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A Note on Ontimal Domains in a Reaction-Diffusion Broblem **A Note on Optimal Domains in a Reaction-Diffusion Problem**  Ecitschrift für Analysis<br>
und three Anwendungen<br>
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A Note on Optimal Domains in a Reaction-Diffusion Problem<br>
C. BANDLE

Es wird untersucht, welches Gefäß für eine chemische Reaktion optimal ist im Sinne, daß keine toten Zonen auftreten und daß der "Wirkungsgrad" möglichst groß wird. Es zeigt sich, da8 bei vorgegebenem Volumen das GefäB moglichst dunn sein mull. Der Beweis stützt sich auf die Methode der Ober- und Unterlosungen und auf die Abschatzung für die Lösung des Dirichletproblems, die vom Inkugelradius abhängt.

Исследуется, какой сосуд при химической реакции оптимален в таком смысле, что нет мертвых зон и что , коеффициент полезного действия" как можно больше. Оказы-<br>вается, что при данном обьеме сосуд должен быть наиболее тонким: Доказательство Dirichletproblems, die vom Inkugelradius abhängt.<br>Исследуется, какой сосуд при химической реакции оптимален в таком смысле, что нет<br>мертвых зон и что ,,коеффициент полезного действия" как можно больше. Оказы-<br>вается, что опирается на метод вверхних и нижних решений и на оценки решений задачи Дирихле, зависящей от радиуса вписанного шара.

A simple model for a chemical reaction is considered. The-vessej where the reaction takes place is optimal if no dead core appears and if effectiveness is high. By means of the method of upper and lower solutions it is shown that, independent of the volume, sufficiently thin domains have this property. The proof is based on an estimate for the solution of the Dirichlet problem. erca, что при данном объеме сосуд должен быть наиболее тонким: Доказательство<br>ирается на метод вверхних и нижних решений и на оценки решений задачи Дирихле,<br>висидей от радиуса вписанного шара.<br>simple model for a chemical  $\begin{aligned}\n\text{Ricar} \text{ factor is considered.} \text{ The expression } \text{Ricar} \text{ factor is } \text{in} \text{ for } \text{the } 1 \text{ and } \text{the } 2 \text{ and } \text{the } 3 \text{ and } \text{the } 4 \text{ and } \text{the } 5 \text{ and } \text{the } 6 \text{ and } \text{the } 6 \text{ and } \text{the } 7 \text{ and } \text{the } 6 \text{ and } \text{the } 7 \text{ and } \text$ increase the reaction bakes place<br>
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lume, sufficiently thin domains<br>
ution of the Dirichlet problem.<br>
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generic point and<br>
<br>
in D. (1.1)

## **L. Introduction**

-

Set  $D \subset \mathbb{R}^N$  be a bounded domain,  $x = (x_1, x_2, ..., x_N)$  a generic point and  $\sum_{N=2}^{N} 1$ 

$$
A = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial}{\partial x_j} \right]
$$

a uniformly, elliptic operator such that

property. The proof is based on an estimate for the solution of the Di-  
\nluction  
\n
$$
\mathbf{R}^N
$$
 be a bounded domain,  $x = (x_1, x_2, ..., x_N)$  a gen'eric-point  
\n
$$
A = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial}{\partial x_j} \right]
$$
\n\nally, elliptic operator such that  
\n
$$
a_{ij} = a_{ji} \in C^1(\overline{D}) \text{ and } \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \ge \sum_{i=1}^N \xi_i^2 \text{ in } D.
$$
\n\nthe deals with the problem  
\n
$$
\begin{bmatrix}\n\frac{\partial u}{\partial x_i} & \frac{\partial u}{\partial x_i
$$

This note deals with the problem

 $a_{ji} \in C^1(\overline{D})$  and  $\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \sum_{i=1}^N a_{ij}(x) \xi_i \xi_j$ <br>s with the problem<br> $-Au = \lambda g(u)$  in  $D \times \mathbb{R}^+$ ,  $\lambda \in \mathbb{R}^+$  $\mathbf{u}^N$  be a bounded domain,  $x = (x_1, x_2, ..., x_N)$ <br>  $\mathbf{u} = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial}{\partial x_j} \right]$ <br>  $\mathbf{y}$ , elliptic operator such that<br>  $\mathbf{u} = a_{ji} \in C^1(\overline{D})$  and  $\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \ge \sum_{i=1}^N \xi_i$ <br>
deals wit (P)  $u(x, 0) = 0$   $'$  $\begin{cases} \n\frac{\partial t}{\partial t} & \text{on } \partial D \times \mathbf{R}^+, \\ \nu(x, 0) = 0 \n\end{cases}$ <br>
the-function  $g = g(\sigma)$  being subject to the following conditions uniformly, elliptic operator such that<br>  $a_{ij} = a_{ji} \in C^1(\overline{D})$  and  $\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \ge \sum_{i=1}^N \xi_i^2$  in  $L$ <br>
is note deals with the problem<br>  $\begin{cases} \frac{\partial u}{\partial t} - Au = \lambda g(u) & \text{in } D \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial D \times \mathbb{R}^+, \end{cases}$ <br>

- (C-1)  $g > 0$  in  $(-\infty, 1)$ ,  $g \in C^{0}(\mathbb{R})$  and  $g \in C^{1}(-\infty, 1)$ ,
- (C-2) g is non-increasing in  $(-\infty, 1)$ ,<br>(C-3)  $g(0) = 1$ ,
- 
- $(C-4)$   $g(\sigma) = 0$  for  $\sigma \ge 1$ ,
- (C-4)  $g(\sigma) = 0$  for  $\sigma \ge 1$ ,<br>
(C-5)  $g(\sigma) = g_0(\sigma) (1 \sigma)_+^p$  for some p,  $0 < p < 1$ , with  $g_0(1 0) > 0$  and  $g_0'(1-0) < \infty \ (\sigma_+ := \max \ (\sigma, 0)).$

