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A Necessary and Sufficient Discreteness Condition for the Spectrum of a two Term Differential Operator of Higher Order

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Im gewichteten Hilbertraum $L_{2,w}(0, \infty)$ werden selbstadjungierte Differentialoperatoren der Ordnung $2n$ betrachtet, die von dem Differentialausdruck $\mathcal{A}y \equiv w^{-1}[(-1)^n (py^{(n)})^{(n)} + qy]$ erzeugt werden. Unter Verwendung von Bedingungen für die Diskretheit des Spektrums solcher Operatoren, wie sie von Kwong und Zettl angegeben worden sind, wird eine notwendige und hinreichende Bedingung für die Diskretheit des Spektrums abgeleitet.

В весовом гильбертовом пространстве $L_{2,w}(0, \infty)$ рассматриваются самосопряженные операторы порядка $2n$, порожденные дифференциальным выражением $\mathcal{A}y \equiv w^{-1}[(-1)^n (py^{(n)})^{(n)} + qy]$. Применяя условия для дискретности спектра, данные Квонгом и Цеттельем, выводится необходимое и достаточное условие для дискретности спектра.

Self-adjoint differential operators of order $2n$ are considered that are associated with the expression $\mathcal{A}y \equiv w^{-1}[(-1)^n (py^{(n)})^{(n)} + qy]$ and the weighted Hilbert space $L_{2,w}(0, \infty)$. By use of discreteness conditions for the spectrum of such operators given by Kwong and Zettl a necessary and sufficient condition for the discreteness of the spectrum is established.

Consider the differential expression

$$\mathcal{A}y \equiv w^{-1}[(-1)^n (py^{(n)})^{(n)} + qy], \qquad 0 \leqq x < \infty, \tag{1}$$

where the weight function w and the coefficients p and q are real-valued and

$$w > 0, \quad p > 0, \quad x \geqq 0, \quad w \in C, \quad p \in W_2^n(0, X), \quad q \in L_2(0, X)$$
$$\text{for all } X > 0.$$

The expression $\mathcal{A}$ determines the symmetric operator A_0,

$$A_0\varphi = \mathcal{A}\varphi, \qquad \varphi \in D(A_0) = C_0^\infty(0, \infty),$$

in the weighted Hilbert space $L_{2,w}(0, \infty)$ of all complex-valued measurable functions f satisfying $\|f\|_w^2 = \int_0^\infty |f|^2 w\, dx < +\infty$. It is known that all self-adjoint extensions A of A_0 have the same essential spectrum $\sigma_e(A)$. In the following we are interested in establishing a necessary and sufficient condition for the case $\sigma_e(A) = \emptyset$, i.e., for the discreteness of the spectrum of A. In the special case where $w = 1$, $p = 1$, and $q \geqq 0$, by a theorem of A. M. Molchanov [8] the spectrum is discrete if and only if

$$\lim_{x \to \infty} \int_x^{x+h} q(t)\, dt = \infty$$

for each $h > 0$. This theorem has been generalized in different directions by several authors. Relating to this we refer to [1, 5] (the case $n = 1$, $w = 1$, $p = 1$), [9]

($n \geqq 1$, $p = 1$, $w = 1$), [2, 7, 4] ($n \geqq 1$, $w = 1$, $q = 0$), [9] ($n = 1$, $w = 1$) and [3] ($n = 1$). In the following by using sufficient conditions for the discreteness of the spectrum of MAN KAM KWONG and A. ZETTL [6] we discuss the case $n \geqq 1$ with general functions w, p, and q. According to [6: Theorem 2] the spectrum of A is bounded below and discrete if the following conditions are fulfilled.

i) There exists a positive function $f(x)$, $0 < x < \infty$, such that

$$\lim_{h\to 0} \left\{ \limsup_{x\to\infty} [hf(x)]^{2(n-1)} \left(\int_x^{x_h} (w + q^-)\, dt \right) \left(\int_x^{x_h} p^{-1}\, dt \right) \right\} = 0,$$

$$x_h = x + hf(x),$$

where $q^- = q^+ - q$ with $q^+(x) = \max\big(q(x), 0\big)$.

ii) There are numbers $X > 0$, $K > 0$, and $h_0 > 0$ such that

$$\int_x^{x_h} (w + q^-)\, dt \leqq K \int_\xi^{\xi_{h'}} (w + q^-)\, dt$$

for all $x \geqq X$, all positive $h \leqq h_0$, and for all ξ with

$$[\xi, \xi_{h'}] = [\xi, \xi + 3^{1-n}hf(\xi)] \subset [x, x + hf(x)], \qquad h' = 3^{1-n}h.$$

iii) For each $h > 0$ we have

$$\lim_{x\to\infty} \left(\int_x^{x_h} (w + q^-)\, dt \right)^{-1} \int_x^{x_h} q^+\, dt = \infty, \qquad x_h = x + hf(x).$$

The condition iii) proves to be also necessary for the discreteness of the spectrum in the case that

$$f(x) = \left(\frac{p(x)}{w(x)} \right)^{1/2n}, \qquad 0 < x < \infty.$$

Theorem: *Let the following conditions be fulfilled.*

1. *The function $w^{-1}q$, $0 < x < \infty$, is bounded below, so that the operator A_0 is bounded from below.*

2. *There exist positive numbers h_0, c_1, c_2, d_1, and d_2, such that*

$$c_1 p(x) \leqq p(\xi) \leqq c_2 p(x), \tag{2}$$

$$d_1 w(x) \leqq w(\xi) \leqq d_2 w(x), \tag{3}$$

when $x \leqq \xi \leqq h_0 f(x)$, $0 < x < \infty$, where $f(x) = \left(\dfrac{p(x)}{w(x)} \right)^{1/2n}$.

