or $\omega = \infty$) of Ω with the following d_M , ε). Our discussion in (i) shows that sue $w_n \in C^{\infty}(\mathbb{R}^N)$. Hence $u \in W_M^{k,p}(\Omega; d_M, \varepsilon)$, a
continuity of the embedding of H in W is clear
2. The embedding $W \hookrightarrow H$
Now we shall deal with embeddings inver 2. The embedding W

Now we shall deal w

more special domain

Definition 2.1:
 $m = 0, 1, ..., N - 1$

on 0.22 if there is an 0

properties:

(i) $M \subset \bigcup_{i=1}^{w} U_i$, an

is disjoint;

(ii) there exists δ :
 $d_M(x) \geq \delta$, *dm(x) 2^* 6, for all x E *U0 ;*

properties:

(i) $M \subset \bigcup_{i=1}^{\omega} U_i$, and there exists $s \in \mathbb{N}$ such that every system of $(s + 1)$ sets U_i

is disjoint; **i=I**

(ii) there exists $\delta > 0$ such that

$$
d_M(x) \geq \delta
$$
, for all $x \in U_0$;

(iii) there are numbers c_1 , c_2 , with $c_2 \geq c_1 > 0$, and a system of one-to-one mappings $T_i:\bar{Q}\to\overline{Q\cap U_i}$ $(i=1,2,..., \omega)$ such that $T_i(Q(m))=M\cap\overline{U}_i$ and $\begin{aligned} &\text{c}, \ &\text{c}} \text{c} \text{ exists } \delta > 0, \ &d_M(x) \geqq \delta, \ \text{for} \ &\text{c} \text{ are number} \ &\text{c} \text{ } \overline{\Omega \cap U}_i \ (i=1,2) \leqq 1, \ &\text{c} \text{ } \overline{\Omega} \text{ and } \overline{\Omega} \text{ and } \overline{\Omega} \text{.} \end{aligned}$ *T* all $x \in U_0$;
 $x c_1, c_2$, with $c_2 \ge c_1 > 0$, and a sys
 $x, 2, ..., \omega$ such that $T_i(Q(m)) = M$
 $T_i(x) - T_i(y)| \leq c_2 |x - y|$
 $T_i(0, ..., \omega)$.

(2.1)

$$
c_1 |x - y| \leq |T_i(x) - T_i(y)| \leq c_2 |x - y|
$$

For all $x, y \in \overline{Q}$ and $i = 1$, ..., α is set M or α *i* α *i* Remark 2.2: In what follows we shall see that this notion of an m -dimensional manifold on $\partial\Omega$ is suitable for dealing with embeddings of Sobolev spaces with weights which are powers of d_M . We use simple idea to avoid technical complications, but theorems of the type of the following Theorem 2.3 hold for more general sets *M.* and

P
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 ϵ and ϵ in particular for the set

$$
M^* := \bigcup_{i=1}^n \{x \in \bar{Q} : x_i = 0 \text{ for } j \neq i\} \subset \partial \Omega,
$$

which is not a one-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

Theorem 2.3: Let $\Omega \subset \mathbb{R}^N$ *be a domain, let* $M \in \partial \Omega$ *be an m-dimensional manifold on* $\partial\Omega$ *, let m* \in {0, 1, ..., *N* - 1} *and suppose that* $p \in (1, \infty)$ *. Then:*

$$
V \hookrightarrow L^p(\Omega; d_M, \varepsilon - p) \tag{2.3}
$$

where

Finded ings of Sobolev Spaces

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and for the set
 $M^* := \bigcup_{i=1}^N \{x \in \overline{Q} : x_i = 0 \text{ for } j \neq i\} \subset \partial\Omega$,

and a one-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

em 2.3: Let $\Omega \subset \mathbb{R}^N$ be a domain, l (i) $V = W^{1,p}(Q; d_M, \varepsilon)$ *if s s* **i** $M^* := \bigcup_{i=1}^N \{x \in \overline{Q} : x_i = 0 \text{ for } j \neq i\} \subset \partial\Omega,$

