On the Solvability of Transonie Potential Flow Problems

M. FEISTAUER and J. NEČAS,

To Prof. Dr. Herbert Beckert on the occasion of his 65th birthday

Wir betrachten Potentialströmungen im schallnahen Bereich. Die Gleichung für das Geschwindigkeitspotential, die eine wirbelfreie, ideale Strömung beschreibt, ist nichtlinear, von zweiter Ordnung und vom gemischten Typ. Bisher ist die Existenz von Lösungen noch nicht bewiesen worden, es gibt nur eine Reihe von numerischen Methoden zur Berechnung solcher Überschallströmungen.

Mit Hilfe des Sekantenverfahrens und einem bequemen Prinzip der optimalen Steuerung konstruieren wir hier ein Funktional *v*, dessen Minimierung zur Lösung des Ausgangsproblems äquivalent ist. Da Schocks auftreten, d. h. Sprunge der Geschwindigkeit, der Dichte und des Drucks, betrachten wir schwache Lösungen im Raum *IV^{1,2}(2)*. Vom physikalischen Standpunkt aus ist die Entropiebedingung entlang des Schocks sehr wichtig. Es gibt verschiedene Möglichkeiten, um diese Bedingung numerisch zu berücksichtigen. Wir betrachten hier eine vereinfachte Form dieser Bedingung, die von Glowinski, Pironneau und anderen benutzt wurde und schlagen gleichzeitig eine kompliziertere Formulierung vor, die sehr natürlich ist...

Wir zeigen, daß diese Bedingungen die zunächst fehlende Kompaktheit kompensieren und können so die Existenz der Lösung im folgenden Sinn beweisen: Wenn die Minimalfolge für das Funktional v der Entropiebedingung genügt und die Geschwindigkeit beschränkt ist, wenn ferner diese Folge schwach gegen die Funktion u konvergiert, dann liegt starke Konvergenz vor, und u ist eine Lösung des Problems im Überschallbereich.

'Diese Arbeit cnthält ferner einige Resultate bezuglich des Unterschallbereichs und der Regularität der Minimalfolge.

Статья посвящена изучению разрешимости задачи околозвукового потенциального течения. Уравнение безвихревого, невязкого, околозвукового течения нелинейно, второго порядка и смешанного типа. Существует ряд численных методов основанных на методах конечных разностей и конечных элементов для расчета околозвуковых течений. Но до сих пор не доказано существование решения этой задачи.

Здесь, воспользуя метод Качанова и удобный принцип оптимального управления, мы конструируем функционал у, минимизация которого эквивалентна решению задачи. Так как разрывы скорости, давления и плотности встречаются в околозвуковых тече-
ниях, мы рассматриваем слабые решения из пространства W^{1,2}(2). С физической точки lever these role summall gegan are valuation at Notwergent, damin legs states tonvergent

Vor, und a ist eine Lösung des Problems im Überschallbereich.

Vor, und a ist eine Lösung des Problems im Überschallbereich des Un зрения очень важно условие энтропии на разрыве. Существуют разные приемы изу--чения этого условия в численных методах. Здесь мы используем простую форму Гловинского, Пиронно и других и кроме того вводим новую естественную, но более сложную формулировку этого условия.

В статье поназано, что эти условия номпенсируют отсутствующую компантность и позволяют доказать существование решения в следующем смысле: Если минимизируюшая последовательность функционала у выполняет (апостериори) условия энтропии II ограниченной скорости и сходится слабо к функции *u*, то она сходится к *u* сильно и *u* нвляется решением задачи.

Нроме того в статье доказаны некоторые результаты касающиеся дозвуковых течений и регулярности минимизирующей последовательности.

The paper is devoted to the study of the solvability of transonic potential flow problems. The velocity potential equation governing irrotational non-viscous transonic flows is nonlinaer,

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second order and of mixed type. There exist a series of numerical methods for the solution of the transonic flows. However, the existence of the solution has not yet been proved.

Here, with the use of the secant-modulus method and a convenient optimal control principle, we construct a functional ψ , whose minimization is equivalent to the solution of the problem. Since the so-called shocks, represented by jumps in the velocity, density and pressure, occur in the flow field, we consider weak solutions from the space $W^{1,2}(\Omega)$. From the physical point of view the entropy condition across the shock is very important. There exist various approaches how to embody this condition into the numerical method. Here we consider its simplified version by Glowinski, Pironneau etc. and besides, we propose its new natural, more complex formulation. aches how to embody this condition into the numerical method. Here we consider its simplified
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We show that these conditions introduce the missing compactness into the problem and of the functional ψ satisfies (a posteriori) the entropy and bounded velocity conditions and • converges weakly to a function n, then it converges strongly to *u* and *u* is a solution of the transonic flow problem.

The paper contains also some results concerning subsonic flows and the regularity of the

Introduction

The study of transonic flows is the centre of general attention for many specialists. It has a great importance in the design of high speed airerafts and highly efficient turbines, compressors and other blade machines with large output.

There exist several mathematical models for the investigation of transonic flows. Very often we meet the model of an irrotational, steady, non-viscous flow, since it is represented by relatively simple equations and gives good results for subsonic stream fields via velocity potential or stream function formulations (cf. e.g. [10, 12, 31]).

In contrast to subsonic irrotational non-viscous flows, where the solvability of boundary value problems for velocity potential (or stream function) can be proved by the monotone operator theory, the mathematical study of transonic flow problems is very difficult. The potential equation is nonlinear and of mixed type, since it is elliptic in a subsonic region and hyperbolic in a supersonic region. The boundary between these regions is not known in advance and is obtained together with the sought solution. Moreover, passing across this boundary is not continuous in general, but it is connected with jumps (called shocks or shock waves) in the velocity, density and pressure. Hence, the concept of a classical solution has no sense and it is necessary to consider generalized weak solutions. If we introduce a weak formulation similarly as in elliptic problems, then the corresponding operator is not monotone and has no compactness properties which would allow the application of the monotone and pseudomonotone operator theory or some compactness results for proving the solvability. Therefore, up to now, there exist no results concerning the existence or uniqueness of the solution to the transonic potential flow problems.

In contrast to the lack of theoretical results, we can meet a- lot of numerical methods for modelling transonic two- and also three-dimensional stream fields past airfoils, cascades of profiles or blades and through channels. Most of these numerical approaches are based on finite-difference methods combined with a successive line over relaxation (SLOR). They are connected with the names COLE, MURMAN, GARA-- REDIAN, KORN. JAMESON, KOZEL, POLAŠEK and others ([6, 16, 20, 21, 23, 24] and the bibliographies therein). Some authors use multigrid techniques for improving the convergence (see e.g. [2, 22]). Remarkable results were obtained by GLOWINSKI, PIRONNEAU, BRISTEAU, PERIAUX, PERRIER and POIRIER ([3, 4, 17-19, 32] and others) who use the finite clement method, least squares and conjugate gradients. Other finite element techniques can be found e.g. in [5, 71.

Numerical results of GLOWLNSKI, P1RONNEAU etc. show that the solutions of the discrete potential transonic flow problems possess nonuniqueness of the solution and that the nonphysical solutions with expansion shocks occur. These solutions are usually very attractive for iterative processes (cf. e.g. [18, 32]). Similar experience with the nonphysical solutions was obtained by FEISTAUER and ORŠULIK [8, 30]. In order to avoid the solutions with nonphysical shocks, we must assume that the velocity .decreases across the shock (entropy condition). This condition is embodied into the numerical methods by various ways. In [16, 20, 21, 23, 24] the 'upwind discretization used in the supersonic region causes an artificial viscosity which depresses the nonphysical shocks. In recent papers $[2, 5, 7]$ an artificial viscosity is introduced by upwinding the density. In $[3, 4, 17-19, 32]$ the entropy condition is considered as a constraint in an optimal control problem and is handled via penalization, regularization, or other optimalization techniques.

Here, in this paper, we try to answer the fundamental question concerning the solvability of the transonic flow problems, formulated for the velocity potential. With the use of the secant modulus methods known also as Katehanov's method, popular in elasticity, we construct a suitable functional whose minimization is equivalent to the solution of the boundary value problem considered. Since this functional is nonconvex, without any compactness properties, it is impossible to apply some known convex analysis results for proving the convergence of a minimizing sequence to a solition. However, if we assume a posteriori that the elements of this sequence satisfy a convenient entropy condition and some regularity assumptions, then we build the missing compactness into the problem and prove the convergence of this sequence to the solution of the transonic flow problem.

This process is constructive. It can be discretized by the finite element method and used as a basis of a new method for the numerical simulation of the transonic flows. This will be the subject-matter of fortheomming papers.

