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Laplace-Gauß Integrals, Gaussian Measure Asymptotic Behaviour and Probabilities of Moderate Deviations

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Für beliebige endlichdimensionale Borelmengen wird eine allgemeine Formel zur Bestimmung ihres Gaußmaßes angegeben, welche die von B. Cavalieri und E. Torricelli entwickelte „Methode der Indivisibeln" widerspiegelt. Darauf aufbauend werden für Mengen, deren Abstand vom Ursprung gegen Unendlich strebt, entsprechende Aussagen über das asymptotische Verhalten ihres Gaußmaßes formuliert und zwei spezielle Grenzwertsätze für Wahrscheinlichkeiten mittlerer Abweichungen von Summen unabhängiger, identisch verteilter Zufallsvektoren abgeleitet.

Для вычичления гауссовской меры любого конечномерного борелевского множества выводится общая формула, которая отражает метод „неделимых величин" Б. Кавальери и Э. Торричелли. На основе этой формулы изучается асимптотическое поведение гауссовской меры на множествах, расстояние которых от начала координат стремится к бесконечности. Кроме того приводятся две специальные предельные теоремы для умеренных уклонений сумм независимых случайных векторов.

There is given a general Gaussian measure representation on arbitrary finite dimensional Borel sets. This representation reflects B. Cavalieri's and E. Torricelli's "indivisibeln method" in a modern language. Based upon it, assertions are derived about the Gaussian measure asymptotic behaviour on Borel sets whose distance from the origin tends to infinity. Also two specific multivariate moderate deviation limit theorems for sums of i.i.d. random vectors are deduced.

1. Introduction

The most general integral limit theorems for probabilities of large deviations of sums of finite dimensional random vectors have been considered by L. Vilkauskas [21], A. A. Borovkov and B. A. Rogozin [2] and A. V. Nagaev and S. K. Sakojan [12]. Their results, however, due to insufficient knowledge about the Gaussian limit law, are not quite practicable at present. B. von Bahr [1: § 8] has pointed out one of the underlying problems, and both he and many authors after him have considered various more or less specific cases.

The main purpose of the present paper is to give a general Gaussian law representation on arbitrary Borel sets and to derive from it some asymptotic properties of the Gaussian law. The underlying idea is a generalisation of B. Cavalieri's and E. Torricelli's indivisibeln method, which will be deduced and formulated here with the help of some standard methods of measure theory. After a discussion of different examples, we will finally formulate two specific multivariate moderate deviation limit theorems as a consequence of the aforementioned results and those in [16]. In [17] we use the results of the present paper for deducing some general assertions related to the structure of multidimensional central limit theorems under moderate deviations.

Let $\mathbf{R}^k$, $k \geqq 1$ be the k-dimensional Euclidean space equipped with its Borel σ-field $\mathfrak{B}^k$ and the usual Euclidean norm $x \to \|x\| = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$, $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. We denote by $\Phi(\cdot)$ the standardized Gaussian law on $\mathfrak{B}^k$ and by $\mu^{(k)}$ the Lebesgue measure on $\mathfrak{B}^k$. Further, put $\lambda A = \{(\lambda x_1, \ldots, \lambda x_k) : x \in A\}$. Then

$$\Phi(\lambda A) = (2\pi)^{-(k/2)} I(\lambda)$$

for all $\lambda > 0$ and $A \in \mathfrak{B}^k$; where the parameter integral

$$I(\lambda) = \int_{\lambda A} \exp\{-\|x\|^2/2\} \mu^{(k)}(dx) \tag{1}$$

will be called a Laplace-Gauß integral.

Finally, let us make reference to the considerations [7, 8] on Brownian motion and renewal theory, where integrals of type (1) or similar also arise.

2. Laplace-Gauß integrals in $\mathbf{R}^k$

With the notation

$$a = \inf\{\|x\|^2 : x \in A\}, \qquad m(\lambda) = a\lambda^2/2, \qquad M(\lambda) = \sup\{\lambda^2 \|x\|^2/2 : x \in A\}$$

and

$$V_{A,\lambda}(c) = \begin{cases} \mu^{(k)}(\{x \in A : \lambda^2 \|x\|^2/2 < c\}) & \text{if } c > m(\lambda), \\ 0 & \text{elsewhere}, \end{cases}$$

we are able to formulate the starting point of our considerations.