Then the spectrum of any self-adjoint extension of A_0 is discrete if and only if

$$\lim_{x\to\infty} [w(x) f(x)]^{-1} \left(\int_x^{x_h} q\, dt \right) = \infty \quad \textit{for each} \quad h > 0 \tag{4}$$

where $x_h = x + hf(x)$.

Proof: The semiboundedness of A_0 easily follows from the hypothesis 1. Now we prove that the conditions i)—iii) from above are fulfilled. By means of conditions 1 and 2 we obtain

$$[hf(x)]^{2(n-1)} \left(\int_x^{x_h} (w + q^-)\, dt\right) \left(\int_x^{x_h} p^{-1}\, dt\right)$$

$$\leqq [hf(x)]^{2(n-1)}\, h^2 f^2(x)\, d_2 w(x) \sup_{0<x<\infty} \left(1 + \frac{q^-}{w}\right) c_1^{-1} p^{-1}(x)$$

$$\leqq C h^{2n} f^{2n}(x)\, w(x)\, p^{-1}(x) = C h^{2n}.$$

Hence, i) is satisfied. ii) is obtained from the estimate

$$\int_x^{x_h} (w + q^-)\, dt \leqq \sup_{0<x<\infty} \left(1 + \frac{q^-}{w}\right) d_2 w(x)\, hf(x) = Chf(x)\, w(x)$$

$$\leqq C \left(\frac{d_2}{c_1}\right)^{1/2n} hf(\xi)\, w(x) \leqq C_1 d_1^{-1} \int_\xi^{\xi_h'} w\, dt \leqq K \int_\xi^{\xi_h'} (w + q^-)\, dt.$$

Finally, in view of condition 1 and (3) we have

$$[w(x)\, f(x)]^{-1} \left(\int_x^{x_h} q\, dt\right) \leqq d_2 h \sup_{0<x<\infty} \left(1 + \frac{q^-}{w}\right) \left(\int_x^{x_h} (w + q^-)\, dt\right)^{-1} \left(\int_x^{x_h} q^+\, dt\right),$$

and by means of (4) it follows that iii) is also fulfilled. Hence, the spectrum of any self-adjoint extension of A_0 is discrete.

Now we prove that the condition (4) is necessary for the discreteness of the spectrum. Without loss of generality we can assume that $q(x) \geqq 0$, $0 < x < \infty$: for the spectrum of the operator A is discrete if and only if the spectrum of the operator $A_{\tilde q}$ is discrete where $A_{\tilde q}$ arises from A by replacing the function q by $\tilde q = q + \mu w$. From $\inf\limits_{0<x<\infty} \frac{q}{w} = -\mu > -\infty$, however, it follows that $\tilde q \geqq 0$. Hence, let q be non-negative in the following.

To prove that condition (4) is necessary for the discreteness of the spectrum we assume that (4) does not hold. Then there exist positive numbers C and h_1 and a sequence of points $(x_j)_{j=1,2,\ldots}$, $0 < x_1 < x_2 < \cdots$, tending to infinity such that

$$[w(x_j)\, f(x_j)]^{-1} \left(\int_{x_j}^{x_{j,h_1}} q\, dt\right) \leqq C, \qquad x_{j,h_1} = x_j + h_1 f(x_j). \tag{5}$$

At this point we may assume that $0 < h_1 \leqq h_0$, because the inequality (5) also holds when h_1 is replaced by any number $h_1' > 0$ smaller than h_1. Now choose $x_{j+1} > x_{j,h_1}$, $j = 1, 2, \ldots$ In the following a sequence of functions $u_j \in D(A_0)$, $j = 1, 2, \ldots$, will be constructed with the properties

$$0 < d_1 \leqq \|u_j\|_w^2 \leqq d_2 \quad \text{and} \quad (A_0 u_j, u_j)_w \leqq C, \qquad j = 1, 2, \ldots$$

Let φ be a real-valued function satisfying

$$\varphi \in C^\infty(-\infty, \infty), \qquad \operatorname{supp} \varphi \subset (0, 1), \qquad \int_{-\infty}^{\infty} \varphi^2\, dx = 1,$$

and set

$$u_j(x) = [h_1 w(x_j)\, f(x_j)]^{-1/2}\, \varphi\big([h_1 f(x_j)]^{-1} (x - x_j)\big), \quad 0 < x < \infty, \quad j = 1, 2, \ldots$$

