Ritz-Regularization versus Least-Square-Regularization. Solution Methods for Integral Equations of the First Kind

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Es werden drei Regularisierungsmethoden diskutiert (Tikhonov-, Ritz- und Galerkin-Methode), und es wird gezeigt, daß im Falle positiv definiter oder selbstadjungierter Operatoren die Ritz- und die Galerkin-Methode (die letzte mit nicht-reellem Parameter) bessere Konvergenzeigenschaften haben als die Tikhonov-Methode.

Обсуждаются три методы регуляризации (методы Тихонова, Ритца и Галеркина) и показывается, что в случае положительно-определенных или самосопряженных операторов методы Ритца и Галеркина (последный - с невещественным параметром) обладают личшим свойством сходимости чем метод Тихонова.

Three principles of regularization methods are discussed (Tikhonov., Ritz- and Galerkinmethod), and it is shown that in the case of positive definite or of selfadjoint operators the Ritz- and the Galerkin-principles (the latter with non-real parameter) have better properties than the Tikhonov-principle.

1. Formulation of the minimum principles

For the (approximate) solution of equations with positive definite operators in a Hilbert space by minimum principles it is common to use a Ritz variational method instead of a least square principle. For linear equations with unbounded operators J. NITSCHE [8] has shown that the Ritz method has better convergence properties than the least square method, for linear equations of the second kind I proved a similar result [9]. For linear equations of the first kind J. N. FRANKLIN [3] discussed both methods. Here I will improve the results of J.N. FRANKLIN and will show that also in the case of the finite dimensional approximations the condition numbers of the systems of linear equations which occur in the computational process have better properties in the Ritz case than in the least square case.

In the sequel H is a real infinite dimensional Hilbert space, W is a dense subspace of H , and

$$
T: H \to H
$$
, $L: W \to H$

are linear operators, both selfadjoint and positive definite, but T is compact and L has a bounded (for instance if $L = I$, $H = W$) or a compact (for instance if L is a differential operator) inverse. For solving equations of the first kind

$$
Tx = i
$$

 (1)

one uses regularization methods to convert the ill-posed problem (1) in a well-posed problem. A. N. TIKHONOV [11] and others considered the positive quadratic functional $Q_a: W \to \mathbb{R}, \alpha > 0$,

$$
Q_a(w) = ||Tw - y||^2 + \alpha ||Lw||^2
$$

and computed the minimum w_a^{LS} of Q_a

$$
Q_{\mathfrak{a}}(w_{\mathfrak{a}}^{LS})=\min\left\{Q_{\mathfrak{a}}(w):w\in W\right\}
$$

or $w_{a,n}^{LS} \in W_n$

 $Q_a(w_{a,n}^{LS}) = \min \{Q_a(w) : w \in W_n\},\$

where W_n is an *n*-dimensional subspace of *W*. In the sequel I will call this method *the least-square-requkriza1ion.*

By *Rilz-regularization. I will* denote the following principle: to compute the minimum of the quadratic functional $(\alpha > 0)$

$$
R_{\alpha}(w)=(Tw, w)-2(\hat{y}, w)+\alpha(Lw, w).
$$

This method is due to M. M. LAURENT'EV $[5]$ and A. B. BAKUSHINSKI $[1]$. For sake of simplicity I will always assume that T_z is injective. Also I will assume that $y \in \text{Range } T$. If \tilde{y} is an arbitrary approximation of y, than it is known that the least-square-regularization is not worse than the Ritz-regularization (see e.g. G . VAT-NIKKO *(121).*

2. Ritz-regularization for commuting operators.

In the case that the operators *T* and *L* commute, I can exactly describe the advan tage of the Ritz-solution. Let (τ_i) be the sequence of eigenvalues of T , (λ_i) the sequence of eigenvalues of L , and (u_i) the orthonormalized sequence of the eigenvectors both of (τ_j) and (λ_j) . The compactness of *T* implies $0 < \tau_j \to 0$, the compactness of L^{-1} or (τ_j) and (λ_j) . The compactness of *L I* implies $0 < \tau_j \rightarrow 0$, the compactness of *L* ² implies $0 < \lambda_j \rightarrow \infty$ or the boundedness of *L*⁻¹ implies $0 < \inf \lambda_j$. In each case *The Exergendariza*
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 $\frac{\tau_j}{\lambda_j} \rightarrow 0$. **ators**
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KKO [12]).

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 $\frac{\tau_j}{\lambda_j} \rightarrow 0$.
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 $\hat{x} = \sum \xi_j u_j$.
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\varepsilon_j = \frac{\gamma}{\lambda_j} \to 0
$$

Let

$$
\hat{x} = \sum \xi_i u_i.
$$

It is well known, that w_a^{IS} and w_a^R have the representations

$$
w_{\alpha}{}^{LS} = (T^2 + \alpha L^2)^{-1} \, Ty = (T^2 + \alpha L^2)^{-1} \, T^2 \hat{x},
$$

$$
w_{\alpha}{}^{R} = (T + \alpha L)^{-1} \, y = (T + \alpha L)^{-1} \, T \hat{x}
$$