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Such problems occur in simple models for reaction-diffusion processes [1, 4, 9] in the case where the reactions are endothermic or isothermal. Here *D* represents the domain where the reaction takes place, and  $1 - u = c$  is the concentration of the reactant. A considerable amount of work has been devoted recently to these equa-'

tions  $[1, 3-5, 9, 10]$ .<br>An important tool in the study of problem  $(P)$  is the method of upper and lower solutions. A function  $\overline{u} = \overline{u}(x, t)$  is called a *upper solution*, if

$$
\overline{u}_t - A\overline{u} \geq \lambda g(\overline{u}) \text{ in } D \times \mathbf{R}^+, \quad \overline{u} \geq 0 \text{ on } \partial D \times \mathbf{R}^+, \quad \overline{u}(x,0) \geq 0.
$$

Conversely,  $u = u(x, t)$  is a *lower solution* if the inequality signs are reversed. The theorem [8] states that, if there exist upper solutions such that  $\underline{u} \leq \overline{u}$  in  $D \times [0, T]$ , then the problem (P) possesses a solution *u* with  $u \leq u \leq \overline{u}$  in  $D \times [0, T]$ . In our case the assumptions are satisfied for  $\overline{u} \equiv 1$  and  $\underline{u} \equiv 0$ . Consequently problem (P) has a solution *u* such that  $0 \leq u(x, t) \leq 1$ , which is unique by virtue of the monotonicity of g (C-2). Observing that  $u(x, t + \Delta t) = \overline{u}(x, t)$  is an upper solution for problen (P) we conclude that *ulete* the reaction takes place, and  $I = u$ <br> *A* considerable amount of work has been<br>  $3-5, 9, 10$ .<br> *uortant tool in the study of problem (P) is*<br> *u*(*x, t)* is called a *upper so*<br>  $\overline{u}_t - A\overline{u} \geq \lambda g(\overline{u})$  in  $D \$ is the concentration of the<br> *ted* recently to these equa-<br> *method of upper and lower*<br> *n*, if<br>  $\times$  **R**<sup>+</sup>,  $\overline{u}(x, 0) \ge 0$ .<br>  $\exists$  *i*  $\forall$  **k**  $\forall$  **E**  $\overline{u}$  in  $D \times [0, T]$ . In our<br>  $\exists$  *u*  $\le \overline{u}$  in  $D \times [0, T$ An important tool in the study of problem<br>solutions. A function  $\overline{u} = \overline{u}(x, t)$  is called a  $up_1$ <br> $\overline{u}_t - A\overline{u} \geq \lambda g(\overline{u})$  in  $D \times \mathbb{R}^+$ ,  $\overline{u} \geq 0$ <br>Conversely,  $\underline{u} = \underline{u}(x, t)$  is a *lower solution* if t *x*  $\equiv \overline{u}$  in  $D \times [0, 1]$ ,<br>  $\leq \overline{u}$  in  $D \times [0, T]$ . In our<br>
Consequently problem (P)<br> *ue by virtue of the mono-*<br> *(1.2)*<br>  $= U(x)$ , where  $U(x)$  is the<br> *re.* It represents the region<br>
te  $\Omega(\infty)$  for the dead core<br>

$$
u(x, t_1) \leq u(x, t_2) \quad \text{for} \quad t_1 \leq t_2. \tag{1.2}
$$

t-±oo solution of the stationary problem.<br>The set  $\Omega(t) := \{x \in D : u(x, t) = 1\}$  is called the *dead core*. It represents the region

Furthermore, by standard arguments we have  $\lim_{t\to\infty} u(x, t) = U(x)$ , where  $U(x)$  is the solution of the stationary problem.<br>The set  $\Omega(t) := \{x \in D : u(x, t) = 1\}$  is called the *dead core*. It represents the region where no reacti where no reaction takes place at the time *I*. We shall write  $\Omega(\infty)$  for the dead core of  $U(x)$ . The special feature about our problem is that für  $\lambda$  sufficiently large,  $\Omega(t)$  $u(x, t_1) \leq u(x, t_2)$  for  $t_1 \leq t_2$ .<br> *Q*(*t*)  $t_1 \leq t_2$ .<br> *Q*(*t*) :  $\leq u(x, t_2)$  for  $t_1 \leq t_2$ .<br> *Co d i*  $\rightarrow \infty$ <br> *Co d i*  $\rightarrow \infty$ <br> *co caction takes place at the time <i>t*. We shall write<br> *Q*(*t*) :  $\equiv \$ ge,  $\Omega(t)$ <br>
(1.3)<br>
[4]<br>
(1.4)

$$
\Omega(t_1) \subset \Omega(t_2) \subset \Omega(\infty) \quad \text{for} \quad t_1 \leq t_2. \tag{1.3}
$$
\n
$$
\text{arly, } \Omega(t) \text{ depends on the geometry of } D \text{ and on } \lambda. \text{ We have for example [4]}
$$
\n
$$
\Omega(t) \subset \Omega_{D_1}(t) \subset \Omega_{D_2}(t) \quad \text{for} \quad D_1 \subset D_2 \text{ and } \lambda \text{ fixed,}
$$
\n
$$
\Omega(t) = \Omega_{D_1}(t) \quad \text{for} \quad t_1 \leq t_2 \quad \text{and} \quad D_1 \text{ fixed,}
$$
\n
$$
\Omega(t) = \Omega_{D_1}(t) \quad \text{for} \quad t_2 \leq t_1 \quad \text{and} \quad D_1 \text{ fixed,}
$$
\n
$$
\Omega(t) = \Omega_{D_1}(t) \quad \text{for} \quad t_2 \leq t_1 \quad \text{and} \quad D_1 \text{ fixed,}
$$
\n
$$
\Omega(t) = \Omega_{D_1}(t) \quad \text{for} \quad t_1 \leq t_2 \quad \text{and} \quad D_1 \text{ fixed,}
$$

