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Ritz-Regularization versus Least-Square-Regularization. Solution Methods for Integral Equations of the First Kind

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Es werden drei Regularisierungsmethoden diskutiert (Tikhonov-, Ritz- und Galerkin-Methode), und es wird gezeigt, daß im Falle positiv definiter oder selbstadjungierter Operatoren die Ritz- und die Galerkin-Methode (die letzte mit nicht-reellem Parameter) bessere Konvergenzeigenschaften haben als die Tikhonov-Methode.

Обсуждаются три методы регуляризации (методы Тихонова, Ритца и Галеркина) и показывается, что в случае положительно-определенных или самосопряженных операторов методы Ритца и Галеркина (последний — с невещественным параметром) обладают лучшим свойством сходимости чем метод Тихонова.

Three principles of regularization methods are discussed (Tikhonov-, Ritz- and Galerkin-method), and it is shown that in the case of positive definite or of selfadjoint operators the Ritz- and the Galerkin-principles (the latter with non-real parameter) have better properties than the Tikhonov-principle.

1. Formulation of the minimum principles

For the (approximate) solution of equations with positive definite operators in a Hilbert space by minimum principles it is common to use a Ritz variational method instead of a least square principle. For linear equations with unbounded operators J. Nitsche [8] has shown that the Ritz method has better convergence properties than the least square method, for linear equations of the second kind I proved a similar result [9]. For linear equations of the first kind J. N. Franklin [3] discussed both methods. Here I will improve the results of J. N. Franklin and will show that also in the case of the finite dimensional approximations the condition numbers of the systems of linear equations which occur in the computational process have better properties in the Ritz case than in the least square case.

In the sequel H is a real infinite dimensional Hilbert space, W is a dense subspace of H, and

$$T : H \to H, \qquad L : W \to H$$

are linear operators, both selfadjoint and positive definite, but T is compact and L has a bounded (for instance if $L = I$, $H = W$) or a compact (for instance if L is a differential operator) inverse. For solving equations of the first kind

$$Tx = y \tag{1}$$

one uses regularization methods to convert the ill-posed problem (1) in a well-posed problem. A. N. Tikhonov [11] and others considered the positive quadratic functional $Q_\alpha : W \to \mathbf{R}$, $\alpha > 0$,

$$Q_\alpha(w) = \|Tw - y\|^2 + \alpha \, \|Lw\|^2$$

and computed the minimum w_α^{LS} of Q_α

$$Q_\alpha(w_\alpha^{LS}) = \min \{Q_\alpha(w) : w \in W\}$$

or $w_{\alpha,n}^{LS} \in W_n$

$$Q_\alpha(w_{\alpha,n}^{LS}) = \min \{Q_\alpha(w) : w \in W_n\},$$

where W_n is an n-dimensional subspace of W. In the sequel I will call this method the *least-square-regularization.*

By *Ritz-regularization* I will denote the following principle: to compute the minimum of the quadratic functional ($\alpha > 0$)

$$R_\alpha(w) = (Tw, w) - 2(\tilde{y}, w) + \alpha(Lw, w).$$

This method is due to M. M. LAURENT'EV [5] and A. B. BAKUSHINSKI [1]. For sake of simplicity I will always assume that T is injective. Also I will assume that $y \in$ Range T. If $\tilde{y}$ is an arbitrary approximation of y, than it is known that the least-square-regularization is not worse than the Ritz-regularization (see e.g. G. VAINIKKO [12]).

2. Ritz-regularization for commuting operators

In the case that the operators T and L commute, I can exactly describe the advantage of the Ritz-solution. Let (τ_j) be the sequence of eigenvalues of T, (λ_j) the sequence of eigenvalues of L, and (u_j) the orthonormalized sequence of the eigenvectors both of (τ_j) and (λ_j). The compactness of T implies $0 < \tau_j \to 0$, the compactness of L^{-1} implies $0 < \lambda_j \to \infty$ or the boundedness of L^{-1} implies $0 < \inf \lambda_j$. In each case $\varepsilon_j = \dfrac{\tau_j}{\lambda_j} \to 0$.

