(1)

Riemannian Manifolds for which a Power of the Radius is k-harmonic

R. SCHIMMING

Es sei $\sigma = \sigma(x, y)$ die Syngesche Funktion einer Riemannschen Mannigfaltigkeit (M, g) beliebiger Signatur. Wir betrachten die Bedingung, daß eine bestimmte Potenz von σ oder der Logarithmus von σ k-harmonisch ist. Dann erweist sich (M, g) in vielen Fällen als flach. Bestimmte Klassen nicht-flacher Mannigfaltigkeiten können durch eine Bedingung vom genannten Typ charakterisiert werden.

Пусть $\sigma = \sigma(x, y)$ — функция Синга Риманого многообразия (M, g) произвольной сигнатуры. Рассматривается условие, что некоторая степень о или догарифм о является k -гармонической функцией. Тогда в многих случаях (M, g) оказывается плоским. Некоторые классы неплоских многообразий могут быть охарактеризованы условием названного типа.

Let $\sigma = \sigma(x, y)$ denote Synge's function of a Riemannian manifold (M, g) of any signature and consider the condition that some power of σ or the logarithm of σ is k-harmonic. Then in many cases (M, g) turns out to be flat. Certain classes of non-flat manifolds can be characterized just by a condition of the aforesaid type.

Introduction

For points x, y of a smooth n-dimensional properly Riemannian manifold (M, g) which are not too far from each other the geodesic distance $r = r(x, y)$ is defined. For fixed y and variable x we call $r = r(x, y)$ shortly the radius. Recently the problem

$$
A^{k}r^{2l}=0
$$

has been posed, i.e. the condition that some power of the radius is k -harmonic. $[2-4, 6, 7]$. Here k denotes a positive integer, l a real number, and

 $\Delta := g^{\alpha\beta} \overline{V}_{\alpha} \overline{V}_{\beta} =$ 'Laplace operator to g, $\nabla = dx^a V_a := \text{Levi-Civita derivative to } g,$ $q = q_{ab} dx^a dx^b =$ Riemannian metric, $(q^{a\beta}) := (q_{a\beta})^{-1}.$

One motivation for the problem comes from the fact that certain classes of manifolds can be characterized just by a condition (1), namely the simply harmonic spaces with $n \geq 3$ by $k = 1$, $2l = 2 - n$, and the 3-dimensional spaces of constant curvature by $k = 2$, $2l = 1$. On the one hand it is an aim to find further such characterizations, on the other hand it is to be expected that for many combinations (k, l, n) the condition (1) implies local flatness. For instance, [4] is devoted to the (still open) conjecture that this should be the case for $k = 2$, $2l = 2 - n < 0$. There

is also another motivation: for $2(k - l) = n$, $l < 0$ from (1) there follows that $A^{k-1}r^{2l}$ is an elementary solution of the Laplace equation. In general it is very difficult to calculate an elementary solution, in our special case we have an explicit ex-236 R. SCHIMNING

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 $24^{k-1}r^{2l}$ is an elementary solution of the Laplace equation. In general it is very

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1. The Riemannian metric g may be of any signature (properly or pseudo-Riemannian). The usual definition of the geodesic distance fails then; instead.Synge's will i

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r:=2\;|\sigma|^{1/2},\qquad e:=\hbox{sign}\;\sigma
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as secondary quantities. It will turn out that 'results are available especially for lorentzian g, i.e. for a signature $(+-\cdots-)$ or $(-+\cdots+)$. *.* S

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too. It is a natural completion of.(1) because

lactionary latter and the method of the new aspects. The usual definition of the geodesic distance fails then; instead if inction $\sigma = \sigma(x, y)$ is to be taken as a primary quantity and $r := 2 |\sigma|^{1/2}$, $e := \text{sign } \sigma$.

lar **III.** There are no upper bounds for the numbers k , l , n in our results. Contrarily, the literature until now concentrated to small values of *k. J.* EIGHTEDN [7] studied also general *k* for definite *g*. as secondary quantities. It will turn out that results are available esp
lorentzian g, i.e. for a signature $(+ - \cdots -)$ or $(- + \cdots +)$.

II. We will study the problem
 $A^k(\log r) = 0$

too. It is a natural completion of (1) beca **III.** There are no upper bounds for the numbers k , l , n in
the literature until now concentrated to small values of k . J
also general k for definite g .
We always consider $y \in M$ as fixed and $x \in M$ as variabl

We always consider $y \in M$ as fixed and $x \in M$ as variable; all differential operations refer to the point x. Let $N(y)$ denote a normal neighbourhood of $y \in M$; in it $\sigma = \sigma(x, y)$ is defined. Let further *4k* and $x \in M$ as fixed and $x \in M$ as variable; all differential operation of $y \in M$ as fixed and $x \in M$ as variable; all differential operation of $y \in M$; in it y) is defined. Let further $N(y)^{-} := \{x \in N(y) | \sigma(x, y) = 0\}$.

$$
N(y)^{-} := \{x \in N(y) | \sigma(x, y) = 0\}.
$$

Now we make precise the

¹³ rob1ens: *Search for quadrupels (M, g, k, 1) consisting in*

- *a manifold M' of class* C^{∞} and of dimension $n =: 2m + 2 \geq 2$,
- $-$ a Riemannian metric q over M of class C^{∞} and of any signature, $N(y)^{-} := \{x \in N(y) \mid \sigma(x, y) = 0\}.$
 $i.e., the probability function is the function of the function $y \in N$. The function $M(x, y, k, l)$ consisting in a *hidden function* $M(x, y, k, l)$ consisting in a *hidden function* $M(x, y)$ *in* $$$
- *a positive integer k*,
- *a real number,l*,

• such that for each $y \in M$

$$
\Delta^k |\sigma|^l = 0 \quad \text{in} \quad N(y)^-.
$$

This problem we will abbreviate by $\Delta^k \sigma^l = 0$.

Search also for tripels (M, g, k) such that analogously for each $y \in M$ *
* $A^k(\log |\sigma|) = 0$ *.*

$$
\Delta^k(\log|\sigma|) = 0.
$$

This problem we will abbreviate by $A^k \log \sigma = 0$.

Here we present a selection of our results:

i **1.** If $\Delta^k \sigma^l = 0$ then one of the numbers l or $l + m$ is an integer between 0 and $k - 1$ *(inclusively).* - a real number.l,

such that for each $y \in M$
 $A^k |\sigma|^l = 0$ and $N(y)^-$.

This problem we will abbreviate by $A^k \sigma^l = 0$

Search also for tripels (M, g, k) such that σ
 $A^k(\log |\sigma|) = 0$.

This problem we will abbreviate b

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1.1. If $\Delta \sigma^l=0$ then $l=0$ or $l=-m$.

1.2. If $A^2 \sigma^1 = 0$ then *l* is one of the numbers 0, 1, $-m$, 1 - *m*.

2. If Δ^k log $\sigma = 0$ then n is an even number and $n \leq 2k$.

(2)

(4)

Examples:

2.1. If Λ log $\sigma = 0$ then $n = 2$.

2.2. If Λ^2 log $\sigma = 0$ then $n = 2$ or $n = 4$.

3. From $\Lambda^k \sigma^l = 0$ there follows local flatness if one of the following additionent is *is fulfilled*:

3.1. $k = 2$, $l = 1$, 3. From $A^k \sigma^l = 0$ there follows local flatness if one of the following additional assump
ns is fulfilled:
3.1. $k = 2$, $l = 1$, $n \le 5$, g definite.
3.2. $k = 2$, $l = -m$, $n \ge 5$, g lorentzian.
3.3. $k = 2$, $l = 1 - m$, $n \$ *tions is fulfilled:* **Examples:**
 Examples:
 Examples:
 2.1. If $\Delta \log \sigma = 0$ then $n = 2$.
 2.2. If $\Delta^2 \log \sigma = 0$ then $n = 2$ or $n = 4$.
 3. *From* $\Delta^k \sigma^l = 0$ there follows local flatness if one of the f

is fulfilled:

3.1. $k = 2,$ Riemannian Manifolds with *k*-harmonic

2.1. If Λ log $\sigma = 0$ then $n = 2$.

2.2. If $\Lambda^2 \log \sigma = 0$ then $n = 2$ or $n = 4$.

3. *From* $\Lambda^k \sigma^l = 0$ there follows local flatness if one of the follows

as is fulfilled:

3.

3.1. $k = 2, l = 1, n \leq 5, g$ definite.
3.2. $k = 2, l = -m, n \geq 5, g$ lorentzian.

3.2. $k = 2, l = -m, n \geq 5, g$ lorentzian.

3.3. $k = 2, l = 1 - m, n \leq 5, g$ definite or lorentzian.

3.4. $k = 3, l = 1, n \leq 5, g$ definite, $R = 0$.

3.5. $k \geq 2, l = k - 1, n = 2$.

3.6. $4 \leq k \leq m, l + m = k - 1, n$ even, g lorentzian.

3.7. 4

3.4. $k = 3$, $l = 1$, $n \leq 5$, g definite, $R = 0$.
3.5. $k \geq 2$, $l = k - 1$, $n = 2$.
3.6. $4 \leq k \leq m$, $l + m = k - 1$, n even, g lorentzian.

3.3. $k = 2, l = 1 - m, n \leq 5, g \text{ definite or } l$

3.4. $k = 3, l = 1, n \leq 5, g \text{ definite, } R = 0,$

3.5. $k \geq 2, l = k - 1, n = 2.$

3.6. $4 \leq k \leq m, l + m = k - 1, n \text{ even, } g \text{ } l$ orentzian.

3.7. $4 \leq k \leq n - 1, l + m = k - 1, n \text{ odd, } g \text{ } l$ orentzian.

3.8. $4 \leq k \$ 3.4. $k = 3$, $l = 1$, $n \le 5$, g definite, $R = 0$,

3.5. $k \ge 2$, $l = k - 1$, $n = 2$.

3.6. $4 \le k \le m$, $l + m = k - 1$, n even, g lorentzian,

3.7. $4 \le k \le n - 1$, $l + m = k - 1$, n odd, g lorentzian,

3.8. $4 \le k \le m + 1$, $l + m$ *3.5.* $k \geq 2, l = k - 1, n = 2$.
 3.6. $4 \leq k \leq m, l + m = k - 1, n$ even, g lorentzian,
 3.7. $4 \leq k \leq n - 1, l + m = k - 1, n$ odd, g lorentzia
 3.8. $4 \leq k \leq m + 1, l + m = k - 2, n$ even, g lorentzia
 3.9. $k \geq 4, l + m = k - 2, n$ odd, g lor

4. $k \geq 4$, $l + m = k - 2$, *n* odd, *g* lorentzian, $R = 0$.
 4. From $\Delta^k \log \sigma = 0$ there follows local flatness if one of the following additional *assumptions is fulfilled:* 3.3. $k = 2, i = 1 - m, n \leq 5, g \text{ } d \text{ } e \text{ } f \text{ } d \text{ } 3.4. \quad k = 3, l = 1, n \leq 5, g \text{ } d \text{ } e \text{ } f \text{ } n \text{ } 3.5. \quad k \geq 2, l = k - 1, n = 2.$

3.5. $k \geq 2, l = k - 1, n = k$.

3.8. $4 \leq k \leq m, l + m = k - 1,$

3.7. $4 \leq k \leq m + 1, l + m = k$.

3.9. $k \geq$

4.1. $k \leq 2, n = 2.$

4.2. $2 \leq k \leq m + 1$, g lorentzian.

4.3. $3 \leq k \leq m + 2$, g definite.

4.4. $k = m + 1$, $n \geq 4$, g definite or lorentzian.