Clearly,  $\mathit{Q}(t)$  depends on the geometry of  $D$  and on  $\lambda$ . We have for example [4]

(i) 
$$
\Omega_{D_1}(t) \subset \Omega_{D_2}(t)
$$
 for  $D_1 \subset D_2$  and  $\lambda$  fixed,

(ii)  $Q_{\lambda_i}(t) \subset Q_{\lambda_i}(t)$  for  $\lambda_1 \leq \lambda_2$  and *D* fixed.

A comparison result of a different nature was obtained by means of rearrangement methods [4] and is expressed as follows.

*Theorem I: Let Q\*(t) he the dead core corresponding to problem (P) with A replaced by the Laplacian and D by the sphere*  $D^*:=\{x: |x| < R$ *, vol*  $D =$  *vol*  $D^*$ *} of the same volume as D. Then* meas  $Q(t) \leq$  meas  $Q^*(t)$ .

Another quantity of physical interest is the so called *effectiveness* defined by

$$
\eta(t) = \frac{1}{\operatorname{vol} D} \int\limits_{D} g[u(x, t)] dx \leq 1.
$$

It is the ratio of -the actual average reaction rate in a region to the rate corresponding to reference values of reactant concentration and temperature. Rearrangement techniques yield [5, 3].

Theorem II: Let  $\eta^*$  be the effectiveness corresponding to problem (P) with  $A = \Delta$ *and D* =  $\frac{1}{\text{vol } D} \int_{D} f(u(x, t)) dx \leq 1$ .<br> *and D* =  $\frac{1}{\text{vol } D} \int_{D} g(u(x, t)) dx \leq 1$ .<br> *alse in a region to the ratio of the actual average reaction rate in a region to the ratio reference values of reactant concentratio* and  $D = D^*$ . If in addition to (C-1) – (C-5) g is concave, then  $\eta \geq \eta^*$ . In the stationary *case*  $(\eta(\infty))$  no concavity on q is required. botume as D. Then meas  $s_2(t) \leq \text{meas}$ <br>
Another quantity of physical inte<br>  $\eta(t) = \frac{1}{\text{vol } D} \int_{D} g[u(x, t)] dx$ <br>
It is the ratio of the actual average re<br>
to reference values of reactant conce<br>
niques yield [5, 3].<br>
Theorem I

For practical purposes it is desirable to have a small dead core and a high effec-

Optimal Domains in a Reaction Diffusion Problem 209<br>If we restrict ourselves to the case where  $A = \Delta$ , then the Theorems I and II imply the following extremal properties of the sphere:

- *(i) Among all domains of given volume the volume of the dead core is biggest for the sphere.'* 
	- *(ii) Among all domains of given volume the sphere has the smallest effectiveness.*

The sphere is in some sense the worst region. In the "International Meeting on Optimization Techniques" at the Wartburg 1983, Professor R. Klötzler asked the question: is there a best domain? Since the worst domain is fat, the beat domain, provided it exists, has to be thin. In fact, it turns out that for any fixed *2* there is always a domain of given volume such that  $\Omega(\infty)$  is empty. Moreover there exists a sequence of domains  $\{D_k\}_{k=1}^\infty$  such that The sphere is in some sense the worst region<br>mization Techniques" at the Wartburg 198<br>tion: is there a best domain? Since the wors<br>it exists, has to be thin. In fact, it turns of<br>domain of given volume such that  $\Omega(\infty)$  these a best domain: since the worst domain is rat, the best domain, provided<br>here a best bin. In fact, it turns out that for any fixed  $\lambda$  there is always a<br>ns  $\{D_k\}_{k=1}^{\infty}$  such that<br> $\text{vol } D_k = V$  and  $\lim_{k \to \infty} \eta_D(t$ 

$$
\operatorname{vol} D_k = V \quad \text{and} \quad \lim_{k \to \infty} \eta_{D_k}(t) = 1.
$$

 

Those statements- follow from counterparts of Theorem I and II, which will be formulated in Section 3.

I'm indebted to Prof. R. Klötzler for having suggested this problem.

## **2. Preliminary bounds for the solution of a Dirichiet problem**

2.1 We proceed now to derive bounds for the solution of the Dirichlet problem

 $A\psi + 1 = 0$  in *D*,  $\psi = 0$  on  $\partial D$ . (2.1)<br>For  $N = 2$  and  $A = \Delta$ ,  $\psi$  coincides up to a constant factor, with the warping function in the torsion problem of a cylindrical beam. A number of inequalities relating  $\psi_{\text{max}}$ to several geometrical quantities has been derived (cf.  $[2, 14-16]$  and the literature cited there) The following lemma extends an inequality of **PAYNE [14].** Its proof is very similar and will be given for the sake of completeness. 1'm indebted to Prof. R. Klötzler for having suggested this probl<br>
2. Preliminary bounds for the solution of a Dirichlet problem<br>
2.1 We proceed now to derive bounds for the solution of the Dirich<br>  $A\psi + 1 = 0$  in D,  $\psi =$ 2.1 We proceed now to derive bounds for the solution  $A\psi + 1 = 0$  in  $D$ ,  $\psi = 0$  on  $\partial D$ .<br>
For  $N = 2$  and  $A = \Delta$ ,  $\psi$  coincides up to a constant in the torsion problem of a cylindrical beam. A nurto several geometrical