or •

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\n $\varepsilon_j = \frac{\tau_j}{\lambda_j} \to 0$.
\nLet
\n $\hat{x} = \sum_i \xi_j u_j$.
\nIt is well known, that w_a ^{*LS*} and w_a ^{*R*} have the representations
\n w_a ^{*LS*} = $(T^2 + \alpha L^2)^{-1} T y = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}$,
\n w_a ^{*R*} = $(T + \alpha L)^{-1} y = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}$,
\nor
\n w_a ^{*LS*} = $\sum \frac{\tau_j^2}{\tau_j^2 + \alpha \lambda_j^2} \xi_j u_j$, w_a ^{*R*} = $\sum \frac{\tau_j}{\tau_j + \alpha \lambda_j} \xi_j u_j$.
\nThe error norms are
\n $||\hat{x} - w_a$ ^{*LS*}||² = $\sum \left(1 - \frac{\tau_j^2}{\tau_j^2 + \alpha \lambda_j^2}\right)^2 \xi_j^2 = \alpha^2 \sum \left(\frac{\lambda_j^2}{\tau_j^2 + \alpha \lambda_j^2}\right)^2 \xi_j^2$
\n $= \alpha^2 \sum \left(\frac{1}{\epsilon_j^2 + \alpha}\right)^2 \xi_j^2$,
\nand
\n $||\hat{x} - w_a$ ^{*R*}||² = $\sum \left(1 - \frac{\tau_j}{\tau_j + \alpha \lambda_j}\right)^2 \xi_j^2 = \alpha^2 \sum \left(\frac{\lambda_j}{\tau_j + \alpha \lambda_j}\right)^2 \xi_j^2$
\n $= \frac{\lambda}{\tau} \sum \left(1 - \frac{\tau_j}{\tau_j + \alpha \lambda_j}\right)^2 \xi_j^2 = \alpha^2 \sum \left(\frac{\lambda_j}{\tau_j + \alpha \lambda_j}\right)^2 \xi_j^2$

$$
w_{a}^{L3} = \sum \frac{1}{\tau_{j}^{2} + \alpha \lambda_{j}^{2}} \xi_{j} u_{j}, \qquad w_{a}^{L} = \sum \frac{1}{\tau_{j} + \alpha \lambda_{j}} \xi_{j} u_{j}.
$$

The error norms are
\n
$$
||\hat{x} - w_{a}^{LS}||^{2} = \sum \left(1 - \frac{\tau_{j}^{2}}{\tau_{j}^{2} + \alpha \lambda_{j}^{2}}\right)^{2} \xi_{j}^{2} = \alpha^{2} \sum \left(\frac{\lambda_{j}^{2}}{\tau_{j}^{2} + \alpha \lambda_{j}^{2}}\right)^{2} \xi_{j}^{2}
$$
\n
$$
= \alpha^{2} \sum \left(\frac{1}{\epsilon_{j}^{2} + \alpha}\right)^{2} \xi_{j}^{2},
$$
\nand
\n
$$
||\hat{x} - w_{a}^{R}||^{2} = \sum \left(1 - \frac{\tau_{j}}{\tau_{j} + \alpha \lambda_{j}}\right)^{2} \xi_{j}^{2} = \alpha^{2} \sum \left(\frac{\lambda_{j}}{\tau_{j} + \alpha \lambda_{j}}\right)^{2} \xi_{j}^{2}
$$
\n
$$
= \alpha^{2} \sum \left(\frac{1}{\epsilon_{j} + \alpha}\right)^{2} \xi_{j}^{2}.
$$

 $\frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{n} \frac{1}{2} \sum_{j=1}^{n$

The linear operator $S_a: H \to H$,

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\text{Ritz-Regularizat}
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\nThe linear operator $S_a: H \to H$,

\n
$$
S_a x = \sum_{i} \frac{\varepsilon_i + \alpha}{\varepsilon_i^2 + \alpha} (x, u_i) u_i,
$$
\nhas the property $\hat{x} - w_a^{LS} = S_a(\hat{x} - w_a^{R})$, therefore

\n
$$
\|\hat{x} - w_a^{LS}\| \leq \|S_a\| \cdot \|\hat{x} - w_a^{R}\|, \qquad \|\hat{x} - w_a\| \leq \varepsilon
$$

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$$
S_a x = \sum \frac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha} (x, u_j) u_j,
$$

property $\hat{x} - w_a{}^{LS} = S_a(\hat{x} - w_a{}^R)$, therefore

$$
\|\hat{x} - w_a{}^{LS}\| \leq \|S_a\| \cdot \|\hat{x} - w_a{}^R\|, \qquad \|\hat{x} - w_a{}^R\| \leq \|S_a{}^{-1}\| \cdot \|\hat{x} - w_a{}^{LS}\|.
$$

Since S_a is selfadjoint the norms of S_a and S_a^{-1} are

$$
S_a x = \sum \frac{1}{\epsilon_i^2 + \alpha} (x, u_i) u_i,
$$

\n
$$
s \text{ the property } \hat{x} - w_a^2 = S_a(\hat{x} - w_a^R), \text{ therefore}
$$

\n
$$
||\hat{x} - w_a^2|| \le ||S_a|| \cdot ||\hat{x} - w_a^R||, \qquad ||\hat{x} - w_a^R|| \le ||S_a^R||.
$$

\n
$$
\text{since } S_a \text{ is selfadjoint the norms of } S_a \text{ and } S_a^{-1} \text{ are}
$$

\n
$$
||S_a|| = \sup \frac{\epsilon_i + \alpha}{\epsilon_i^2 + \alpha} \quad \text{and} \quad ||S_a^{-1}|| = \sup \frac{\epsilon_i^2 + \alpha}{\epsilon_i + \alpha}.
$$

Since S_a is selfadjoint the norms of S_a and S_a^{-1} are
 $\|\S_a\| = \sup_{\epsilon_j^2 + \alpha} \frac{\epsilon_j + \alpha}{\epsilon_j^2 + \alpha}$ and $\|S_a^{-1}\| = \sup_{\epsilon_j^2 + \alpha} \frac{\epsilon_j^2 + \alpha}{\epsilon_j + \alpha}$.