1. Fundamental concepts and equations

Let us consider an *irrotational, steady, adiabatic and isentropic flow of a non-viscous, compressible fluid* in a bounded domain $\Omega \subset \mathbb{R}_n$ ($n = 2$ or $n = 3$). We assume that the boundary $\partial\Omega$ of Ω is Lipschitz-continuous and Ω is simply connected. It means that every two piecewise differentiable curves connecting in Ω arbitrary points x, $y \in \Omega$ are mutually transformable in Ω by a homotopy. The closure of Ω is denoted by Ω . basis of a new method f
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compressible fluid in a bounded domain $\Omega \subseteq \mathbf{R}_n$ ($n = 2$ or $n = 3$). We assume that

the boundary $\partial \Omega$ of Ω is Lipschitz

The flow considered is described by the following equations:

 (1.1)

Now considered is described by the following equations:

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\text{div } (\varrho \mathbf{v}) = 0,
$$
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$$
\sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} = \dots - \frac{1}{\varrho} \frac{\partial p}{\partial x_i}, \quad i = 1, \dots, n,
$$
\n
$$
p = p(\varrho) = c\varrho^{\mathbf{x}},
$$
\n(1.3)

$$
p = p(\varrho) = c\varrho^*,\tag{1.3}
$$

$$
\operatorname{rot} \mathbf{v} = 0,\tag{1.4}
$$

considered in Ω . We use the notation: $v -$ velocity vector with the components (dependent on $x = (x_1, ..., x_n) \in \overline{\Omega}$) continuously differentiable in Ω . In following

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considerations we shall also admit discontinuities of these quantities. Let us remark that (1.1) is the *continuity equation, (1.2)* are the *Euler equations of motion, (1.3) is the condition of the adiabatic, isentropic process* and (1.4) represents the *condition of* 308 M. FEISTAUER and J. NECAS

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Let us introduce the *local speed of sound a* by the relation

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a^2 = \frac{dp}{dq} = p' \tag{1.5}
$$

the irrotational flow.
\nLet us introduce the local speed of sound a by the relation
\n
$$
a^2 = \frac{dp}{d\varrho} = p'
$$
\n(1.5)
\nand the so-called pressure function
\n
$$
\mathcal{P}(\varrho) = \int_{\varrho}^{\varrho} \frac{p'(r)}{\tau} d\tau.
$$
\nWe denote by ϱ_0 , ϱ_0 , a_0 the values of the density, pressure and speed of sound, re-
\nspectively, corresponding to the velocity $\mathbf{v} = 0$. It is easy to see that
\ngrad $\mathcal{P} = \frac{1}{\varrho}$ grad ϱ .
\nFrom this, the relation
\n
$$
\sum_{j=1}^{n} v_j \frac{\partial \mathbf{v}}{\partial x_j} = \frac{1}{2} \text{ grad } |\mathbf{v}|^2 - \mathbf{v} \times \text{rot } \mathbf{v}
$$
\n(1.8)
\nand (1.4) the identity grad $(\mathcal{P} + \frac{1}{2} |\mathbf{v}|^2) = 0$ follows, which implies

We denote by ϱ_0 , ϱ_0 , a_0 the values of the density, pressure and speed of sound, respectively, corresponding to the velocity $v = 0$. It is easy to see that

$$
\text{grad } \mathcal{P} = \frac{1}{\varrho} \text{ grad } p. \tag{1.7}
$$

From this, the relation

By, corresponding to the velocity
$$
\mathbf{v} = 0
$$
. It is easy to see that
\n
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\text{grad } \mathcal{P} = \frac{1}{\varrho} \text{ grad } p.
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$$
\sum_{j=1}^{n} v_j \frac{\partial \mathbf{v}}{\partial x_j} = \frac{1}{2} \text{ grad } |\mathbf{v}|^2 - \mathbf{v} \times \text{rot } \mathbf{v}
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$$
\text{the identity grad } \left(\mathcal{P} + \frac{1}{2} |\mathbf{v}|^2 \right) = 0 \text{ follows, which implies}
$$
\n
$$
\mathcal{P} + \frac{1}{2} |\mathbf{v}|^2 = 0.
$$
\n(1.9)

 $\left(\begin{matrix} \mathbf{\mathbf{\mathcal{P}}}\mathbf{+}\frac{1}{2}\left|\mathbf{v}\right| \end{matrix}\right)$ spectively, corresponding to the grad $\mathcal{P} = \frac{1}{\varrho}$ grad p .

From this, the relation
 $\sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j} = \frac{1}{2}$ grad $|$

and (1.4) the identity grad $(\mathcal{P}$
 $\mathcal{P} + \frac{1}{2} |v|^2 = 0$.

If we use (1.3), (1.

The identity grad
$$
(v^2 + \frac{1}{2} |v|^2) = 0
$$
 follows, when implies

$$
v^2 + \frac{1}{2} |v|^2 = 0.
$$
 (1.9)

If we use (1.3) , (1.5) , (1.6) and (1.9) , we derive the *relation between the density and*

$$
\mathcal{P} + \frac{1}{2} |v|^2 = 0.
$$
\n
$$
P = \frac{1}{2} |v|^2 = 0.
$$
\n
$$
P = \rho_0 \left(1 - \frac{z - 1}{2a_0^2} |v|^2 \right)^{\frac{1}{z - 1}}.
$$
\n
$$
P = \frac{1}{2} |v|^2 \left(\frac{1}{2} \right)^{\frac{1}{z - 1}}.
$$
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P = \frac{1}{2} |v|^2 \left(\frac{1}{2} \right)^{\frac{1}{z - 1}}.
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P = \frac{1}{2} |v|^2 \left(\frac{1}{2} \right)^{\frac{1}{z - 1}}.
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P = \frac{1}{2} |v|^2 \left(\frac{1}{2} \right)^{\frac{1}{z - 1}}.
$$
\n(1.10)

²

² grad $|v|^2 - v \times \text{rot } v$
 p $|v|^2$ $- v \times \text{rot } v$
 *2a*² $|v|^2$ $- 0$ follows, which implies
 *2a*₀² $|v|^2$ $\Big|^{x-1}$.
 *2a*₀² $|v|^2$ $\Big|^{x-1}$.
 *2a*₀² $|v|^2$ $\Big|^{x-1}$.
 *2a*₀² $|v|^2$ $\Big|^{x-1$ On the basis of (1.4) and the assumptions concerning the domain Ω we can introduce the velocity potential due the velocity potential due the velocity potential of $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\varrho(|\nabla \cdot \mathbf{From} | (1.10) \text{ we see } \nabla |\mathbf{v}|^2 \leq \frac{2a_0^2}{\varkappa - 1}$

If $|\mathbf{v}|^2 \leq \frac{2a_0^2}{\varkappa + 1}$ then $|\mathbf{v$ $\varrho = \varrho_0 \left(1 - \frac{z-1}{2a_0^2} |v|^2\right)^{k-1}$. (1.10)

in the basis of (1.4) and the assumptions concerning the domain Ω we can intro-

the velocity potential u : $\Omega \to \mathbb{R}_1$ such that grad $u = \nabla u = \mathbf{v}$. By substit

and (1.4) the identity grad
$$
(\mathcal{P} + \frac{1}{2} |v|^2) = 0
$$
 follows, which implies
\n
$$
\mathcal{P} + \frac{1}{2} |v|^2 = 0.
$$
\n(1.9)
\nIf we use (1.3), (1.5), (1.6) and (1.9), we derive the relation between the density and velocity of the form
\n
$$
\varrho = \varrho_0 \left(1 - \frac{z - 1}{2a_0^2} |v|^2\right)^{\frac{1}{\kappa - 1}}
$$
\n(1.10)
\nOn the basis of (1.4) and the assumptions concerning the domain Ω we can introduce the velocity potential $u : \Omega \to \mathbb{R}_1$ such that grad $u = \nabla u = v$. By substituting into (1.1) we get the equation for the potential u :
\n
$$
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\varrho(|\nabla u|^2) \frac{\partial u}{\partial x_i}\right] = 0.
$$
\n(1.11)
\nFrom (1.10) we see that the density is defined for
\n
$$
|v|^2 \le \frac{2a_0^2}{\kappa - 1}.
$$
\nIf
\n
$$
|v|^2 \le \frac{2a_0^2}{\kappa + 1} \quad \text{in} \quad \Omega_1 \subset \Omega,
$$
\n(1.12)
\nthen $|v| \le u$ in Ω_1 and we say that the flow is subsonic in Ω_1 . On the other hand, if
\n
$$
|v|^2 > \frac{2a_0^2}{\kappa + 1} \quad \text{in} \quad \Omega_2 \subset \Omega,
$$
\n(1.14)

From (1.10) we see that the density is defined for

$$
|z_1 \partial x_1| \overset{c}{\leq} \cdots \overset{c}{\leq} \partial x_i|
$$
\n(1.10) we see that the density is defined for\n
$$
|v|^2 \leq \frac{2a_0^2}{z-1}.
$$
\n
$$
|v|^2 \leq \frac{2a_0^2}{z+1} \quad \text{in} \quad \Omega_1 \subset \Omega,
$$
\n
$$
\leq a \text{ in } \Omega_1 \text{ and we say that the flow is subsonic in } \Omega_1. \text{ On the other hand, if}\n
$$
|v|^2 > \frac{2a_0^2}{z+1} \quad \text{in} \quad \Omega_2 \subset \Omega,
$$
\n(1.14)
$$

if

$$
|\mathbf{v}|^2 \le \frac{2a_0^2}{\mathbf{x} + 1} \quad \text{in} \quad \Omega_1 \subset \Omega,\tag{1.13}
$$

If
 $|v|$
then $|v| \le$

$$
\leq a \text{ in } \Omega_1 \text{ and we say that the flow is subsonic in } \Omega_1. \text{ On the other hand, if}
$$

$$
|v|^2 > \frac{2a_0^2}{\varkappa + 1} \text{ in } \Omega_2 \subset \Omega,
$$
 (1.14)

then $|v| > a$ in Ω_2 and the *flow is supersonic* in Ω_2 . The equation (1.11) is elliptic or hyperbolic in the subsonic or supersonic region, respectively.