is not a one-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

sorem 2.3: Let $\Omega \subset \mathbb{R}^N$ be a domain, let $M \in \partial\Omega$ be an m-dimensional m $M^* := \bigcup_{i=1}^{N} \{x \in \bar{Q} : x_j = 0 \text{ for } j \neq i\} \subset \partial\Omega,$

which is not a one-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

Theorem 2.3: Let $\Omega \subset \mathbb{R}^N$ be a domain, let $M \in \partial\Omega$ be an m-dimensional manif *i* is not a on-dimensional manifold on $\partial\Omega$ in the sense of Definition 2.1.

eorem 2.3: Let $\Omega \subset \mathbb{R}^N$ be a domain, let $M \in \partial\Omega$ be an m-dimensional manifold, let $m \in \{0, 1, ..., N-1\}$ and suppose that $p \in (1, \infty)$. l *number*.
 \geq covering {{ ϕ_i , $i = 0, 1, ...$ }
 l u , such the

(ii)
$$
\sqrt{V} = W_M^{1,p}(\Omega; d_M, \varepsilon) \, \text{if}
$$

 $\varepsilon + p + m - N;$ (2.6)

(iii)
$$
V = W_0^{1,p}(\Omega; d_M, \varepsilon)
$$
 if $m < N - 1$ and ε is an arbitrary real number.

Proof: Let $(\phi_i)_{i=0}^{\infty}$ be a partition of unity subordinate to the covering ${U_i}_{i=0}^{\infty}$ mentioned in Definition 2.1. Let $u \in W^{1,p}(\Omega; d_M, \varepsilon)$ and put $u_i = u\phi_i$, $i = 0, 1, ..., \omega$. It is enough to prove that there is a constant c , independent of u , such that for *•* $f(x) = W_0^{1,p}(\Omega; d_M, \varepsilon)$ *if* $m < N - 1$ *and* ε *is an arbitrary real number.*
 (2.6)
 (2.7)
 (2.7)
 $\varepsilon \geq m - N$;
 $= W_M^{1,p}(\Omega; d_M, \varepsilon)$ if
 $\varepsilon + p + m - N$;
 $= W_0^{1,p}(\Omega; d_M, \varepsilon)$ if $m <$
 \therefore Let $(\phi_i)_{i=0}^{\infty}$ be a partit

of in Definition 2.1. Let us ugh to prove that there
 \ldots, ω ,
 $\int |u_i(x)|^p d_M^{t-p}(x) dx \leq$
 $u_i \wedge \$ **•** ed in Definition 2.1. Let $u \in W^{1,p}(\Omega; d_M, d_M)$

wugh to prove that there is a constant \cdots , ω ,
 $\int_{U_1 \cap \Omega} |u_i(x)|^p d_M^{e-p}(x) dx \leq c ||u_i||_W$.
 $0, (2.7)$ holds trivially in view of (2.1). N

implicity omit the subscript

$$
i = 0, 1, ..., \omega,
$$

$$
\int_{U_i \cap \Omega} |u_i(x)|^p d_M^{s-p}(x) dx \le c ||u_i||_W.
$$

For $i = 0$, (2.7) holds trivially in view of (2.1). Now suppose that $i > 0$, and for the

$$
c_1 d_{Q(m)}(y) \leq d_M(T(y)) \leq c_2 d_{Q(m)}(y) \tag{2.8}
$$

For $i = 0$, (2.1) holds trivially in view of (2.1) . Now suppose the sake of simplicity omit the subscript *i* on u_i and U_i . By (2.2)
 $c_1 d_{Q(m)}(y) \le d_M(T(y)) \le c_2 d_{Q(m)}(y)$