The function $f:[0,1] \to \mathbb{R}$ with $f(\epsilon) = \frac{\epsilon + \alpha}{\epsilon^2 + \alpha}$ has its maxim and it has the value $||S_{\alpha}|| \leq f(\varepsilon_0) = \frac{1}{2}(\sqrt{\alpha^2 + \alpha} - \alpha)^{-1} = O(\alpha^{-1/2})$. From max $||\hat{x} - w_a^{LS}|| \leq ||S_a|| \cdot ||\hat{x} - w_a^{R}||$, wherevers
 $||\hat{x} - w_a^{LS}|| \leq ||S_a|| \cdot ||\hat{x} - w_a^{R}||$, $||\hat{x} - w_a^{R}|| \leq ||S_a^{-1}|| \cdot ||S_a^{R}||$
 $||S_a|| = \sup_{\xi_i} \frac{\varepsilon_i + \alpha}{\varepsilon_i^2 + \alpha}$ and $||S_a^{-1}|| = \sup_{\xi_i} \frac{\varepsilon_i^2 + \alpha}{\varepsilon_i + \alpha}$.

function $f:[0,1] \to$ *•* $\frac{1}{f(s)} = 1$ it follows $||S_a^{-1}|| \leq 1$. This proves the following theorem. $||S_{\alpha}|| = \sup \frac{1}{\epsilon_{f}^{2} + \alpha}$ and $||S_{\alpha}^{-1}|| = \sup \frac{1}{\epsilon_{f} + \alpha}$.

tion $f:[0,1] \to \mathbb{R}$ with $f(\epsilon) = \frac{\epsilon + \alpha}{\epsilon^{2} + \alpha}$ has its maximum for $\epsilon_{0} =$

nas the value $||S_{\alpha}|| \le f(\epsilon_{0}) = \frac{1}{2}(\sqrt{\alpha^{2} + \alpha} - \alpha)^{-1} = O(\alpha^{-1/2})$

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Theorem 1: For the errors $\hat{x} - w_a^R$ and $\hat{x} - w_a^L$ hold •

$$
\|\hat{x} - w^{LS}\| \leq \frac{1}{2} \left(\sqrt{\alpha^2 + \alpha} - \alpha \right)^{-1} \|\hat{x} - w_a^R\|, \quad \|\hat{x} - w_a^R\| \leq \|\hat{x} - w_a^L\|.
$$

Since, $\lim ||S_{\alpha}|| = \infty$ the Ritz-solution is always better than the least-squaresolution. Of course these estimates are the best possible ones. J. N. FRANKLIN [3] compared the minima of the functionals R_a and Q_a but since the regularization methods are important for $\lim \alpha = 0$ and the numerical problems occur for small α , it is adequate to compare *w. LS* with

The computation of w_a^R and w_a^L are also infinite dimensional problems as the solution of (1), therefore one has to ask for the properties of finite dimensional approximations. The optimal n -dimensional approximation of selfadjoint operators is given by approximations which use the eigenfunctions. So if we will compare the $||x - w^{LS}|| \leq \frac{1}{2} \left(\sqrt{x^2 + \alpha} - \alpha \right)^{-1} ||x - w_a^R||, \quad ||x - w_a||$

Since $\lim ||S_a|| = \infty$ the Ritz-solution is always better

solution. Of course these estimates are the best possible on

compared the minima of the functionals R_a optimal least-square solution with the optimal Ritz-solution we have to specialize Since $\lim_{n \to \infty} \|S_{\alpha}\| = \infty$ the Ritz-solution is always better the solution. Of course these estimates are the best possible one compared the minima of the functionals R_a and Q_a , but simethods are important for $\lim_{$ • mations which use the eigenfunctions. So if we will compare obtain the optimal Ritz-solution we have to spe
 \ldots, u_n).
 $W_n = \text{span } \{u_1, \ldots, u_n\}$ for the *n*-dimensional Ritz-solution
 $u_{\alpha,n}^{LS}$ hold
 $\leq \frac{1}{2} \left(\sqrt{\alpha$

Corollary 2: *In* $W_n = \text{span } \{u_1, ..., u_n\}$, *for the n-dimensional Ritz-solution w*

$$
\|\hat{x} - w_{\alpha,n}^{LS}\| \leq \frac{1}{2} \left(\sqrt{\alpha^2 + \alpha} - \alpha \right)^{-1} \|\hat{x} - w_{\alpha,n}^R\|, \qquad \|\hat{x} - w_{\alpha,n}^R\| \leq \|\hat{x} - w_{\alpha,n}^{LS}\|.
$$

Proof: Let $P_n : H \to W_n$ be the orthogonal projection.' Then by the same com-
tation as in the proof of Theorem 2 one gets
 $\|\hat{x} - w_{\alpha,n}^R\|^2 = \alpha^2 \sum_{n=1}^n \left(\frac{1}{\alpha + \alpha^n} \right)^2 \xi_i^2 + \|(I - P_n) \hat{x}\|^2$, putation as in the proof of Theorem 2 one gets

Corollary 2: In
$$
W_n = \text{span}\{u_1, ..., u_n\}
$$
, for the *n*-dimensional Ritz-solution u
and the *n*-dimensional least-square-solution $w_{\alpha,n}^{LS}$ hold

$$
\|\hat{x} - w_{\alpha,n}^{LS}\| \leq \frac{1}{2} \left(\sqrt{\alpha^2 + \alpha} - \alpha \right)^{-1} \|\hat{x} - w_{\alpha,n}^R\|, \qquad \|\hat{x} - w_{\alpha,n}^R\| \leq \|\hat{x} - w_{\alpha,n}^{LS}\|
$$
Proof: Let $P_n : H \to W_n$ be the orthogonal projection. Then by the same co
putation as in the proof of Theorem 2 one gets

$$
\|\hat{x} - w_{\alpha,n}^R\|^2 = \alpha^2 \sum_{j=1}^n \left(\frac{1}{\epsilon_j + \alpha} \right)^2 \xi_j^2 + \| (I - P_n) \hat{x} \|^2,
$$

$$
\|\hat{x} - w_{\alpha,n}^{LS} \|^2 = \alpha^2 \sum_{j=1}^n \left(\frac{1}{\epsilon_j^2 + \alpha} \right) \xi_j^2 + \| (I - P_n) \hat{x} \|^2.
$$

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The linear operator $S_{a,n}: H \to H$,

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$$
. S $S_{a,n}: H \to H$
\n- 2. $S_{a,n}u_j = \begin{cases} \frac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha}, & 1 \leq j \leq n \\ u_j, & j > n \end{cases}$
\n- 3. U $i < j \leq n$
\n- 4. u_j $j > n$
\n- 5. u_k $j < n$
\n- 6. u_k $j < n$
\n- 7. $u_{k,n} = S_{a,n}(\hat{x} - w_{a,n}^R)$
\n