3.1. $k = 2, i = 1, n \geq 5, g \,\text{derivative}.$

3.2. $k = 2, l = -m, n \geq 5, g \,\text{derivative or } \text{fore} \,\text{arrows}.$

3.3. $k = 2, l = 1 - m, n \leq 5, g \,\text{definite or } \text{fore} \,\text{arrows}.$

3.4. $k = 3, l = 1, n \leq 5, g \,\text{definite, } R = 0.$

3.5. $k \geq 2, l = k - 1, n = 2.$

3.6. $4 \leq k \leq m, l + m = k -$ **5.** *There exist non-flat manifolds of any dimension n* \geq 4 (namely simply harmonic *manifolds*) satisfying for any $k \Delta^k \sigma^{k-1} = 0$ and $\Delta^k \sigma^{k-m-1} = 0$.

6. There exist non-flat manifolds of any even dimension $n \geq 4$ (namely simply har-

2.6. $4 \le k \le m, l + m = k - 1, n$ even, g lorentzian.
 3.7. $4 \le k \le m - 1, l + m = k - 1, n$ odd, g lorentzian.
 3.8. $4 \le k \le m + 1, l + m = k - 2, n$ even, g lorentzian, $R = 0$,
 3.9. $k \ge 4, l + m = k - 2, n$ odd, g lorentzian, $R = 0$,
 4. 7. There exist non-flat manifolds of any dimension $n \geq 3$ and of any non-definite signature.(namely generalizations of the plane gravitational waves) satisfying $A^2\sigma = 0$. **5.** There exist non-flat manual folds) satisfying for any
6. There exist non-flat manual for any
6. There exist non-flat manual monic manifolds) satisfying Δ
7. There exist non-flat manual manual for Δ
8. T **5.** There exist non-flat manifolds of any dimension $n \geq 4$ (namely simply harmonic unifolds) satisfying for any $k \Delta^k \sigma^{k-1} = 0$ and $\Delta^k \sigma^{k-m-1} = 0$.
 6. There exist non-flat manifolds of any even dimension $n \geq$ 7. There exist non-flat manifolds of any dim
signature (namely generalizations of the plane g
8. The 3-dimensional manifolds of constant c
terized by $A^2 |\sigma|^{1/2} = 0$.
9. If $\Delta^k |\sigma|^{k-m-1} = 0$ and $\binom{m}{k} \neq 0$ then Δ

S. The 3-dimensional manifolds of constant curvature (of any signature) are charac-

ized by $\Delta^2 |\sigma|^{1/2} = 0$.

9. If $\Delta^k |\sigma|^{k-m-1} = 0$ and $\binom{m}{k} \neq 0$ then $\Delta^{k-1} |\sigma|^{k-m-1}$ is an elementary solution of
 i. Laplace equation. This is logarithm-free for even *n*.

10. If $\Delta^{m+1}(\log |\sigma|) = 0$ and

the Laplace equation. This is logarithm-free for even n.

Let us shortly review results in the literature on the problem $A^k \sigma^l = 0$ for the properly Riemannian case: R. CADDEO [2] proved that $A^2r = 0$ if and only if $n = 1$ or if (M, g) is a 3-dimensional manifold of constant curvature. R. CADDEO and P. MATZEU [3] showed that $A^{2r-1} = 0$ if and only if (M, q) is locally flat or is a 3-dimensional manifold of constant curvature. In the papers [3, 4] the problems' $A^{2}r^{2} = 0$ and $A^{2}r^{2-n} = 0$ are considered. Necessary conditions are derived; under certain additional assumptions these conditions imply local flatness. Especially, R. CADDEO and L. VANHECKE [4] have shown that an odd-dimensional (M, g) satisfying $A^{2}r^{2-n} = 0$ is a harmonic manifold. J. EICHHORN [6] proved that a harmonic, manifold satisfying $A^{2}r^2 = 0$ is simply harmonic, which implies local flatness. In [7] he initiated the consideration of arbitrary large *k* and became 'able to discuss the Laplace equation. This is logarithm-free for even n

10. If $\Delta^{m+1}(\log |\sigma|) = 0$ and $n \ge 4$ is even then \angle

mentary solution of the Laplace equation.

Let us shortly review results in the literature of

properly Rie Let us shortly review results in the literature on the problem $A^4\sigma^l = 0$ operly Riemannian case: R. CADDEO [2] proved that $A^2r = 0$ if and only if (M, g) is a 3-dimensional manifold of constant curvature. R. CADDIMime

The three problems to characterize all Riemannian manifolds *(M, g)* which

 $\dot{=}$ satisfy $\Delta^k \sigma^i = 0$ or $\Delta^k \log \sigma = 0$,

— are harmonic ones,

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respectively have much in common. They are of comparable difficulty and none of the problems is solved' yet. Possibly, future progress in one of the problems will influence the other two'problems.

Our main tool will be a version of the "method of coincidence limits", which has been initiated-by J. L. SYNGE [13], combined with the calculus of symmetric differential forms. It 'is to be noted that the method is not yet exhausted by the results which are presented here. respectively have much in common. They are of comparable difficulty and of the problems is solved yet. Possibly, future progress in one of the problems influence the other two problems.

Our main tool will be a version of *U* = *U*P *=*

I wish to express my thanks to A. GRAY and to J. EICHHORN for suggesting the problem and to the authors of $[2-4]$ for giving me insight into their work.

A symmetric differential form of degree p

$$
u = u_p = u_{\alpha_1 \alpha_2 \ldots \alpha_p} dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p}
$$

is a special notation for, a.symmetric covariant tensor field of degree *p* with local which are presented here.

I wish to express my thanks to A. GRAY and to J. EICHHORN for sugg

problem and to the authors of $[2-4]$ for giving me insight into their work

§ 1. Symmetric differential forms. The two-point *x').* Apart from the usual tensorial operations there are specific operations for symmetric $\begin{array}{r}\n \text{is a sp} \\
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 \text{from } 1 \\
 \text{forms:} \\
 1.8\n \end{array}$ and to the authors of $[2-4]$ for giving me i
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 metric differential forms. The two-point functions σ and μ
tric differential form of degree p
 $u = u_p = u_{a_1a_1...a_p} dx^{a_1} dx^{a_2} \dots dx^{a_p}$
ial notation for a symmetric covariant tensor field of degree p with loc
that

forms:
1. Symmetric product of a.p-form u_p and a q-form v_q

$$
u_n v_n := u_{\alpha_1 \ldots \alpha_n} v_{\beta_1 \ldots \beta_n} dx^{\alpha_1} \ldots dx^{\alpha_p} dx^{\beta_1} \ldots dx^{\beta_q}.
$$

2. Trace = tr with respect to the metric
$$
g
$$

\ntr $u_p := g^{a\beta} u_{\alpha\beta\sigma_1... \alpha_p} dx^{\alpha_1} \dots dx^{\alpha_p}$ for $p \ge 3$,
\ntr $u_0 := 0$, tr $u_1 := 0$, tr $u_2 := g^{a\beta} u_{\alpha\beta}$.

$$
\operatorname{tr} u_0 := 0, \quad \operatorname{tr} u_1 := 0, \quad \operatorname{tr} u_2 := g^{\alpha \beta} u_{\alpha \beta}.
$$

3. Trace-free part u_p of u_p with respect to g. For $p \leq 4$ one has

forms:
\n1. Symmetric product of a
$$
p
$$
-form u_p and a q -form v_q
\n $u_p v_q := u_{a_1...a_p} v_{\beta_1... \beta_q} dx^{a_1} ... dx^{a_p} dx^{\beta_1} ... dx^{\beta_q}$.
\n2. Trace = tr with respect to the metric g
\ntr $u_p := g^{a\beta} u_{a\beta a_1...a_p} dx^{a_1} ... dx^{a_p}$ for $p \ge 3$,
\ntr $u_0 := 0$, tr $u_1 := 0$, tr $u_2 := g^{a\beta} u_{a\beta}$.
\n3. Trace-free part $-u_p$ of u_p with respect to g . For $p \le 4$ one has
\n $-u_0 = u_0$, $-u_1 = u_1$, $-u_2 = u_2 - \frac{1}{n} g$ tr u_2 ,
\n $(n+2) (-u_3 - u_3) = -3g \cdot tr u_3$,
\n $(n+2) (n+4) (-u_4 - u_4) = -6(n+2) g \cdot tr u_4 + 3g^2 \cdot tr^2 u_4$.
\n4. Symmetric differential d built by means of V
\n $du_p := V_a u_{a_1...a_p} dx^a dx^{a_1} ... dx^{a_p}$.
\nLateron we will make use of the formula
\n $(p+2) (p+1) tr^k (gu_p)$

$$
(n+2)(n+4)(-u_4-u_4)=-6(n+2)g\cdot\operatorname{tr} u_4+3g^2\cdot\operatorname{tr}^2 u_4.
$$

$$
du_n := \mathcal{V}_n u_{\alpha_1 \ldots \alpha_n} dx^{\alpha} dx^{\alpha_1} \ldots dx^{\alpha_p}.
$$

Lateron we will make use of the formula

$$
(p+2)(p+1)\,\text{tr}^k(gu_p)
$$

 $=4k(m+p-k+2)$ tr^{k-1} $u_p+(p+2-2k)(p+1-2k)$ q tr^k u_p . (1.1) Lateron we will make use of the formula
 $(p + 2) (p + 1) \text{ tr}^k (gu_p)$
 $= 4k(m + p - k + 2) \text{ tr}^{k-1} u_p + (p - k)$

We choose the notations and conventions
 $Riem = R_{n} (dx^n \wedge dx^p) (dx^n \wedge dx^n) =$

4. Symmetric differential d built by means of
$$
\bar{V}
$$
\n
$$
du_p := \bar{V}_a u_{a_1...a_p} dx^a dx^{a_1} \ldots dx^{a_p}.
$$
\nLateron we will make use of the formula

\n
$$
(p + 2)(p + 1) \operatorname{tr}^k (gu_p)
$$
\n
$$
= 4k(m + p - k + 2) \operatorname{tr}^{k-1} u_p + (p + 2 - 2k) (p + 1 - 2k) g \operatorname{tr}^k
$$
\nWe choose the notations and conventions

\n
$$
Riem = R_{a\beta\mu\nu}(dx^a \wedge dx^\beta) (dx^\mu \wedge dx^\nu) = \text{curvature tensor},
$$
\n
$$
(\bar{V}_a \bar{V}_\beta - \bar{V}_\beta \bar{V}_a) v^\mu =: R_{a\beta}{}^\mu v^\nu \text{ for vector fields } v = v^a \partial_a,
$$
\n
$$
Ric = R_{a\beta} dx^a dx^\beta = \text{Ricci tensor} := g^{\mu\nu} R_{a\mu\nu} dx^a dx^\beta,
$$
\n
$$
R = \text{scalar curvature} := g^{a\beta} R_{a\beta},
$$
\n
$$
Weyl = W_{a\beta\mu\nu}(dx^a \wedge dx^\beta) (dx^\mu \wedge dx^\nu) = \text{conformal curvature tensor.}
$$

For quadratic expressions in the curvature we adopt special notations:

$$
\begin{aligned}\n\text{Riemannian Manifolds with } k \cdot \text{} \\
\text{dratic expressions in the curvature we adopt} \\
|Ric|^2 &:= R_{\alpha\beta} R^{\alpha\beta}, \qquad |Riem|^2 := R_{\alpha\beta\mu\tau} R^{\alpha\beta\mu\tau}, \\
(Riem)^2 &:= R_{\alpha\beta\alpha} R_{\alpha\beta\sigma}^{\alpha\beta} dx^{\alpha\beta} dx^{\beta\sigma} dx^{\nu}.\n\end{aligned}
$$

$$
(Riem)^2:=R_{\alpha\lambda\varrho\beta}R_{\mu}^{\ \lambda\varrho},dx^{\alpha}dx^{\beta}dx^{\mu}dx^{\nu}.
$$

(Riemannian Manifold

ratic expressions in the curvature w
 $|Ric|^2 := R_{\alpha\beta}R^{\alpha\beta}$, $|Riem|^2 := R_{\alpha\beta\beta}$
 $(Riem)^2 := R_{\alpha k\beta}R_{\mu}^{\ \lambda\varrho}$, $dx^{\alpha} dx^{\beta} dx^{\mu} dx^{\nu}$.