Lemma 2.1: Let  $\omega_N$  be the volume of the unit sphere and let  $A$  satisfy (1.1). Then

\n if a par is a function of the graph of the graph of the graph of the graph of the graph. The equation of (2.1) is given by:\n 
$$
\psi_{\text{max}}^{1+N/2} \leq \frac{2+N}{2\omega_N} \cdot (2N)^{-N/2} \int_{\mathcal{D}} \psi \, dx.
$$
\n

Proof: The proof is based on rearrangement techniques developed in [2, 17]. For  $\psi_{\text{max}}^{1+N/2} \leq \frac{2+N}{2\omega_N} (2N)^{-N/2} \int \psi \, dx.$ <br> *Equality holds for*  $D = D^*$  *and*  $A = \triangle$ .<br>
Proof: The proof is based on rearrangement techniques<br>
this purpose let us introduce the notation<br>  $D(\tilde{\psi}) = \{x \in D : \tilde{\psi}(x) \geq \til$ 

$$
D(\tilde{\psi}) = \{x \in D : \tilde{\psi}(x) \geq \tilde{\psi}\}, \quad a(\tilde{\psi}) = \text{meas } D(\tilde{\psi}).
$$

By Schwarz's inequality and  $(1.1)$  we have

Equality holds for 
$$
D = D^*
$$
 and  $A = \Delta$ .  
\nProof: The proof is based on rearrangement techniques developed in [2, 17]. For  
\nthis purpose let us introduce the notation  
\n
$$
D(\tilde{\psi}) = \{x \in D : \tilde{\psi}(x) \geq \tilde{\psi}\}, \quad a(\tilde{\psi}) = \text{meas } D(\tilde{\psi}).
$$
\nBy Schwarz's inequality and (1.1) we have  
\n
$$
\left\{\frac{1}{d\tilde{\psi}} \int_{D(\tilde{\psi}) - D(\tilde{\psi} + d\tilde{\psi})} |\mathcal{F}\psi| dx\right\}^2 \leq \left\{\frac{1}{d\tilde{\psi}} \int_{D(\tilde{\psi}) - D(\tilde{\psi} + d\tilde{\psi})} |\mathcal{F}\psi|^2 dx \frac{a(\tilde{\psi}) - a(\tilde{\psi} + d\tilde{\psi})}{d\tilde{\psi}}\right\}
$$
\n
$$
\leq \left\{\frac{1}{d\tilde{\psi}} \int_{D(\tilde{\psi}) - D(\tilde{\psi} + d\tilde{\psi})} \left(\sum_{j=1}^N a_{ij} \psi_x \psi_x\right) dx \frac{a(\tilde{\psi}) - a(\tilde{\psi} + d\tilde{\psi})}{d\tilde{\psi}}\right\}.
$$
\nIn view of Fleming-Rishel's formula  
\n
$$
-\frac{d}{d\tilde{\psi}} \int_{D(\tilde{\psi})} |\mathcal{F}\psi| dx = P(\tilde{\psi}), \tag{2.3}
$$
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$$
-\frac{d}{d\tilde{\psi}} \int\limits_{D(\tilde{\psi})} |\tilde{V}\psi| dx = P(\tilde{\psi}), \tag{2.3}
$$

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\nwhere 
$$
P(\tilde{\psi})
$$
 is the perimeter of  $D(\tilde{\psi})$  in the sense of de Giorgi. In addition  
\n
$$
\frac{d}{d\tilde{\psi}} \int_{D(\tilde{\psi})} \left( \sum_{i,j=1}^{N} a_{ij} \psi_{x_i} \psi_{x_j} \right) dx = \int_{D(\tilde{\psi})} A\psi dx = -a(\tilde{\psi}).
$$
\n2.4)  
\nIf we let  $d\tilde{\psi}$  tend to zero, then (2.2) combined with (2.3) and (2.4) yields  
\n $P^2(\tilde{\psi}) \leq -a(\tilde{\psi}) a'(\tilde{\psi})$  for almost all  $\tilde{\psi}$ .  
\nFrom the isoperimetric inequality  $P(\tilde{\psi}) \geq N \omega_N^{1/N} a(\tilde{\psi})^{1-1/N}$ , it follows that  $N^2 \omega_N^{2/n}$ 

$$
d\tilde{\psi} \text{ tend to zero, then (2.2) combined with (2.3) and (2.4) yields}
$$
  

$$
P^2(\tilde{\psi}) \leq -a(\tilde{\psi}) a'(\tilde{\psi}) \text{ for almost all } \tilde{\psi}.
$$
 (2.5)