It is known, that in a purely subsonic flow, where the equation (1.11) is elliptic, the velocity potential is continuously differentiable. However, if we consider a transonic flow, then the domain Ω can be divided into a subsonic region Ω_1 and a supersonic region Ω_2 . The boundary between Ω_1 and Ω_2 contains usually shocks with jumps in v, p, ρ . It means that the velocity potential u is no more continuously differentiable in Ω . then $|\mathbf{v}| > a$ in Ω_2 and the *flow is supersity*
hyperbolic in the subsonic or supersonic

It is known, that in a purely subsonic fl
velocity potential is continuously differer

flow, then the domain Ω can be divi by with the domain of the domain of the domain of the domain of the bound
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the shock when shock with the shock with $\frac{\partial u}{\partial t}$ = $\frac{\partial u}{\partial t}$ = $\frac{\partial u}{\partial t}$ ($|\nabla u|^2$) $\frac{\partial u}{\partial t}$ then $|\mathbf{v}| > a$ in Ω_2 and the *flow is supersonic* in Ω_2 . The equation (
It is known, that in a purely subsonic flow, where the equation (1
It is known, that in a purely subsonic flow, where the equation (1
region

Across the shock we consider *Prandtl's conditions*

(a)
$$
\frac{\partial u}{\partial t} \Big|^{-1} =
$$

(b)
$$
\varrho(|\nabla u|^2)
$$

 $=\frac{\partial u}{\partial t}\Big|^{+}$
 $\left|\frac{\partial u}{\partial n}\right|^{-} = \varrho(|\nabla u|^{2}) \frac{\partial w}{\partial n}\Big|^{+},$

where $-$ or $+$ denotes the quantities in front of the shock or behind the shock; respectively. By $\partial/\partial t$ and $\partial/\partial n$ we denote the derivative with respect to the tangential and normal directions to the shock, respectively. Very important is the *entropy condiom*, then the domain *D* can be divided into a subsonic region *S₁* are α , α , β . The boundary between *Ω₁* and *Ω₂* contains usually shocks in *g₁*, *D₅ D₁* means that the velocity potential *u* $\begin{vmatrix} \frac{\partial u}{\partial t} \end{vmatrix} = \frac{\partial u}{\partial t} \end{vmatrix}$
 $\frac{\partial u}{\partial n} \begin{vmatrix} -\frac{\partial u}{\partial n} \end{vmatrix} = o(|\nabla u|^2) \frac{\partial w}{\partial n} \end{vmatrix}^+$,

or + denotes the quantities in front of the shock

ely. By $\partial/\partial t$ and $\partial/\partial n$ we denote the derivative with r
 $\begin{aligned}\n\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial t}\n\end{aligned}$
 $\begin{aligned}\n\frac{\partial u}{\partial n} &= \frac{\partial u}{\partial n} \Big|^{+}, \\
\frac{\partial v}{\partial n} &= \frac{\partial (|\nabla u|^2)}{\partial n} \frac{\partial w}{\partial n} \Big|^{+}, \\
\frac{\partial v}{\partial t} &= \frac{\partial v}{\partial t} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} \frac$ where $-$ or $+$ denotes the quantities in front of the shock or the spectively. By $\partial/\partial t$ and $\partial/\partial n$ we denote the derivative with respectively and normal directions to the shock, respectively. Very important if this a

$$
\mathbf{v}\Vert_{\mathbb{L}}\gg\Vert\mathbf{v}\Vert_{\mathbb{L}}
$$

(the velocity must decrease). In this paper we shall consider the entropy condition formulated with the use of the velocity potential in the form

respectively. By
$$
\partial/\partial t
$$
 and $\partial/\partial n$ we denote the derivative with respect to the tangential
and normal directions to the shock, respectively. Very important is the *entropy con-*
dition across the shock:\n
$$
|v||^{-} > |v||^{+}
$$
\n(1.16)\n\n(the velocity must decrease). In this paper we shall consider the entropy condition
formulated with the use of the velocity potential in the form\n
$$
\int_{\Omega} \rho'(|\nabla u|^2) |\nabla u|^2 \nabla u \cdot \nabla h \, dx \leq M \int h \, dx
$$
\n(1.17)\n
$$
\forall h \in \mathcal{D}(\Omega), \quad h \geq 0
$$
\n(natural form) and also\n
$$
- \int_{\Omega} \nabla u \cdot \nabla h \, dx \leq M \int h \, dx
$$
\n(1.18)\n
$$
\int_{\Omega} \rho' \cdot \nabla h \, dx \leq M \int h \, dx
$$
\n(1.19)\n
$$
\int_{\Omega} \rho u \cdot \nabla h \, dx \leq M \int h \, dx
$$
\n(1.18)\n
$$
\int_{\Omega} \rho h \in \mathcal{D}(\Omega), \quad h \geq 0
$$
\n(simplified form). We denote by $\mathcal{D}(\Omega)$ the set of all functions from $C^{\infty}(\overline{\Omega})$ with compact supports in Ω . $M \in \mathbb{R}_1$ is a convenient constant. The condition (1.18) was used as ρ in [3, 18, 32]. Its advantage is linearity.

$$
-\int_{\Omega} \nabla u \cdot \nabla h \, dx \leq M \int_{\Omega} h \, dx
$$

$$
\forall h \in \mathcal{D}(\Omega), \quad h \geq 0
$$

(simplified form). We denote by $\mathcal{D}(\Omega)$ the set of all functions from $C^{\infty}(\overline{\Omega})$ with compact supports in Ω . $M \in \mathbb{R}_1$ is a convenient constant. The condition (1.18) was used e.g. in [3, 18, 32]. Its advantage is linearity.

Let us remark that in a real flow the transition across the shock is connected with the increase of the entropy and with the rise of the vorticity. It means that our model of the irrotational, isentropic flows can be applied, if we confine ourselves to stream fields with the Mach number $M = |v|/a < 1.6$, where the so-called *weak shocks* occur only. Then the changes in the entropy and the production of the vorticity on the shocks are negligible. α
 $\forall h \in \mathcal{D}(\Omega)$,

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oorts in Ω . $M \in \mathbf{R}_1$ is a convenient constant. The condition (1.18) was used

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ith the rise of the vorticity. It means the

ws can be applied, if we confine ourselv
 $M = |v|/a < 1.6$, where the so-called

the entropy and the production of the

i

This is important for the dependence of the density on the velocity. In the following we shall assume that this dependence is given by a function ϱ with the following

 ϱ and ϱ' are continuous in $[0, +\infty)$,

$$
\varrho(s) = \varrho_0 \left(1 - \frac{\varkappa - 1}{2a_0^2} s \right)^{\frac{1}{\varkappa - 1}} \text{ for } s \in [0, s^*].
$$

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\nwith
$$
s^* \in \left(\frac{2a_0^2}{\alpha+1}, \frac{2a_0^2}{\alpha-1}\right)(s^* \text{ is close to } \frac{2a_0^2}{\alpha-1})
$$
 and
\n $0 < \varrho_{\infty} \le \varrho(s) \le \varrho_0 < +\infty,$
\n $|\varrho'(s) (1 + s)| \le c_0,$
\n $\varrho'(s) \le 0$ (fundamental condition)
\nfor all $s \in [0, +\infty)$. ϱ_{∞} , ϱ_0 and c_0 are constants.
\nThe existence of such ϱ is evident. It can be obtained e.g. by extending
\n $|\begin{bmatrix} 1 & 1 & 2a_0^2 \end{bmatrix}|$

for all $s \in [0, +\infty)$. ϱ_{∞} , ϱ_{0} and c_{0} are constants.