sition 1.1: A lorentzian metric g

lat. **Proposition 1.1:** *A lorentzian metric q with* $n \geq 4$ and $\lceil (Weyl)^2 \rceil = 0$ *is conformally flat.*

EXECUTE:

At the anisometric system is the curvature we adopt special notations:
 $|Ric|^2 := R_a \beta R^a{}^{\beta}$, $|Riem|^2 := R_{a\beta\mu}R^{\beta\mu}$,
 $(Riem)^2 := R_{a\lambda\beta}R^{\lambda\beta}$, $dx^a dx^b dx^a$,
 $Riem)^2 := R_{a\lambda\beta}R^{\lambda\beta}$, $dx^a dx^b dx^a dx^b$.
 \therefore Prop A proof follows from a close inspection of the arguments which are given by A. LICHNEROWICZ and A. G. WALKER in [10], and also slightly more explicit by **Example 18.** Riemannian Manifolds with k-harmonic Power of Radius 239

For quadratic expressions in the curvature we adopt special notations:
 $|Ric|^2 := R_{a\beta}R^{a\beta}$, $|Riem|^2 := R_{a\beta\mu}R^{a\beta\mu}$,
 $(Riem)^2 := R_{a\lambda\beta}R_{\mu}^{b\beta}dx^a dx$ Thus the proposition may be considered as known, though to our knowledge not explicitly stated before. Note that similar $-$ but not identical $-$ arguments occur in the context of the Bel-Robinson tensor in general relativity $[1, 12]$ *i* $[Re|^2 := R_{a\beta}R^{a\beta}, \qquad |Riem|^2 := R_{a\beta\mu}R^{a\beta\mu},$
 $(Riem)^2 := R_{a\beta\mu}R^{a\beta}, dx^a dx^b dx^c dx^c$.
 Proposition 1.1: *A lorentzian metric g with* $n \ge 4$ and $-(Weyl)^2 = 0$ as coformally flat.

A proof follows from a close inspection of H. S. RUSE, A. G. WALKER and T. J. WILLER

H. S. RUSE, A. G. WALKER and T. J. WILLER

Thus the proposition may be considered as

explicitly stated before. Note that similar –

in the context of the Bel-Robinson tensor in
 proposition may be considered as known, thought is the stated before. Note that similar — but not identext of the Bel-Robinson tensor in general relative sition 1.2: A lorentzian metric g with $n \geq 3$ and -1 tant curva

Proposition 1.2: *A lorentzian metric g with* $n \geq 3$ *and* $Ric = 0$ *and* $-(Riem)^2 = 0$ *of constant curvature.*

Proof: For $n = 3$ the assertion follows from $-Ric = 0$. For $n \geq 4$ we apply Prosition 1.1. Namely, from Ric Proof: For $n = 3$ the assertion follows from $Ric = 0$. For $n \ge 4$ we apply Proposition 1.1. Namely, from $Ric = 0$ there follows

$$
(Riem)^2 = (Weyl)^2 + (n-1)K^2g^2 \quad \text{with} \quad n(n-1)K := R,
$$

and as a conclusion $-(Riem)^2 = -(Weyl)^2$

Definition 1.1: Let $N(y)$ be a normal neighbourhood of $y \in M$, i.e. the expo-

is a diffeomorphism out of the tangential space of y onto $N(y)$ or, equivalently, $N(y)$. $\exp_y : x^* \mapsto x$

comorphism out of the tangential space

main of a normal coordinate system
 $x \mapsto x^* \mapsto (x^{*a}) \equiv (x^{*1}, x^{*2}, ..., x^{*n}).$

is the domain of a normal coordinate system

$$
x\mapsto x^*\mapsto (x^{*\alpha})\equiv (x^{*\alpha}, x^{*\alpha}, \ldots, x^{*\alpha}).
$$

The quantity

1.1. Namely, from
$$
-Ric = 0
$$
 there follows\n $(Riem)^2 = (Weyl)^2 + (n-1) K^2g^2$ with $n(n-1) K := R$, conclusion $-(Riem)^2 = -(Weyl)^2$ \n\nIition 1.1: Let $N(y)$ be a normal neighbourhood of $y \in M$, i.e. the exponential distribution is given by $x \leftrightarrow x$ and $y \leftrightarrow x^* \rightarrow x^*$. The differential space of y onto $N(y)$, or, equivalently, $N(y)$ and $x \leftrightarrow x^* \mapsto (x^*x) \equiv (x^*1, x^*2, ..., x^*n)$.\n\nIt is given by $x \leftrightarrow x$ and $y \leftrightarrow x$ and <math display="</p>

is called *Synge's two-point function*. From σ there are derived

$$
e = e(x, y) := \text{sign } \sigma(x, y), \qquad \mu = \mu(x, y) := \frac{1}{2} (\Delta \sigma - n).
$$
 (1.3)

The two-point scalar fields σ and μ are ingredients of the "method of coincidence" limits" which is due to J. L. SYNGE [13]. The limit for $x \rightarrow y$, if existing, of a twopoint quantity is called *coincidence limit*. The equality of the coincidence limits is an equivalence relation of two-point quantities and will be denoted by \triangleq . One-point quantities and constants may be looked upon as special two-point quantities. The two-point scalar fields σ and μ are in
inits" which is due to J. L. SYNGE [13]. T
int quantity is called *coincidence limit*. T
equivalence relation of two-point quantit
antities and constants may be looked up
L *• • •Coincidence limits:* point function. From σ there are derived

= sign $\sigma(x, y)$, $\mu = \mu(x, y) := \frac{1}{2} (A\sigma - n)$. (

ar fields σ and μ are ingredients of the "method of coincide

to J. L. SYNGE [13]. The limit for $x \rightarrow y$, if existing, of a *point junction.* From σ there are derived
 $:=$ sign $\sigma(x, y)$, $\mu = \mu(x, y) := \frac{1}{2} (A\sigma - n)$. (1.3)

lar fields σ and μ are ingredients of the "method of coincidence

to J. L. SYNGE [13]. The limit for $x \rightarrow y$, if ex

Let us recall some properties of σ and μ following [9, 13, 11, 12]:

$$
Symmetry: \sigma(x, y) = \sigma(y, y).
$$

aience relation of two-point quantities and win be
is and constants may be looked upon as special to
recall some properties of
$$
\sigma
$$
 and μ following [9, 1
letry: $\sigma(x, y) = \sigma(y, x)$.
dence limits:
 $\sigma = 0$, $\vec{V}_a \sigma = 0$, $\vec{V}_a \vec{V}_\beta \sigma = g_{\alpha\beta}$,
 $\vec{V}_a \vec{V}_\beta \vec{V}_\gamma \sigma = 0$, $-3 \vec{V}_a \vec{V}_\beta \vec{V}_\gamma \vec{V}_\delta \sigma = R_{\gamma\alpha\beta\delta} + R_{\delta\alpha\beta\gamma}$.

R. SCHIMMING

Differential equation: $q^{\alpha\beta}V_{\alpha}\sigma V_{\beta}\sigma = 2\sigma$. Ledger's formula:

$$
-(p+1) d^p\sigma_{\alpha\beta} = p(p-1) d^{p-2}(R_{\gamma\alpha\beta\delta} dx^{\gamma} dx^{\delta}) + \sum_{q=2}^{p-2} {p \choose q} d^q\sigma_{\alpha\gamma} d^{p-q}\sigma_{\beta}^{\gamma} \quad (1.5)
$$

for $p \ge 2$, where the sum for $p = 2$ and for $p = 3$ is to be taken as zero and where we abbreviate $\sigma_{\alpha} := \bar{V}_{\alpha}\sigma, \ \sigma_{\alpha\beta} := \bar{V}_{\alpha}\bar{V}_{\beta}\sigma.$

Conclusions:

$$
d\sigma \triangleq 0, \qquad d^2\sigma \triangleq g, \qquad d\sigma^2 = dx^2, d^p\sigma \triangleq 0 \quad \text{for} \quad p \ge 3, \qquad d^p\sigma^2 \triangleq 0 \quad \text{for} \quad p \ge 2.
$$
 (1.6)

Trace version of Ledger's formula:

$$
-2(p+1) d^{p}\mu \triangleq p(p-1) d^{p-2} Ric + \sum_{q=2}^{p-2} {p \choose q} d^{q} \sigma_{\alpha\beta} d^{p-q} \sigma^{\alpha\beta}.
$$
 (1.7)

for $p \geq 2$. Conclusions:

$$
\mu = 0, \quad d\mu = 0, \quad -3d^2\mu = Ric,
$$

\n
$$
-4d^3\mu = 3d Ric, \quad -15d^4\mu = 18d^2 Ric + 4(Riem)^2,
$$

\n
$$
-3\Delta\mu = R, \quad -2d\Delta\mu = dR,
$$

\n
$$
-15d^2\mu = 12\Delta R + 2(|Riem|^2 - |Ric|^2).
$$
\n(1.8)

Special version of Ledger's formula for $n = 2$:

$$
-2(p+1) dp \mu \triangleq {p \choose 2} g d^{p-2} R + 4 dp \mu2.
$$
 (1.9)

We proceed with some technical preparations.

Proposition 1.3: For a regular two-point function $f = f(x, y)$

$$
\varDelta(|\sigma|^l f) = 2 |\sigma|^{l-1} D_l f \tag{1.10}
$$

with the linear differential expression of second order

$$
D_{i}f := l(l+m+ \mu)f + l\sigma^{2}V_{\bullet}f + \frac{1}{2} \sigma \Delta f.
$$
 (1.11)

Proof: Insert $f_1 = |\sigma|^l$, $f_2 = f$ into the product rule

$$
\Delta(f_1 f_2) = (\Delta f_1) f_2 + f_1 \Delta f_2 + 2g^{\alpha \beta} V_{\alpha} f_1 V_{\beta} f_2
$$

and use $e\Delta |\sigma|^l = 2 |\sigma|^{l-1} l(l+m+\mu)$

Proposition 1.4: In the coincidence limit

$$
d^p D_l f = l \sum_{r=2}^p {p \choose r} d^r \mu d^{p-r} f + l(l+m+p) d^p f + \frac{1}{2} {p \choose 2} g d^{p-2} \Delta f \ \text{for} \ \ p \geq 2
$$
\n(1.12)

 and

$$
-d^{p} A^{q} D_{i} f \doteq st^{-} d^{p} A^{q} f + l(-d^{p} A^{q} \mu) f + \cdots
$$

for $p \ge 2$, $s := l + q$, $t := l + m + p + q$, (1.13)

where \ldots indicates terms which are of a differential order in f greater than 0 and less than $p + 2q$.