for all $y \in \overline{Q}$; and from [3: Chap. 2, for all $y \in \overline{Q}$; and from [3: Chap. 2, Lemma 3.1] it follows that there are positive constants c_3 , c_4 , depending only on c_1 , c_2 , p and N , such that

$$
\begin{aligned}\n\cdots, \omega, \\
\int |u_i(x)|^p d_M^{t-p}(x) dx &\leq c \|u_i\|_{\mathcal{W}}.\n\end{aligned}
$$
\n(2.7)
\n
$$
0, (2.7) \text{ holds trivially in view of (2.1). Now suppose that } i > 0, \text{ and for the\nimplicitly omit the subscript } i \text{ on } u_i \text{ and } U_i. \text{ By (2.2)}\n\end{aligned}
$$
\n
$$
c_1 d_{Q(m)}(y) \leq d_M(T(y)) \leq c_2 d_{Q(m)}(y) \tag{2.8}
$$
\n
$$
\in \overline{Q}; \text{ and from [3: Chap. 2, Lemma 3.1] it follows that there are positive\nso c_3 , c_4 , depending only on c_1 , c_2 , p and N , such that\n
$$
\begin{aligned}\nc_3 \int |w(x)|^p dx &\leq \int |w(T(y))|^p dy \leq c_4 \int |w(x)|^p dx \\
&\leq L^p(\Omega \cap U).\n\end{aligned}
$$
\n
$$
\in L^p(\Omega \cap U).
$$
\n
$$
\begin{aligned}\n\text{and } v(y) &= u(T(y)), \text{ and introduce the "cylindrical" coordinates}\n\end{aligned}
$$
$$

It is enough to prove that there is a constant c, independent of u, such that for $i = 0, 1, ..., \omega$,
 $\int |u_i(x)|^p d_{M}^{t-p}(x) dx \le c ||u_i||_W$. (2.7)

For $i = 0, (2.7)$ holds trivially in view of (2.1). Now suppose that $i > 0$, and for For all $y \in Q$ put $v(y) = u(T(y))$, and introduce the "cylindrical" coordinates or all $w \in L^p(\Omega \cap U)$.

For all $y \in Q$ put $v(y) = u(T(y))$, and introduce the "cylindrical" coordinates
 $y = (y', y'') \mapsto (y', \theta, r), \quad y' = (y_1, ..., y_m), \quad \theta \in \Xi := \left(0, \frac{\pi}{2}\right)^{N-m-1}, \quad r \in (0, R(\theta)),$ $=(y', y'') \mapsto (y', 0)$
= $d_{Q(m)}(y) = \left(\sum_{i=m}^{N} \frac{dy}{dx}\right)$ t $v(y) = u(T(y))$, and introduce the "cyline".

, r), $y' = (y_1, ..., y_m)$, $\theta \in \Xi := \left(0, \frac{\pi}{2}\right)^{N-k}$
 $\left(y_i^2\right)^{1/2}$. The corresponding Jacobian is $\left(\frac{\pi}{D}\right)^{1/2}$. $\text{drical'' } \text{ co} \ \begin{aligned} &\text{m--1} \quad \quad \cdot \quad \quad r \in \ \frac{D(y)}{(y',\,\theta,\,r)} \end{aligned}$ exponding Jacobian is $\left| \frac{D(y)}{D(y', \theta, r)} \right| = r^{N-m-1}$ $\mathbf{B}(y) \leq d_M(T(y)) \leq c_2 d_{Q(m)}(y)$

and from [3: Chap. 2, Lemma 3.1] it follows that the

depending only on c_1 , c_2 , p and N , such that
 $\mathbf{B}(x)|^p dx \leq \int_{Q} |w(T(y))|^p dy \leq c_4 \int_{\Omega_1} |w(x)|^p dx$
 $\qquad \qquad$
 $\qquad \qquad$ $\qquad \qquad$ sake of simplicity omit the
 $c_1d_{Q(m)}(y) \leq d_M(T$

for all $y \in \overline{Q}$; and from [3]

constants c_3 , c_4 , depending
 $c_3 \int |w(x)|^p dx \leq$
 $c_3 \int |w(x)|^p dx$

or all $w \in L^p(\Omega \cap U)$.