Let

$$\hat{x} = \sum \xi_j u_j .$$

It is well known, that w_α^{LS} and w_α^R have the representations

$$w_\alpha^{LS} = (T^2 + \alpha L^2)^{-1}\, Ty = (T^2 + \alpha L^2)^{-1}\, T^2 \hat{x},$$

$$w_\alpha^R = (T + \alpha L)^{-1}\, y = (T + \alpha L)^{-1}\, T\hat{x}$$

or

$$w_\alpha^{LS} = \sum \frac{\tau_j^2}{\tau_j^2 + \alpha\lambda_j^2}\, \xi_j u_j, \qquad w_\alpha^R = \sum \frac{\tau_j}{\tau_j + \alpha\lambda_j}\, \xi_j u_j .$$

The error norms are

$$\|\hat{x} - w_\alpha^{LS}\|^2 = \sum \left(1 - \frac{\tau_j^2}{\tau_j^2 + \alpha\lambda_j^2}\right)^2 \xi_j^2 = \alpha^2 \sum \left(\frac{\lambda_j^2}{\tau_j^2 + \alpha\lambda_j^2}\right)^2 \xi_j^2$$

$$= \alpha^2 \sum \left(\frac{1}{\varepsilon_j^2 + \alpha}\right)^2 \xi_j^2,$$

and

$$\|\hat{x} - w_\alpha^R\|^2 = \sum \left(1 - \frac{\tau_j}{\tau_j + \alpha\lambda_j}\right)^2 \xi_j^2 = \alpha^2 \sum \left(\frac{\lambda_j}{\tau_j + \alpha\lambda_j}\right)^2 \xi_j^2$$

$$= \alpha^2 \sum \left(\frac{1}{\varepsilon_j + \alpha}\right)^2 \xi_j^2 .$$

The linear operator $S_\alpha : H \to H$,

$$S_\alpha x = \sum \frac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha} (x, u_j) u_j,$$

has the property $\hat{x} - w_\alpha^{LS} = S_\alpha(\hat{x} - w_\alpha^R)$, therefore

$$\|\hat{x} - w_\alpha^{LS}\| \leqq \|S_\alpha\| \cdot \|\hat{x} - w_\alpha^R\|, \qquad \|\hat{x} - w_\alpha^R\| \leqq \|S_\alpha^{-1}\| \cdot \|\hat{x} - w_\alpha^{LS}\|.$$

Since S_α is selfadjoint the norms of S_α and S_α^{-1} are

$$\|S_\alpha\| = \sup \frac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha} \quad \text{and} \quad \|S_\alpha^{-1}\| = \sup \frac{\varepsilon_j^2 + \alpha}{\varepsilon_j + \alpha}.$$

The function $f:[0,1] \to \mathbf{R}$ with $f(\varepsilon) = \dfrac{\varepsilon + \alpha}{\varepsilon^2 + \alpha}$ has its maximum for $\varepsilon_0 = \sqrt{\alpha^2 + \alpha} - \alpha$ and it has the value $\|S_\alpha\| \leqq f(\varepsilon_0) = \dfrac{1}{2} \left(\sqrt{\alpha^2 + \alpha} - \alpha\right)^{-1} = O(\alpha^{-1/2})$. From max $\dfrac{1}{f(\varepsilon)} = 1$ it follows $\|S_\alpha^{-1}\| \leqq 1$. This proves the following theorem.

Theorem 1: *For the errors $\hat{x} - w_\alpha^R$ and $\hat{x} - w_\alpha^{LS}$ hold*

$$\|\hat{x} - w^{LS}\| \leqq \frac{1}{2} (\sqrt{\alpha^2 + \alpha} - \alpha)^{-1} \|\hat{x} - w_\alpha^R\|, \qquad \|\hat{x} - w_\alpha^R\| \leqq \|\hat{x} - w_\alpha^{LS}\|.$$

Since $\lim \|S_\alpha\| = \infty$ the Ritz-solution is always better than the least-square-solution. Of course these estimates are the best possible ones. J. N. Franklin [3] compared the minima of the functionals R_α and Q_α, but since the regularization methods are important for $\lim \alpha = 0$ and the numerical problems occur for small α, it is adequate to compare w_α^{LS} with w_α^R.