The proof of (1.12) makes use of (1.6) , (1.8) . For the proof of (1.13) replace p in (1.12) by $p + 2q$ and apply the iterated trace operator tr^q with the help of (1.1):

Riemannian Manifolds with k-harmonic Power of Radius
\nThe proof of (1.12) makes use of (1.6), (1.8). For the proof of (1.13) replace
\n(1.12) by
$$
p + 2q
$$
 and apply the iterated trace operator \mathbf{t}^{q} with the help of (1.
\n $\mathbf{t}^{q} d^{p+2q} D_{l} f = d^{p} d^{q} D_{l} f + \cdots,$
\n
$$
\begin{pmatrix} p + 2q \\ 2 \end{pmatrix} - \mathbf{t}^{rq} (qd^{p+2q-2} d f) = 2q(m + p + q)^{-} \mathbf{t}^{rq-1} (d^{p+2q-2} d f) + \cdots
$$
\n
$$
= 2q(m + p + q)^{-} d^{p} d^{q} f + \cdots.
$$
\nHere \cdots indicates terms which do not contribute to the expressions (1.13). The
\nefficient of $\neg d^{p} d^{q} f$ becomes
\n $l(l + m + p + 2q) + q(m + p + q) = (l + q) (l + m + p + q) = st$
\n8.2. Derivation of the necessary conditions
\nProposition 2.1: Define two-point scalar fields $f_{k}^{l} = f_{k}^{l}(x, y)$ recursively
\nrespect to $k = 0, 1, 2, \ldots$ by
\n $f_{0}^{l} := 1, \quad f_{k+1}^{l} := D_{l-k} f_{k}^{l}.$
\nThen
\n $(e d)^{k} | \sigma|^{l} = 2^{k} | \sigma|^{l-k} f_{k}^{l}.$
\nThe proof is done by mathematical induction with respect to k and by me:
\n(1.10) **1**
\nConclusion: $d^{k} \sigma^{l} = 0$ if and only if $f_{k}^{l} = 0$.
\nExamples:
\n $f_{1}^{l} = l(l + m + \mu),$

Hereindicates terms which do not contribute to the expressions (1.13). The coefficient of $\frac{d^p A^{q}}{dp}$ becomes

§ 2. Derivation of the necessary conditions

l(l±mp+2q)+q(m+p+q)=(i±q)(l+m+p+q)St **^I** *11._I II* 7-i *II* Proposition 2.1: Define two-point scalar fields $f_k^i = f_k^i(x, y)$ recursively with *respect to* $k = 0, 1, 2, ...$ *by*

$$
f_0^l := 1, \qquad f_{k+1}^l := D_{l-k} f_k^l. \tag{2.1}
$$

$$
(e\Delta)^k \left| \sigma \right|^l = 2^k \left| \sigma \right|^{l-k} f_k^l. \tag{2.2}
$$

The proof is done by mathematical induction with respect to k and by means of $l(l+m+p+2q) + q(m+p+q) = (l+q)(l+m+p+q) = st$

§ 2. Derivation of the necessary conditions

Proposition: 2.1: Define two-point scalar fields $f_k^l = f_k^l(x, y)$ recursively

respect to $k = 0, 1, 2, ...$ by
 $f_0^l := 1, \quad f'_{k+1} := D_{l-k}f_k^l$.

Th 2. Derivation of the necessary conditions

roposition 2.1. Define two-point scalar fields $f_k^1 = f_k^1(x, y)$ recursively with

repeat to $k = 0, 1, 2, ...$ by
 $f_0^1 := 1, \quad f_{k+1}^1 := D_{l-k}f_k^1$.

(ed)* $|\sigma|^1 = 2^k |\sigma|^{1-k} f_k^1$.

(2. */1 (i)* α *1 (i)* α *1 (i) 1 1 1 <i>1 t* $\begin{align*}\n&= f_k^{-1}(x, y) \quad \text{recursively} \quad \text{with} \quad \text{(2.1)}\n\end{align*}$

spect to k and by means of
 $\sigma^{\alpha} \overline{V}_{\alpha} \mu + \frac{1}{2} l \sigma \Delta \mu. \qquad \text{(2.4)}\n\end{align*}$ reposition 2.1: *Define two-point seasar figures* $f_k = f_k$

reposition 2.1: *Define two-point seasar figures* $f_k = f_k$

reposition $f_0^{l} := 1$, $f_{k+1}^{l} := D_{l-k}f_k^{l}$.

The proof is done by mathematical induction with respe (2.1)

(2.2)

ion with respect to k and by means of

0.

(2.3)
 $(1 + \mu) + \sigma^{\alpha} \sqrt{\mu} + \frac{1}{2} \ln \sqrt{\mu}.$ (2.4)

(2.5) *and with these there holds - - -*

- - f **:=** limf'/l *for k >* 1 *x* $A^k \sigma^l = 0$ *if and only if* $f_k^l = 0$.

ples:
 $f_1^l = l(l + m + \mu)$,
 $f_2^l = l(l - 1) [(l + m + \mu) (l + m - 1 + \mu)]$

sition 2.2: There exist the limits
 $f_k := \lim_{l \to 0} f_k^l / l$ for $k \ge 1$

these there holds
 $A^k(\log |\sigma|) = 2^k \sigma^{-k} f_k$.

$$
f_1^{\ \ l} = l(l+m+\mu), \tag{2.3}
$$

$$
u1u3sin: 2nσn = 0 ij and only ij Jkr = 0.
$$
\n
$$
f1t = l(l + m + μ),
$$
\n
$$
f2t = l(l - 1) [(l + m + μ) (l + m - 1 + μ) + σαΓαμ] + $\frac{1}{2}$ lσΔμ. (2.4)
$$

$$
f_k := \lim_{l \to 0} f_k^l / l \quad \text{for} \quad k \ge 1 \tag{2.5}
$$

$$
\Delta^k(\log|\sigma|) = 2^k \sigma^{-k} f_k. \tag{2.6}
$$

Jk₀¹ = 0 *if* and only *if* $f_k^1 = 0$.

ples:
 $f_1^1 = l(l+m + \mu)$,
 $f_2^1 = l(l-1) [(l+m+\mu) (l+m-1+\mu) + \sigma^a \bar{V}_a \mu] + \frac{1}{2} l \sigma \Delta \mu$. (2.4)

sition 2.2: There exist the limits
 $f_k := \lim_{t \to 0} f_k^1/l$ for $k \ge 1$

these there hold Proof: Mathematical induction with respect to k shows that the sequence of functions which is recursively defined by $f_1^l = l(l+m+\mu),$
 $f_2^l = l(l-1)[(l+m+\mu)(l+m-1+\mu)$

Proposition 2.2: There exist the limits
 $f_k := \lim_{l\to 0} f_k^l/l$ for $k \ge 1$

and with these there holds
 $A^k(\log |\sigma|) = 2^k \sigma^{-k} f_k$.

Proof: Mathematical induction with respect to him μ] + $\frac{1}{2}$ $l\sigma\Delta\mu$. (2.4)

(2.5)

(2.6)

that the sequence of

(2.7) f_u¹ = $l(l + m + \mu)$,

f₂ⁱ = $l(l - 1)$ [$(l + m + \mu)$ $(l + m - 1 + \mu) + \sigma^{\alpha} V_{\alpha} \mu$] + $\frac{1}{2}$ lod μ .

Proposition 2.2: There exist the limits

f_k := $\lim_{l \to 0} f_k^l |l$ for $k \ge 1$

and with these there holds
 $A^k(\log |\sigma|)$

$$
f_1 := m + \mu, \qquad f_{k+1} := D_{-k} f_k \tag{2.7}
$$

-Proposition 2.3: *Ii the coincidence limit - - - . -. -- - - (2.8) 4 k!(k -* **1)!** (71). **(2.9)**

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Proof: By passing to the coincidence limits the differential recursions (2.1) shrink 242 R. SCHIMMING

Proof: By passing to the coincidence limits the differential recursions (2.1) shrin

to the algebraic recursions
 $f_0^l = 1$, $f_{k+1}^l = (l - k) (l + m - k) f_k^l$,
 $f_1 = m$, $f_{k+1} = -k(m - k) f_k$, *z*
 to the coincidence limits the differential recursions (2.1) s
 *f*_{k+1} $\stackrel{!}{=}$ (*l* - *k*) (*l* + *m* - *k*) *f*_k^{*t*},
 *f*_{k+1} $\stackrel{?}{=}$ - *k*(*m* - *k*) *f*_k,

just by (2.8), (2.9)

also write

2 R. SCHIMMING
\nProof: By passing to the coincidence limits the d
\nthe algebraic recursions
\n
$$
f_0^l = 1, \quad f_{k+1}^l \geq (l-k) (l+m-k) f_k^l,
$$
\n
$$
f_1 \doteq m, \quad f_{k+1} \doteq -k(m-k) f_k,
$$
\nd these are solved just by (2.8), (2.9)

and these are solved just by (2.8) , (2.9)

R. SCHIMMING
\n
$$
f: By passing to the coincidence limits the differential recursions (2.1) shrink gebrai c recursions\n
$$
f_0^l = 1, \t f_{k+1}^l = (l - k) (l + m - k) f_k^l,
$$
\n
$$
f_1 = m, \t f_{k+1} = -k(m - k) f_k,
$$
\n
$$
\text{where } k \geq 1 \text{ one can also write}
$$
\n
$$
f_k^l = l(l - 1) \dots (l - k + 1) (l + m) (l + m - 1) \dots (l + m - k + 1),
$$
\n
$$
f_k^l = (-1)^k (k - 1)! m(m - 1) \dots (m - k + 1).
$$
\n
$$
(2.10)
$$
\n
$$
f_k = (-1)^k (k - 1)! m(m - 1) \dots (m - k + 1).
$$
\n
$$
(2.11)
$$
\n
$$
f_k = 1 \dots (l + 1) \dots (l + m - k + 1).
$$
\n
$$
f_k^l = 0 \text{ then the power exponent } l \text{ is an integer with } 0 \leq l \leq k - 1
$$
\n
$$
l \geq 0 \text{ then the power exponent } l \text{ is an integer with } 0 \leq l \leq k - 1
$$
\n
$$
l \geq 0 \text{ then the dimension } n
$$
$$

$$
f_k = (-1)^k (k-1)! \, m(m-1) \dots (m-k+1). \tag{2.11}
$$

As a direct conclusion we get the

Theorem 2.1: If $\Delta^k \sigma^l = 0$ then the power exponent l is an integer with $0 \leq l \leq k - 1$ Theorem 2.1: *If* $\Lambda^k \sigma^l = 0$ then the power exponent *l* is an integer with $0 \leq l \leq k - 1$
or $l + m$ is an integer with $0 \leq l + m \leq k - 1$. *If* $\Lambda^k \log \sigma = 0$ then the dimension *n is even and* $2 \leq n \leq 2k$. *is an integer*
d $2 \le n \le 2$
sition 2.4:
 $-d^p \Delta^q f_{k+1}^{\perp} =$ (c) $k-1$)! $m(m-1) \ldots (m-k+1)$.

(we get the
 $k\sigma^l = 0$ then the power exponent l is an integer with $0 \le l+m \le k-1$. If $\varDelta^k \log \sigma = 0$ then the αk .

In the coincidence limit

(s - k) $(t-k)$ $\negthinspace d^p \varDelta^q f_k^l + (l-k)$ $\negthinspace d^p$

Proposition 2.4: *In the coincidence limit*

$$
-d^{p} \Delta^{q} f_{k+1}^{l} \triangleq (s-k) (t-k) - d^{p} \Delta^{q} f_{k}^{l} + (l-k) (-d^{p} \Delta^{q} \mu) f_{k}^{l} + \cdots,
$$

for $p, q \ge 0$, $s := l + q$, $t := l + m + p + q$, (2.12)

where ... indicates terms which are of a differential order in /' greater than 0 *and less than p + 2q. For p + 2q* \leq *3 and for p + 2q = 4, Ric = 0 these residual terms vanish.*

Proof: Proposition 1.4 is applied to the present situation; *1* is to be replaced by $1 - k$. For $p + 2q = 4$ the individual terms are to be inspected. Especially, one vanish.