The existence of such ρ is evident. It can be obtained e.g. by extending **[0**, $\frac{1}{9}$ s^* . $+\infty$). ϱ_{∞} , ϱ_{0} and c_{0} are constants.

nee of such ϱ is evident. It can be obtained e.g.
 $+\frac{2a_{0}^{2}}{x-1}$ to $[0, +\infty)$ by a positive constant $\varrho_{\infty} = \varrho\left(\frac{1}{2}, \frac{1}{2}\right)$ $\int_{-1}^{1} (2a_0^2) \sqrt{1}$ 310 M. FEISTAUER and J. NECAS

with $s^* \in \left(\frac{2a_0^2}{\kappa+1}, \frac{2a_0^2}{\kappa-1}\right)(s^*$ is close to $\frac{2a_0^2}{\kappa-1}$ and
 $0 < \varrho_{\infty} \le \varrho(s) \le \varrho_0 < +\infty$,
 $|\varrho'(s)| (1 + s)| \le c_0$,
 $\varrho'(s) \le 0$ (fundamental condition)

for all If s^* is close enough $\frac{2a_0^2}{\pi}$, then for $|v|^2 \in [0, s^*]$ the corresponding Mach number varies from 0 The
 ϱ $\left[\begin{array}{cc} 0, & \dots \\ 0, & \dots \end{array}\right]$
 $\begin{array}{c}\text{the} & \text{the} \\ \text{the} & \text{the} \\ \text{the} & \text{the} \\ \text{the} & \text{the} \end{array}$ $\begin{aligned} \n\varrho & \left(\frac{0}{2} \left(s^* + \frac{-\omega}{\varkappa - 1} \right) \right) \text{ to } [0, +\infty) \text{ by a positive constant } \varrho_{\infty} = \varrho \left(\frac{1}{2} \left(s^* + \frac{2\omega_0}{\varkappa - 1} \right) \right) \text{ and then, by a suitable smoothing in the interval } \left[s^*, \frac{2a_0^2}{\varkappa - 1} \right]. \text{ If } s^* \text{ is close enough to } \frac{2a_0^2}{\varkappa - 1}, \text{ then for } |v|^2 \in [0, s^*] \text{$ of the point $\frac{2a_0^2}{x-1}$ and the extension of ϱ to $[0, +\infty)$ is not significant for the validity of our model from the physical point of view.

In the following, some considerations will be restricted to the str validity of our model from the physical point of view. For a consideration is the obtained eig. by extending
 $\left[\begin{array}{c} 1 & \text{if } 0 & \$

In the following, some considerations will be restricted to the stream fields with the velocity satisfying the condition

$$
|v|^2 \le s_1 < \frac{6a_0^2}{x+1} \tag{1.24}
$$

If $x = 1.4$ (the flow of the air), then (1.24) represents the restriction to flow fields with $M \in [0, 2.23)$.

2. Formulation of the boundary value problem

Since $\partial\Omega$ is Lipschitz-continuous, we can define the $(n - 1)$ -dimensional Lebesgue measure μ_{n-1} on $\partial\Omega$. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup S \cup \mathfrak{M}$, where Γ_i and S are open sets in $\partial\Omega$ and $\mu_{n-1}(\mathfrak{M}) = 0$. We shall consider the *velocity potential equation* **(10)** of the boundary value problem

S. Lipschitz continuous, we can define the $(n - 1)$ -dimensional Lebesgue
 $\begin{aligned}\n\mu_{-1} &\text{ on } \partial\Omega. \text{ Let } \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup S \cup \mathfrak{M}, \text{ where } \Gamma_i \text{ and } S \text{ are open sets in } \partial\Omega \\
\mu_{-1} &\geq 0. \text{ We shall consider the velocity potential equation} \\
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 \therefore

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where Γ_i and S and

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 (1.23) and th **Example 2**
 Example 2
 (a)
 with $M \in [0, 2.23)$.
 2. Formulation of the boundary value problem

Since $\partial \Omega$ is Lipschitz-continuous, we can define to

measure μ_{n-1} on $\partial \Omega$. Let $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup S \cup \mathfrak{M}$, we

and $\mu_{n-1}(\mathfrak{M}) = 0$. We [0, 2.23).
 ulation of the boundary value of the boundary value of the boundary value of μ_{n-1} **on** $\partial \Omega$ **. Let** $\partial \Omega = \Gamma_1 \cup \mathfrak{M}$ **) = 0. We shall consider** $\sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[\varrho(|\nabla u|^2) \frac{\partial u}{\partial x_i} \right] = 0$ **

functi**

$$
\mu_{n-1} \text{ on } \partial \Omega. \text{ Let } \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup S \cup \mathcal{W}, \text{ where } \Gamma_i \text{ and } S \text{ are open sets in } \partial \Omega
$$

\n
$$
(\mathcal{W}) = 0. \text{ We shall consider the velocity potential equation}
$$

\n
$$
\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[\varrho(|\nabla u|^2) \frac{\partial u}{\partial x_i} \right] = 0 \quad \text{in} \quad \Omega
$$

\n
$$
\text{function } \varrho \text{ that has the properties (1.19)–(1.23)) and the boundary condition}
$$

\n
$$
u = 0 \quad \text{on} \quad \Gamma_1,
$$

\n
$$
\varrho(|\nabla u|^2) \frac{\partial u}{\partial n} = g \quad \text{on} \quad S \cup \Gamma_2.
$$

\n
$$
\text{on } S \text{ and } g < 0 \text{ on } \Gamma_2, \text{ then we get the situation corresponding to the flow}
$$

(with the function ϱ that has the properties $(1.19) - (1.23)$) and the *boundary condi*-

(2.2)

$$
\varrho(|\nabla u|^2) \frac{\partial u}{\partial n} = g \quad \text{on} \quad S \cup \Gamma_2. \tag{2.3}
$$

If $g = 0$ on S and $g < 0$ on Γ_2 , then we get the situation corresponding to the flow in a channel whose impermeable walls form the set S and are parallel in a neighbourhood of the outlet Γ_1 , which is normal to S. Γ_2 denotes the inlet. Usually, we assume that $\mu_{n-1}(F_1) > 0$. Sometimes we also admit the possibility $\mu_{n-1}(F_1) = 0$ and then we consider the boundary condition $(|\nabla u|^2) \frac{\partial u}{\partial n} = g$ on $S \cup \Gamma_2$.
 $\ln S$ and $g < 0$ on Γ_2 , then we get therefore whose impermeable walls form therefore whose impermeable walls form that $\mu_{n-1}(\Gamma_1) > 0$. Sometimes we also a onsider the boundary (2.3)

corresponding to the flow

d are parallel in a neigh-

otes the inlet. Usually, we

ossibility $\mu_{n-1}(T_1) = 0$ and

(2.4)

$$
\varrho(|\nabla u|^2)\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega. \tag{2.4}
$$

In this case we assume that

Transonic Potential Flow Problems

\n311

\nbase we assume that

\n
$$
\int_{\partial\Omega} g \, ds = 0.
$$
\n(2.5)

\nif the shocks occur in the stream field, then we assume that the conditions

\n
$$
-b
$$
 are satisfied. (The entropy condition will be considered later.)\nder to introduce a weak formulation of this boundary value problem, we use space

\n
$$
V = \{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\},
$$
\n(2.6)

\n'₁)
$$
> 0
$$
 and the conditions (2.2) and (2.3) are considered, and

- Further, if the shocks occur in the stream field, then we assume that the conditions $(1.15, a - b)$ are satisfied. (The entropy condition will be considered later.)

In order to introduce a weak formulation of this boundary value problem, we define the space

$$
V = \{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\},\tag{2.6}
$$

if $\mu_{n-1}(I_1) > 0$ and the conditions (2.2) and (2.3) are considered, and

Case we assume that

\n
$$
\int g \, ds = 0.
$$
\n(2.5)

\nif the shocks occur in the stream field, then we assume that the conditions

\n-b) are satisfied. (The entropy condition will be considered later.)

\nler to introduce a weak formulation of this boundary value problem, we

\nwe space

\n
$$
V = \{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\},
$$
\n(2.6)

\n1) > 0 and the conditions (2.2) and (2.3) are considered, and

\n
$$
V = \begin{cases} v \in W^{1,2}(\Omega) : \int_a^b v \, dx = 0 \\ 0 \end{cases},
$$
\n(2.7)

\n7) = 0 and we use the condition (2.4). $W_2^1(\Omega)$ is the well-known Sobolev

\nFor the definitions of all spaces used, see e.g. the books of NEAs [27, 28]).

Example that
 $s = 0$.

shocks occure satisfied. (

introduce a

example to $\{v \in W^{1,2}(\Omega)$

and the con
 $\begin{cases} v \in W^{1,2}(\Omega) \\ v \in \text{definitions} \\ v \in L^{\infty}(\Gamma_2 \cup S) \end{cases}$ if $\mu_{n-1}(\Gamma_1) = 0$ and we use the condition (2.4). $W_2^1(\Omega)$ is the well-known Sobolev space. (For the definitions of all spaces used, see e.g. the books of *NEAS* [27, 28]). We assume, that $g \in L^{\infty}(\Gamma_2 \cup S)$ (or $g \in L^{\infty}(\partial \Omega)$). In both cases (2.6) and (2.7), the space *V* can be equipped with the norm r to introduce a weak formulati
space
 $V = \{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_1 \text{ is } 0 \text{ and the conditions (2.2) and } V = \left\{v \in W^{1,2}(\Omega) : \int v dx = 0\right\},$
 $V = \left\{v \in W^{1,2}(\Omega) : \int v dx = 0\right\},$
 $v = 0 \text{ and we use the condition}$
or the definitions of all spaces used
nat $g \in L^{\infty$ $V = \left\{ v \in W^{1,2}(\Omega) : \int v \, dx = 0 \right\},$
 f $\mu_{n-1}(F_1) = 0$ and we use the condition (2.4). $W_2^1(\Omega)$ is th

space. (For the definitions of all spaces used, see e.g. the books

sssume, that $g \in L^{\infty}(\Gamma_2 \cup S)$ (or $g \in L^{\infty$ (2.7)

vell-known Sobolev

NEČAS [27, 28]). We

1 (2.7), the space V

(2.8)

em, if

(2.9)

ons (2.2), (2.3).)