Then we have

$$\|u_j\|_w^2 = [h_1 w(x_j)\, f(x_j)]^{-1} \int_{x_j}^{x_{j,h_1}} \varphi^2\big([h_1 f(x_j)]^{-1}(x - x_j)\big)\, w(x)\, dx$$

$$= [w(x_j)]^{-1} \int_0^1 \varphi^2(x)\, w[x_j + h_1 f(x_j)\, x]\, dx$$

and by using the inequality (3)

$$d_1 \leqq \|u_j\|_w^2 \leqq d_2. \tag{6}$$

Further we have

$$[u_j^{(n)}(x)]^2 \leqq C_n h_1^{-(2n+1)} f^{-(2n+1)}(x_j)\, w^{-1}(x_j) \tag{7}$$

and by virtue of (3), (5), and (7) we get

$$(A_0 u_j, u_j)_w = \int_{x_j}^{x_{j,h_1}} [p(u_j^{(n)})^2 + q u_j^2]\, dx \tag{8}$$

$$\leqq C_{n,h_1} w^{-1}(x_j)\, f^{-2n}(x_j)\, c_2 p(x_j) + C_{0,h_1}[w(x_j)\, f(x_j)]^{-1} \int_{x_j}^{x_{j,h_1}} q\, dt$$

$$\leqq C_{n,h_1} c_2 + C_{0,h_1} C = C^*.$$

The constant C^* is independent of j. Consequently, by a theorem of Rellich (see [10]) from (6) and (8) it follows that the essential spectrum of a self-adjoint extension of A_0 is not empty. This proves the Theorem ∎

In the case where $n = 1$ condition (4) is closely related to the corresponding condition of D. B. HINTON [3].

Remark: The hypotheses (2) and (3) are fulfilled if there exists a number M such that

$$f(x)\, \frac{|p'|}{p} \leqq M \quad \text{and} \quad f(x)\, \frac{|w'|}{w} \leqq M, \qquad 0 < x < \infty. \tag{9}$$

This can be seen as follows. From $f = \left(\frac{p}{w}\right)^{1/2n}$ and (9) we have

$$f'(x) = \frac{1}{2n}\, f(x) \left(\frac{p'}{p} - \frac{w'}{w}\right) \quad \text{and} \quad -M \leqq f'(x) \leqq M, \qquad 0 < x < \infty.$$

By integration we obtain $-M(\xi - x) \leqq f(\xi) - f(x) \leqq M(\xi - x)$ where ξ is restricted by $x \leqq \xi \leqq x + h_0 f(x)$, $h_0 > 0$. Hence we have

$$(1 - Mh_0)\, f(x) \leqq f(\xi) \leqq (1 + Mh_0)\, f(x).$$

Choosing the number h_0 sufficiently small we see that there are positive numbers γ_1 and γ_2 such that

$$\gamma_1 f(x) \leqq f(\xi) \leqq \gamma_2 f(x), \quad x \leqq \xi \leqq x + h_0 f(x), \quad 0 < x < \infty. \tag{10}$$

Further from (9) it follows that $-M \leqq f(t)\dfrac{p'(t)}{p(t)} \leqq M$, $0 < t < \infty$, and because of (10) we have

$$-M\gamma_1^{-1}f^{-1}(x) \leqq \frac{p'(t)}{p(t)} \leqq M\gamma_1^{-1}f^{-1}(x), \qquad x \leqq t \leqq x + h_0 f(x).$$

By integration from x to ξ we get

$$-M\gamma_1^{-1}f^{-1}(x)\,(\xi - x) \leqq \ln\frac{p(\xi)}{p(x)} \leqq M\gamma_1^{-1}f^{-1}(x)\,(\xi - x)$$

and

$$-M_1 \leqq \ln\frac{p(\xi)}{p(x)} \leqq M_1, \qquad x \leqq \xi \leqq x + h_0 f(x).$$

Hence we have

$$e^{-M_1}p(x) \leqq p(\xi) \leqq e^{M_1}p(x), \qquad x \leqq \xi \leqq x + h_0 f(x).$$

Concerning the case $n = 1$ the condition (9) was established by D. B. Hinton [3].

Examples: 1. In the case

$$\mathcal{A}y = x^{-\alpha}[(-1)^n\,(x^\beta y^{(n)})^{(n)} + qy], \qquad 1 \leqq x < \infty,$$

we have $f(x) = x^{\frac{\beta-\alpha}{2n}}$. The hypotheses (2) and (3) are fulfilled when $\beta - \alpha \leqq 2n$. Then the spectrum is discrete if and only if

$$\lim_{x\to\infty} x^{\frac{(1-2n)\alpha-\beta}{2n}} \int_x^{x_h} q\,dt = \infty, \qquad x_h = x + hx^{\frac{\beta-\alpha}{2n}}, \quad \text{for each } h > 0.$$

The special case $\alpha = 0$ is handled in [9].

2. In the case

$$\mathcal{A}y = p^{-1}[(-1)^n\,(py^{(n)})^{(n)} + qy], \qquad 0 < x < \infty, \qquad |p'| \leqq Mp.$$

the spectrum is discrete if and only if

$$\lim_{x\to\infty} p^{-1}(x) \int_x^{x+h} q\,dt = \infty \text{ for each } h > 0.$$

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