Lemma 1: *There exists a $\mu^{(1)}$-zero-set $E \subset \mathbf{R}^1$ such that on $\mathbf{R}^1 \setminus E$ there exists a derivative $V'_{A,\lambda}$ of the function $V_{A,\lambda}$. Putting $V'_{A,\lambda}(c) = 0$ for $c \in E$, the function $V'_{A,\lambda}$ is Lebesgue integrable on $\mathbf{R}^1$ and*

$$I(\lambda) = \lambda^k \exp\{-\lambda^2 a/2\} \int_0^{M(\lambda)-m(\lambda)} V'_{A,\lambda}(c + m(\lambda))\, e^{-c} \mu^{(1)}(dc), \tag{2}$$

for all $\lambda > 0$.

Define $S_r = \{x \in \mathbf{R}^k : \|x\| = r\}$, $r \geqq 0$. For $x \in \mathbf{R}^k$ with $\|x\| = r$ and $x/r = \theta$, $\theta \in S_1$ write $x = (r, \theta)$. Let ω denote the uniform distribution on S_1 (with $\omega(S_1) = 1$). Put

$$I_{A,c}(\theta) = \begin{cases} 1 \text{ if } x = (c, \theta) \in A, \\ 0 \text{ elsewhere}, \end{cases}$$

for all $c \geqq 0$, $\theta \in S_1$ and

$$\omega_k = 2\pi^{k/2}/\Gamma(k/2),$$

where Γ denotes the gamma function. Now we are in a position to formulate the main result of the present paper.

Theorem 1: *For all $\lambda > 0$ it holds that*

$$I(\lambda) = \omega_k \lambda^{k-2} e^{-\lambda^2 a/2} \times \int_0^{M(\lambda)-m(\lambda)} (2c/\lambda^2 + a)^{k/2-1} e^{-c} \mathfrak{F}\left(\sqrt{\frac{2c}{\lambda^2} + a}\right) \mu^{(1)}(dc), \tag{3}$$

where

$$\mathfrak{F}(c) = \int_{S_1} I_{A,c}(\theta)\, \omega(d\theta), \quad c \geqq 0.$$

Remark 1: Because of

$$\mathfrak{F}(c) = \omega(\{\theta \in S_1 : x = (c, \theta) \in A\}), \qquad c \geqq 0$$

one can explain $\mathfrak{F}(c)$ as the "percentage" of $A \cap S_c$ with respect to S_c. The consideration of the function $\mathfrak{F}$ reflects B. Cavalieri's and E. Torricelli's "indivisibeln method" [13, 22] in a modern language. Namely we extend this method to a multidimensional measure which is not a usual volume but a Gaussian measure. One may say that two solids have equal (centred) Gaussian volumes if they are situated between the same two fixed spheres, centred at zero, of radius' r_1 and r_2, $0 \leqq r_1 < r_2 \leqq \infty$ and if for almost all spheres centred at zero and having radius r, $r \in [r_1, r_2]$, their values $\mathfrak{F}(r)$ coincide. To evaluate the actual (centred) Gaussian volume of a given solid one has to multiply the respective function

$$c \to \mathfrak{F}(\sqrt{2c/\lambda^2 + a}), \qquad c \geqq 0$$

by some dimension dependent constant and by a weight function

$$c \to (2c/\lambda^2 + a)^{k/2-1}\, \mathrm{e}^{-c}, \qquad c \geqq 0$$

before integrating this product with respect to the Lebesgue measure.

Remark 2: Because of $\int_0^{\varepsilon} c^{-1/2}\, \mathrm{e}^{-c}\, dc \leqq 2\sqrt{\varepsilon}$, $\varepsilon > 0$ the integral in relation (3) is also well defined in the case $a = 0$, $k = 1$.

Proof of Lemma 1: Replacing x by λx and applying integral transformation formulae [9: p. 163], we obtain

$$I(\lambda) = \lambda^k \int_A \mathrm{e}^{-\lambda^2\|x\|^2/2} \mu^{(k)}(dx) = \lambda^k \int_{m(\lambda)}^{M(\lambda)} \mathrm{e}^{-c}\, dV_{A,\lambda}(c).$$

With the notation

$$\nu_{A,\lambda}(B) = \mu^{(k)}(\{x \in A : \lambda^2 \|x\|^2/2 \in B\})$$

for $B \in \mathfrak{B}^1$ it follows that

$$V_{A,\lambda}(c) = \nu_{A,\lambda}(\{t \in \mathbf{R}^1 : t < c\}), \qquad c \in \mathbf{R}^1.$$

That is why $V_{A,\lambda}$ is an absolutely continuous function iff the measure $\nu_{A,\lambda}$ is absolutely continuous with respect to the measure $\mu^{(1)}$ [9: p. 181]. We shall now show the validity of the aforementioned relation between $\nu_{A,\lambda}$ and $\mu^{(1)}$: As the product measure $\omega_k \omega(d\theta)\, r^{k-1} \mu^{(1)}(dr)$ coincides with the Lebesgue measure $\mu^{(k)}(dx)$ [4: pp. 36, 63], it holds that

$$\nu_{A,\lambda}(B) = \int_{r^2 \in 2B/\lambda^2} \int_{\theta \in S_1} I_{A,r}(\theta)\, \omega_k \omega(d\theta)\, r^{k-1} \mu^{(1)}(dr).$$