280 E. Schock

The linear operator $S_{a,n}: H \to H$,
 $S_{a,n}u_j = \begin{cases} \frac{\epsilon_j + \alpha}{\epsilon_j^2 + \alpha}, & 1 \leq j \leq n \\ u_j, & j > n \end{cases}$

has the property $\hat{x} - w_{a,n}^{LS} = S_{a,n}(\hat{x} - w_{a,n}^R)$. Since $||S_{a,n}|| \leq ||S_a||$, $||S_{a,n}^{-1}|| \leq 1$, the corollary fol has the property $\hat{x} - w_{\alpha,n}^{LS} = S_{\alpha,n}(\hat{x} - w_{\alpha,n}^R)$. Since $||S_{\alpha,n}|| \le ||S_{\alpha,n}|| \le 1$, the corollary follows from Theorem 2 **I**

3. The general case

If *T* and *L* do not commute then the exact behavior of the errors is not so easy to compute. But it is possible to obtain a result which shows that also in the general case the Ritz-solution behaves better than the leas compute. But it is possible to obtain a result which shows that also in the general case the Ritz-solution behaves better than the least-square-solution. $S_{a,n}u_j = \begin{cases} \frac{\varepsilon_j + \alpha}{\varepsilon_j^2 + \alpha}, & 1 \leq j \leq n \\ u_j, & j > n \end{cases}$
 Theorem 3: $w_{a,n}^{LS} = S_{a,n}(x - w_{a,n}^R)$ *.* Since $||S_{a,n}|| \leq ||S_a||$, $||S_{a,n}|| \leq ||S_a||$.
 The general case
 T and *L* do not commute then the exact behavior of $(u_j, \quad j > n$

thas the property $\hat{x} - w_{\epsilon,n}^{LS} = S_{\epsilon,n}(\hat{x} - w_{\epsilon,n}^R)$. Since $||S_{\epsilon,n}||$

corollary follows from Theorem 2 \blacksquare

3. The general case

If T and L do not commute then the exact behavior of the compute. But it

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Theorem 3: For the errors
$$
\hat{x} - w_a^R
$$
 and $\hat{x} - w_a^L$ hold.

$$
\hat{x} - w_a{}^{LS} = S_a(\hat{x} - x_a{}^R), \qquad \hat{x} - x_a{}^R = S_a{}^{-1}(\hat{x} - x_a{}^{LS})
$$

Proof: We have again the representations

$$
w_a^{LS} = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}, \qquad w_a^R = (T + \alpha L)^{-1} T \hat{x}
$$

where $S_{\alpha} = I + B_{\alpha} \cdot C$ with $||B_{\alpha}|| \ge \frac{1}{\alpha}$.

Proof: We have again the representations
 $w_{\alpha}{}^{LS} = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}$, $w_{\alpha}{}^R = (T + \alpha L)^{-1} T \hat{x}$.

The linear operator $S_{\alpha} : H \to H$ with the property $\hat{x} - w_{\alpha}{}$ $\hat{x} - w_a{}^{LS} = S_a(\hat{x} - x_a{}^R), \quad \hat{x} - x_a{}^R = S_a{}^{-1}(\hat{x} - x_a{}^{LS})$

where $S_a = I + B_a \cdot C$ with $||B_a|| \ge \frac{1}{\alpha}$.

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 $w_a{}^{LS} = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}, \quad w_a{}^R = (T + \alpha L)^{-1} T \hat{x}$.

The linear operator

 $S_{\alpha} = (T^2 + \alpha L^2)^{-1} (LT + \alpha L^2) = I + (\alpha I + L^{-2}T^2)^{-1} (L^{-1}T - L^{-2}T^2)$
 $= I + B_{\alpha} \cdot C.$

compute. But it is possible

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Theorem 3: For the error
 $\hat{x} - w_a^{LS} = S_a(\hat{x}$

where $S_a = I + B_a \cdot C$ wi

Proof: We have again
 $\hat{w}_a^{LS} = (T^2 + \alpha I)$

The linear operator S_a : *I*

fined by the equat The spectrum of $L^{-2}T^2$ is contained in the non-negative reals, since from $L^{-2}T^2v = \lambda v$ follows $T^2v = \lambda L^2v$, $||Tv||^2 = \lambda ||Lv||^2$, therefore the norm of $B_a = (\alpha I + L^{-2}T^2)^{-1}$ $x - w_a^{2.5} = S_a(x - x_a^n)$, $x - x_a^n = S_a^{-1}(\bar{x} - x_a^{1.5})$

where $S_a = I + B_a \cdot C$ with $||B_a|| \ge \frac{1}{\alpha}$.

Proof: We have again the representations
 $w_a^{LS} = (T^2 + \alpha L^2)^{-1} T^2 \hat{x}$, $w_a^R = (T + \alpha L)^{-1} T \hat{x}$.

The linear operator $S_a : H \to H$ w

A similar result is possible for the finite dimensional approximations. Let $W_n \subset W$ be an arbitrary *n*-dimensional subspace of *W* and $P_n : H \to W_n$ the orthogonal. be an arbitrary *n*-dimensional subspace of *W* and $P_n : H \to W_n$ the orthogonal projection. Then the minimal solutions of Q_a resp. R_a in W_n have the representations $w_{a,n}^{LS} = (P_n T^2 P_n + \alpha P_n L^2 P_n)^{-1} P_n T^2 \hat{x}$,
 $w^R = (P_T$ *L,* $\mathbb{E}_{\mathbf{P}}$ and $\mathbf{E}_{\mathbf{P}}$ and $\mathbf{E}_{\mathbf{P}}$ and $\mathbf{E}_{\mathbf{P}}$ and $\mathbf{E}_{\mathbf{P}}$ is $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ and $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ $\mathbf{E}_{\mathbf{P}}$ A similar result is possible for the finite dimensional approximations. Let