From the isoperimetric inequality  $P(\tilde{\psi}) \geq N \omega_N^{1/N} a(\tilde{\psi})^{1-1/N}$ , it follows that  $N^2 \omega_N^{2/n}$  $(\frac{1}{2})^{\frac{1}{2}-1/N}$ , it 210 C. BANDLE<br>
Where  $P(\tilde{\psi})$  is the perimeter of  $D(\tilde{\psi})$  in the sense of de Giorgi. In addition<br>  $\frac{d}{d\tilde{\psi}} \int_{\nu(\tilde{\psi})} \left( \sum_{j=1}^{N} a_{ij} \psi_{z_i} \psi_{z_j} \right) dx = \int_{D(\tilde{\psi})} A\psi dx = -a(\tilde{\psi})$ . (2.4)<br>
If we let  $d\tilde{\psi}$  tend to 0 C. BANDLE<br>
here  $P(\tilde{\psi})$  is the perimeter of  $D(\tilde{\psi})$  in the sense of de Giorgi.<br>  $\frac{d}{d\tilde{\psi}} \int_{D(\tilde{\psi})} \left( \sum_{i,j=1}^{N} a_{ij} \psi_{x_i} \psi_{x_j} \right) dx = \int_{D(\tilde{\psi})} A\psi dx = -a(\tilde{\psi})$ .<br>
we let  $d\tilde{\psi}$  tend to zero, then (2.2) combin  $\leq -a'(\tilde{\psi})$ . Integra<br>
e get<br>  $\psi_{\text{max}} \leq \left(\frac{A}{\omega_{\lambda}}\right)^{2/N} \cdot \frac{1}{2N}$ If we let  $d\tilde{\psi}$  tend to zero, then (2.2) combined with (2.3) and (2.4) yields<br>  $\int P^2(\tilde{\psi}) \leq -a(\tilde{\psi}) a'(\tilde{\psi})$  for almost all  $\tilde{\psi}$ .<br>
From the isoperimetric inequality  $P(\tilde{\psi}) \geq N\omega_N^{1/N} a(\tilde{\psi})^{1-1/N}$ , it follows  $\begin{aligned} &\mathcal{L}(\psi) \leq -a(\psi) a'(\psi) \quad \text{for almost all } \bar{\psi}. \end{aligned}$ <br>
isoperimetric inequality  $P(\tilde{\psi}) \geq N \omega_N^{-1/N} a(\tilde{\psi})^1$ <br>  $\leq -a'(\tilde{\psi})$ . Integration gives  $\tilde{\psi} \geq \psi_{\text{max}} - \left(\frac{a}{\omega_N}\right)^{2/N}$ <br>
get<br>  $\max_{\max} \leq \left(\frac{A}{\omega_N}\right)^{2/N} \cdot \frac{1}{2N$ 

$$
\psi_{\max} \leqq \left(\frac{A}{\omega_{\scriptscriptstyle N}}\right)^{2/N} \cdot \frac{1}{2N} \,. \tag{2.6}
$$

*Hence, setting*  $\tilde{\psi}(t) := \inf \{\tilde{\psi} : a(\tilde{\psi}) < t\}$  we find

$$
\int\limits_{0}^{\infty}\tilde{\psi}\,da\geq \int\limits_{0}^{\infty}\tilde{\psi}\,da\geq \int\limits_{0}^{\infty}\left\{\psi_{\text{max}}-\left(\frac{a}{\omega_{N}}\right)^{2/N}\cdot\frac{1}{2N}\right\}da;\qquad\gamma=(2N\psi_{\text{max}})^{N/2}\,\omega_{N}.
$$

/

From

Hence, setting 
$$
\tilde{\psi}(t) := \inf \{\tilde{\psi} : a(\tilde{\psi}) < t\}
$$
 we find  
\n
$$
\int_{0}^{t} \tilde{\psi} da \geq \int_{0}^{t} \tilde{\psi} da \geq \int_{0}^{t} \left[\psi_{\text{max}} - \left(\frac{a}{\omega_{N}}\right)^{2/N} \cdot \frac{1}{2N}\right] da; \qquad \gamma = (1 + \sqrt{N})
$$
\nFrom\n
$$
\int_{0}^{t} \left\{\psi_{\text{max}} - \left(\frac{a}{\omega_{N}}\right)^{2/N} \cdot \frac{1}{2N}\right\} da = (2N)^{N/2} \omega_{N} \frac{2}{N+2} \psi_{\text{max}}^{1+N/2}
$$
\nand\n
$$
\int_{0}^{t} \tilde{\psi} da = \int_{0}^{t} \psi dx
$$
\nthe assertion follows  
\nIf we multiply (2.1) by  $\psi$  and integrate we obtain, observing (1.1),\n
$$
\int_{0}^{t} \psi dx = \int_{0}^{N} \left(\sum_{i=1}^{N} a_{ij} \psi_{x_i} \psi_{x_j}\right) dx \geq \int_{0}^{N} |\nabla \psi|^{2} dx.
$$

and

$$
\int\limits_{0}^{A}\tilde{\psi}\;da=\int\limits_{D}\psi\;dx
$$

From  
\n
$$
\int_{0}^{z} \left\{ \psi_{\text{max}} - \left( \frac{a}{\omega_{N}} \right)^{2/N} \cdot \frac{1}{2N} \right\} da = (2N)^{N/2} \omega_{N} \frac{2}{N+2} \psi_{\text{max}}^{1+N/2}
$$
\nand  
\n
$$
\int_{0}^{4} \tilde{\psi} da = \int_{D} \psi dx
$$
\nthe assertion follows  
\nIf we multiply (2.1) by  $\psi$  and integrate we obtain, observing (1.1),  
\n
$$
\int_{D} \psi dx = \int_{D} \left( \sum_{i,j=1}^{N} a_{ij} \psi_{x_i} \psi_{x_j} \right) dx \ge \int_{D} |\nabla \psi|^2 dx.
$$
\nSchwarz's inequality together with (2.7) implies  
\n
$$
\int_{D} \psi dx \le \int_{D} \frac{\left( \int_{D} \psi dx \right)^2}{\int_{D} |\nabla \psi|^2 dx} \le V \frac{\int_{D} \psi^2 dx}{\int_{D} |\nabla \psi|^2 dx},
$$
\nwhere  $V$  stands for the volume of  $D$ . Moreover, by the Berleich principle

$$
\int_{D} \psi \, dx = \int_{D} \left( \sum_{i,j=1}^{N} a_{ij} \psi_{x_i} \psi_{x_j} \right) dx \geq \int_{D} |V \psi|
$$
\nis inequality together with (2.7) implies