The computation of w_α^R and w_α^{LS} are also infinite dimensional problems as the solution of (1), therefore one has to ask for the properties of finite dimensional approximations. The optimal n-dimensional approximation of selfadjoint operators is given by approximations which use the eigenfunctions. So if we will compare the optimal least-square solution with the optimal Ritz-solution we have to specialize $W_n = \operatorname{span}\{u_1, u_2, \ldots, u_n\}$.

Corollary 2: *In $W_n = \operatorname{span}\{u_1, \ldots, u_n\}$, for the n-dimensional Ritz-solution $w_{\alpha,n}^R$ and the n-dimensional least-square-solution $w_{\alpha,n}^{LS}$ hold*

$$\|\hat{x} - w_{\alpha,n}^{LS}\| \leqq \frac{1}{2} \left(\sqrt{\alpha^2 + \alpha} - \alpha\right)^{-1} \|\hat{x} - w_{\alpha,n}^R\|, \qquad \|\hat{x} - w_{\alpha,n}^R\| \leqq \|\hat{x} - w_{\alpha,n}^{LS}\|.$$

Proof: Let $P_n : H \to W_n$ be the orthogonal projection. Then by the same computation as in the proof of Theorem 2 one gets

$$\|\hat{x} - w_{\alpha,n}^R\|^2 = \alpha^2 \sum_{j=1}^{n} \left(\frac{1}{\varepsilon_j + \alpha}\right)^2 \xi_j^2 + \|(I - P_n)\hat{x}\|^2,$$

$$\|\hat{x} - w_{\alpha,n}^{LS}\|^2 = \alpha^2 \sum_{j=1}^{n} \left(\frac{1}{\varepsilon_j^2 + \alpha}\right) \xi_j^2 + \|(I - P_n)\hat{x}\|^2.$$

The linear operator $S_{\alpha,n} : H \to H$,

$$S_{\alpha,n} u_j = \begin{cases} \dfrac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha}, & 1 \leqq j \leqq n \\ u_j, & j > n \end{cases}$$

has the property $\hat{x} - w_{\alpha,n}^{LS} = S_{\alpha,n}(\hat{x} - w_{\alpha,n}^R)$. Since $\|S_{\alpha,n}\| \leqq \|S_\alpha\|$, $\|S_{\alpha,n}^{-1}\| \leqq 1$, the corollary follows from Theorem 2 ∎

3. The general case

If T and L do not commute then the exact behavior of the errors is not so easy to compute. But it is possible to obtain a result which shows that also in the general case the Ritz-solution behaves better than the least-square-solution.

Theorem 3: *For the errors $\hat{x} - w_\alpha^R$ and $\hat{x} - w_\alpha^{LS}$ hold*

$$\hat{x} - w_\alpha^{LS} = S_\alpha(\hat{x} - x_\alpha^R), \qquad \hat{x} - x_\alpha^R = S_\alpha^{-1}(\hat{x} - x_\alpha^{LS})$$

where $S_\alpha = I + B_\alpha \cdot C$ with $\|B_\alpha\| \geqq \dfrac{1}{\alpha}$.

Proof: We have again the representations

$$w_\alpha^{LS} = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}, \qquad w_\alpha^R = (T + \alpha L)^{-1} T\hat{x}.$$

The linear operator $S_\alpha : H \to H$ with the property $\hat{x} - w_\alpha^{LS} = S_\alpha(\hat{x} - w_\alpha^R)$ is defined by the equation $I - (T^2 + \alpha L^2)^{-1} T^2 = S_\alpha\big(I - (T + \alpha L)^{-1} T\big)$. Therefore

$$S_\alpha = (T^2 + \alpha L^2)^{-1} (LT + \alpha L^2) = I + (\alpha I + L^{-2}T^2)^{-1} (L^{-1}T - L^{-2}T^2)$$
$$= I + B_\alpha \cdot C.$$