For all $y \in Q$ put $v(y)$
 $y = (y', y'') \mapsto (y',$ for all $y \in \overline{Q}$; and from [3: Chap. 2, Lemma 3.1] it follows that there are

constants c_3 , c_4 , depending only on c_1 , c_2 , p and N , such that
 $c_3 \int |w(x)|^p dx \leq \int |w(T(y))|^p dy \leq c_4 \int |w(x)|^p dx$

or all $w \in L^p(\Omega$ (y', y'')
 $d_{Q(m)}(y)$
 (y', θ) , e
 \int
 $\frac{1}{Q}$
 $=$
 $\frac{1}{Q}$ ϵ_3 , ϵ_4 , depending only on ϵ_1 , ϵ_2 , p and N , such that
 ϵ_3 $\int_{\partial_1 U} |w(x)|^p dx \leq \int_{\partial_1 U} |w(x)|^p dx$ (2.9)
 $\epsilon_1 F(Q \cap U)$.
 $\int_{\partial_1 U} f(Q \cap U)$.
 $\int_{\partial_2 U} f(Q \cap U)$.
 $\int_{\partial_1 U} f(Q \cap U)$.
 $\int_{\partial_2 U} f(Q \cap U)$.

$$
\int\limits_{Q} |v(y)|^p \ d_{Q(m)}^{t-p}(y) \ dy
$$
\n
$$
R(0)
$$

$$
\times \Phi(y', \theta), \text{ and}
$$
\n
$$
\int_{Q} |v(y)|^p d\xi_{(m)}^{-p}(y) dy
$$
\n
$$
= \int_{Q(m)} \int_{\mathcal{E}} \Phi(y', \theta) \int_{Q(m)} |v(y', \theta, r)|^p r^{e-p+N-m-1} dr d\theta dy'. \qquad (2.10)
$$
\n
$$
\text{Since } v(y', \theta, r) = 0 \text{ for almost all } y' \in Q(m), \theta \in \mathcal{E} \text{ and for all } r \text{ in a neighbourhood}
$$

of $R(\theta)$, we can extend the function *v* by zero for $r \geq R(\theta)$ with preservation of all

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differentiability properties. Application of the Hardy inequality (see; for example, [2: Chap. 5]) to the inner integral on the right-hand side of (2.10) now shows that for almost all $y' \in Q(m)$, $\theta \in \mathcal{Z}$,

D. E. EDMGNDS, A. KUFNER and J. RÁKOSNÍK
\nentiability properties. Application of the Hardy inequality (see, for example,
\nmap. 5]) to the inner integral on the right-hand side of (2.10) now shows that for
\nst all
$$
y' \in Q(m)
$$
, $\theta \in \Xi$,
\n
$$
\int_{0}^{\infty} [v(y', \theta, r)]^p r^{\epsilon-p+N-m-1} dr \leq c_H \int_{0}^{\infty} \left| \frac{\partial y}{\partial r} (y', \theta, r) \right|^p r^{\epsilon+N-m-1} dr
$$
\n
$$
\epsilon + N - m - 1 > p - 1
$$
\n
$$
\lim_{\to \infty} v(y', \theta, r) = 0 \text{ (this condition is satisfied trivially), or if}
$$
\n
$$
\epsilon + N - m - 1 < p - 1
$$
\n
$$
\lim_{r \to 0+} v(y', \theta, r) := v_0(y', \theta) = 0.
$$
\n(2.13)
\n
$$
\lim_{r \to 0+} v(y', \theta, r) := v_0(y', \theta) = 0.
$$
\n(2.14)
\nconstant c_H in (2.11) is given by $c_H = p/|\epsilon - p + N - m|$. Integrating the in-
\nity (2.11) over $Q(m) \times \Xi$ and passing back to Cartesian coordinates we obtain