Hence

$$\nu_{A,\lambda}(B) \leqq (\omega_k/2) \int_{2B/\lambda^2} z^{k/2-1} \mu^{(1)}(dz).$$

Consequently $\mu^{(1)}(B) = 0$ yields $v_{A,\lambda}(B) = 0$ for any $B \in \mathfrak{B}^1$. Now, from

$$I(\lambda) = \lambda^k \int_{m(\lambda)}^{M(\lambda)} e^{-c} v_{A,\lambda}(c)\, \mu^{(1)}(dc)$$

after replacing c by $c + m(\lambda)$, equation (2) follows ∎

First proof of Theorem 1: If $c \in \mathbf{R}^1 \setminus E$ then

$$v_{A,\lambda}(c) = \lim_{h\to 0} [V_{A,\lambda}(c+h) - V_{A,\lambda}(c-h)]/(2h)$$

$$= \omega_k \lambda^{-2} \lim_{h\to 0} h^{-1} \int_{\sqrt{2c/\lambda^2 - h}}^{\sqrt{2c/\lambda^2 + h}} r^{k-1} \mathfrak{F}(r)\, \mu^{(1)}(dr).$$

From Lebesgue's theorem [10: p. 277] it follows that

$$\lim_{h\to 0} h^{-1} \int_{2c/\lambda^2}^{2c/\lambda^2 + h} \mathfrak{F}(\sqrt{z})\, z^{k/2-1} \mu^{(1)}(dz) = (2c/\lambda^2)^{k/2-1} \mathfrak{F}(\sqrt{2c/\lambda^2}).$$

Hence

$$V'_{A,\lambda}(c) = \omega_k \lambda^{-2} (2c/\lambda^2)^{k/2-1}\, \mathfrak{F}(\sqrt{2c/\lambda^2}).$$

Putting $z = r^2$ and replacing c by $c + m$, Lemma 1 now yields (3) ∎

Second proof of Theorem 1: Since

$$V_{A,\lambda}(c) = \omega_k \int_{S_1 \times (0, \sqrt{2c}/\lambda)} I_{A,r}(\theta)\, \omega(d\theta)\, r^{k-1} \mu^{(1)}(dr),$$

by Fubini's theorem [4: p. 35] it follows that

$$V_{A,\lambda}(c) = \omega_k \int_0^{\sqrt{2c}/\lambda} r^{k-1} \mathfrak{F}(r)\, \mu^{(1)}(dr).$$

If $c \in \mathbf{R}^1 \setminus E$ then

$$V'_{A,\lambda}(c) = \omega_k (\sqrt{2c}/\lambda)^{k-1} (\lambda \sqrt{2c})^{-1}\, \mathfrak{F}(\sqrt{2c}/\lambda).$$

Finally, Lemma 1 implies

$$I(\lambda) = \omega_k\, 2^{k/2-1}\, e^{-\lambda^2 a/2}$$

$$\times \int_0^{M(\lambda)-m(\lambda)} \left(c + \frac{\lambda^2 a}{2}\right)^{k/2-1} e^{-c} \mathfrak{F}\left(\sqrt{\frac{2c}{\lambda^2} + a}\right) \mu^{(1)}(dc) \quad ∎$$

In the following let $\{A(\lambda)\}_{\lambda>0}$ be a system of Borel sets in $\mathbf{R}^k$. Further, define the quantities $a = a(\lambda)$, $m(\lambda)$, $M(\lambda)$ and $\mathfrak{F}(\cdot) = \mathfrak{F}(\lambda, \cdot)$, $\lambda > 0$ in the same way as above and put

$$I_1(\lambda) = \int_{\lambda A(\lambda)} e^{-\|x\|^2/2} \mu^{(k)}(dx),\ \lambda > 0.$$

We shall now formulate some consequences of Theorem 1 concerning the asymptotic behaviour of the integrals $I_1(\lambda)$ when λ tends to infinity.

Corollary 1: *Suppose that* $\mathfrak{F}(\lambda, c) \equiv \mathfrak{F}_0$ *for all* $c > 0$ *and all* $\lambda > 0$. *Also, assume that* $a(\lambda) \geqq a_0$ *for some positive* a_0 *and all* $\lambda > 0$. *Then*

$$I_1(\lambda) \sim \omega_k \mathfrak{F}_0 \big(\lambda \sqrt{a(\lambda)}\big)^{k-2} \, e^{-\lambda^2 a(\lambda)/2} \quad as \quad \lambda \to \infty. \tag{4}$$

Here $f(\lambda) \sim g(\lambda)$ *means* $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = 1$.