be an arbitrary *n*-dimensional subspace of W and $P_n : H \to W_n$ the or

projection. Then the minimal solutions of Q_a resp. R_a in W_n have the A similar result is possible for the finite dimensional approximations. Let $W_n \subseteq W$
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rojection. Then the minimal solutions of Q_a resp. R_a in

$$
w_{a,n}^{LS} = (P_n T^2 P_n + \alpha P_n L^2 P_n)^{-1} P_n T^2 \hat{x}
$$

$$
w_{a,n}^R = (P_n T P_n + \alpha P_n L P_n)^{-1} P_n T \hat{x}.
$$

The operator $S_{a,n} : H \to H$ with $\hat{x} - w_{a,n}^{LS} = S_{a,n}(\hat{x} - w_{a,n}^R)$ is defined by
 $S_{a|_{(I-P_n)H}} = I - P_n$,

and using the abbreviations $T_a = P T P_a + T a = P T a P_a$ $I_a = P$

$$
S_{\mathfrak{a}|_{(I-P_n)H}} = I - P_n,
$$

and, using the abbreviations $T_n = P_n T P_n$, $T_n^2 = P_n T^2 P_n$, $L_n = P_n L P_n$, $w_{a,n}^R = (P_n T P_n + \alpha P_n L P_n)^{-1} P_n T \hat{x}.$

berator $S_{a,n} : H \to H$ with $\hat{x} - w_{a,n}^{LS} = S_{a,n} (\hat{x} - w_{a,n}^R)$ is de
 $S_{a|_{(I-P_n)H}} = I - P_n,$

using the abbreviations $T_n = P_n T P_n, T_n^2 = P_n'$.
 $P_n L^2 P_n$, for $x \in P_n H$
 $I - (T_n^2 + \alpha L_n^2)^{-1} T_n^2$

$$
I - (T_n^2 + \alpha L_n^2)^{-1} T_n^2 = S_{\alpha,n} (I - (T_n + \alpha L_n)^{-1} T_n).
$$

Then again
\n
$$
S_{a,n} = I + (\alpha I + L_n^{-2}T_n^2)^{-1} (L_n^{-1}T_n - L_n^{-2}T_n^2)
$$
\nand

in
\n
$$
S_{a,n} = I + (\alpha I + L_n^{-2}T_n^2)^{-1} (L_n^{-1}T_n - L_n^{-2}T_n^2)
$$
\n
$$
||B_{a,n}|| = ||(\alpha I + L_n^{-2}T_n^2)^{-1}|| \ge \frac{1}{\alpha}.
$$

Ritz-Regularization for Integral Equations

4. Galerkin-regularization

In the case of linear equations of the second kind the Ritz-method leeds to tequations

8. Galerkin method for the ametion. But the following example In the case of linear equations of the second kind the Ritz-method leeds to the same equations as the Galerkin-method. But the following example shows that the usual Galerkin method for the equation

$$
Tw + \alpha Lw = y
$$

does not give a convergent set of solutions w_a with $\lim w_a = \hat{x}$.

Example: Let *T* be selfadjoint, injective and compact with an infinite number of positive and an infinite number of negative eigenvalues. For $\alpha \neq 0$, $\alpha \in \mathbf{R}$ the equation **In-regularization**

se of linear equations of the second kind the Ritz-method leeds to the sa

s as the Galerkin-method. But the following example shows that the us

method for the equation
 $Tw + \alpha Lw = y$

give a convergent

$$
Tw + \alpha w = y
$$

has a solution w_a , if $-\alpha$ is not an eigenvalue of T, so it is nonsense to ask for $\lim w_a$ also in the general case.

But I will show that, there is a convergent regularization method for non-real parameter α . For $L = I$ this method is due to A. B. BAKUSHINSKI [1]. Let $T : H \rightarrow H$ be compact and selfadjoint, $L: W \to H$ continuous invertible and selfadjoint, $0 + \alpha \in \mathbf{\hat{R}}$. Then I call the following method *Galerkin-regularization*: To compute

a solution $w_{i\alpha}^G \in W + iW$ resp. $w_{i\alpha,n}^G \in W_n + iW_n$ from
 $(Tw + i\alpha Lw - y, v) = 0$

for all $v \in W + iW$ resp. $v \in W_n + iW_n$.

In the case o a solution $w_{i\alpha}^G \in W + iW$ resp. $w_{i\alpha}^G \in W_n + iW_n$ from Experimentative converger
 W \rightarrow *H* converger
 W \rightarrow *H* converger
 W \rightarrow *H* converger
 w \rightarrow *W n*,
 i \rightarrow *iW*_{*n*},
 i \rightarrow *iW*_{*n*},
 i \rightarrow *iW*_{*n*},
 i \rightarrow *iW*_{*n*},
 i \rightarrow *i i v* rat there is a convergent regularizat *I* this method is due to A. B. BARUSH
fadjoint, $L: W \rightarrow H$ continuous invalled the following method *Galerkin-reg*
iW resp. $w_{ia,n}^G \in W_n + iW_n$ from
 $-y, v) = 0$
ssp. $v \in W_n + iW_n$;
nuting

$$
(Tw + i\alpha Lw - y, v) = 0
$$

for all $v \in W + i\overline{W}$ resp. $v \in W_n + iW_n$,

$$
(Tw + i\alpha Lw - y, v) = 0
$$

for all $v \in W + i\overline{W}$ resp. $v \in W_n + iW_n$,
In the case of commuting operators T and L as in § 2 one g

$$
w_{ia}^G = (T + i\alpha L)^{-1} T\hat{x} = \sum_{j=1}^{\infty} \frac{\tau_j}{\tau_j + i\alpha \lambda_j} \xi_j u_j.
$$

$$
r_a = \text{Re } w_{ia}^G \text{ and } s_a = -\text{Im } w_{ia}^G \text{ have the representations}
$$