if

$$
\varepsilon + N - m - 1 > p - 1 \tag{2.12}
$$

and $\lim v(y', \theta, r) = 0$ (this condition is satisfied trivially), or if

$$
+ N - m - 1 < p - 1 \tag{2.13}
$$

and

$$
\lim_{t \to \infty} v(y', \theta, r) := v_0(y', \theta) = 0. \tag{2.14}
$$

and
 $\lim_{r \to 0+} v(y', \theta, r) := v_0(y', \theta) = 0.$ (2.13)

The constant c_H in (2.11) is given by $c_H = p/|\varepsilon - p + N - m|$. Integrating the in-

equality (2.11) over $Q(m) \times \overline{\mathcal{Z}}$ and passing back to Cartesian coordinates we obtain equality (2.11) over $Q(m) \times \mathcal{Z}$ and passing back to Cartesian coordinates we obtain

$$
\varepsilon + N - m - 1 < p - 1
$$
\n(2.13)
\n
$$
\lim_{r \to 0+} v(y', \theta, r) := v_0(y', \theta) = 0.
$$
\n(2.14)
\nThe constant c_H in (2.11) is given by $c_H = p/|\varepsilon - p + N - m|$. Integrating the inequality (2.11) over $Q(m) \times \mathbb{Z}$ and passing back to Cartesian coordinates we obtain
\n
$$
\int_{0}^{\infty} |v(y)|^p d_{Q(m)}^p(y) dy \leq c_H \int \left| \frac{\partial v}{\partial r}(y) \right|^p d_{Q(m)}^p(y) dy
$$
\n
$$
\leq c_5 \sum_{j=m+1}^N \int \left| \frac{\partial v}{\partial y}(y) \right|^p d_{Q(m)}^p(y) dy.
$$
\nUse of (2.9) with $w = u d_M^{(i/p)-1}$ and with $w = \frac{\partial u}{\partial x_j} d_M^{i/p}$, the estimates (2.8) and the inequality
\n
$$
\left| \frac{\partial v}{\partial y}(y) \right| \leq c_2 \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(T(y)) \right|
$$
 which follows from (2.2) now gives (2.7) with
\na constant c which depends only on p, ε , N, c_1 and c_2 .
\nAll that remains is to discuss the validity of the assumptions (2.12)–(2.14). The
\ninequalities (2.4) and (2.12) are equivalent. If (2.5) holds, then (2.13) is satisfied.

inequality $\left|\frac{\partial v}{\partial u_i}(y)\right|\leq c_2 \sum_{i=1}^N \left|\frac{\partial u}{\partial u_i}\left(T(y)\right)\right|$ which follows from (2.2) now gives (2.7) with **a** constant *c* which depends only on p, ε , N, c_1 and c_2 . $\int_{Q} |v(y)|^p d\xi_{(m)}^{-p} (y) dy \leq c_H \int_{Q} \left| \frac{\partial v}{\partial r} (y) \right|^p d\xi_{(m)}(y) dy$
 $\leq c_5 \sum_{j=m+1}^{N} \int_{Q} \left| \frac{\partial v}{\partial y_j} (y) \right|^p d\xi_{(m)}(y) dy.$

Use of (2.9) with $w = ud_M^{(i/p)-1}$ and with $w = \frac{\partial u}{\partial x_j} d_M^{i/p}$, the estimates (2.8) and the

 $\begin{vmatrix} \frac{\partial y_i}{\partial y_i} & x_i(y_j) \end{vmatrix}$, which follows from (2.2) if

only on p, ε , N, c_1 and c_2 .