Proof: Let ϱ be a function with the properties

$$\lim_{\lambda\to\infty} \varrho(\lambda) = \infty \quad \text{and} \quad \lim_{\lambda\to\infty} \varrho(\lambda)/\lambda^2 = 0.$$

Then the following two relations hold as $\lambda \to \infty$:

$$\int_0^{\varrho(\lambda)} \big(2c/\lambda^2 + a(\lambda)\big)^{k/2-1} \, e^{-c} \, dc \sim a(\lambda)^{k/2-1}$$

and

$$\int_{\varrho(\lambda)}^{\infty} \big(2c/\lambda^2 + a(\lambda)\big)^{k/2-1} \, e^{-c} \, dc = o\big(a(\lambda)^{k/2-1}\big).$$

Here $f(\lambda) = o\big(g(\lambda)\big)$ means $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = 0$. Hence, (3) implies (4) ∎

Corollary 2: *It holds that*

$$\Phi\big(\lambda A(\lambda)\big) = o(1) \, \lambda^{k-2} \, e^{-\lambda^2 a(\lambda)/2} \tag{5}$$

as $\lambda \to \infty$ *if and only if*

$$\int_0^{M(\lambda)-m(\lambda)} \big(2c/\lambda^2 + a(\lambda)\big)^{k/2-1} \, e^{-c} \mathfrak{F}\big(\lambda, \big(\sqrt{2c/\lambda^2 + a(\lambda)}\big) \, \mu^{(1)}(dc) = o(1) \tag{6}$$

as $\lambda \to \infty$.

Remark 3: Condition (6) can be understood both as a condition concerning $M(\lambda) - m(\lambda)$ and as a condition concerning the "percentage" of the volume of the set $\{x \in A(\lambda) : y \leqq \|x\| \leqq y + \delta\}$ with respect to the volume of the set $\{x \in \mathbf{R}^k : y \leqq \|x\| \leqq y + \delta\}$, where $y + \delta = \big(2c/\lambda^2 + a(\lambda)\big)^{1/2} \sim a(\lambda)^{1/2} = y$ as $\lambda \to \infty$. Further note that condition (6) plays an importand role in limit theorems for probabilities of moderate deviations. Namely it describes all cases in which the function d from [16: (3)] satisfies $x^2\big(d(x) - a\big) \to \infty$ as $x \to \infty$. In all these cases [16: (11)] cannot correspond to [16: (12)].

Corollary 3: Let ϱ *be a function with* $\lim_{\lambda\to\infty} \varrho(\lambda) = \infty$ *and put*

$$J(\lambda) = \int_0^{\varrho(\lambda)} \big(2c/\lambda^2 + a(\lambda)\big)^{k/2-1} \, e^{-c} \mathfrak{F}\big(\lambda, \sqrt{2c/\lambda^2 + a(\lambda)}\big) \, \mu^{(1)}(dc).$$

Then it holds that

$$I_1(\lambda) \sim \omega_k \lambda^{k-2} \, e^{-\lambda^2 a(\lambda)/2} J(\lambda) \tag{7}$$

as $\lambda \to \infty$ *if and only if*

$$\int_{\varrho(\lambda)}^{\infty} \big(2c/\lambda^2 + a(\lambda)\big)^{k/2-1} \, e^{-c} \mathfrak{F}\big(\lambda_0, \sqrt{2c/\lambda^2 + a(\lambda)}\big) \, \mu^{(1)}(dc) = o\big(J(\lambda)\big) \tag{8}$$

as $\lambda \to \infty$.

Remark 4: Condition (8) will be fulfilled if the function $\mathfrak{F}(\lambda, \cdot)$ satisfies suitable lower and upper inequalities for $c \in \left[\sqrt{a(\lambda)}, \sqrt{a(\lambda) + 2\varrho(\lambda)/\lambda^2}\right]$ and $c > \sqrt{a(\lambda) + 2\varrho(\lambda)/\lambda^2}$ respectively. If $M(A(\lambda)) = \{x \in \mathbf{R}^k: xx^T = a(\lambda)\}$ then (8) expresses how large a neighbourhood of $M(A(\lambda))$ must be in order to determine the exact asymptotic behaviour of $\Phi(\lambda A(\lambda))$ as $\lambda \to \infty$.

3. Examples of Gaussian measure asymptotic behaviour

In this chapter we shall derive some assertions about the Gaussian measure asymptotic behaviour on various Borel sets whose distance from the origin, generally spoken, tends to infinity.