\n
$$
\int_{D} \psi \, dx \leq \frac{\left( \int_{D} \psi \, dx \right)^2}{\int_{D} |V \psi|^2 \, dx} \leq V \frac{\int_{D} \psi^2 \, dx}{\int_{D} |V \psi|^2 \, dx},
$$
\nstends for the volume of D. Moreover,  $\phi$ .

where  $V$  stands for the volume of  $D$ . Moreover by the Rayleigh principle

$$
\int_{D} \psi^2 dx \le \frac{1}{\lambda_1} \int_{D} |\nabla \psi|^2 dx,
$$
  
is the smallest eigenvalue of  
 $\Delta w + \lambda_1 w = 0$  in *D*,  $w = 0$  on  $\partial D$ .  
so the following estimate for  $w_{\text{max}}$ . (2.8)

where  $\lambda_1$  is the smallest eigenvalue of

$$
\Delta w + \lambda_1 w = 0 \text{ in } D, \qquad w = 0 \text{ on } \partial D.
$$

This leads to the following estimate for  $\psi_{\text{max}}$ 

$$
\psi_{\max}^{1+N/2} \leq \frac{2+N}{2\omega_N} (2N)^{-N/2} \frac{V}{\lambda_1}.
$$

Optima Domains in a Reaction-Diffusion-Problem 211<br>
Requality is never sharp. Clearly any lower bound for  $\lambda_1$  will pro-Remark: This inequality is never sharp. Clearly any lower bound for  $\lambda_1$  will provide an upper bound for  $\psi_{\text{max}}$ .

In the next section we describe classes of domains of fixed volume whose first eigenvalue exceeds any given number. Optima Domains<br>
Remark: This inequality is nev<br>
vide an upper bound for  $\psi_{\text{max}}$ .<br>
In the next section we describe<br>
eigenvalue exceeds any given num<br>
2.2 The best known bound for<br>
inequality [2, 15]<br>  $\lambda_1 \geq \left(\frac{\omega_N}{V}\right$ 

2.2 The best known bound for  $\lambda_1$  is obtained from the Rayleigh-Faber-Krahn inequality [2, 15]

$$
\lambda_1 \geqq \left(\frac{\omega_N}{V}\right)^{2/N} \cdot j_{(N-2/2)} \qquad (j_k \text{ first zero of the Bessel function } J_k),
$$

equality holding only for the sphere. Obviously this estimate is not of great use for our purpose. The only chance to achieve our goal is to restrict ourselves to "thin" domains. Fram ark: This inequality is never sharp. Clearly any lower bound for  $\lambda_1$  vide an upper bound for  $\psi_{\text{max}}$ .<br>
In the next section we describe classes of domains of fixed volume where<br>
eigenvalue exceeds any given numb  $r_1 \geq {\left(\frac{\omega_N}{V}\right)}^{2/N} \cdot j_{(N-2/2)}$  (*j<sub>k</sub>* first zero of the Bessel functionally only for the sphere. Obviously this estimate is<br>se. The only chance to achieve our goal is to restrict<br>ow introduce the following classes o the sphere. Obviously this estimate is not of great use for<br>
ance to achieve our goal is to restrict ourselves to "thin"<br>
the following classes of domains<br>
ying between two parallel  $(N - 1)$ -dimensional hyperplane<br>
t a dis

 $\mathscr{P} = \{D \subset \mathbb{R}^N \text{ lying between two parallel } (N-1)\text{-dimensional hyperplane}\}$ at a distance  $2\varrho$ , that is all domains of breadth  $\leq 2\varrho$ ,

 $\mathcal{A}_k = \{\text{plane domains of inradius } \varrho \text{ and connectivity } k\}.$ 

If we compare the first eigenvalue of a domain  $D \in \mathcal{P}$  with the corresponding eigen- $\mathscr{E}_X = \{\text{convex domains in } \mathbb{R}^N \text{ of } \text{inradius } \varrho\},$ <br>  $\mathscr{A}_k = \{\text{plane domains of } \text{inradius } \varrho \text{ and } \text{connectivity } k\}.$ <br>
If we compare the first eigenvalue of a domain  $D \in \mathscr{P}$  with the corresponding eigenvalue for the "strip" and use the monotonicity domain [6] we get increases to achieve our government to achieve our government of the following classes of c<br>wing between two paralt a distance  $2\varrho$ , that is<br>promains in  $\mathbb{R}^N$  of inradiu<br>mains of inradius  $\varrho$  and<br>genvalue of a do  $\mathscr{P} = \{D \subset \mathbb{R}^N \text{ lying between two  
\nat a distance } 2\varrho,$ <br>  $\mathscr{E}_X = \{\text{convex domains in } \mathbb{R}^N \text{ of } \mathscr{A}_k = \{\text{plane domains of inradius}\} \}$ <br>
If we compare the first eigenvalue of a do<br>
value for the "strip" and use the monoton<br>
domain [6] we get<br>  $\lambda_1(D) \geqq \left(\frac{\pi}{2$ genvalue of a domain  $D \in \mathcal{P}$  with the corresponding eigen-<br>use the monotonicity of the eigenvalues with respect to the<br>for  $D \in \mathcal{P}$ . (2.10)<br>d the same result holds for convex domains in the plane,<br>for  $D \in \mathcal{E}_2$ . our purpose. The only chance to achieve our goal is to restrict ourselves<br>our purpose. The only chance to achieve our goal is to restrict ourselves t<br>domains.<br> $\mathscr{P} = \{D \subseteq \mathbb{R}^N \text{ lying between two parallel } (N-1)\text{-dimensional hy} \text{ at a distance } 2\rho$ , that is