the problem $(2,1)$ to

$$
||u|| = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{1/2}.
$$
\n(2.8)

We say that u is a *weak solution* of the velocity potential problem, if

$$
||u|| = \left(\int_{\alpha} |Vu|^2 dx\right)^{1/2}.
$$
\nWe say that *u* is a *weak solution* of the velocity potential problem, if\n(a) $u \in V$,\n(b) $\int_{\alpha} \rho(|\nabla u|^2) \nabla u \cdot \nabla v dx = \int_{\partial \Omega} g v ds \quad \forall v \in V.$ \n(2.9)\n(b) $\int_{\alpha} \rho(|\nabla u|^2) \nabla u \cdot \nabla v dx = \int_{\partial \Omega} g v ds \quad \forall v \in V.$ \n(2.9)\n(b) By the use of Green's theorem, it is possible to show that both the problem (2.1) to (2.3), (1.15, a - b) and the problem (2.1), (2.4), (1.15, a - b) are formally equivalent to (2.9, a - b). The details are contained in [30].\nLet us put\n
$$
R(s) = \int_{0}^{s} \rho(t) dt \quad \text{for} \quad s \in [0, +\infty)
$$
\nand define the functional $\Phi : V \to \mathbf{R}_1$:\n
$$
\phi(u) = \frac{1}{2} \int_{0}^{u} R(|\nabla u|^2) dx \quad \text{for} \quad u \in V.
$$
\n(2.11)

(We can put e:g. $g = 0$ on Γ_1 in the case of the boundary conditions (2.2), (2.3).)

By the use of Green's theorem it is possible to show that both the problem (2.1) to (2.3), (1.15, a-b) and the problem (2.1) , (2.4) , $(1.15, a-b)$ are formally equivalent to $(2.9, a-b)$. The details are contained in [30]. ^{*s*},
 *u*₂? $\nabla u \cdot \nabla v \, dx = \int_{\partial \Omega} g v \, ds \quad \forall v \in V.$ (2.9)
 g. $g = 0$ on Γ_1 in the case of the boundary conditions (2.2), (2.3).)
 Green's theorem it is possible to show that both the problem (2.1) to
 b) and

Let us put

$$
R(s) = \int_{0}^{s} \varrho(t) dt \quad \text{for} \quad s \in [0, +\infty)
$$
 (2.10)

$$
\varPhi(u) = \frac{1}{2} \int\limits_{\Omega} R(|\nabla u|^2) dx \quad \text{for} \quad u \in V. \tag{2.11}
$$

Let $u \in V$. By $w = w(u)$ we denote the solution of the linear problem

(2.10)
\nto (2.9, a - b). The details are contained in [30].
\nLet us put
\n
$$
R(s) = \int_{0}^{s} \varrho(t) dt \text{ for } s \in [0, +\infty)
$$
\nand define the functional $\Phi : V \to \mathbf{R}_1$:
\n
$$
\Phi(u) = \frac{1}{2} \int_{\Omega} R(|\nabla u|^2) dx \text{ for } u \in V.
$$
\n(2.11)
\nLet $u \in V$. By $w = w(u)$ we denote the solution of the linear problem
\n(a) $w \in V$,
\n(b) $\int_{\Omega} \varrho(|\nabla u|^2) \nabla w \cdot \nabla v dx = \int_{\partial \Omega} g v ds \forall v \in V$
\nand define the functional $\psi : V \to \mathbf{R}_1$ by the relation
\n
$$
\psi(u) = \Phi(u) - \Phi(w(u)) - \int_{\partial \Omega} g u ds + \int_{\partial \Omega} g w(u) ds, u \in V.
$$
\n(2.13)

and define the functional $\psi: V \to \mathbf{R}_1$ by the relation

$$
\psi(u) = \Phi(u) - \Phi\big(w(u)\big) - \int_{\partial\Omega} gu \, ds + \int_{\partial\Omega} gw(u) \, ds, \quad u \in V. \tag{2.13}
$$

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It is evident that a fixed point *u* of the mapping $u \to w(u)$ is a solution of the problem (2.9, a—b): The *fundamental idea* how to find this solution *u* is based on a convenient optimal control principle. Here, we shall *minimize the functional* ψ on the space V. It. means that we seek $u \in V$ such that *(FEISTAUER and J. NECAS*
 (i) (I) (

$$
\psi(u) = \min_{\xi \in V} \psi(\xi). \tag{2.14}
$$

If we find this u , then we have to verify whether the condition (1.12) *(or better* (1.24) *)* and the entropy condition considered, i.e. (1.17) or (1.18) are satisfied, in order to be sure that *u* is a physical solution.

From the numerical point of view, it will be probably suitable to minimize the functional ψ in the set of all $u \in V$ satisfying the finite velocity condition (i.e. (1.12) or (1.24) and the entropy condition $((1.17)$ or (1.18)).

Remark 2.15: Studying the channel flow with the supersonic inlet Γ_2 ; it is necesand the entropy condition considered, i.e. (1.17) or (1.18) are satisfied, in order to
be sure that u is a physical solution.
From the numerical point of view, it will be probably suitable to minimize the
functional ψ On Reinark 2.15: Studying the channel flow with the supersonic inlet Γ_2 , it is necessary to consider both the Neumann condition $\varrho \frac{\partial u}{\partial n} \Big| \Gamma_2 = g$ and the Dirichlet condition $u | \Gamma_2 = u_2$ (with given g and u_2) a dition $u \mid \Gamma_2 = u_2$ (with given g and u_2) at the inlet Γ_2 . Then it is convenient to extend. entropy condition considered, i.e. (1.17) or (1.18) are sa
hat u is a physical solution.
the numerical point of view, it will be probably suitabl
 $|l| \psi$ in the set of all $u \in V$ satisfying the finite velocity co
and the or (1.24)) and the entropy condition ((1.17) or

Reinark 2.15: Studying the channel flow w

sary to consider both the Neumann condition

dition $u | \Gamma_2 = u_2$ (with given g and u_2) at the in

the functional y by the iden sary to consider both the Neumann condition $\varrho \frac{\partial}{\partial n} \left| \int_{2}^{n} = g$ and the Dirichl
dition $u | \Gamma_{2} = u_{2}$ (with given g and u_{2}) at the inlet Γ_{2} . Then it is convenient to
the functional ψ by the identificat

$$
I(u) = \int_{\Gamma_1} |u - u_2|^2 \, ds, \tag{2.16}
$$

added to the right-hand side in (2.13). For simplicity, we shall not deal with this case in the following.

Remark 2.17: The reason for the choice of the functional ψ as the cost function in our optimal control problem will be cleared up in the following section (see Remark

This method, known also as Katchanov's method, is described in a series of papers (e.g. [141), among others also in books [28] and [29]. It plays a fundamental role in our considerations. We shall explicate it in its abstract version according to the cited references.

in the following.

The reason for the choice of the functional ψ as the cost function in

our optimal control problem will be cleared up in the following section (see Remark

3.27).

3. Secant modulus meth **IF** modulus method

in modulus method

in a series of papers

in books [28] and [29]. It plays a fundamental role in

derations. We shall explicate it in its abstract version according to the cited

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be a Hilbert spa *Doming the modulus method*
 Doming others also in books [28] and [29]. It plays a fundamental role in
 Doming others also in books [28] and [29]. It plays a fundamental role in
 B B B **E** *B* **E** *B B B D*

Let V be a Hilbert space with the norm $\|\cdot\|$ and $\Phi: V \to \mathbb{R}_1$ be a functional that has the Gâteaux differential $D\Phi(u, \cdot)$ at every $u \in V$. For each $u \in V$ let us consider a form $B(u, v, w)$, bilinear and symmetric in $v, w \in V$, with the following pro*perties* $(u, v, w \in V)$: *Figure 1.1* and $\Phi: V \to \mathbf{R}_1$ be a fundamental role in
 e shall explicate it in its abstract version according to the cited
 differential DO(u, .) at every $u \in V$. For each $u \in V$ let us con-
 v), bilinear and sy Let V or a time from ill if and $\theta: V \rightarrow \mathbf{R}_1$ be a tunctional
that has the Gateaux differential $D\Phi(u, \cdot)$ at every $u \in V$. For each $u \in V$ let us con-
sider a form $B(u, v, w)$, bilinear and symmetric in $v, w \in V$, with

$$
|B(u, v, w)| \leq c_1 \|v\| \|w\|,
$$
\n(3.1)

$$
B(u, v, v) \geq c_2 ||v||^2, \quad c_2 > 0,
$$
\n(3.2)

$$
D\Phi(u,v) = B(u,u,v), \tag{3.3}
$$

$$
\frac{1}{2} B(u, v, v) - \frac{1}{2} B(u, u, u) - \Phi(v) + \Phi(u) \ge 0.
$$
\n(3.4)

Let us consider a continuous linear functional *f* defined on *V* (i.e. $f \in V^*$, $V^* = \text{dual}$ to *V*). If $u \in V$, then we denote by $w = w(u)$ the solution of the equation

$$
B(u, w, v) = \langle f, v \rangle \ \forall \ v \in V. \tag{3.5}
$$

(The symbol $\langle f, v \rangle$ for $f \in V^*$, $v \in V$ denotes the duality between *V* and V^*). From the Lax-Milgram lemma the existence and uniqueness of such *w* follow.