$$
r_a = \sum \frac{\tau_j^2}{\tau_j^2 + \alpha^2 \lambda_j^2} \xi_j u_j, \qquad s_a = \alpha \sum \frac{\lambda_j \tau_j}{\tau_j^2 + \alpha^2 \lambda_j^2} \xi_j u_j.
$$
This shows
a) $r_a = w_{ai}^{LS}$, b) $\lim_{s \to 0} r_a = \hat{x}$, c) $\lim_{s \to 0} s_a = 0$.
In the general case a similar result holds.
Theorem 4: Let $T : H \to H$, $L : W \to H$ be selfadjoint linear
injective L continuous invertible and L^{-1} compact and positive

case of commuting operators
$$
T
$$
 and L as in § 2 one ge
\n
$$
w_{i\alpha}^G = (T + i\alpha L)^{-1} T\hat{x} = \sum_{j=1}^{\infty} \frac{\tau_j}{\tau_j + i\alpha \lambda_j} \xi_j u_j.
$$
\n
$$
w_{i\alpha}^G
$$
 and $s_{\alpha} = -\text{Im } w_{i\alpha}^G$ have the representations
\n
$$
r_{\alpha} = \sum \frac{\tau_j^2}{\tau_j^2 + \alpha^2 \lambda_j^2} \xi_j u_j, \qquad s_{\alpha} = \alpha \sum \frac{\lambda_j \tau_j}{\tau_j^2 + \alpha^2 \lambda_j^2} \xi_j u_j.
$$
\n
$$
w_{\beta}
$$
\nas\n
$$
r_{\alpha} = w_{\alpha}^{LS}, \qquad b) \lim_{\alpha \to 0} r_{\alpha} = \hat{x}, \qquad c) \lim_{\alpha \to 0} s_{\alpha} = 0.
$$

This shows

a)
$$
r_a = w_a^{LS}
$$
, b) $\lim_{\alpha \to 0} r_a = \hat{x}$, c) $\lim_{\alpha \to 0} s_a = 0$.

In the general case a similar result holds.

Theorem 4: Let $T : H \to H$, $L : W \to H$ be selfadjoint linear operators, T compact *injective, L continuous invertible and L⁻¹ compact and positive definit. For real* $\alpha \neq 0$ $w_{ia}^c = (T + i\alpha L)^{-1} T\hat{x} = \sum_{j=1}^{\infty} \frac{\tau_j}{\tau_j + i\alpha \lambda_j} \xi_j u_j$.
 $r_a = \text{Re } w_{ia}^c$ and $s_a = -\text{Im } w_{ia}^c$ have the representations
 $r_a = \sum \frac{\tau_j^2}{\tau_j^2 + \alpha^2 \lambda_j^2} \xi_j u_j$, $s_a = \alpha \sum \frac{\lambda_j \tau_j}{\tau_j^2 + \alpha^2 \lambda_j^2}$

This shows

a) $r_a =$ ℓ *let* $w_{i\alpha}^G \in W + iW$ be the solution of the equation vs

a) $r_a = w_a^{LS}$, b) $\lim_{\alpha \to 0} r_a = \hat{x}$, c) $\lim_{\alpha \to 0}$

neral case a similar result holds.

em 4: Let $T : H \to H$, $L : W \to H$ be .

L continuous invertible and L^{-1} compa
 $W + iW$ be the solution of the equation
 $(Tw_{is}^G$

$$
(Tw_{i\alpha}^G + i\alpha L w_{i\alpha}^G - y, v) = 0
$$

for all $v \in W + iW$. Then

$$
W + iW \tbe the solution of the equation
$$

\n
$$
(Tw_{i_{\alpha}}^G + i\alpha Lw_{i_{\alpha}}^G - y, v) = 0
$$

\n
$$
\in W + iW. \text{ Then}
$$

\n
$$
\text{Re } w_{i_{\alpha}}^G = [(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T)^2 \dot{x},
$$

\n
$$
\lim_{\alpha \to 0} \text{Re } w_{i_{\alpha}}^G = \dot{x}, \qquad \lim_{\alpha \to 0} \text{Im } w_{i_{\alpha}}^G = 0.
$$

Proof: Let $r_a = \text{Re } w_a^c$, $s_a = -\text{Im } w_a^c$. Then $T + i\alpha L$ is continuous invertible. Since $L^{-1}T$ is compact. $L^{-1}T + i\alpha I$ is not invertible only in the case if $-i\alpha$ is an eigenvalue of *L*⁻¹*T*. But $(Tu, u) + i\alpha(Lu, u) = 0$ contradicts the condition of

S

S -

(2)

selfadjointness. The equation $(T + i\alpha L)$ $(r_a - i s_a) = T\hat{x}$ is equivalent to $Tr_a + \alpha L s_a$ $=\hat{T}x, \ \alpha Lr_s - Ts_s = 0.$ With $r_s = \frac{1}{\alpha} L^{-1}Ts_s$ one get $TL^{-1}Ts_s + \alpha^2Ls_s = \alpha T\hat{x}$ and $\begin{align} \begin{aligned} \n\text{d} \alpha L) \left(r_{\sf a} - i s_{\sf a} \right) & = T \hat{x} \text{ is equivalent to} \ \frac{1}{\pi} \, L^{-1} T s_{\sf a} \text{ one get } TL^{-1} T s_{\sf a} + \alpha^2 L s_{\sf a} \ \n\text{d} T \left(r \right) \hat{x}, \ \nT)^2 \, \hat{x}. \ \n\text{d} T \left(r \right) & = \alpha^2 [(L^{-1} T)^2 + \alpha^2 I] \n\end{aligned} \end{align}$ 282 E. ScHOCK

selfadjointness. The equation $(T + i\alpha L) (r_a - i s_a) = T\dot{x}$ is equivalent to T ?
 $= T\dot{x}$, $\alpha Lr_a - Ts_a = 0$. With $r_a = \frac{1}{\alpha} L^{-1}Ts_a$ one get $TL^{-1}Ts_a + \alpha^2 Ls_a =$
 $s_a = [\Delta L^{-1}T]^2 + \alpha^2 I]^{-1} (L^{-1}T)^2 \dot{x}$.