Lemma 2: *It holds that*

$$\Phi(\{x \in \mathbf{R}^k: \|x\| \geqq \lambda\}) \sim 2^{1-k/2}[\Gamma(k/2)]^{-1} \lambda^{k-2} e^{-\lambda^2/2} \tag{9}$$

as $\lambda \to \infty$.

Proof: By Theorem 1

$$I(\lambda) = \omega_k \lambda^{k-2} e^{-\lambda^2/2} \int_0^\infty (2c/\lambda^2 + 1)^{k/2-1} e^{-c} \, dc$$

so that

$$\Phi(\{x \in \mathbf{R}^k: \|x\| \geqq \lambda\}) \sim (2\pi)^{-k/2} \omega_k \lambda^{k-2} e^{-\lambda^2/2} \quad \blacksquare$$

Lemma 3: *Let f and g be positive functions with $f(\lambda) \to 0$ and $g(\lambda) \to 0$ as $\lambda \to \infty$. Assume ξ to be a function satisfying $\xi(\lambda) \geqq \xi_0$ for some positive ξ_0. Also, suppose that $\xi(\lambda) f(\lambda) \to 0$, $\xi(\lambda) g(\lambda) \to 0$ as $\lambda \to \infty$. If*

$$A(\lambda) = \{x \in \mathbf{R}^k: \xi(\lambda) - g(\lambda) \leqq \|x\| \leqq \xi(\lambda) + f(\lambda)\}$$

then

$$\Phi(A(\lambda)) \sim 2^{1-k/2}[\Gamma(k/2)]^{-1} \left(f(\lambda) + g(\lambda)\right) \xi(\lambda)^{k-1} e^{-\xi^2(\lambda)/2} \tag{10}$$

as $\lambda \to \infty$.

Proof: First let $\lambda = 1$ in definition (1) of the integral I. Next put $A = A(\lambda)$ in (1). Then we get $a = \left(\xi(\lambda) - g(\lambda)\right)^2$, $m(1) = a/2$ and $M(1) = \left(\xi(\lambda) + f(\lambda)\right)^2/2$. From Theorem 1 it follows that

$$\Phi(A(\lambda)) = (2\pi)^{-k/2} \omega_k e^{-(\xi(\lambda) - g(\lambda))^2/2} \times \int_0^{\theta(\lambda)} \left[2c + \left(\xi(\lambda) - g(\lambda)\right)^2\right]^{k/2-1} e^{-c} \, dc,$$

where

$$\theta(\lambda) = \left[\left(\xi(\lambda) + f(\lambda)\right)^2 - \left(\xi(\lambda) - g(\lambda)\right)^2\right]/2 = \xi[f + g] + [f^2 - g^2]/2.$$

From Lebesgue's theorem [10: p. 277] we get

$$\int_0^{\theta(\lambda)} \left[2c + \left(\xi(\lambda) - g(\lambda)\right)^2\right]^{k/2-1} e^{-c} \, dc \sim \left[\xi(\lambda) \left(f(\lambda) + g(\lambda)\right)\right] \xi(\lambda)^{k-2} \quad \blacksquare$$

Lemma 4 (see Figure 1): *Let $k = 2$, $s > 0$ be a constant and t be a function with the properties $0 < t_0 \leqq t(\lambda) < s$ and $\left(s - t(\lambda)\right) \lambda^2 \to \infty$ as $\lambda \to \infty$. Put $A(\lambda) = \{(x, y)$*

$\in \mathbf{R}^2 \colon x^2/s + y^2/t(\lambda) \geqq 1\}$. *Then*

$$\Phi(\lambda A(\lambda)) \sim \sqrt{2s}\,\left[\sqrt{\pi\cdot t(\lambda)\cdot(s-t(\lambda))}\;\lambda\, e^{\lambda^2 t(\lambda)/2}\right]^{-1} \tag{11}$$

as $\lambda \to \infty$.

Proof: It is easy to check that $a = t(\lambda)$ and $\sup\{x^2+y^2\colon (x,y)\in A\} = s$. If $c \geqq (s-t(\lambda))\,\lambda^2/2$ then $\mathfrak{F}(\lambda, \sqrt{2c/\lambda^2+a}) \equiv 1$ and if $0 \leqq c \leqq (s-t(\lambda))\,\lambda^2/2$ then $\mathfrak{F}(\lambda, \sqrt{2c/\lambda^2+a})$ equals $4\int\limits_0^{x_0}\sqrt{1+y_0'^2(x)}\,dx$ divided by the circumference of the circle $x^2+y^2 = t + 2c/\lambda^2$, where $x_0 = (2sc/[(s-t)\,\lambda^2])^{1/2}$. Using the same method as in the preceding example, we obtain (11) ∎