cuss the validity of the assumptions

are equivalent. If (2.5) holds, then
 $Q(m)$, $\theta \in \mathcal{F}$ and for $r, h > 0$ we hav All that remains is to discuss the validity of the assumptions $(2.12) - (2.14)$. The inequalities (2.4) and (2.12) are equivalent. If (2.5) holds, then (2.13) is satisfied.

a constant c which depends only on p, e, N, c₁ and c₂.
\nAll that remains is to discuss the validity of the assumptions (2.12)–(2.14). The inequalities (2.4) and (2.12) are equivalent. If (2.5) holds, then (2.13) is satisfied.
\nFurther, for almost all
$$
y' \in Q(m)
$$
, $\theta \in \mathcal{Z}$ and for $r, h > 0$ we have, by Hölder's inequality,
\n
$$
|v(y', 0, r + h) - v(y', \theta, r)| = \begin{vmatrix} r + h & v \\ \frac{\partial v}{\partial r} & v' & \theta \\ r & \frac{\partial v}{\partial r} \end{vmatrix} \begin{vmatrix} r + h & v \\ \frac{\partial v}{\partial r} & v' & \theta \\ r & \frac{\partial v}{\partial r} \end{vmatrix} e^{t + N - m - 1} d\rho
$$
\n
$$
\leq \left(\int_{r}^{r + h} \left| \frac{\partial v}{\partial r} & (y', \theta, \varrho) \right|^{p} e^{t + N - m - 1} d\rho \right)^{(p-1)/p} = o(1)
$$
\nas $h \to 0$, since the former of the last two integrals is finite and the exponent in the latter one is positive. Hence the function $v(y', 0, \cdot)$ is uniformly continuous in a neighborhood of the origin and the limit $v_0(y', \theta)$ in (2.14) exists. Since $h(v)$
\n
$$
\int_{0}^{h(v)} |v(y', 0, r)| r^{t + N - m - 1} dr < \infty,
$$
\nthe assumption (2.5) yields (2.14). Thus assertion (i) holds.

-

as $h \to 0$, since the former of the last two integrals is finite and the exponent in the latter one is positive. Hence the function $v(y', \theta, \cdot)$ is uniformly continuous in a neighbourhood of the origin and the limit $v_0(y', \theta)$ in (2.14) exists. Since

$$
\int\limits_{0}^{R(\theta)}|v(y',\theta,r)|\,r^{\epsilon+N-m-1}\,dr<\infty\,,
$$

If $V = W_{M}^{1,p}(\Omega; d_M, \varepsilon)$, it is enough to deal with functions *u* in $C_M^{\infty}(\Omega)$, so that supp $v \subset Q$; and condition (2.14) is then trivially satisfied. The condition (2.6) means that (2.12) or (2.13) holds. Embeddings of Sobolev Spaces 33

If $V = W_M^{1,p}(\Omega; d_M, \varepsilon)$, it is enough to deal with functions u in $C_M^{\infty}(\Omega)$, so that

supp $v \subset Q$; and condition (2.14) is then trivially satisfied. The condition (2.6)

means that (2. *W*_M^{1,p}(2; d_M, ε), it is enough to deal with Q ; and condition (2.14) is then trivially
at (2.12) or (2.13) holds.
I, suppose that $m < N - 1$. If $\varepsilon \neq p + m - e$ $W_0^{1,p}(\Omega; d_M, \varepsilon) \subset W_M^{1,p}(\Omega; d_M, \varepsilon)$. Thus $e \$