The spectrum of $L^{-2}T^2$ is contained in the non-negative reals, since from $L^{-2}T^2 v = \lambda v$ follows $T^2 v = \lambda L^2 v$, $\|Tv\|^2 = \lambda \|Lv\|^2$, therefore the norm of $B_\alpha = (\alpha I + L^{-2}T^2)^{-1}$ is for $\alpha > 0$ by [2: Cor. 3/p. 566] $\|B\| \geqq \dfrac{1}{\alpha}$ ∎

A similar result is possible for the finite dimensional approximations. Let $W_n \subset W$ be an arbitrary n-dimensional subspace of W and $P_n : H \to W_n$ the orthogonal projection. Then the minimal solutions of Q_α resp. R_α in W_n have the representations

$$w_{\alpha,n}^{LS} = (P_n T^2 P_n + \alpha P_n L^2 P_n)^{-1} P_n T^2 \hat{x},$$
$$w_{\alpha,n}^R = (P_n T P_n + \alpha P_n L P_n)^{-1} P_n T \hat{x}.$$

The operator $S_{\alpha,n} : H \to H$ with $\hat{x} - w_{\alpha,n}^{LS} = S_{\alpha,n}(\hat{x} - w_{\alpha,n}^R)$ is defined by

$$S_\alpha|_{(I-P_n)H} = I - P_n,$$

and, using the abbreviations $T_n = P_n T P_n$, $T_n^2 = P_n T^2 P_n$, $L_n = P_n L P_n$, $L_n^2 = P_n L^2 P_n$, for $x \in P_n H$

$$I - (T_n^2 + \alpha L_n^2)^{-1} T_n^2 = S_{\alpha,n}\big(I - (T_n + \alpha L_n)^{-1} T_n\big).$$

Then again

$$S_{\alpha,n} = I + (\alpha I + L_n^{-2}T_n^2)^{-1} (L_n^{-1}T_n - L_n^{-2}T_n^2)$$

and

$$\|B_{\alpha,n}\| = \|(\alpha I + L_n^{-2}T_n^2)^{-1}\| \geqq \frac{1}{\alpha}.$$

4. Galerkin-regularization

In the case of linear equations of the second kind the Ritz-method leeds to the same equations as the Galerkin-method. But the following example shows that the usual Galerkin method for the equation

$$Tw + \alpha Lw = y$$

does not give a convergent set of solutions w_α with $\lim_{\alpha \to 0} w_\alpha = \hat{x}$.

Example: Let T be selfadjoint, injective and compact with an infinite number of positive and an infinite number of negative eigenvalues. For $\alpha \neq 0$, $\alpha \in \mathbf{R}$ the equation

$$Tw + \alpha w = y$$

has a solution w_α, if $-\alpha$ is not an eigenvalue of T, so it is nonsense to ask for $\lim_{\alpha \to \infty} w_\alpha$ also in the general case.

But I will show that there is a convergent regularization method for non-real parameter α. For $L = I$ this method is due to A. B. Bakushinski [1]. Let $T : H \to H$ be compact and selfadjoint, $L : W \to H$ continuous invertible and selfadjoint, $0 \neq \alpha \in \mathbf{R}$. Then I call the following method *Galerkin-regularization*: To compute a solution $w_{i\alpha}^G \in W + iW$ resp. $w_{i\alpha,n}^G \in W_n + iW_n$ from

$$(Tw + i\alpha Lw - y, v) = 0$$

for all $v \in W + iW$ resp. $v \in W_n + iW_n$.