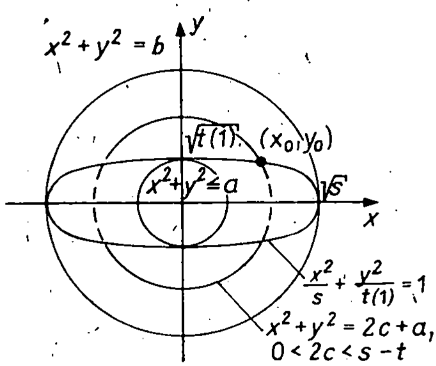

Fig. 1: ($\lambda = 1$)

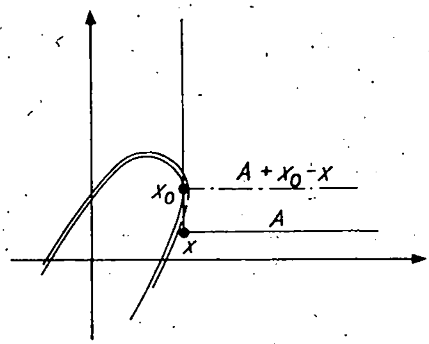

Fig. 2

We return now to an arbitrary finite dimension k. Let B be a symmetric positive definite $k \times k$-matrix. Put

$$\Phi_B(M) = [(2\pi)^k \det B]^{-1/2} \int\limits_M \exp\{-yB^{-1}y^T/2\}\,dy, \qquad M \in \mathfrak{B}^k,$$

$$x = (x_1, \ldots, x_k) \quad \text{with} \quad x_i > 0, \qquad i = 1, \ldots, k,$$

$r = (r_1, \ldots, r_k) = xB^{-1T}$ (B^{-1} denotes the inverse of B) and

$$Q(\lambda, x) = [(2\pi\lambda^2)^k \det B]^{-1/2}\left(\prod_{i=1}^k |r_i|^{-1}\right)\exp\{-\lambda^2 x r^T/2\}.$$

Further, let

$$A = \left(\mathop{\times}_{i=1}^{j} [x_i, \infty)\right) \times \left(\mathop{\times}_{i=j+1}^{k} (-\infty, x_i]\right) \quad \text{for some } j \in \{1, \ldots, k-1\}.$$

Lemma 5: *The condition*

$$\operatorname{sign} r_i = \operatorname{sign} x_i, \qquad i = 1, \ldots, k \tag{12}$$

[15: p. 56/*Bedingung* $\mathfrak{D}$] *is necessary and sufficient for*

$$\Phi_B(\lambda A) \sim Q(\lambda, x), \qquad \lambda \to \infty. \tag{13}$$

Proof: First suppose that (12) holds. Substituting $q_i = |y_i - \lambda x_i| \lambda r_i$, $i = 1, \ldots, k$, we get $\Phi_B(\lambda A) = Q(\lambda, x) J(\lambda)$, where

$$J(\lambda) = \int_0^\infty \ldots \int_{-\infty}^0 \exp\left\{-q_1 - \cdots - q_j + q_{j+1} + \cdots + q_k - (q_i/|r_i|)_{\overline{1,k}} B^{-1} (q_i/|r_i|)^T_{\overline{1,k}} \Big/ (2\lambda^2)\right\} dq.$$

Because of (12) there exists $C_0 > 0$ with $|r_i| \geqq C_0$ for $i = 1, \ldots, k$ so that $J(\lambda) \sim 1$, $\lambda \to \infty$.

Now, suppose (13) holds. Condition (12) means that the quadratic form $yB^{-1}y^T$ has over the set A a uniquely determined minimum at the point x. Assume (12) does not hold. Then there exists $x_0 \neq x$ such that $x_0 B^{-1} x_0^T = \inf\{zB^{-1}z^T : z \in A\} < xB^{-1}x^T$ (see Figure 2). One can show then that

$$\Phi_B(\lambda A) \asymp \lambda^{-1} e^{-\lambda^2 x_0 B^{-1} x_0^T/2}, \quad \lambda \to \infty \tag{14}$$

so that

$$Q(\lambda, x) \asymp \lambda^{-k} e^{-\lambda^2 x B^{-1} x^T/2} = o\big(\Phi_B(\lambda A)\big).$$

Here $f(\lambda) \asymp g(\lambda)$ means $0 < \varliminf_{\lambda\to\infty} f(\lambda)/g(\lambda) \leqq \varlimsup_{\lambda\to\infty} f(\lambda)/g(\lambda) < \infty$. However, the last relation is in contradiction to (13) ∎