Proof: Proposition 1.4 is applied to the $l - k$. For $p + 2q = 4$ the individual the obtains inductively $df_k^l = 0$ for all k, l. 2.4: In the coincidence limit
 $+1 = (s - k) (t - k)^{-d} 2^{d} t^{l} + (l - k) (-d^{p} 4^{q} \mu) t^{l} + \cdots$
 $1 \ge 0, s := l + q, t := l + m + p + q,$ (2.12)

es terms which are of a differential order in t^{l} greater than 0 and less

or $p + 2q \le 3$ and for *compary which are of a differential order in* f_k^l *greater than 0 and less*
 $p + 2q \leq 3$ and for $p + 2q = 4$, Ric = 0 these residual terms

on 1.4 is applied to the present situation; *l* is to be replaced by
 $d f_k^l \stackrel$

Proposition 2.5: *Define numbers* $c_k^{\mu} = c_k^{\mu}(p, q)$ by

$$
c_k^l := \sum_{r=1}^k \left[(k-r)! \right]^2 \binom{s-r}{k-r} \binom{t-r}{k-r} (l-r+1) f_{r-1}^l. \tag{2.13}
$$

(Take the coincidence limit of f_{r-1}^l *on the right-hand side.) Then for* $1 \leq p + 2q \leq 3$ *and for* $p + 2q = 4$, Ric = 0 obtains inductively
 Proposition 2.5
 $c_k^i := \sum_{r=1}^k [i]$

(Take the coincidence

and for $p + 2q = 4$,
 $-d^p \Delta^q f_k^i =$

Proof: By ignorin

$$
-d^p \Delta^q f_k^l \doteq c_k^l(p,q) - d^p \Delta^q \mu. \tag{2.14}
$$

 $-d^p A^q f_k^l \doteq c_k^l(p, q) \cdot d^p A^q \mu.$ (2.14)

Proof: By ignoring the residual terms ... the system (2.12) becomes an algebraic

cursion system for the coincidence limits. This is reduced by the ansatz (2.14) to

e recursion recursion system for the coincidence limits. This is reduced by the ansatz (2.14) to $c_k' := \sum_{r=1}^{\infty} [(k-r)!]^2 (k-r) (k-r)^{(l-r+1)}$
(Take the coincidence limit of f_{r-1}^l on the right-hand side.) Thand for $p + 2q = 4$, Ric = 0
 $-d^p 4^q f_k^l = c_k^l (p, q)^{-d^p 2^q \mu}$.
Proof: By ignoring the residual terms ... the sys *c*^{*x*}₂^{*v*}_{**j**}^{*i*} \leq *c*_i^{*t*}(*o, q*) *^{<i>d*}²*A*^{*v*}_{*u*}</sub>. (2.12) becomes an algebraic system for the coincidence limits. This is reduced by the ansatz (2.14) to sion system for the c_k^{*i*} $c'_{k+1} \le$

$$
c_{k+1}^{l} \triangleq (s-k) (t-k) c_k^{l} + (l-k) f_k^{l}, \qquad c_0^{l} = 0,
$$

and the latter is solved just by (2.13) the recursion system for the c_k^l
 $c_{k+1}^l \triangleq (s-k) (t-k) c_k$

and the latter is solved just by (2

Conclusion: *If* $\Delta^k \sigma^l = 0$ then

$$
c_k{}^l(0,1)\,R=0\,,\quad c_k{}^l(2,0)\,{}^-\!Ric=0\,,\tag{2.15}
$$

$$
c_k{}^l(1,1)\,d\,R=0\,,\ \ \, c_k{}^l(3,0)\,\, \, d\,Ric=0\,.
$$

 $I \Delta^k \sigma^l = 0$ and $Ric = 0$ then

$$
c_k^{(0)}(2) | Riem|^2 = 0, \qquad c_k^{(0)}(4,0)^{-(Riem)^2} = 0. \qquad (2.16)
$$

Let us classify - for given *n*, *k*, *p*, *q* - the values of *l* with $f_k^l \triangleq 0$:

(i) $q \leq k - 1$ and *l* is an integer with k -narmonic integer with $q \leq k - 1$ and *l* is an integer with $0 \leq l \leq k - q - 1$.

(i) $q \geq 1$ and *l* is a positive integer with $k - q \leq l \leq k - q$

Let us classify $-$ for given *n*, *k*, *p*, *q* $-$ the values of *l* with $f_k^l \stackrel{.}{=} 0$:

(i) $q \le k - 1$ and *l* is an integer with $0 \le l \le k - q - 1$.

(ii) $q \ge 1$ and *l* is a positive integer with $k - q \le l \le k - 1$.

(iii (iii) $1 \leq p + q \leq k - 1$ and $l + m$ is an integer with $0 \leq l + m \leq k - p - q - 1$.
(iv) $1 \leq p + q \leq k$ and $l + m$ is a positive integer with $k - p - q \leq l + m$
 $\leq k - 1$. Examinant Manifolds with

Let us classify $-$ for given n , k , p , q $-$ the v

(i) $q \le k - 1$ and l is an integer with $0 \le l$

(ii) $q \ge 1$ and l is a positive integer with k $-$

(iii) $1 \le p + q \le k - 1$ and $l + m$ Riemannian Manifolds with *k*-harmonic Power

Let us classify - for given *n*, *k*, *p*, *q* - the values of *l* with f_k^l

(i) $q \le k - 1$ and *l* is an integer with $0 \le l \le k - q - 1$.

(ii) $q \ge 1$ and *l* is a positive in **ICCT** *ICC* **ICCC ICCC ICCC**

/

 $c_{\mathbf{k}}^{i'}(p,q) = 0$ in the cases (i), (iii),

 c_k ^{*'*}(*p, q*) > 0 *in the case* (ii),

$$
c_k{'}(p,q) \sim \sum_{r=1}^k {s-r \choose q}{t-r \choose p+q-1}
$$
 in the case (iv).

If n is odd or $l < 0$ *then the proportionality factor is non-zero here.*

 $c_k(t(p, q) \sim \sum_{r=1}^k {s-r \choose q} {t-r \choose p+q-1}$ in the case (iv).
 n is odd or $l < 0$ then the proportionality factor is non-zero l

Proof: (i): By assumption $s = l + q < k$ and thus $\begin{pmatrix} s - l \\ k - l \end{pmatrix}$ $r(r)$ = 0 for $1 \le r \le k$. (iii): By assumption $t = l + m + p + q < k$ and thus $\binom{l - r}{k - r} = 0$ for $1 \le r \le k$. (iii): By assumption $t = l + m + p + q < k$ and thus $\begin{pmatrix} l - r \\ k - r \end{pmatrix} = 0$ for $1 \le r \le k$.
(ii): By assumption $l < k$ and because of $(l - r + 1) f_{r-1} = 0$ for $r > l$ the summation in (2.13) stops at $r = l$ and each of the remaining su (ii): By assumption $l < k$ and because of $(l - r + 1) f_{r-1} = 0$ for $r > l$ the summation in (2.13) stops at $r = l$ and each of the remaining summands is positive as a product of five positive factors. (iv): By assumption $c_k^{\mathcal{T}}$ can-be transformed into *i*, *g*) > 0 *in the case* (ii),
 i, *g*) $\sim \sum_{r=1}^{k} {s-r \choose q} {t-r \choose p+q-1}$ *in the case* (iv).
 odd or $l < 0$ *then the proportionality factor is non-zero here.*
 of: (i): By assumption $s = l + q < k$ and thus ${k-r \choose k-r} = 0$ $c_k^{-1}(p, q) = 0$ in the cases (i), (iii),
 $c_k^{-1}(p, q) > 0$ in the case (ii),
 $c_k^{-1}(p, q) > \sum_{i=1}^{k} {s-r \choose q-i} {t-r \choose q+q-1}$ in the case (iv).

If n is odd or $l < 0$ then the proportionality factor is non-zero her.

Proof: (i): B

on in (2.13) stops at
$$
r = l
$$
 and each of the remaining summands is positive
\nuct of five positive factors. (iv): By assumption c_k^l can be transform
\n
$$
c_k^l = k!(k-1)! \binom{k}{q}^{-1} \binom{k-1}{p+q-1}^{-1} \binom{l}{k-q} \binom{l+m}{k-p-q}
$$
\n
$$
\times \sum_{r=1}^k \binom{s-r}{q} \binom{t-r}{p+q-1}.
$$
\n*l is half-integer or negative then*\n
$$
\binom{l}{k-q} \neq 0.
$$
\nThe other factors in
\nProposition 2.7: Define numbers $c_k = c_k(p, q)$ by
\n
$$
c_k = [(k-1)!]^2 \sum_{r=1}^k \binom{k-1}{r-1}^{-2} \binom{q-r}{k-r} \binom{t-r}{k-r} (-1)^{r-1} \binom{m}{r-1}
$$
\n*with* $t := m + p + q$. Then for $1 \leq p + 2q \leq 3$ and for $p + 2q = 4$,
\n
$$
-d^p A^q I_k = c_k(p, q) - d^p A^q \mu.
$$
\nProof: Pass to the limit $l \to 0$ in Proposition 2.5 according to (2.5),
\nystem for the $c_k = c_k(p, q)$ reads $c_{k+1} = (q - k) (t - k) c_k - k I_k$,
\n
$$
c_k = \sum_{r=1}^k (q - k) (t - k) c_k - k I_k,
$$

If *l* is half-integer or negative then $\begin{pmatrix} l \\ k-q \end{pmatrix}$ + 0. The other factors in front of the sum are non-zero

$$
c_k^l = k!(k-1)! \binom{k}{q}^{-1} \binom{k-1}{p+q-1}^{-1} \binom{l}{k-q} \binom{l+m}{k-p-q}
$$

\n
$$
\times \sum_{r=1}^k \binom{s-r}{q} \binom{t-r}{p+q-1}.
$$

\nl is half-integer or negative then $\binom{l}{k-q} + 0$. The other factors in front of the
\nm are non-zero **1**
\nProposition 2.7: *Define numbers* $c_k = c_k(p, q)$ *by*
\n
$$
c_k = [(k-1)!]^2 \sum_{r=1}^k \binom{k-1}{r-1}^{-2} \binom{q-r}{k-r} \binom{t-r}{k-r} (-1)^{r-1} \binom{m}{r-1}.
$$

\n*the* $t := m + p + q$. Then for $1 \leq p + 2q \leq 3$ and for $p + 2q = 4$, *Ric* = 0

with $t := m + p + q$. Then for $1 \leq p + 2q \leq 3$ and for $p + 2q = 4$, Ric $= 0$

$$
-d^p \Delta^q f_k \triangleq c_k(p,q) - d^p \Delta^q \mu. \tag{2.18}
$$

Proof: Pass to the limit $l \to 0$ in Proposition 2.5 according to (2.5). The recursion with $t := m + p + q$. Then for $1 \leq p + 2q \leq 3$ and for $p + 2q = 4$, Ric = 0
 $-d^p 4^q k = c_k(p, q) d^p 4^q \mu$. (2.18)

Proof: Pass to the limit $l \to 0$ in Proposition 2.5 according to (2.5). The recursion

system for the $c_k = c_k(p, q)$ solved just by (2.17) *l* is half-integer or negative then $\binom{r}{k-q} \neq 0$

m are non-zero \blacksquare

Proposition' 2.7: Define numbers $c_k = c_k(p,$
 $c_k = \left[(k-1)! \right]^2 \sum_{r=1}^k {k-1 \choose r-1}^{-2} {q-r \choose k-r}$

th $t := m + p + q$. Then for $1 \leq p + 2q \leq 3$
 $-d^p \Delta^q f$ system for the $c_k = c_k(p, q)$ reads $c_{k+1} = (q - k)(t - k)c_k - k f_k$, $c_1 = 1$, and is Example 1: Pass to the limit $l \to 0$ in Proposition:

corporation: $l_f \Delta^k \log \sigma = 0$ then

ck(0, 1) $R = 0$, $c_k(2, 0)'$ -Ric = 0,

c_k(1, 1) $dR = 0$, $c_k(3, 0)$ id Ric = $c_k = [(k-1)!]^2 \sum_{r=1}^k {k-1 \choose r-1}^{-2} {q-r}$
 $m+p+q$. Then for $1 \leq p+2q$
 $-d^p \Delta^q f_k \doteq c_k(p, q)^{-d^p} \Delta^q \mu$.
 \therefore Pass to the limit $l \to 0$ in Propos

or the $c_k = c_k(p, q)$ reads $c_{k+1} \leq$

st by (2.17)

usion: If $\Delta^k \log \sigma = 0$ $c_k = [(k-1)!]^2 \sum_{r=1}^{\infty} {k \choose r-1} \left(\frac{q-r}{k-r}\right) \left(\frac{r-r}{k-r}\right) (-1)^r$
 IIf $t := m + p + q$ *. Then for* $1 \leq p + 2q \leq 3$ *and for* $p + 2$
 $d^p 2^q k = c_k(p, q) - d^p 2^q \mu$.