In the case of commuting operators T and L as in § 2 one get the solution

$$w_{i\alpha}^G = (T + i\alpha L)^{-1}\, T\hat{x} = \sum_{j=1}^{\infty} \frac{\tau_j}{\tau_j + i\alpha\lambda_j}\, \xi_j u_j .$$

$r_\alpha = \operatorname{Re} w_{i\alpha}^G$ and $s_\alpha = -\operatorname{Im} w_{i\alpha}^G$ have the representations

$$r_\alpha = \sum \frac{\tau_j^2}{\tau_j^2 + \alpha^2\lambda_j^2}\, \xi_j u_j , \qquad s_\alpha = \alpha \sum \frac{\lambda_j \tau_j}{\tau_j^2 + \alpha^2 \lambda_j^2}\, \xi_j u_j .$$

This shows

$$\text{a) } r_\alpha = w_{\alpha^2}^{LS}, \qquad \text{b) } \lim_{\alpha \to 0} r_\alpha = \hat{x}, \qquad \text{c) } \lim_{\alpha \to 0} s_\alpha = 0 .$$

In the general case a similar result holds.

Theorem 4: *Let $T : H \to H$, $L : W \to H$ be selfadjoint linear operators, T compact injective, L continuous invertible and L^{-1} compact and positive definit. For real $\alpha \neq 0$ let $w_{i\alpha}^G \in W + iW$ be the solution of the equation*

$$(Tw_{i\alpha}^G + i\alpha L w_{i\alpha}^G - y, v) = 0$$

for all $v \in W + iW$. Then

$$\begin{aligned} &\operatorname{Re} w_{i\alpha}^G = [(L^{-1}T)^2 + \alpha^2 I]^{-1}\, (L^{-1}T)^2\, \hat{x}, \\ &\lim_{\alpha \to 0} \operatorname{Re} w_{i\alpha}^G = \hat{x}, \qquad \lim_{\alpha \to 0} \operatorname{Im} w_{i\alpha}^G = 0 . \end{aligned} \tag{2}$$

Proof: Let $r_\alpha = \operatorname{Re} w_{i\alpha}^G$, $s_\alpha = -\operatorname{Im} w_{i\alpha}^G$. Then $T + i\alpha L$ is continuous invertible. Since $L^{-1}T$ is compact. $L^{-1}T + i\alpha I$ is not invertible only in the case if $-i\alpha$ is an eigenvalue of $L^{-1}T$. But $(Tu, u) + i\alpha(Lu, u) = 0$ contradicts the condition of selfadjointness. The equation $(T + i\alpha L)(r_\alpha - is_\alpha) = T\hat{x}$ is equivalent to $Tr_\alpha + \alpha Ls_\alpha = T\hat{x}$, $\alpha Lr_\alpha - Ts_\alpha = 0$. With $r_\alpha = \frac{1}{\alpha} L^{-1}Ts_\alpha$ one get $TL^{-1}Ts_\alpha + \alpha^2 Ls_\alpha = \alpha T\hat{x}$ and

$$s_\alpha = \alpha[(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T)\,\hat{x},$$
$$r_\alpha = [(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T)^2\,\hat{x}.$$

From

$$\hat{x} - r_\alpha = \left(I - (L^{-1}T)^2 [(L^{-1}T)^2 + \alpha^2 I]^{-1}\right)\hat{x} = \alpha^2[(L^{-1}T)^2 + \alpha^2 I]^{-1}\,\hat{x}$$

it follows

$$(L^{-1}T)^2 (\hat{x} - r_\alpha) = \alpha^2 r. \tag{3}$$

Since L^{-1} is positive definit, it follows

$$\begin{aligned}\alpha^2(Lr_\alpha, r_\alpha) &\leqq \alpha^2(Lr_\alpha, r_\alpha) + \left(L^{-1}T(\hat{x} - r_\alpha), T(\hat{x} - r_\alpha)\right)\\ &\leqq \alpha^2(Lr_\alpha, r_\alpha) + \left(L(L^{-1}T)^2 (\hat{x} - r_\alpha), \hat{x} - r_\alpha\right)\\ &\leqq \alpha^2(Lr_\alpha, r_\alpha) + \alpha^2(Lr_\alpha, \hat{x} - r_\alpha) \leqq \alpha^2(Lr_\alpha, \hat{x}).\end{aligned}$$