(ii), since $\overline{W_0}^{1,p}(\Omega; d_M, \varepsilon) \subset W_M^{1,p}(\Omega; d_M, \varepsilon)$. Thus suppose that $\varepsilon = p + m - N$, and let $u \in C_0^{\infty}(\Omega)$. Since $m < N - 1$, we can write *2* Q ; and condition (2.14) is then trivially satisfied
at (2.12) or (2.13) holds.
*2 W*₀^{1,*p*}(Q ; d_M , ε) $\subset W_M^{1,p}(\Omega; d_M, \varepsilon)$. Thus suppose
 $\in C_0^{\infty}(\Omega)$. Since $m < N - 1$, we can write
 $2^{-1/2}(y_l + \varrho_l) \leq$

$$
2^{-1/2}(y_l + \varrho_l) \leq d_{Q(m)}(y) \leq y_l + \varrho_l, \quad \varrho_l^2 = \sum_{j=m+1,j+l}^{N} y_j^2
$$

(*l* = *m* + 1, ..., *N*; *y* \in *Q*)

and

$$
\int_{Q} |v(y)|^{p} d\xi_{(m)}^{-p} (y) dy \leq c_{6} \int_{(0,1)^{N-1}} \int_{0}^{1} |v(y)|^{p} (y_{1} + \varrho_{l})^{\epsilon-p} dy_{l} dy^{(l)},
$$

(*l* = *m* + 1, ..., *N*; *y* $\in Q$)

and
 $\int_{Q} |v(y)|^p d\xi_{(m)}^{-p}(y) dy \leqq c_6 \int_{(0,1)^{N-1}0}^{1} |v(y)|^p (y_1 + \rho_1)^{s-p} dy_1 dy_1^{(l)}$,

where $y^{(l)} = (y_1, ..., y_{l-1}, y_{l+1}, ..., y_N)$. Taking into account the fact that supp $v \subset Q$

and $\varepsilon = p$ Lemma 5.3]) to the inner integral on the right-hand side and obtain

$$
(l = m + 1, ..., N; y \in Q)
$$
\n
$$
\int_{Q} |v(y)|^p d\xi_{(m)}^{-p} (y) dy \leq c_6 \int_{(0,1)^{N-1}} \int_{0}^{1} |v(y)|^p (y_l + \varrho_l)^{\epsilon - p} d\varrho
$$
\n
$$
v = (y_1, ..., y_{l-1}, y_{l+1}, ..., y_N).
$$
\nTaking into account to $p + m - N < p - 1$, we apply the generalized H. 5.3]) to the inner integral on the right-hand side and
$$
\int_{Q} |v(y)|^p d\overline{\xi_{(m)}} (y) dy \leq c_7 \int_{Q} \left| \frac{\partial v}{\partial y_l} (y) \right|^p d\xi_{(m)} (y) dy.
$$

It is now sufficient to use (2.9) with $w = u d_M^{(e/p)-1}$ and with $w = \frac{\partial u}{\partial x} d_M^{e/p}$; the assertion (iii) and consequently Theorem 2.3 are proved \blacksquare *where* $\int_{0}^{10(9)} \frac{\log_{10}(9)}{\log_{10}(9)} dy = c_7 \int_{0}^{10(9)} \frac{dy}{dy}$ *with* $w = u d_M^{(t/p)-1}$ and with $w = \frac{\partial u}{\partial x_i} d_M^{t/p}$; the assertion (iii) and consequently Theorem 2.3 are proved **1**
3. Concluding remarks
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3. Concluding remarks

We present two natural consequences of the work in $\S 1$ and $\S 2$.

Corollary 3.1: Let Ω be a bounded domain in \mathbb{R}^N , let M be an m-dimensional *manifold on* $\partial\Omega$ *where* $m \in \{0, 1, ..., N-1\}$ *, and let* $p \in (1, \infty)$ *,* $k \in \mathbb{N}$ *. Then*

$$
V\hookrightarrow H^{k,p}(\Omega;d_M,\varepsilon),
$$

(i) $V = W^{k,p}(Q; d_M, \varepsilon)$ *if*

or

assertion (iii) and consequently Theorem 2.3 are proved **I**

\n3. Concluding remarks

\nWe present two natural consequences of the work in § 1 and § 2.