Proof: Pass to the limit $l \to 0$ in Proposition 2.5 according

sys $m + p + q$. Then for $1 \leq p + 2q \leq 3$ and for $p + 2q = 4$, Ric = 0
 $-d^p 2^q 4^k = c_k(p, q) - d^p 2^q 4^k$. (2.18)
 \therefore Pass to the limit $l \to 0$ in Proposition 2.5 according to (2.5). The recursion

or the $c_k = c_k(p, q)$ reads c_{k

$$
u(t) = \begin{cases}\n\text{u} & \text{if } t \leq 2.17, \\
u(t) & \text{if } t \leq 1/2 \end{cases}
$$
\n
$$
c_k(0, 1) \quad R = 0, \qquad c_k(2, 0)^t - Ric = 0,
$$
\n
$$
c_k(1, 1) \quad d \quad R = 0, \qquad c_k(3, 0) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
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$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$
\n
$$
c_k(0, 1) \quad d \quad R = 0, \qquad c_k(0, 1) \quad d \quad Ric = 0.
$$

$$
c_k(1, 1) d R = 0, \qquad c_k(3, 0) d Ric = 0.
$$

$$
c_k(0, 2) | Riem|^2 = 0, \qquad c_k(4, 0) - (Riem)^2 = 0. \tag{2.20}
$$

16*

Proposition 2.8: If $n \geq 3$ and $q + 1 \leq k \leq m + p + q$ then $c_k(p, q) \neq 0$. Proof: By assumption $c_k = c_k(p, q)$ can be transformed into

$$
c_k = (-1)^{k-1} \left[(k-1)! \right]_{r=q+1}^2 {k-1 \choose r-1}^{-2} {k-q-1 \choose k-r} {t-r \choose k-r} {m \choose r-1}.
$$

Here all summands are positive I

§ 3. Results for $l + m = k - 1$. Elementary solutions of the Laplace equation

Let us remind the notion of an elementary solution following [9, 11]. We denote by N a neighbourhood of the diagonal in $M \times M$ in which Synge's function $\sigma = \sigma(x, y)$ is defined and

$$
N^- := \{(x, y) \in N \mid \sigma(x, y) = 0\}.
$$

Definition 3.1: Let $n \geq 3$. A two-point function $u = u(x, y)$ which is defined and C^{∞} in a set N^{-} and satisfies

$$
du = 0 \text{ in } N^2,
$$
\n
$$
\lim_{\alpha \to 0} |\sigma|^m u = \text{const.} \neq 0
$$
\n(3.1)

 $\lim_{n \to \infty}$ σ \mathbb{Z}^{m} $u = \text{const.} \neq 0$

is called an elementary solution of the Laplace equation.

For even $n \geq 4$ such an elementary solution has the form

$$
u = \sigma^{-m} \mathcal{U} + \mathfrak{U} \log |\sigma| \tag{3.3}
$$

with C^{∞} functions $\mathcal{U} = \mathcal{U}(x, y)$, $\mathcal{U} = \mathcal{U}(x, y)$. The rare situations in which the logarithmic part $\mathfrak U$ log $|\sigma|$ is missing are especially interesting.

Definition 3.2: Let $n \geq 4$ be even and let the two-point function $\mathcal{U} = \mathcal{U}(x, y)$ be defined and C^{∞} in a set N. An elementary solution of the Laplace equation of the form

$$
u = u(x, y) = \sigma^{-m} \mathcal{U}(x, y) \quad \text{in} \quad N^{-}
$$
\n
$$
(3.4)
$$

 (3.5)

 (3.6)

is called *logarithm-free*.

For a lorentzian metric g there exists an interpretation in other terms: The Laplace equation then admits a logarithm-free elementary solution if and only if Huygens' principle in the sense of $[9, 5, 12]$ is valid.

Theorem 3.1: Let

$$
\varDelta^k \, |\sigma|^{k-m-1} = 0
$$

and for even n additionally $k \leq m$. Then

$$
u:=\varDelta^{k-1}\,\vert\sigma\vert^{k-m-1}
$$

is an elementary solution of the Laplace equation and for even n it is logarithm-free.

Proof: The Propositions 2.1 and 2.3 in fact realize the wanted formulas (3.4) and (3.2) :

$$
\Delta^{k-1} |\sigma|^{k-m-1} = (2e)^{k-1} |\sigma|^{-m} \frac{k-m-1}{k-1},
$$

$$
\int_{k-1}^{k-m-1} \stackrel{=} (-1)^{k-1} (k-1)!^2 {m-1 \choose k-1}
$$

Riemannian Manifolds with k -ha
 Theorem 3.2: Let $n \geq 4$ *be even and*
 $\Delta^{m+1}(\log |\sigma|) = 0.$ Riemannian M $d^{m+1}(\log |\sigma|) = 0.$
 $u := \Delta^m(\log |\sigma|)$ *0S (3.7)* Riemannian Manifolds with *k*-harmonic Power of R
 Theorem 3.2: Let $n \ge 4$ be even and
 $\Delta^{m+1}(\log |\sigma|) = 0$.
 Then
 $u := \Delta^m(\log |\sigma|)$

is a logarithm-free elementary solution of the Laplace equation. Riemannian Manifolds with *k*-harmonic Power of Radius 245

Theorem 3.2: Let $n \ge 4$ be even and
 $\Delta^{m+1}(\log |\sigma|) = 0.$ (3.7)

Then
 $u := \Delta^m(\log |\sigma|)$ (3.8)

a logarithm-free elementary solution of the Laplace equation.

Proof Riemannian Manifolds with *k*-harmonic Power of Ramoneum 3.2: Let $n \ge 4$ be even and
 $A^{m+1}(\log |\sigma|) = 0$.

Then
 $u := A^m(\log |\sigma|)$

is a logarithm-free elementary solution of the Laplace equation.

Proof: The Propositions 2.2 *u* : = $\Delta^{m}(\log |\sigma|)$
 a logarithm-free elementary solution of the

Proof: The Propositions 2.2 and 2.3 is
 $\Delta^{m}(\log |\sigma|) = 2^{m} \sigma^{-m} f_{m}$, $\frac{1}{m} =$

Proposition 3.1: From the assumption

•

Proof: The Propositions 2.2 and 2.3 in fact realize the wanted formulas .(3.4) and (3.2): *(in - free elementary solution of the Laplace equation.*
 2.2° and 2.3 in fact realize the wanted formulas (3.4)
 $\Delta^{m}(\log |\sigma|) = 2^{m} \sigma^{-m} f_{m}$, $f_{m} = (-1)^{m-1} m! (m - 1)!$
 $\Delta^{m}(\log |\sigma|) = 2^{m} \sigma^{-m} f_{m}$, $f_{m} = (-1)^{m-1} m!$ *hm-free elementary solution of the Laplace equation.*

The Propositions 2.2 and 2.3 in fact realize the wanted formulas (3.4)
 ${}^{m}(\log |\sigma|) = 2^{m} \sigma^{-m} f_{m}$, $f_{m} = (-1)^{m-1} m! (m - 1)!$

ition 3.1: *From the assumption of Theo* $z_0 \in \mathbb{R}$ and (3.2) :
 $\Delta^m(\log |\sigma|) = 2^m \sigma^{-m} f_m$, $\frac{1}{m} = (-1)^{m-1} m! (m$

Proposition 3.1: From the assumption of Theorem 3.
 $(n - k - 1) R = 0$ for $k \ge 1$, Theorem 3.

From the assumption of Theorem 3.1 and Ric = 0 there

$$
\Delta^m(\log|\sigma|) = 2^m \sigma^{-m} f_m, \qquad f_m = (-1)^{m-1} m! (m-1)! \quad \blacksquare
$$

Proposition 3.1: *From the assumption of Theorem* 3.1 *there follows*

$$
(n-k-1) R = 0 \text{ for } k \ge 1, \qquad \text{Ric} = 0 \text{ for } k \ge 2. \tag{3.9}
$$

From the assumption of Theorem 3.1 and $Ric = 0$ *there follows*

$$
-(Riem)^2 = 0 \quad \text{for} \quad k \ge 4. \tag{3.10}
$$

Proof: For $1 \leq p + q \leq k$ we have the case (iv) of Proposition 2.6 with a non-

$$
-(Riem)^2 = 0 \text{ for } k \ge 4.
$$
 (3.10)
Proof: For $1 \le p + q \le k$ we have the case (iv) of Proposition 2.6 with a non-zero proportionality factor. Especially we get

$$
c_k^l(0, 1) \sim 2 \sum_{r=1}^k (s-r) = k(k - n + 1) \text{ for } k \ge 1,
$$

$$
c_k^l(2, 0) \sim \sum_{r=1}^k (t-r) = {t \choose 2} = {k + 1 \choose 2} \text{ for } k \ge 2,
$$

$$
c_k^l(4, 0) \sim \sum_{r=1}^k {t-r \choose 3} = {t \choose 4} = {k+3 \choose 4} \text{ for } k \ge 4,
$$
and we apply now the conclusion following Proposition 2.5 **■**
Theorem 3.3: If $A^k \sigma^{k-m-1} = 0$ and if g is *localzian* and $4 \le k \le m$ for even n
or $4 \le k \ne n - 1$ for odd n respectively then (M, g) is flat.
The proof is composed by the Proposition 3.1 and 1.2 **■**
Proposition 3.2: From the assumption of Theorem 3.2 there follows
 $Ric = 0$, $-(Riem)^2 = 0$, $|Riem|^2 = 0$. (3.11)
Proof: For $m \ge q$ and $p + q \ge 1$ we have the situation of Proposition 2.8. For
the special case $n = 4$, $p = 0$, $q = 2$ we calculate directly $c_2(0, 2) = 1$. The assertion
follows now from the conclusion to Proposition 2.7 **■**

and we apply now the conclusion following Proposition 2.5 \blacksquare

 $A^k \sigma^{k-m-1} = 0$ and if

odd n respectively th

osed by the Proposi

From the assumption
 $A^-(Riem)^2 = 0$,

and $m + a \ge 1$ we

The proof is composed by the Propositions 3.1 and 1.2 . $\frac{1}{2}n - 1$ *for*
 $\frac{1}{2}n - 1$ *for*
 $\frac{1}{2}n + 1$
 $\frac{1}{2}n - 1$

Proposition 3.2: *From the assumption of Theorem* 3.2 *there follows*

$$
Ric = 0, \t (Riem)^2 = 0, \t |Riem|^2 = 0.
$$
\t(3.11)

Proof: For $m \geq q$ and $p + q \geq 1$ we have the situation of Proposition 2.8. For the special case $n=4, p=0, q=2$ we calculate directly $c_2(0, 2)=1.$ The assertion $c_k^l(4,0) \sim \sum_{r=1}^k {l-r \choose 3} = {l \choose 4} = {k+3 \choose 4}$ for $k \ge 4$,
and we apply now the conclusion following Proposition 2.5 **a**
Theorem 3.3: If $A^k \sigma^{k-m-1} = 0$ and if g is lorentzian and $4 \le$
or $4 \le k \ne n-1$ for odd n respec $\frac{2}{3}$ **Proposition 3.2:** From the assume $Ric = 0$, $\qquad (Riem)^2 = 0$,

Proof: For $m \ge q$ and $p + q \ge 1$

the special case $n = 4$, $p = 0$, $q = 2$

follows now from the conclusion to Pr

Theorem 3.4: If $\Delta^{m+1} \log \sigma = 0$

even then $(M, g$ $\begin{align} \n 1 & \text{if } |Riem|^2 \n 1 & \text{if } Riem \n 1 & \text{if } Riem \n 2 & \text{if } Riem \n 3 & \text{if } Riem \n 4 & \text{if } Riem \n 5 & \text{if } Riem \n 6 & \text{if } Riem \n 7 & \text{if } Riem \n 8 & \text{if } Riem \n 1 & \text{$ From $Ric' = 0$, \therefore $(Riem)^2 = 0$, \therefore $|Riem|^2 = 0$.