From $\dot{x} - r_a$

$$
s_{\alpha} = \alpha [(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T) \hat{x},
$$

$$
r_{\alpha}=[(L^{-1}T)^2+\alpha^2I]^{-1}(L^{-1}T)^2\,\hat{x}.
$$

$$
s_{\alpha} = \alpha [(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T) \hat{x},
$$

\n
$$
r_{\alpha} = [(L^{-1}T)^2 + \alpha^2 I]^{-1} (L^{-1}T)^2 \hat{x}.
$$

\nFrom
\n
$$
\hat{x} - r_{\alpha} = (I - (L^{-1}T)^2 [(L^{-1}T)^2 + \alpha^2 I]^{-1}] \hat{x} = \alpha^2 [(L^{-1}T)^2 + \alpha^2 I]^{-1} \hat{x}
$$

\nit follows
\n
$$
(L^{-1}T)^2 (\hat{x} - r_{\alpha}) = \alpha^2 r.
$$

\nSince L^{-1} is positive definite, it follows

(3)

From

•

 $\frac{1}{2}$

$$
(L^{-1}T)^2 (\hat{x} - r_{\alpha}) = \alpha^2 r.
$$

$$
= T\hat{x}, \ \alpha Lr_a - Ts_a = 0. \ \text{With } r_a = \frac{1}{\alpha} L^{-1}Ts_a \text{ one get } TL^{-1}Ts_a + c
$$
\n
$$
s_a = \alpha[(L^{-1}T)^2 + \alpha^2I]^{-1} (L^{-1}T) \hat{x},
$$
\n
$$
r_a = [(L^{-1}T)^2 + \alpha^2I]^{-1} (L^{-1}T)^2 \hat{x}.
$$
\nFrom\n
$$
\hat{x} - r_a = (I - (L^{-1}T)^2 [(L^{-1}T)^2 + \alpha^2I]^{-1}) \hat{x} = \alpha^2[(L^{-1}T)^2 - \alpha^2I]^{-1} \hat{x}.
$$
\n
$$
= \alpha^2[(L^{-1}T)^2 (\hat{x} - r_a) = \alpha^2r.
$$
\nSince L^{-1} is positive definite, it follows\n
$$
\alpha^2(Lr_a, r_a) \leq \alpha^2(Lr_a, r_a) + (L^{-1}T(\hat{x} - r_a), T(\hat{x} - r_a))
$$
\n
$$
\leq \alpha^2(Lr_a, r_a) + (L(L^{-1}T)^2 (\hat{x} - r_a), \hat{x} - r_a)
$$
\n
$$
\leq \alpha^2(Lr_a, r_a) + \alpha^2(Lr_a, \hat{x} - r_a) \leq \alpha^2(Lr_a, \hat{x}).
$$
\nTherefore for all $\alpha > 0$ \n
$$
w_a \in K := \{w \in W : (Lw, w) \leq (Lw, \hat{x})\}.
$$

V

Therefore for all $\alpha > 0$

$$
w_{\alpha} \in K := \{ w \in W : (Lw, w) \leq (Lw, \hat{x}) \}.
$$

But *K* is contained in the relative compact set $\{w \in W : (Lw, w) \leq (L\hat{x}, \hat{x})\}$ because Therefore for all $\alpha > 0$
 $w_{\alpha} \in K := \{w \in W : (Lw, w) \leq (Lw, \hat{x})\}.$

But *K* is contained in the relativ compact set $\{w \in W : (Lw, w) \leq (L\hat{x}, \hat{x})\}$ because
 $w \in K$ implies $||L^{1/2}w||^2 \leq (L^{1/2}w, L^{1/2}\hat{x}) \leq ||L^{1/2}w|| \cdot ||L^{1/$ $w_a \in K := \{w \in W : (Lw, w) \leq (Lw, \hat{x})\}.$
But K is contained in the relative compact set $\{w \in W : (Lw, w) \leq (L\hat{x}, \hat{x})\}$ becau $w \in K$ implies $||L^{1/2}w||^2 \leq (L^{1/2}w, L^{1/2}\hat{x}) \leq ||L^{1/2}w|| \cdot ||L^{1/2}\hat{x}||$. The injective mappin $(L^{-1}$ $(L^{-1}T)^2|\vec{g}: \vec{K} \to (L^{-1}T)^2 K$ is continuous invertible, therefore (3) implies $\lim_{\Delta s} x = \hat{x}$ and therefore $\lim_{\Delta s} s = 0$ $\leq \alpha^2(Lr_a, r) \leq \alpha^2(Lr_a, r)$

erefore for all $\alpha > 0$
 $w_a \in K := \{w \in W : (I \in K \text{ is contained in the rel}) \in K \text{ implies } \frac{||L^{1/2}w||^2}{||L^{1/2}w||^2} \leq (L^{1/2} -1T)^2|\overline{\kappa} : \overline{K} \to (L^{-1}T)^2 \overline{K} \text{ is of the} \text{therefore.}$

The representation
 $w_a^{LS} = [L^{-2}T^2 + \alpha^2 I$ $w_a \in K := \{w \in W : (Lw, w) \leq (Lw, \hat{x})\}.$

But K is contained in the relativ compact set $\{w \in W : (Lw, w) \leq (L\hat{x}, \hat{x})\}$ because $w \in K$ implies $||L^{1/2}w||^2 \leq (L^{1/2}w, L^{1/2}\hat{x}) \leq ||L^{1/2}w|| \cdot ||L^{1/2}\hat{x}||$. The injective mapping $(L$

$$
w_{\mathfrak{a}^1}^{LS} = [L^{-2}T^2 + \alpha^2 I]^{-1} L^{-2} T^2 \hat{x}
$$

shows that r_a differs from w_a^{LS} only by a quantity which is given by the measure of non-commutativity of *L* and *T.*

C. W. GROETSCH [4] discusses the convergence of the finite dimensional approximations $w_{\alpha,n}^{LS}$ in the case $L = I$. In this case $r_{\alpha,n} = w_{\alpha,n}^{LS}$, so his convergence proper-
ties also hold in the case of finite dimensional Galerkin-approximations.