Theorem 3.6: Let Φ and B have the above properties. Then

Transonic Potential Flow Problems 313
\n*bold* (*f*, *v*) for
$$
f \in V^*
$$
, $v \in V$ denotes the duality between *V* and V^*). From
\nMilgram lemma the existence and uniqueness of such *w* follow.
\n $em 3.6$: Let Φ and *B* have the above properties. Then
\n
$$
\frac{1}{2} c_2 ||u - w(u)||^2 \leq \psi(u) := \Phi(u) - \Phi(w(u)) - \langle f, u - w(u) \rangle.
$$
\n(3.7)
\n $u \in V.$

Proof: Let us put

\n
$$
\text{nbol } \langle f, v \rangle \text{ for } f \in V^*, v \in V \text{ denotes the duality between } V \text{ and } V^* \text{). From Milgram lemma, the existence and uniqueness of such } w \text{ follow.}
$$
\n

\n\n
$$
\text{rem 3.6:} \text{ Let } \Phi \text{ and } B \text{ have the above properties. Then}
$$
\n

\n\n
$$
\frac{1}{2} c_2 \|\|u - w(u)\|^2 \leq \psi(u) := \Phi(u) - \Phi(w(u)) - \langle f, u - w(u) \rangle.
$$
\n

\n\n
$$
u \in V.
$$
\n

\n\n
$$
\text{Let us put}
$$
\n

\n\n
$$
\pi(v) = \Phi(u) - \langle f, v \rangle + \frac{1}{2} B(u, v, v) - \frac{1}{2} B(u, u, u)
$$
\n

\n\n
$$
C(u) = \Phi(u) - \langle f, u \rangle.
$$
\n

\n\n
$$
\text{Im } (3.4) \text{ for } v := w = w(u) \text{ we get}
$$
\n

\n\n
$$
\text{C}(u) = \mathcal{L}(\mathcal{L}) \times \mathcal{L
$$

for every
$$
u \in V
$$
.
\nProof: Let us put
\n
$$
\pi(v) = \Phi(u) - \langle f, v \rangle + \frac{1}{2} B(u, v, v) - \frac{1}{2} B(u, u, u)
$$
\nand
\n
$$
C(u) = \Phi(u) - \langle f, u \rangle.
$$
\n(3.8)

Theorem 3.6: Let
$$
\Phi
$$
 and B have the above properties. Then
\n
$$
\frac{1}{2} c_2 ||u - w(u)||^2 \leq \psi(u) := \Phi(u) - \Phi(w(u)) - \langle f, u - w(u) \rangle.
$$
\n(3.7)
\nfor every $u \in V$.
\nProof: Let us put
\n
$$
\pi(v) = \Phi(u) - \langle f, v \rangle + \frac{1}{2} B(u, v, v) - \frac{1}{2} B(u, u, u)
$$
\n(3.8)
\nand
\n
$$
C(u) = \Phi(u) - \langle f, v \rangle.
$$
\n(3.9)
\nThen, from (3.4) for $v := w = w(u)$ we get
\n
$$
C(w) \leq C(w) + \Phi(u) - \Phi(w) + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u)
$$
\n
$$
= \Phi(u) - \langle f, w \rangle + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u) = \pi(w).
$$
\n(3.10)
\nFurther, if we use the last relation, the equation (3.5) (with $v := u$ or $v := w$) and the properties of B, then
\n
$$
\pi(w) = \Phi(u) - \langle f, w \rangle + \langle f, u \rangle - \langle f, u \rangle + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u)
$$
\n
$$
= C(u) + B(u, w, u) - B(u, w, w) + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u)
$$

Further, if we use the last relation, the equation (3.5) (with $v := u$ or $v := \hat{w}$) and the properties of B , then

$$
\pi(w) = \Phi(u) - \langle f, w \rangle + \langle f, u \rangle - \langle f, u \rangle + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u)
$$

\n
$$
= C(u) + B(u, w, u) - B(u, w, w) + \frac{1}{2} B(u, w, w) - \frac{1}{2} B(u, u, u)
$$

\n
$$
= C(u) - \frac{1}{2} B(u, u - w, u - w) \le C(u) - \frac{1}{2} c_2 ||u - w||^2.
$$
 (3.11)
\nHence, $C(w) \le \pi(w) \le C(u) - \frac{1}{2} c_2 ||u - w||^2$, which immediately gives the assertion
\nof the theorem **1**.
\nTheorem 3.12: If Φ and B , have the above properties and moreover, if
\n $D\Phi(u + h, h) - D\Phi(u, h) \ge c_3 ||h||^2 \forall u, h \in V$ (3.13)
\nwith a constant $c_3 > 0$ (independent of u, h), then there exists a unique critical point u
\nof the functional C, $C(u) = \Phi(u) - \langle f, u \rangle$, h), the define a sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative
\nprocess: " $u \in V$ (is arbitrary) and $u = w(u)$."

Hence, $C(w) \leq \pi(w) \leq C(u) - \frac{1}{2} c_2 ||u - w||^2$, which immediately gives the assertion of the theorem **I**.

Theorem 3.12: $If \Phi$ and B , have the above properties and moreover, if

$$
D\Phi(u+h,h) - D\Phi(u,h) \ge c_3 ||h||^2 \forall u, h \in V. \tag{3.13}
$$

with a constant $c_3 > 0$ (independent of u, h), then there exists a unique critical point u *of the functional C,* $C(u) = \Phi(u) - \langle f, u \rangle$ *. If we define a sequence* $\{u_n\}_{n=0}^{\infty}$ by the iterative *process* " $u_0 \in V$ (*is arbitrary*) and $u_{n+1} = w(u_n)$ for $n \geq 0$ ", then $u_n \to u$ in V, if $n \to +\infty$, and u is a unique minimum point of C.

Proof: The assumptions layed on Φ imply that the functional *C* is coercive, weakly lower semi-continuous, strictly convex and bounded from below. Hence, there exists exactly one critical point *u* of *C*. (See e.g. [15] or [33].) Moreover, *u* is its mini-

- mum point. Now, we have $D\Phi(u, h) = \langle f, h \rangle$ for all $h \in V$ and

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\nmuum point. Now, we have
$$
D\Phi(u, h) = \langle f, h \rangle
$$
 for all $h \in V$ and
\n
$$
c_3 ||u_n - u||^2 \leq D\Phi(u_n, u_n - u) - D\Phi(u, u_n - u)
$$
\n
$$
= B(u_n, u_n, u_n - u) - \langle f, u_n - u \rangle
$$
\n
$$
= B(u_n, u_n, u_n - u) - B(u_n, u_{n+1}, u_n - u)
$$
\n
$$
= B(u_n, u_n - u_{n+1}, u_n - u) \leq c_1 ||u_n - u_{n+1}|| ||u_n - u||.
$$
\nThis yields the estimate
\n
$$
||u_n - u|| \leq \frac{c_1}{c_3} ||u_n - u_{n+1}||.
$$
\n(3.14)
\nFrom (3.7) we see that $0 \leq \frac{1}{2} c_2 ||u_n - u_{n+1}||^2 \leq C(u_n) - C(u_{n+1}).$ Since C is bounded

$$
||u_n - u|| \leq \frac{c_1}{c_3} ||u_n - u_{n+1}||. \tag{3.14}
$$

from below, necessarily $C(u_n) - C(u_{n+1}) \to 0$ for $n \to +\infty$ and hence, $u_n - u_{n+1} \to 0$.
From this and (3.14) we get $u_n \to u$, if $n \to +\infty$ This yields the estimate
 $||u_n - u|| \leq \frac{c_1}{c_3} ||u_n - u_{n+1}||$.