[6] we get  
\n
$$
\lambda_1(D) \ge \left(\frac{\pi}{2_0}\right)^2 \quad \text{for} \quad D \in \mathcal{P}.
$$
\n(2.10)

As **HERSCH [12]** observed the same result holds for convex domains in the plane,  
that is  

$$
\lambda_1(D) \ge \left(\frac{\pi}{2\varrho}\right)^2 \quad \text{for} \quad D \in \mathcal{E}_2.
$$
 (2.11)

Using Cheeger's method OSSERMAN [13] proved that

$$
\lambda_1(D) \ge \left(\frac{1}{2\varrho}\right)^2 \quad \text{for} \quad D \in \mathcal{E}_N \tag{2.12}
$$

$$
\lambda_1(D) \geq \left(\frac{\pi}{2\varrho}\right) \quad \text{for} \quad D \in \mathcal{P}.
$$
\n(2.10)

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$$
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\nUsing Cheeger's method OSEERMAX [13] proved that

\n
$$
\lambda_1(D) \geq \left(\frac{1}{2\varrho}\right)^2 \quad \text{for} \quad D \in \mathcal{E}_N
$$
\n(2.12)

\nand

\n
$$
\lambda_1(D) \geq \left(\frac{1}{2\varrho}\right)^2 \quad \text{for} \quad D \in \mathcal{A}_1 \quad \text{and} \quad D \in \mathcal{A}_2.
$$
\n(2.13)

\nThe best result at the time for  $D \in \mathcal{A}_k$ ,  $k > 2$ , is due to CROKE [7], namely

\n
$$
\lambda_1(D) \geq \frac{1}{2k\varrho^2} \quad \text{for} \quad D \in \mathcal{A}_k, \quad k \geq 2.
$$
\n(2.14)

\nLittle is known for higher dimensions except that, no lower bound depending only on the inradius  $\varrho$  can be expected to hold without further assumptions.

The best result at the time for  $D \in \mathcal{A}_k$ ,  $k>2$ , is due to CROKE [7], namely

result at the time for 
$$
D \in \mathcal{A}_k
$$
,  $k > 2$ , is due to **CrORE [7]**, namely  
\n
$$
\lambda_1(D) \ge \frac{1}{2k\varrho^2} \quad \text{for} \quad D \in \mathcal{A}_k, \qquad k \ge 2.
$$
\n(2.14)

Little is known for higher dimensions except that, no lower bound dependig only on the inradius  $\rho$  can be expected to hold without further assumption on the geometry of *D.* As HAYMAN [11] points out, the first eigenvalue of a ball doesn't change much if narrow, inward pointing spiches are removed, whereas  $\varrho$  tends to zero. Little is known for higher di<br>on the inradius  $\rho$  can be expanding the expansion of D. As HAYMAN [11]<br>much if narrow, inward poin<br>14\*

**/** 

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Inserting these estimates into  $(2.9)$  we get the

*Theorem 2.1: Let A satisfy* **(1. 1),** p *be the solutiow of (2.1) and D be a domain of volume V. Then the following estimates hold.*  haves lino (2.3) we get the<br> *A satisfy* (1.1),  $\psi$  be the solution of (2.1) and *D* be a domain of<br> *D*  $\frac{2N}{\pi^2 \omega_N}$   $V_0^2$  for all *D* in  $\mathcal P$  or in  $\mathcal E_2$ .<br>  $\frac{2N}{\pi^2 \omega_N}$   $(2N)^{-N/2} \cdot V_0^2$  for all *D* in

C. BANDLE  
\ninserting these estimates into (2.9) we get the  
\n'heorem 2.1: Let A satisfy (1.1), 
$$
\psi
$$
 be the solution of (2.1) a  
\nune V. Then the following estimates hold.  
\n(i)  $\psi_{\text{max}}^{1+N/2} \leq \frac{(4+2N) (2N)^{-N/2}}{\pi^2 \omega_N} V \varrho^2$  for all D in  $\mathcal{P}$  or in  $\mathcal{E}_2$ .  
\n(ii)  $\psi_{\text{max}}^{1+N/2} \leq \frac{4+2N}{\omega_N} (2N)^{-N/2} \cdot V \varrho^2$  for all D in  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and  $\mathcal{U}_2$   
\n $k(2+N)$ 

$$
\text{(ii) } \psi_{\text{max}}^{1+N/2} \leq \frac{4+2N}{\omega_N} \, (2N)^{-N/2} \cdot V \varrho^2 \text{ for all } D \text{ in } \mathcal{A}_1, \mathcal{A}_2 \text{ and } \mathcal{E}_N.
$$

\n- 2. C. BANDLE
\n- Inserting these estimates into (2.9) we get the Theorem 2.1: Let A satisfy (1.1), 
$$
\psi
$$
 be the solution of (2.1) and  $\lim_{M \to \infty} V$ . Then the following estimates hold.
\n- (i)  $\psi_{\text{max}}^{1+ N/2} \leq \frac{(4 + 2N) (2N)^{-N/2}}{\pi^2 \omega_N} V \varrho^2$  for all D in  $\mathcal{P}$  or in  $\mathcal{E}_2$ .
\n- (ii)  $\psi_{\text{max}}^{1+N/2} \leq \frac{4 + 2N}{\omega_N} (2N)^{-N/2} \cdot V \varrho^2$  for all D in  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and  $\mathcal{E}_N$ .
\n- (iii)  $\psi_{\text{max}}^{1+N/2} \leq \frac{k(2+N)}{\omega_N} (2N)^{-N/2} \cdot V \varrho^2$  for all D in  $\mathcal{A}_k$  ( $k > 2$ ).
\n- Remark: In the case  $A = \triangle$ , SPERB [16] derived the estimate  $\psi_{\text{max}} \leq \frac{\varrho^2}{2}$  for all  $D \in \mathcal{E}_2$ .
\n- early (2.15) is not comparable to Theorem 2.1 (i).
\n

Remark: In the case  $A \doteq \Delta$ , SPERB [16] derived the estimate

$$
\psi_{\max} \leqq \frac{\varrho^2}{2} \quad \text{ for all } \ D \in \mathcal{E}_2.
$$

Clearly  $(2.15)$  is not comparable to Theorem 2.1 (i).