5. Condition numbers \mathcal{L} .

The approximate solutions $w_{\alpha,n}^{LS}$, $w_{\alpha,n}^R \in W_n$ resp. $w_{\alpha,n}^G \in W_n + iW_n$ of the leastsquare-regularizatioh, Ritz-regularization resp. Galerkin-regularization are determined by systems of linear equations. Here I will discuss the conditions of the systems. $w_4^{L_8} = [L^{-2}T^2 + \alpha^2I]^{-1} L^{-2}T^2\dot{x}$

shows that r_a differs from $w_4^{L_8}$ only by a quantity which is a

non-commutativity of L and T.

C. W. GROTSCH [4] discusses the convergence of the fini-

imations $w_{a,n}^{L_8$ The condition of a matrix A is defined by
 $\varkappa = ||A|| \cdot ||A^{-1}||.$ 5. **Condition numbers**

The approximate solutions $w_{n,n}^{LS}$, $w_{n,n}^R \in W_n$ resp. $w_{n,n}^C \in W_n + iW$

square-regularization, Ritz-regularization resp. Galerkin-regulariza

mined by systems of linear equations. Here I will d

$$
\varkappa=\|A\|\cdot\|A^{-1}\|.
$$

In the case of Hilbert space operators it is adequate to use the spectral norms of the matrices.

For arbitrary subspaces W_n and non-commuting operators T and L it seems to be .impossible to find a satisfactory estimate of the conditions. But it is easy to corn- pute the conditions in the case of optimal choice of W_n for commuting operators and to compare these optimal conditions. differently subspaces W_n and hon-commuting possible to find a satisfactor state of the case of Hilbert space operators it is adequated atrices.
For arbitrary subspaces W_n and non-commuting possible to find a satisfact

Lemma 5 : Let $S: H \rightarrow H$ be a selfadjoint compact positive definite linear operator. Let (μ_n) be the monotone ordered sequence of eigenvalues of S. Let P_n be the orthogonal projection onto the space of the first n eigenvectors and P_n an orthogonal projection onto an arbitrary n-dimensional subspace with

$$
||S|| = ||P_n S P_n|| = ||P_n S P_n||
$$

Then the optimal condition of the projected operator is

$$
\varkappa_{\rm opt} = \varkappa(\hat{P}_n S \hat{P}_n) = \frac{\mu_1}{\mu_n} \leq \varkappa(P_n S P_n).
$$

Proof: The extremal principle of Poincaré and Fischer states

$$
u_n = \max_{H_n \subset H} \min_{x \in H_n} \frac{(Sx, x)}{(x, x)} \ge \min_{x \in P_n H} \frac{(Sx, x)}{(x, x)} = \min_{x \in P_n H} \frac{(P_n S P_n x, x)}{(x, x)} = \mu_n (P_n S P_n)
$$

To compute the conditions with respect to the solution \hat{x} I assume the following

assumptions and normalizations: Let T and L commute, the quotients $\varepsilon_i =$ monotone, $\varepsilon_1 = 1$, $W_n \subset H$ with $u_1 \in W_n$. Then the following theorem is true.

Theorem 6: For positive definite operators L and T the condition $\mathsf{x}(R_{a,n})$ of the Ritz-system

$$
is
$$

$$
\sum_{i=1}^{n} \zeta_i[(Tw_i, w_k) + \alpha(Lw_i, w_k)] = \langle \hat{x}, Tw_k \rangle, \qquad k = 1, 2, ..., n
$$

$$
\varkappa(R_{\alpha,n})\geq \varkappa_{\text{opt}}(R_{\alpha,n})=\frac{1+\alpha\varepsilon_n^{-1}}{1+\alpha},
$$

the condition $\kappa(LS_{\alpha,n})$ of the least-square-system

$$
\sum_{i=1}^n \zeta_i[(Tw_i, Tw_k) + \alpha(Lw_i, Lw_k)] = (\hat{x}, T^2w_k), \qquad k = 1, 2, ..., n
$$

is

$$
\kappa(LS_{\alpha,n})\geq \kappa_{\text{opt}}(LS_{\alpha,n})=\frac{1+\alpha\epsilon_n^{-2}}{1+\alpha},
$$

and the condition $\kappa(G_{\alpha,n})$ of the Galerkin-system

$$
\sum_{j=1}^n \zeta_j[(T\hat{w}_j, w_k) + i\alpha(Lw_j, w_k)] = (\hat{x}, Tw_k), \qquad k = 1, 2, ..., n
$$

$$
is
$$

$$
\varkappa(G_{\alpha,n}) \geq \varkappa_{\text{opt}}(G_{\alpha,n}) = \sqrt{\frac{1+\alpha^2\varepsilon_n^{-2}}{1+\alpha^2}}.
$$

Proof: By Lemma 5 the optimal condition is obtained by using the first n eigenfunctions, so the systems of linear equations have diagonal matrices with the elements

$$
(1+\alpha \varepsilon_j^{-1}) \, \delta_{jk}, \qquad (1+\alpha^2 \varepsilon_j^{-2}) \, \delta_{jk}, \qquad (1+i\alpha \varepsilon_j^{-1}) \, \delta_{jk}
$$

for the Ritz-, least-square- resp. Galerkin-system. So the modules of the eigenvalues are obvious l

I remark that it follows

$$
\epsilon_{\rm opt}(G_{\alpha,n})=\sqrt{\varkappa_{\rm opt}(LS_{\alpha^*,n})}\approx \varkappa_{\rm opt}(R_{\alpha,n}).
$$

This shows again the advantage of the Ritz- and the Galerkin-regularization.

283

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