From (3.7) we see that $0 \leq \frac{1}{2} c_2 ||u_n - u_{n+1}||^2 \leq C(n+1)$

From below, necessarily $C(u_n) - C(u_{n+1}) \to 0$ for $n -$

From this and (3.14) we get $u_n \to u$, if $n \to$

Remark 3.15: If the functionals Φ and *B* have the properties $(3.1) - (3.4)$ (the condition (3.13) need not be satisfied in general) and C is bounded from below, then in virtue of (3.7), for $u_{n+1} = w(u_n)$, $u_0 \in V$, we have $\psi(u_n) = C(u_n) - C(u_{n+1}) \to 0$ and $u_n - u_{n+1} \to 0$, if $n \to +\infty$. *f* $u_n - u \| \leq \frac{c_1}{c_3} \|u_n - u_{n+1}\|$
 \Rightarrow v we see that $0 \leq \frac{1}{2} c_2 \|u_n - u_{n+1}\|^2 \leq C(u_n) - C(u_{n+1})$. Since
 w , necessarily $C(u_n) - C(u_{n+1}) \to 0$ for $n \to +\infty$ and hence, u
 u and (3.14) we get $u_n \to u$, if $n \to +\infty$ *B(u, v, w) =f* o(1 Vu1 ²) Vv . Vw *dx, .* V *u, v, w* E *V.*

Example 3.16 (application to the potential compressible flow): Let us consider the functional Φ defined by (2.10) and (2.11), where the function ρ satisfies (1.19)–(1.23). We put or $u_{n+1} = w(u_n)$, $u_0 \in V$, we have $\psi(u_n) = C(u_n) - C(u_{n+1}) \to 0$

if $n \to +\infty$.
 which the potential compressible flow): Let us consider the

by (2.10) and (2.11), where the function ϱ satisfies (1.19)-(1.23).
 w ds,

$$
\langle f, v \rangle = \int_{\partial \Omega} g v \, ds, \quad v \in V; \tag{3.17}
$$

and define the form *B* by the relation

$$
B(u, v, w) = \int_{\Omega} \varrho(|\nabla u|^2) \nabla v \cdot \nabla w \, dx, \quad \forall \ u, v, \ w \in V. \tag{3.18}.
$$

It is easy to see that B **is** *bilinear and symmetric with respect to v and w* and that *the conditions* $(3.1) - (3.3)$ *are satisfied* (concerning (3.3)) – see Lemma 3.21). Let us verify (3.4). This condition can be written in the form

$$
\int_{\Omega} \left[\varrho (|\nabla u|^2) \left(|\nabla v|^2 - |\nabla u|^2 \right) - R(|\nabla v|^2) + R(|\nabla u|^2) \right] dx \ge 0 \tag{3.19}
$$

 $\text{for arbitrary } u, v \in V.$ It will do to show that

$$
\begin{aligned}\n\text{ary } u, \, v \in V. \text{ It will do to show that} \\
\varrho(\alpha) \, (\beta - \alpha) - R(\beta) + R(\alpha) \geq 0, \quad \forall \, \alpha, \beta \in [0, +\infty). \n\end{aligned} \tag{3.20}
$$

In view of the relation $R'(s) = \varrho(s)$ in $[0, +\infty)$, (3.20) is satisfied if and only if the function R is concave. However, this is true, since $R''(s) = \varrho'(s) \leq 0$ in $[0, +\infty)$ (cf. (1.23)). It is evident that the equation $(2.9, b)$ *can be written in the form* $B(u, u, v)$ $=$ $\langle f, v \rangle$ for all $v \in V$. $\int_{\Omega} [Q(|\nabla u|^2) \cdot (|\nabla v|^2 - |\nabla u|^2) - R(|\nabla v|^2) + R(|\nabla u|^2)]$
for arbitrary $u, v \in V$. It will do to show that
 $\varrho(\alpha) (\beta - \alpha) - R(\beta) + R(\alpha) \ge 0, \quad \forall \alpha, \beta \in [0, +]$
In view of the relation $R'(s) = \varrho(s)$ in $[0, +\infty)$, (3.20) is s
fu $c_4 = \cos t$, $\forall u, v \in V$.
 $c_4 = \cos t$, $\forall u, v \in V$, $\forall u, v \in V$, $\forall v, v \in V$, $\forall v \in V$, $\forall v \in V$, $\forall v \in V$, and $P'(s) = \rho'(s) \leq \rho'(s) \leq \rho'(s) \leq \rho'(s) \leq \rho'(s) \leq \rho'(s) \leq \rho'(s)$
 $c_5 = \sqrt{2}$, $c_6 = \sqrt{2}$, $c_7 = \sqrt{2}$, $c_8 = \sqrt{2}$, c

In the following we shall deal with some properties of Φ , C and B from the above example and of the corresponding functional ψ defined by (2.13).

Lemma 3.21: 1. The functional $C(u) = \Phi(u) - \langle f, u \rangle$ is bounded on every bounded *set in V.*

Lemma 3.21: 1. *1*
 in V.

2. $||w(u)|| \leq c_2^{-1}||f||_1$

3. $\Phi(u) \geq \frac{1}{2} e^{-\omega} ||u||_1$ $\ddot{}$, $\forall u \in V.$ 2. $||w(u)|| \le \frac{1}{2}$
3. $\Phi(u) \ge \frac{1}{2}$ e
4. $|C(w(u))| \le$

3.
$$
\Phi(u) \geq \frac{1}{2} \log ||u||^2
$$
, $\forall u \in V$.
4. $|\psi(u_1u_2)|| \leq c = \text{const}$ $\forall u \in V$

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5. *The functional C-is coercive (i.e.* $C(u) \rightarrow +\infty$ *, if* $u \in V$ *and* $||u|| \rightarrow +\infty$ *), <i>Lipschitz-continuou8 and bounded from below.* $C \text{ is } \text{co}$
 bounded
 $u \in V \rightarrow$
 ψ is coercy
 ψ ($\sqrt{\nabla u}$)
 ϕ
 $\int_{\Omega} \phi(|\nabla u|)^2$ **Example 11** Transonic Potential Flow Problems

al C is coercive $(i.e., C(u) \rightarrow +\infty, if u \in V \text{ and } ||u|| \rightarrow +\infty), Lip$

and bounded from below.
 $\begin{aligned}\n\therefore u \in V \rightarrow w(u) \in V^n \text{ is continuous.} \\
u, v \text{ is continuous and } v(u) \ge 0 \text{ for all } u \in V.\n\end{aligned}$ the Gâteaux differentials

6. The mapping $u \in V \rightarrow w(u) \in V$ is continuous.

7. The functional ψ *is coercive, continuous and* $\psi(u) \geq 0$ *for all* $u \in V$ *.*

8. (1) and C have the Gdteaux differentials

$$
D\Phi(u, v) = \int_{Q} \varrho(|\nabla u|^2) \nabla u \cdot \nabla v \, dx = B(u, u, v),
$$

and

\n- \n 5. The functional C is coercive (i.e.,
$$
C(u) \rightarrow +\infty
$$
, if $u \in V$ and $||u|| \rightarrow +\infty$). L bitz-continuous and bounded from below.\n
\n- \n 6. The mapping " $u \in V \rightarrow w(u) \in V$ " is continuous.\n
\n- \n 7. The functional v is coercive, continuous and $v(u) \geq 0$ for all $u \in V$.\n
\n- \n 8. Φ and C have the Gâteaux differentials\n
$$
D\Phi(u, v) = \int_{\Omega} \varrho(|\nabla u|^2) \, \nabla u \cdot \nabla v \, dx = B(u, u, v),
$$
\n
\n- \n 4.
$$
DC(u, v) = \int_{\Omega} \varrho(|\nabla u|^2) \, \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} g v \, ds
$$
\n
$$
= B(u, u, v) - \langle f, v \rangle, \qquad u, v \in V.
$$
\n
\n- \n Proof: 1. We have\n
$$
|C(u)| = |\Phi(u) - \langle f, u \rangle| \leq \frac{1}{2} \int_{\Omega} R(|\nabla u|^2) \, dx + ||f||_{V^*} ||u||.
$$
\n
\n- \n om (1.21) and (2.10), we have\n
$$
\varrho_{\infty} s \leq |R(s)| \leq \varrho_0 s \text{ in } [0, +\infty) \text{ and thus, } |C(u)| = |C(u) - \langle f, u \rangle| \leq \frac{1}{2} \int_{\Omega} R(|\nabla u|^2) \, dx + |U||u||.
$$
\n
\n

$$
|C(u)| = |\Phi(u) - \langle f, u \rangle| \leq \frac{1}{2} \left| \int_{\Omega} R(|\nabla u|^2) dx \right| + ||f||_{V^{\bullet}}||u||.
$$

From (1.21) and (2.10), we have $\varrho_{\infty} s \leq |R(s)| \leq \frac{1}{2} \int_{\Omega} R(|\nabla u|^2) dx + ||f||_{V^*} ||u||.$

From (1.21) and (2.10), we have $\varrho_{\infty} s \leq |R(s)| \leq \varrho_0 s$ in $[0, +\infty)$ and thus, $|C(u)| \leq \frac{1}{2} \varrho_0 ||u||^2 + ||f||_{V^*} ||u||.$ $\begin{aligned}\nD C(u, v) &= \int_{\Omega} \varrho(|\nabla u|^2) \, \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} g v \, ds \\
&= B(u, u, v) - \langle f, v \rangle, \qquad u, v \in V.\n\end{aligned}$ Proof: 1. We have
 $|C(u)| = |\varPhi(u) - \langle f, u \rangle| \leq \frac{1}{2} \left| \int_{\Omega} R(|\nabla u|^2) \, dx \right| + ||f||_{V^*} ||u||.\n\end{aligned}$ pom (1.21) and (2.10) $\begin{aligned} \|u\|^2 + \|f\|_{V^\bullet}\|u\|. \end{aligned}$ is an immediate consequence of $c_2 \|w(u)\|^2 \leq B\big(u,w(u),w(u)\big) = \langle f, w(u) \rangle \leq 0$ 2 | α