# **3. Main results**

Let us go back to problem (P) with the assumptions made in Section 1. It is imme diate that  $\overline{u} = \lambda \psi(x)$ ,  $\psi$  being the solution of (2.1), is an upper solution for problem (P) and thus<br>  $u(x, t) \leq \lambda \psi_{\text{max}}$  for all  $(x, t) \in D \times \mathbb{R}^+$ . (3.1) (P) and thus **results**<br>b back to p<br>t  $\overline{u} = \lambda \psi(x)$ <br>hus<br> $u(x, t) \leq \lambda$ <br>uuality toge  $\overline{\omega_N}$  (2N)<sup>-N/2</sup> ·  $V \rho^2$  *for all D in A<sub>k</sub> (k >* case  $A = \Delta$ , SPERB [16] derived the esting for all  $D \in \mathcal{E}_2$ .<br>
comparable to Theorem 2.1 (i).<br>
roblem (P) with the assumptions made in , y being the solution of ( results<br>
o back to problem (P) with the as<br>
it  $\overline{u} = \lambda \psi(x)$ ,  $\psi$  being the solution<br>
thus<br>  $u(x, t) \leq \lambda \psi_{\text{max}}$  for all  $(x, t) \in L$ <br>
quality together with the estimate<br>
rem 3.1: Let D belong to either cla<br>
max  $u(x, t) \leq$ ons made in Sect<br> *i*, is an upper sol<br>
<br>
<br>  $\mathcal{E}_N$  or  $\mathcal{A}_k$ . Then<br>  $\mathcal{E}_N$  or  $\mathcal{A}_k$ . Then<br>  $\mathcal{E}_N$  or  $\mathcal{A}_k$ . Then<br>  $\mathcal{E}_N$  on  $\mathcal{E}_N$ . Then<br>  $\mathcal{E}_N$  on  $\mathcal{E}_N$ . Then As an immediate consequence we have the<br>  $u(x, t) \leq \lambda \psi_{\text{max}}$  for all  $(x, t) \in D \times \mathbb{R}^+$ . (3<br>  $u(x, t) \leq \lambda \psi_{\text{max}}$  for all  $(x, t) \in D \times \mathbb{R}^+$ . (3<br>
is inequality together with the estimates of Theorem 2.1 yields<br>
Theorem

$$
u(x, t) \le \lambda \nu_{\text{max}} \quad \text{for all} \quad (x, t) \in D \times \mathbb{R}^+.
$$
 (3.1)

This inequality together with the estimates of Theorem 2.1 yields *Theorem 3.1: Let D belong to either classes*  $\mathcal{P}, \mathcal{E}_N$ <br>Theorem 3.1: Let D belong to either classes  $\mathcal{P}, \mathcal{E}_N$ <br> $\mathcal{F}_N = \sum_{x} \mathcal{F}_N \mathcal{F}_N(x, t) \leq cV c^2$ 

Theorem 3.1: Let D belong to either classes  $\mathcal{P}, \mathcal{E}_N$  or  $\mathcal{A}_k$ . Then

quality together with  
rem 3.1: Let D below  

$$
\max_{x \in \overline{D}} u(x, t) \leq cV e^2,
$$

*where c depends only on the number of dimensions N and on A. There exists there/ore a*  where c a<br>value  $\stackrel{.}{\varrho_0}$ ppends only on the number of dimensions  $P = \varrho_0(\lambda, N, V)$  such that<br>meas  $\Omega(t) \leq$  meas  $\Omega(\infty) = 0$  for  $\varrho \leq \varrho_0$  $u(x, t) \leq \lambda \psi_{\text{max}}$  for all  $(x, t) \in D \times \mathbb{R}^+$ .<br>
This inequality together with the estimates of Theorem<br>
Theorem 3.1: Let D belong to either classes  $\mathcal{P}, \mathcal{E}_N$  comax  $u(x, t) \leq cV\varrho^2$ ,<br>
where c depends only on the

Theorem 3.2: Let the assumptions of the previous theorem hold and suppose that  $vol D = V$ . Then  $\eta(t) \to 1$  as  $\varrho \to 0$ .

An optimal domain has to be thin in the sense that its.inradius is small.

Remark: The Theorems 3.1 and 3.2 hold also if  $g(u)$  is replaced by  $g(x, u)$ . Of all assumptions (C-1)--(C-5) we have only used (C-3), (C-4), the fact that  $g(\sigma)$  takes values in [0, 1] for  $\sigma \in [0, 1]$  and the continuity of g. **•**<br> **•**<br> **•**<br> **•**<br> **•**<br> **•**<br> **• Physical domain has to be thin in the sense the mark:** The Theorems 3.1 and 3.2 hold also if mptions  $(C-1) - (C-5)$  we have only used  $(C-3)$  es in [0, 1] for  $\sigma \in [0, 1]$  and the continuity of  $g$ <br>FERENCES<br>ARIS, R.: The ma

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 $\mathbf{r}_i=\mathbf{r}_i+\mathbf{r}_i+\mathbf{r}_i$  , where  $\mathbf{r}_i=\mathbf{r}_i+\mathbf{r}_$ 

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