\nCorollary 3.1: Let Ω be a bounded domain in **R**^N, let M be an m-dimensional manifold on
$$
\partial\Omega
$$
 where $\hat{m} \in \{0, 1, ..., N - 1\}$, and let $p \in (1, \infty)$, $k \in \mathbb{N}$. Then

\n $V \hookrightarrow H^{k,p}(\Omega; d_M, \varepsilon),$

\nwhere

\n(i) $V = W^{k,p}(\Omega; d_M, \varepsilon)$ if

\n $\varepsilon > kp + m - N$

\n(3.1)

\n(ii) $V = W_M^{k,p}(\Omega; d_M, \varepsilon)$ if

\n $\varepsilon \neq ip + m - N, \quad j = 1, 2, ..., k;$

\n(iii) $V = W_M^{k,p}(\Omega; d_M, \varepsilon)$ if

\n $m < N - 1$ and $\varepsilon \in \mathbb{R}.$

\nProof: Use Theorem 2.3 successively for $D^{\beta}u$ with $|\beta| = k$, $|\beta| = k - 1, ...$

\nCorollary 3.2: Let the assumptions of Corollary 3.1 be satisfied. If ε satisfies (3.1) or (3.2), then

\n $W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon) = W^{k,p}(\Omega; d_M, \varepsilon).$

$$
\varepsilon + jp + m - N, \qquad j = 1, 2, ..., k;
$$
 (3.3)

(iii)
$$
V = W_0{}^{k,p}(\Omega; d_M, \varepsilon)
$$
 if

 $m < N - 1$ and $\varepsilon \in \mathbb{R}$. (3.4)
Proof: Use Theorem 2.3 successively for $D^{\beta}u$ with $|\beta| = k$, $|\beta| = k - 1$, ... **I**

 Corollary 3.2: *Let the assumptions of Corollary* **3.1** *be satisfied. If* ε *satisfies* (3.1) *We Theorem 2.3 successively for* $D^p u$ *with* $|\beta| = \arg 3.2$ *: Let the assumptions of Corollary 3.1 be sation.*
 $W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon) = W^{k,p}(\Omega; d_M, \varepsilon).$

$$
W_M{}^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon) = W^{k,p}(\Omega; d_M, \varepsilon).
$$

³Analysis Bd. 4, Heft 1 (19&>) • *•*

If ε *satisfies* (3.3), then

$$
W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon).
$$
\n(3.5)

*w*_{*M*}*kp*(2; *d_M, e)* = *H*^{*k*}*p*(2; *d_M, e)*. (3.5) *(3.6) (3.6) (3.6) holds for all* $\epsilon \in \mathbb{R}$ *with* W_0 ^{*k*}*p*(*Q*; *d_M*, *ε*) *instead of* W_M ^{*k*}*p*(*Q*; *d*_{*M*} *c*) *(3.5) (3.6) (3. If* $m < N - 1$, then (3.5) holds for all $\epsilon \in \mathbb{R}$ with $W_0^{k,p}(\Omega; d_M, \epsilon)$ instead of $W_M^{k,p}(\Omega; d_M, \epsilon)$ d_M , ε). 34 D. E. EDMUNDS, A. KUFNER and J. RA
 $If \varepsilon$ satisfies (3.3), then
 $W_M^{k,p}(\Omega; d_M, \varepsilon) = H^{k,p}(\Omega; d_M, \varepsilon)$.
 $If \, m < N - 1$, then (3.5) holds for all $\varepsilon \in \mathbb{R}$
 d_M, ε .

Proof: The assertions follow immediately

REF $N M$ N (se, am, e) $=$ Δ (se, am, e).
 $n < N - 1$, then (3.5) holds for all $\varepsilon \in \mathbb{R}$ with $W_0 k.p(\Omega; d_M, \varepsilon)$
 ε).

Proof: The assertions follow immediately from Corollary 3.1 ε

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Proof: The assertions follow immediately from Corollary 3.1 and Theorem 1.2

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