It follows from Lemma 2 and (14) that the considerations in [6] are incorrect. The following example shows that even in the case where (12) holds, the assertions in [6] are not right. Let

$$I(a_1, a_2) = \int_{a_1}^\infty \int_{a_2}^\infty \exp\left\{-\frac{z_1^2 - 2tz_1z_2 + z_2^2}{2(1-t^2)}\right\} dz, \qquad -1 < t < 0.$$

A straightforward integration by parts [14] then yields

$$I(a_1, a_2) \sim I_1 = \frac{(1-t^2)^2}{(a_2 - ta_1)(a_1 - ta_2)} \exp\left\{-\frac{a_1^2 - 2ta_1a_2 + a_2^2}{2(1-t^2)}\right\}$$

as $a_1 \to \infty$, $a_2 \to \infty$, while [6: Theorem 3] asserts that

$$I(a_1, a_2) \sim I_2 = \frac{(1-t^2)^2 (a_1^2 + a_2^2)^2}{(a_1^2 - 2a_1a_2t + a_2^2) a_1 a_2} \exp\left\{-\frac{a_1^2 - 2ta_1a_2 + a_2^2}{2(1-t^2)}\right\}.$$

In the case $a_2 = 3a_1$, $t = -1/2$ one has $I_2/I_1 = 875/39$, which is a contradiction to $I \sim I_1$ and $I \sim I_2$.

For a slight generalisation of the preceding example let $B = B(\lambda)$ be positive definite,

$$x(\lambda) = \big(x_1(\lambda), \ldots, x_k(\lambda)\big), \qquad x_i(\lambda) \to \infty \text{ as } \lambda \to \infty,$$

$$A(\lambda) = \left(\mathop{\times}_{i=1}^{j} [x_i(\lambda), \infty)\right) \times \left(\mathop{\times}_{i=j+1}^{k} (-\infty, -x_i(\lambda)]\right)$$

and

$$r(\lambda) = \big(r_1(\lambda), \ldots, r_k(\lambda)\big) = x(\lambda) B^{-1}(\lambda).$$

Lemma 6: *Put $x = x(\lambda)$, $B = B(\lambda)$, $r = r(\lambda)$, $Q = Q_\lambda$ and suppose that both* (12) *and*

$$|r_i(\lambda)| \geqq C_0 > 0, \quad \text{for all } \lambda > 0 \tag{15}$$

are satisfied. Then

$$\Phi_{B(\lambda)}\big(A(\lambda)\big) \sim Q_\lambda\big(1, x(\lambda)\big), \quad \lambda \to \infty. \tag{16}$$

4. Probabilities of moderate deviations in special sets of R^k

The aim of this chapter is to show with two examples how to apply the results from Chapters 2 and 3 to the moderate deviation limit theorems from [16].

Let $X_1, X_2, \ldots$ be a sequence of independent random vectors, each having the same distribution as X, being defined on a probability space $(\Omega, \mathfrak{A}, P)$ and having values in $\mathbf{R}^k$. Assume that there exist second order moments of X. Let $EX = \mathfrak{o} = (0, \ldots, 0) \in \mathbf{R}^k$ and $EX^TX = B$, where B is a symmetric, positive definite $k \times k$-matrix. Put $Z_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$ and let E_k denote the $k \times k$-unit matrix.

Theorem 2 (see Figure 1): *Let $k = 2$, $B = E_2$, $0 < t < s$ and $|x| = (x_1^2/s + x_2^2/t)^{1/2}$ for $x = (x_1, x_2) \in \mathbf{R}^2$. Then the asymptotic relation*

$$P(|Z_n| \geqq \lambda(n) \sqrt{n}) = \Phi(\{x \in \mathbf{R}^2 \colon |x| \geqq \lambda(n)\})(1 + o(1)), \qquad n \to \infty \tag{17}$$

holds uniformly with respect to $\lambda(n) \in [1, c\sqrt{\log n}]$, $c > 0$ if and only if for any norm $|||\cdot|||$ it holds that

$$P(|||X||| \geqq y) = o(1)\,(\log y)^{1/2 + c^2 t/2}\, y^{-2-c^2 t}, \qquad y \to \infty. \tag{18}$$

Remark 5: Note that condition (18) does not depend on s in an explicit way. For easy comparison with the case $s = t$ we quote the corresponding conclusion from [16: Theorem 5] (see also [20] or [18]): If $s = t$ then (17) is valid if and only if it holds that

$$P(|||X||| \geqq y) = o(1)\,(\log y)^{1 + c^2 t/2}\, y^{-2-c^2 t}, \qquad y \to \infty. \tag{19}$$

Note that (18) is somewhat sharper than (19).