Therefore for all $\alpha > 0$

$$w_\alpha \in K := \{w \in W : (Lw, w) \leqq (Lw, \hat{x})\}.$$

But K is contained in the relativ compact set $\{w \in W : (Lw, w) \leqq (L\hat{x}, \hat{x})\}$ because $w \in K$ implies $\|L^{1/2}w\|^2 \leqq (L^{1/2}w, L^{1/2}\hat{x}) \leqq \|L^{1/2}w\| \cdot \|L^{1/2}\hat{x}\|$. The injective mapping $(L^{-1}T)^2|_{\bar{K}} : \bar{K} \to \overline{(L^{-1}T)^2 K}$ is continuous invertible, therefore (3) implies $\lim r_\alpha = \hat{x}$ and therefore $\lim s_\alpha = 0$ ∎

The representation

$$w_\alpha^{LS} = [L^{-2}T^2 + \alpha^2 I]^{-1} L^{-2}T^2\hat{x}$$

shows that r_α differs from w_α^{LS} only by a quantity which is given by the measure of non-commutativity of L and T.

C. W. GROETSCH [4] discusses the convergence of the finite dimensional approximations $w_{\alpha,n}^{LS}$ in the case $L = I$. In this case $r_{\alpha,n} = w_{\alpha^2,n}^{LS}$, so his convergence properties also hold in the case of finite dimensional Galerkin-approximations.

5. Condition numbers

The approximate solutions $w_{\alpha,n}^{LS}$, $w_{\alpha,n}^{R} \in W_n$ resp. $w_{i\alpha,n}^{G} \in W_n + iW_n$ of the least-square-regularization, Ritz-regularization resp. Galerkin-regularization are determined by systems of linear equations. Here I will discuss the conditions of the systems. The condition of a matrix A is defined by

$$\varkappa = \|A\| \cdot \|A^{-1}\|.$$

In the case of Hilbert space operators it is adequate to use the spectral norms of the matrices.

For arbitrary subspaces W_n and non-commuting operators T and L it seems to be impossible to find a satisfactory estimate of the conditions. But it is easy to compute the conditions in the case of optimal choice of W_n for commuting operators and to compare these optimal conditions.

The following general lemma is folklore.

Lemma 5: *Let $S : H \to H$ be a selfadjoint compact positive definite linear operator. Let (μ_n) be the monotone ordered sequence of eigenvalues of S. Let $\hat{P}_n$ be the orthogonal projection onto the space of the first n eigenvectors and P_n an orthogonal projection onto an arbitrary n-dimensional subspace with*

$$\|S\| = \|\hat{P}_n S \hat{P}_n\| = \|P_n S P_n\|.$$

Then the optimal condition of the projected operator is

$$\varkappa_{\text{opt}} = \varkappa(\hat{P}_n S \hat{P}_n) = \frac{\mu_1}{\mu_n} \leqq \varkappa(P_n S P_n).$$

Proof: The extremal principle of Poincaré and Fischer states

$$\mu_n = \max_{H_n \subset H} \min_{x \in H_n} \frac{(Sx, x)}{(x, x)} \geqq \min_{x \in P_n H} \frac{(Sx, x)}{(x, x)} = \min_{x \in P_n H} \frac{(P_n S P_n x, x)}{(x, x)} = \mu_n(P_n S P_n) \ ∎$$

To compute the conditions with respect to the solution $\hat{x}$ I assume the following assumptions and normalizations: Let T and L commute, the quotients $\varepsilon_j = \frac{\tau_j}{\lambda_j}$ monotone, $\varepsilon_1 = 1$, $W_n \subset H$ with $u_1 \in W_n$. Then the following theorem is true.

Theorem 6: *For positive definite operators L and T the condition $\varkappa(R_{\alpha,n})$ of the Ritz-system*

$$\sum_{j=1}^{n} \zeta_j[(Tw_j, w_k) + \alpha(Lw_j, w_k)] = (\hat{x}, Tw_k), \qquad k = 1, 2, \ldots, n$$

is

$$\varkappa(R_{\alpha,n}) \geqq \varkappa_{\text{opt}}(R_{\alpha,n}) = \frac{1 + \alpha\varepsilon_n^{-1}}{1 + \alpha},$$