Proof: For $m \ge q$ and $p + q \ge 1$ we have the situation

the special case $n = 4$, $p = 0$, $q = 2$ we calculate directly c_2 follows now from the conclusion to Proposition

Theorem 3.4: *If* Δ^{m+1} log $\sigma = 0$ and *if* g *is definite or lorentzian and* $n \geq 4$ *is n* then (M, g) *is flat.*

follows now from the conclusion to Proposition 2.7

Theorem 3.4: If $\Delta^{m+1} \log \sigma = 0$ and if g is definite or lorentzian and $n \ge 4$ is

even then (M, g) is flat.

The proof is composed by the Propositions 3.2 and 1.2

L Theorem 3.4: If $\Delta^{m+1} \log \sigma = 0$ and if g is definite or lorentzian and $n \ge 4$ is
even then (M, g) is flat.
The proof is composed by the Propositions 3.2 and 1.2 \blacksquare
Let us now specialize $k = 1$. This is only for th **Proof:** For $m \ge q$ and $p + q \ge 1$ we have the special case $n = 4$, $p = 0$, $q = 2$ we calculation follows now from the conclusion to Proposi
Theorem 3.4: If $\Delta^{m+1} \log \sigma = 0$ and if
even then (M, g) is flat.
The proof is co Let us now specialize $k = 1$. This is only for the sake of completeness — the well-
known simply harmonic manifolds will emerge [11]. A simply harmonic manifold

$$
\mu = 0
$$
 for any $n \ge 2$, $d\sigma^{-m} = 0$ for $n \ge 3$, $d \log \sigma = 0$ for $n = 2$.

Theorem 3.5: If $\Delta \sigma^l = 0$ with $l = 0$ then $l = -m$ and (M, g) is a simply har*monic manifold of a dimension n* ≥ 3 . If Λ log $\sigma = 0$ then (M, g) is a simply harmonic *manifold of dimension* $n = 2$. *z*₁ *z*₁ *z*₁ *z*₁ *d*₁ = 0 *with* $l + 0$ *then* $l = -m$ *and* (*M*, *g*) *i n nipold of a dimension* $n = 2$.
 2 2of **i** *nimension n* = 2.
 2of **i** = $2\sigma^{l-1}l(l + m + \mu)$, Δ log $\sigma = 2\sigma^{-1}(m +$ R. SCHIMMING

em 3.5: If $\Delta \sigma' = 0$ with $l + 0$ then $l = -m$ and (M, g) is a simply har-

mifold of a dimension $n \geq 2$. If $\Delta \log \sigma = 0$ then (M, g) is a simply harmonic

of dimension $n = 2$.

oof follows immediately from
 (Right) 2 *(Right)* **2** *(Right) (Right) (A) (a) (i) (a) (i)* *****a) (i) (i) (i) (i) (i) (i) (i) (i)* *****(i) (i) (i) (i) (i)* *****(i) (i) (i*

The proof follows immediately from

$$
\varDelta \sigma^l = 2\sigma^{l-1}l(l+m+\mu), \qquad \varDelta \log \sigma = 2\sigma^{-1}(m+\mu) \quad \blacksquare
$$

Let us now specialize $k = 2$.

Proposition 3.3: If
$$
\Delta^2 \sigma^{1-m} = 0
$$
 and $m(m-1) \neq 0$ then

$$
(n-3) R = 0, \t ^{-1} Ric = 0, \t (3.12)
$$

$$
f(Riem)^2 = 0, \qquad n(n-1) |Riem|^2 = 2R^2. \tag{3.13}
$$

Proof: While (3.12) is a specialization of (3.9), we derive (3.13) more directly from (with a non-zero proportionality factor)

$$
0 = d^4 l_2^{1-m} \cdot m[6(d^2\mu)^2 + 5d^4\mu] - 3gd^2\Delta\mu
$$

Theorem 3.6: *If* $\Delta^2 \sigma^{1-m} = 0$ *and n* ≥ 5 *and if g is definite or lorentzian then (M, g) is flat.*

The proof is composed by the Propositions 3.3 and 1.2. (For definite q the result is already known $[3, 7]$)

§ **4.** Results **for other situations**

We will omit the proofs in 'the following as long as they run along the same lines as in *§* 3.

Proposition 4.1: If $\Delta^k \sigma^{k-m-2} = 0$ and $k \geq 2$ and for even n additionally $k \leq m + 1$ *then* $0 = a^2 I_2^* \approx m [0(a^2 \mu)^2 + 5a^2 \mu] - 3ga^2 \mu$

em 3.6: If $A^2 a^{1-m} = 0$ and $n \ge 5$ and if g is definite or lorentzian then

flat.

coof is composed by the Propositions 3.3 and 1.2. (For definite g the result

known [3, 7])
 oof is composed by the Propositions 3.3 and 1.2. (For definite g the result

^{*z*} known [3, 7]). ■
 alts for other situations
 mit the proofs in the following as long as they run along the same lines as

sition 4.1: in § 3.

Proposition 4.1: If $A^k \sigma^{k-m-2} = 0$ and $k \ge 2$ and
 then
 $Ric = 0$.

If $A^k \sigma^{k-m-2} = 0$ and $R = 0$ and $k \ge 4$ and for eve
 $\Gamma(kiem)^2 = 0$.

Theorem 4.1: If $A^k \sigma^{k-m-2} = 0$ and if g is loren

for even n additio

If
$$
\Delta^k \sigma^{k-m-2} = 0
$$
 and $R = 0$ and $k \ge 4$ and for even n additionally $k \le m+1$ then

$$
F(Riem)^2=0
$$

Theorem 4.1: *If* $A^k \sigma^{k-m-2} = 0$ and if g is lorentzian and $R = 0$ and $k \geq 4$ and *for even n additionally* $k \leq m + 1$ *then* (M, g) *is flat.*

Proposition 4.2: *If* $A^k \sigma^{k-m-3} = 0$ *and Ric* $= 0$ *and k* ≥ 4 *and for even n addi*-Theorem 4.1: If $A^k \sigma^{k-m-2} = 0$ and
for even *n* additionally $k \leq m + 1$ then (
Proposition 4.2: If $A^k \sigma^{k-m-3} = 0$ at
tionally $k \leq m + 2$ then (4.2) holds true.
Proposition 4.3: If $A^k \sigma^{k-m-4} = 0$.

Proposition 4.3: If $\Delta^k \sigma^{k-m-4} = 0$ and Ric = 0 and $k \ge 4$ and for even n additionally $k \le m+3$ then (4.2) holds true. *Richard 4.2: If* $A^k \sigma^{k-m-3} = 0$ *and* $Ric = 0$ *and* $k \ge 4$ *and for even n addi*
 $\leq m + 2$ *then* (4.2) *holds true.*
 Ricc $= 0$ *and* $k \ge 4$ *and for even n addi*
 $\leq m + 3$ *then* (4.2) *holds true.*
 Riccon and $k \ge 4$ and for even *n* addi-
 $R = 0$. If $\Delta^k \sigma^{k-1} = 0$ and $k \ge 2$

2 then (M, g) is flat.

Ric = 0 then $|Riem|^2 = 0$.

(4.3)

(4.4)

Proposition 4.4: *If* $A^k \sigma^{k-1} = 0$ and $k \geq 2$ then $R = 0$. If $A^k \sigma^{k-1} = 0$ and $k \geq 2$ *and* $Ric = 0$ *then* $|Riem|^2 = 0$.

Theorem 4.2: If
$$
\Delta^k \sigma^{k-1} = 0
$$
 and $n = 2$ and $k \ge 2$ then (M, g) is flat.

Proposition 4.5: If $\Delta^k \sigma^{k-2} = 0$ and $k \geq 3$ and Ric = 0 then $|Riem|^2 = 0$.

Proposition 4.6: *If* $4^2\sigma^{-m} = 0$ and $n \geq 3$ then

$$
Ric = 0, \qquad (Riem)^2 = 0, \tag{4.3}
$$

$$
3(n-4) | Riem|2 = (5n2 - 2n - 12) | Ric|2.
$$
 (4.4)

Proof: While (4.3) follows from

Riemannian Manifolds with k-harmonic?
\nf: While (4.3) follows from
\n
$$
c_2^{-m}(2,0) = m(m+1), \qquad c_2^{-m}(4,0) = 3m(m+1),
$$
\n
$$
c_2^{(4,4)} = 3m(m+1), \qquad c_2^{-m}(4,0) = 3m(m+1),
$$

we derive (4.4) more directly from (with a non-zero proportionality factor)

$$
0 = d^4 f_2^{-m} \cdot n[2(d^2\mu)^2 + d^4\mu] - 2g d^2\Delta\mu.
$$

(For definite g the results (4.4) and $Ric = 0$ are already known [3]) !

Theorem. 4.3: If $A^2\sigma^{-m} = 0$ and $n \geq 3$ and if g is lorentzian then (M, g) is flat.

Proof: From the Propositions 4.6 and 1.2 there follows that (M, g) is of constant Riemannian Manifolds with

Proof: While (4.3) follows from
 $c_2^{-m}(2, 0) = m(m + 1),$ $c_2^{-m}(4, 0) =$

we derive (4.4) more directly from (with a non-ze
 $0 = d^4/2^{-m} \cdot n[2(d^2\mu)^2 + d^4\mu] - 2g d^2\Delta\mu$

(For definite g the results (: While (4.3) follows from
 $c_2^{-m}(2, 0) = m(m + 1),$ $c_2^{-m}(4, 0) = 3$
 $\geq (4.4)$ more directly from (with a non-zer
 $0 = d^4t_2^{-m} \cdot n[2(d^2\mu)^2 + d^4\mu] - 2g d^2\mu$,

mite g the results (4.4) and $Ric = 0$ are \geq
 e em 4.3: If

Proposition 4.7: *If* $A^2\sigma = 0$ *then*

$$
R = 0, \t|Riem|^2 = |Ric|^2. \t(4.5)
$$

The proof follows from $A^2\sigma = 2\Delta\mu$ and (1.8). (Again, for definite g the result is known $[3]$ \blacksquare Fraction 4.7: If $A^2\sigma = 0$ then
 $R = 0$, $|Riem|^2 = |Ric|^2$.
 $\sigma = 0$, $|Riem|^2 = |Ric|^2$.
 $\sigma = 0$ and $\sigma = 2d\mu$ and (1.8).
 $\Gamma = \frac{d^2\sigma}{d\sigma} = 0$ and $\sigma = 0$ and $\sigma = 0$ if $\sigma = 0$ and $\sigma = 0$ if $\sigma = 0$.

From (4.5) there foll **Fig. 1** (1.8). (Again, for definite g the result is
 Fig. 1)
 F
 Fig. 1) F
 Fig. 1) F
 Fig. 1) F
 Fig. 1 ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{9}$ is definite and $n \le 5$ or $Ric = 0$ then (M, g)
 $A^2\$

Theorem 4.4: If $\Delta^2 \sigma = 0$ and if g is definite and $n \leq 5$ or Ric = 0 then (M, g) *is flat. If* $\Delta^2 \sigma = 0$ *and if q is definite and n = 6 then* (M, g) *is conformally flat.* From 1.4: If $\Delta^2 \sigma = 0$ and if g is definite and $n \leq 5$ or $Ric = 0$ then (M, g)
flat. If $\Delta^2 \sigma = 0$ and if g is definite and $n = 6$ then (M, g) is conformally flat.
Proof: From (4.5) there follows $(n - 2) |Weyl|^2 = (n - 6) |R$

definiteness arguments yield the assertion I

Proposition 4.8: If $A^{3}\sigma = 0$ then

$$
|Riem|^2 - |Ric|^2 + 6\Delta R = 0. \tag{4.6}
$$

The proof follows from $A^3\sigma = 2A^2\mu$ and (1.8) I

Theorem 4.5: If $A^3\sigma = 0$ and if g is definite and

$$
F := 6n(n-1) \Delta R - (n-3) R^2 \geq 0
$$

then (M, g) *satisfies* $F=0$ *and is*

of constant curvature for $n = 3$, *flat for* $n=4$ *and for* $n=5$, *conformally flat for* $n = 6$.