We return now to the case of an arbitrary finite dimension k. Let M_1, M_2 be sets of natural numbers with

$$M_1 \cup M_2 = \{1, \ldots, k\} \quad \text{and} \quad M_1 \cap M_2 = \emptyset.$$

Using the rectangle

$$A_1 = \{(z_1, \ldots, z_k) \colon z_i \geqq x_i,\ i \in M_1,\ z_i \leqq -x_i,\ i \in M_2\}$$

for some positive $x_1, \ldots, x_k$ and the complement of an ellipsoid

$$A_2(B) = \{z \in \mathbf{R}^k \colon zB^{-1}z^T \geqq la(B)\}, \qquad l > 1$$

we construct the set $A(B) = A_1 \cup A_2(B)$. Here $a(B) = \inf\{zB^{-1}z^T \colon z \in A_1\}$. Note that in the case $B = E_k$ one has $a(B) = x_1^2 + \cdots + x_k^2$ and condition (12) is satisfied. If (12) does not hold, then $B \neq E_k$.

Theorem 3 (see Figure 3): *Assume that condition* (12) *does not hold. Further, put $A = A(B)$ (where $B \neq E_k$). If*

$$P(X \in yA) = o(1)\,(\log y)^{(1+c^2 a)/2}\, y^{-2-c^2 a}, \qquad y \to \infty \tag{20}$$

then the asymptotic relation

$$P(Z_n \in \lambda(n)\sqrt{n}\,A) = \Phi_B(\lambda(n)\,A)(1 + o(1)), \qquad n \to \infty \tag{21}$$

holds uniformly for $\lambda(n) \in [1, c\sqrt{\log n}]$, $c > 0$.

For the proof of Theorems 2 and 3 we shall use the following auxilary results, which can be checked by straightforward calculations.

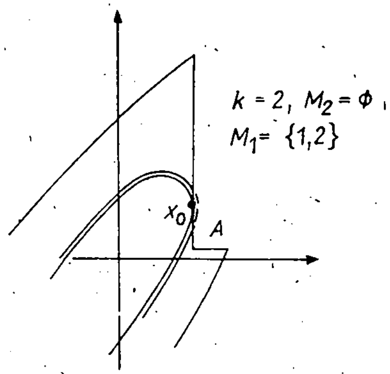

Fig. 3

Lemma 7: *The solution* $h = h(y)$ *of the equation*

$$h \exp\{h^2/(2c^2)\} = y, \qquad y > 0$$

satisfies the asymptotic ($y \to \infty$) *relation*

$$h(y) = c\sqrt{2 \log y}\,\big(1 + O(\log\log y/\log y)\big).$$

Here $f(y) = O\big(g(y)\big)$ *means* $\overline{\lim\limits_{y\to\infty}} f(y)/g(y) < \infty$.

Lemma 8: If

$$d(\lambda) = a + \psi \log \lambda/\lambda^2\big(1 + O(1/\log \lambda)\big), \qquad \lambda \to \infty \tag{22}$$

then the (probability-tail-) function

$$T(k, y) = (\log y)^{k+c^2d(h(y))/2}\, y^{-2-c^2d(h(y))}$$

from [16] *satisfies the asymptotic* ($y \to \infty$) *relation*

$$T(k, y) \asymp (\log y)^{[k+c^2a-\psi/2]/2}\, y^{-2-c^2a}. \tag{23}$$

Proof of Theorem 2: Putting $A = \{x \in \mathbf{R}^2 \colon |x| \geqq 1\}$, with the notation of [16], it holds that $A(\lambda) = \lambda A$. From [16: Theorem 1] it follows that (because of $k = 2$)

$$\Phi(\lambda A) \sim \theta_0(2\pi)^{-1} \exp\{-\lambda^2 d(\lambda)/2\}, \qquad \lambda \to \infty, \tag{24}$$

where θ_0 is some positive constant and d is some function with the properties

$$d(\lambda) \geqq \inf\{\|x\|^2 \colon |x| = 1\} = t \quad \text{and} \quad d(\lambda) \to t \quad \text{for} \quad \lambda \to \infty.$$

The relations (11) and (24) imply (22) with $a = t$ and $\psi = 2$, so that (23) is valid. Substituting (23) into [16: (11)], it follows from [16: Theorem 4] that (17) is equivalent to (18) ∎

Proof of Theorem 3: From (14) and (24) it follows that

$$\lambda^{-1} \exp\{-\lambda^2 x_0 B^{-1} x_0^T/2\} \asymp \lambda^{k-2} \exp\{-\lambda^2 d(\lambda)/2\}, \qquad \lambda \to \infty.$$

This yields (22) with $a = x_0 B^{-1} x_0^T$ and $\psi = 2(k - 1)$. Substituting (23) into [16: (11)], one gets (20) and if follows from [16: Theorem 6], that (20) implies (21) ∎

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