the condition $\varkappa(LS_{\alpha,n})$ of the least-square-system

$$\sum_{j=1}^{n} \zeta_j[(Tw_j, Tw_k) + \alpha(Lw_j, Lw_k)] = (\hat{x}, T^2 w_k), \qquad k = 1, 2, \ldots, n$$

is

$$\varkappa(LS_{\alpha,n}) \geqq \varkappa_{\text{opt}}(LS_{\alpha,n}) = \frac{1 + \alpha\varepsilon_n^{-2}}{1 + \alpha},$$

and the condition $\varkappa(G_{\alpha,n})$ of the Galerkin-system

$$\sum_{j=1}^{n} \zeta_j[(Tw_j, w_k) + i\alpha(Lw_j, w_k)] = (\hat{x}, Tw_k), \qquad k = 1, 2, \ldots, n$$

is

$$\varkappa(G_{\alpha,n}) \geqq \varkappa_{\text{opt}}(G_{\alpha,n}) = \sqrt{\frac{1 + \alpha^2\varepsilon_n^{-2}}{1 + \alpha^2}}.$$

Proof: By Lemma 5 the optimal condition is obtained by using the first n eigenfunctions, so the systems of linear equations have diagonal matrices with the elements

$$(1 + \alpha\varepsilon_j^{-1})\,\delta_{jk}, \qquad (1 + \alpha^2\varepsilon_j^{-2})\,\delta_{jk}, \qquad (1 + i\alpha\varepsilon_j^{-1})\,\delta_{jk}$$

for the Ritz-, least-square- resp. Galerkin-system. So the modules of the eigenvalues are obvious ∎

I remark that it follows

$$\varkappa_{\text{opt}}(G_{\alpha,n}) = \sqrt{\varkappa_{\text{opt}}(LS_{\alpha^2,n})} \approx \varkappa_{\text{opt}}(R_{\alpha,n}).$$

This shows again the advantage of the Ritz- and the Galerkin-regularization.

REFERENCES

[1] БАКУШИНСКИЙ, А. Б.: Об одном численном методе решения интегральных уравнений Фредгольма I рода. Ж. вычисл. мат. и мат. физ. **5** (1965), 744–749.
[2] DUNFORD, N., and J. T. SCHWARTZ: Linear Operators. I: General Theory. New York—London: Interscience Publ. 1958.
[3] FRANKLIN, J. N.: Minimum Principles for Ill-Posed Problems. SIAM J. Math. Anal. **9** (1978), 638–650.
[4] GROETSCH, C. W.: On a Regularization-Ritz-Method for Fredholm Equations of the First Kind. J. Integral Equations **4** (1982), 173–182.
[5] ЛАВРЕНТЬЕВ, М. М.: О некоторых некорректных задачах математической физики. Новосибирск: Изд-во Сиб. отд-ия Акад. Наук СССР 1962.
[6] LUCAS, M. A.: Regularization. In: The Application and Numerical Solution of Integral Equations (Eds.: R. A. Anderssen, F. R. de Hoog, M. A. Lucas). Amsterdam: Noordhoff 1980.
[7] MARTI, J. T.: On the Convergence of an Algorithm Computing Minimum-Norm Solutions of Ill-Posed Problems. Math. Comp. **34** (1980), 521–527.
[8] NITSCHE, J.: Zur Konvergenz des Ritz'schen Verfahrens und der Fehlerquadratmethode. In: Numerische Mathematik, Differentialgleichungen, Approximationstheorie (Eds.: L. Collatz, G. Meinardus, H. Unger). Basel: Birkhäuser-Verlag 1968.
[9] SCHOCK, E.: Modifizierte Ritz-Verfahren. Computing **7** (1971), 46–52.
[10] STOER, J.: Einführung in die Numerische Mathematik I. Berlin—Heidelberg—New York: Springer-Verlag 1972.
[11] ТИХОНОВ, А. Н.: О регуляризации некорректно поставленных задач. Докл. Акад. Наук СССР **153** (1963), 49–52.
[12] ВАЙНИККО, Г.: Методы решения линейных некорректно поставленных задач. Тарту (СССР): Тартуский Ун-т 1982.

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