Proof: For $n \geq 3$ the condition (4.6) can be transformed into

$$
n(n-1) (n-2) |Weyl|^2 + n(n-1) (6-n) |{}^{\text{-}}Ric|^2 + (n-2) F = 0.
$$

Hence from $F \ge 0$ there follows by definiteness arguments $Weyl = 0$, $(n - 6)$ *Ric* $= 0$, $F = 0$ **I** $|Riem|^2 - |Ric|^2 + 6dR = 0.$

The proof follows from $A^3\sigma = 2A^2\mu$ and (1.8) ■

Theorem 4.5: If $A^3\sigma = 0$ and if g is definite and
 $F := 6n(n - 1) \Delta R - (n - 3) R^2 \ge 0$

then (M, g) satisfies $F = 0$ and is

of constant curvature f *m*(*n* - 1) (*n* - 2) |*Weyl*|² + *n*(*n* - 1)
 m(*n* - 1) (*n* - 2) |*Weyl*|² + *n*(*n* - 1)
 m F \geq 0 there follows by definiten
 n R = 0 for 2 \leq *k* \leq *m* + 1,
 m dR = 0 for 2 \leq *k* $\$ *m*(*n* - 1) (*n* - 2) |*Weyl* $f' + n(n-1)$ (*o* - *n*) | *ndc* $f' + (n-2)$ *F*
 som $F \ge 0$ there follows by definiteness arguments $Weyl = 0$, (*n*
 mR = 0 *for* 2 $\le k \le m + 1$, This is not 2.
 mR = 0 *for* 2 $\le k \le m +$ *If at ion* $n = 4$ *and for* $n = 5$,
 If at for $n = 4$ *and for* $n = 5$.

Proof: For $n \ge 3$ the condition (4.6) can
 $n(n - 1) (n - 2) |Weyl|^2 + n(n - 1)$

Hence from $F \ge 0$ there follows by definite
 $= 0, F = 0$ **I**

Proposition

Proposition 4.9: *If* \mathcal{A}^k log $\sigma = 0$ *then*

 $Ric = 0$ for $k \leq m+2$,

 $m \cdot dR = 0$ for $2 \le k \le m+2$, $-d \text{ Ric} = 0$ for $k \le m+3$.

If $d^k \log \sigma = 0$ and $Ric = 0$ then
 $|Riem|^2 = 0$ for $3 \le k \le m+2$, $-(Riem)^2 = 0$ for $k \le m+4$.

Theorem 4.6: *If* A^k log $\sigma = 0$ and $3 \leq k \leq m + 2$ and if g is definite then (M, g) *is flat. If* $\Delta^k \log \sigma = 0$ *and* $2 \leq k \leq m + 1$ *and if g is lorentzian then* (M, g) *is flat.*

 $\label{eq:1} \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)^{2}d\mu$

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Theorem 4.7: If $\Delta^2 \log \sigma = 0$ and $n = 2$ Theorem 4.7: If $\Delta^2 \log \sigma = 0$ and $n = 2$ then (M, g) is flat.

Proof: We evaluate (with non-zero proportionality factors)
 $0 = d^4f_2 \ge 2(d^2\mu)^2 + d^4\mu - g d^2\Delta\mu$,
 $0 = d^5f_2 \ge 4(5d^2\mu d^3\mu + d^5\mu) - 5g d^3\Delta\mu$

th the help o

Proof: We evaluate (with non-zero proportionality factors)

: We evaluate (with non-zero propor
\n
$$
0 = d^4f_2 \approx 2(d^2\mu)^2 + d^4\mu - g d^2d\mu
$$
,
\n $0 = d^5f_2 \approx 4(5d^2\mu d^3\mu + d^5\mu) - 5g d^3$
\nhelp of (1.9). The resulting equation
\n $9dR + 7R^2 = 0$, $3dAR + 5R dR$

$$
0 = d^5f_2 \approx 4(5d^2\mu \ d^3\mu + d^5\mu) - 5g \ d^3\Delta\mu
$$

with the help of (1.9). The resulting equation systeh

has $R = 0$ as the unique solution \blacksquare

§ 5. Non-flat manifolds for which a power of the radius is k-harmonic

A simply harmonic manifold with one *f* the additional properties o

$$
n\leqq 3,
$$

g is definite or lorentzian,•

g is conformally flat

is known to be flat [11]. Otherwise, for each dimension $n \geq 4$ and each signature of *g* different from the definite or the lorentzian one there exist non-flat simply harmonic manifolds [11]. These provide examples for our problems. to be flat [11]. Otherwise, for
from the definite or the lorent
s [11]. These provide example:
rem 5.1: A simply harmonic m
 $A^k \sigma^{k-1} = 0$, $A^k \sigma^{k-m-1} = 0$. *m* is a stringular manifold with one of the additional properties,
 $n \leq 3$,
 g is definite or lorentzian,
 g is conformally flat

to be flat [11]. Otherwise, for each dimension $n \geq 4$ and each signature of *g*

f *estimate or localitarity*
 estimated that
 z flat [11]. Otherwise, for each dimension $n \ge 4$ and each the definite or the lorentzian one there exist non-flat s.
 i. These provide examples for our problems.
 i.

Theorem 5. 1: A simply harmonic manifold (M, g) fulfills for any positive integer k

$$
\Delta^k \sigma^{k-1} = 0, \qquad \Delta^k \sigma^{k-m-1} = 0. \tag{5.1}
$$

A simply harmonic manifold (M, q) *of even dimension* $n = 2m + 2$ *fulfills*

$$
A^{m+1} \log \sigma = 0. \qquad \qquad
$$

Proof: From $\mu = 0$ there follows

y harmonic manifold
$$
(M, g)
$$
 of even dimensa
\n $\Delta^{m+1} \log \sigma = 0$.
\nf: From $\mu = 0$ there follows
\n
$$
\Delta^k \sigma^l = 2^k \sigma^{l-k}(k!)^2 {l \choose k} {l+m \choose k},
$$
\n
$$
\Delta^k \log \sigma = 2^k \sigma^{-k}(-1)^{k-1} k! (k-1)! {m \choose k}
$$
\ngives the assertion
\n
$$
\Gamma
$$
rem 5.2: To any dimension $n \geq 3$ and an
\none there exist non-flat manifolds (M, g) what
\n $\Delta^2 \sigma = 0$.
\n: The metrics of the form
\n
$$
g = 2dx^1 dx^2 + g_{ij}(x^1) dx^i dx^j
$$
 $(i, j = 3$

and this gives the assertion I

Theorem 5.2: To any dimension $n \geq 3$ *and any signature of g different from the inite one there exist non-flat manifolds* (M, g) which satisfy
 $A^2 \sigma = 0$. (5.3) *dc/mite one there exist non-flat manifolds (M, g) which satisfy* Theorem 5.2: To any dimension $n \geq 3$ and any signature of g different from the
definite one there exist non-flat manifolds (M, g) which satisfy
 $A^2\sigma = 0$.
Proof: The metrics of the form
 $g = 2dx^1 dx^2 + g_{ij}(x^1) dx^i dx^j$ $(i, j$ $A^k \log \sigma = 2^k \sigma^{-k} (-1)^{k-1} k! (k-1)! {m \choose k}$
gives the assertion \blacksquare
rem 5.2: To any dimension $n \geq 3$ and any signature
me there exist non-flat manifolds (M, g) which satisfy
 $A^2 \sigma = 0$.
: The metrics of the form
 $g = 2dx^$ $\Delta^k \log \sigma = 2^k \sigma^{-k} (-1)^{k-1} k! (k-1)! \binom{m}{k}$
gives the assertion \blacksquare
rem 5.2: To any dimension $n \geq 3$ and any signature
me there exist non-flat manifolds (M, g) which satisfy
 $\Delta^2 \sigma = 0$.
: The metrics of the form
 $g =$

$$
= 0. \tag{5.3}
$$

Proof: The metrics of the form

$$
q = 2dx^1 dx^2 + g_{ii}(x^1) dx^i dx^j \qquad (i, j = 3, 4, ..., n)
$$

ties: $g = 2dx^1 dx^2 + g_{ij}(x^1) dx^i dx^j$ (*i*, *j* = 3, 4, ..., *n*)

are generalizations of the plane gravitational waves and have the following pr

ties:
 $\Delta \sigma$ depends only on x^1 and not on the x^i ,
 $\Delta f(x^1) = 0$ for any smoo

$$
sign (g_{\alpha\beta}) = (+, -, sign (g_{ij})).
$$

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Proposition 5.1: A 3-dimensional manifold of constant curvature K satisfies.
 $A^2\sigma = -8K$.

The proof follows from the formulas $\begin{aligned} \text{Proposit} \ A^2 \sigma &= -8K. \ \text{The proof} \end{aligned}$ Fremannian Manifolds with *k*-harmonic Power of Radius 249

Proposition 5.1: A 3-dimensional manifold of constant curvature *K* satisfies
 $A^2\sigma = -8K$.

The proof follows from the formulas
 $\mu = -1 + (2K\sigma)^{1/2} \cot (2K\sigma)^{1/2}$ **Example 10**
 Example 10

The proof follows from the formulas

$$
\mu = -1 + (2K\sigma)^{1/2} \cot (2K\sigma)^{1/2} =: f(\sigma), \qquad (5.4)
$$

$$
\Delta^2 \sigma = 2 \Delta \mu = 4 \sigma f''(\sigma) + 4(\mu + m + 1) f'(\sigma). \tag{5.5}
$$

(For definite g the result is known $[2, 3]$ and with its help in $[3]$ higher-dimensional non-flat manifolds satisfying $d^2\sigma = 0$ are constructed) \blacksquare

Theorem 5.3: The condition $\Delta^2 \sigma^{1/2} = 0$ characterizes 3-dimensional manifolds of

Riemannian *N*

Proposition 5.1: A 3-dimension
 $A^2\sigma = -8K$.

The proof follows from the formul
 $\mu = -1 + (2K\sigma)^{1/2} \cot (2K\sigma)$
 $A^2\sigma = 2\Delta\mu = 4\sigma f''(\sigma) + 4(\mu)$

(For definite g the result is known [2,

non-flat manifolds sat Proof: From $4^2\sigma^{1/2} = 0$ there follows $n = 3$ by Theorem 2.1 and $Ric = 0$ by Proposition *3.3.* Conversely, a calculation based on (5:4) and (2.4) *shows* that for a 3-dimensional manifold of constant curvature there holds $f_2^{1/2} = 0$. (Again, the result is already known for definite $g [2, 3, 7]$) $A_j^{\epsilon} \sigma = -8K$.

The proof follows from the formulas
 $\mu = -1 + (2K\sigma)^{1/2} \cot (2K\sigma)^{1/2} =: f(\sigma)$,
 $A^2 \sigma = 2\Delta \mu = 4\sigma f''(\sigma) + 4(\mu + m + 1) f'(\sigma)$.

(For definite g the result is known [2, 3] and with its help in [3] higher-dime

no $\mu = -1 + (2K\sigma)^{1/2} \cot (2K\sigma)^{1/2} =: f(\sigma),$
 $\Delta^2 \sigma = 2\Delta \mu = 4\sigma f''(\sigma) + 4(\mu + m + 1) f'(\sigma).$

(For definite g the result is known [2, 3] and with its help in [3] hig

non-flat manifolds satisfying $\Delta^2 \sigma = 0$ are constructed) \blacksquare definite g the result is known [2, 3] and with its l
flat manifolds satisfying $\Delta^2 \sigma = 0$ are constructed)
heorem 5.3: The condition $\Delta^2 \sigma^{1/2} = 0$ characteriz
cant curvature.
roof: From $\Delta^2 \sigma^{1/2} = 0$ there follows dant curvature.

ant curvature.

roof: From $\Delta^2 \sigma^{1/2} = 0$ there fo

oosition 3.3. Conversely, a calc

dimensional manifold of constant

it is already known for definite

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Verlag 1968. The convention of constant curvature there holds $f_2^{1/2} = 0$. (Again, the esult is already known for definite g [2, 3, 7])

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