3. Similarly as in the assertion 1 we have $\hat{v}_{\infty}s \leq |R(s)| \leq \alpha$

3. Similarly as in the assertion 1 we have

4. $\Phi(u) = \frac{1}{2} \int R(v|u)|^2 \leq B(u, w(u), w(u)) = \langle f, w(u) \rangle \leq$

4. Similarly as in the assertion 1 we have

4. *(u)*] = $|\psi(u) - \langle f, u \rangle| \geq \frac{1}{2} \int_{\Omega} |f(|\nabla u|^2) dx + ||f||_{V^*} ||u||.$

1) and (2.10) we have $\hat{\psi}_{\infty} s \leq |R(s)| \leq \hat{\psi}_{0} s$ in $[0, +\infty)$ and t
 $|\hat{f} + ||f||_{V^*} ||u||.$

assertion is an immediate consequence of (3.2) and (3.5):
 $|C(u)| = |\Phi(u) - \langle f, u \rangle| \leq \frac{1}{2} \left| \int_{\Omega} R(|\nabla u|^2) dx \right| + ||f||_{V^*}||u||.$

com (1.21) and (2.10), we have $\varrho_{\infty} s \leq |R(s)| \leq \varrho_{0} s$ in $[0, +\infty)$ and thus, $|C(t)| \leq \frac{1}{2} \varrho_{0} ||u||^{2} + ||f||_{V^*}||u||.$

2. This assertion is an im

2. This assertion is an immediate consequence of (3.2) and (3.5) :

$$
c_2||w(u)||^2\leq B\big(u,w(u),w(u)\big)=\langle f,w(u)\rangle\leq ||f||_{V^*}||w(u)||.
$$

multiply as in the assertion 1 we have
\n
$$
\Phi(u) = \frac{1}{2} \int_{\Omega} R(|\nabla u|^2) dx \geq \frac{1}{2} \varrho_{\infty} ||u||^2.
$$
\nThis is a consequence of assertion 1 and 2.
\nWe have
\n
$$
C(u) = \Phi(u) - \langle f, u \rangle \geq \frac{1}{2} \varrho_{\infty} ||u||^2 - ||f||
$$
\n
$$
u + \infty
$$
 This and assertion 1 imply the

5. We have

s is a consequence of assertion 1 and 2.
\nhave
\n
$$
C(u) = \Phi(u) - \langle f, u \rangle \geq \frac{1}{2} \varrho_{\infty} ||u||^2 - ||f||_{V^*} ||u|| \to +\infty,
$$

if $||u|| \rightarrow +\infty$. This and assertion 1 imply the boundedness from below of *C*. From the definition of Φ and C and the properties of ρ it follows that both Φ and C are Lipschitz-continuous.

6. The continuity of $w(u)$ is a consequence of results contained in [27: Ch. 3, \S 6]. If $u_n \to u$ in V, then $\varrho(|\nabla u_n|^2) \to \varrho(|\nabla u|^2)$ in measure (in Ω). Moreover, the functions $\varrho(|\nabla u_n|^2)$ are uniformly bounded in Ω . The functions $w(u)$ or $w(u_n)$ are the solutions of the problems (3.5), where we put $u := u$ or $u := u_n$, respectively. I.e., $C(u) = \Phi(u) - \langle f, u \rangle \ge \frac{1}{2} \varrho_{\infty} ||u||^2 - ||f||_{V^*} ||u|| \to +\infty,$
 $\to +\infty$. This and assertion 1 imply the boundedness from bel

tinition of Φ and C and the properties of ϱ it follows that bot

itz-continuous.

he con 5. We have
 $C(u) = \varphi(u) - \langle f, u \rangle \ge \frac{1}{2} \varrho_{\infty} ||u||^2 - ||f||_{V^*} ||u|| \to +\infty$,

if $||u|| \to +\infty$. This and assertion 1 imply the boundedness from below of

the definition of φ and C and the properties of ϱ it follows that $\mathcal{P}(u) - \langle f, u \rangle \geq \frac{1}{2} e^{\frac{1}{2} \omega ||u||^2} - ||f||_{V^*} ||u|| \to +\infty,$

is and assertion 1 imply the boundedness from below of $C_{\mathbf{f}}$ From
 \mathcal{P} and C and the properties of ϱ it follows that both \mathcal{P} and C are
 in $|u_{11}| \rightarrow +\infty$. In a and assertion 1 imply the boundedness if the definition of Φ and C and the properties of ϱ it follows the Lipschitz-continuous.

For $u_n \rightarrow u$ in V , then $\varrho(|\nabla u_n|^2) \rightarrow \varrho(|\nabla u|^2)$ in m

$$
\int_{\Omega} \varrho(|\nabla u|^2) \nabla w(u) \cdot \nabla v \, dx = \langle f, v \rangle \qquad \forall v \in V
$$

$$
\int_{2} \varrho(|\nabla u_{n}|^{2}) \nabla w(u_{n}) \cdot \nabla v \, dx = \langle f, v \rangle \qquad \forall v \in V.
$$

and
 $\int_{\Omega} \varrho(|\nabla u_n|^2) \nabla w(u_n) \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in V.$

Now, by the direct application of [27: Ch. 3, § 6] we get the convergence $w(u_n) \to w(u)$

in V.

7. This assertion follows from the assertions 4–6 and (3.7).

8. This assertion is based on a simple calculation. We leave it to the reader **I**

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Theorem 3:22: *Under the assumption and notation from Example* 3.16 *it holds: 1. There exists a minimizing sequence of the functional*

**2. Every minimizing sequence of the function from Example 3.16 it holds:

2.** *Every minimizing sequence of the functional y.*
 2. *Every minimizing sequence* $\{u_n\}_{n=0}^{+\infty}$ of y is bounded, $\|u_n - w(u_n)\| \to 0$, if $\begin{aligned} \n\text{For } \text{e} \text{ exists a } \text{min} \ \n\text{for } \text{sum} \ x, \text{ and } \{u_n\}_{n=0}^{\infty} \text{ is } \n\text{B}(u_n, u_n, v) = \n\end{aligned}$ 2. Every minimizing sequen
 $w \to +\infty$, and $\{u_n\}_{n=0}^{\infty}$ is generic
 $B(u_n, u_n, v) = \langle f, v \rangle +$

where $F_n \in V^*$ and $||F_n||_{V^*} \to 0$
 Proof : 1 As an example of the

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\nrem 3.22: Under the assumption and notation from Example 3.16 it holds:
\nere exists a minimizing sequence of the functional
$$
\psi
$$
.
\nery minimizing sequence $\{u_n\}_{n=0}^{+\infty}$ of ψ is bounded, $||u_n - w(u_n)|| \to 0$, if
\no, and $\{u_n\}_{n=0}^{\infty}$ is generic, i.e.
\n $B(u_n, u_n, v) = \langle f, v \rangle + \langle F_p, v \rangle \qquad \forall v \in V,$ (3.23)
\n $\{v_n \in V^* \text{ and } ||F_n||_{V^*} \to 0 \text{ for } n \to +\infty.$
\nf: 1. As an example of the minimizing sequence to ψ we can use the sequence

where $F_n \in V^*$ and $||F_n||_{V^*} \to 0$ for $n \to +\infty$.

Proof: 1. As an example of the minimizing sequence to ψ we can use the sequence. $\{u_n\}_{n=0}^{+\infty}$ defined by the iterative process $u_{n+1} = w(u_n)$ with an arbitrary $u_0 \in V$. In Remark 3.15 we have already shown that $\psi(u_n) \to 0$, if $n \to +\infty$. It means that $(u_n) \rightarrow \inf \psi(u) = 0$, since $\psi \geq 0$. Theorem 3.22: Under the assumption and notation from Example 3

1. There exists a minimizing sequence of the functional ψ .

2. Every minimizing sequence $\{u_n\}_{n=0}^{\infty}$ of ψ is bounded, $\|u_n - \to +\infty$, and $\{u_n\}_{n$ im 3.22: Under the assumption and notation from Example 3.16 it holds:

exists a minimizing sequence of the functional ψ .

f minimizing sequence $\{u_n\}_{n=0}^{\infty}$ of ψ is bounded, $||u_n - w(u_n)|| \rightarrow 0$, if and $\{u_n\}_{n=0}^$ $\begin{aligned}\n\langle w, w \rangle &= \langle f, v \rangle + \langle F_n, v \rangle \quad \forall v \in V, \\
\langle w, v \rangle &= \langle f, v \rangle + \langle F_n, v \rangle \quad \forall v \in V,\n\end{aligned}$ (3.23)

and $\|F_n\|_{V^*} \to 0$ for $n \to +\infty$.

an example of the minimizing sequence to ψ we can use the sequence

by the iterativ

2. Let $u_n \in V$,

3.15 we have already shown that
$$
\psi(u_n) \to 0
$$
, if $n \to +\infty$. It means that
\n
$$
\inf_{u \in V} \psi(u) = 0, \text{ since } \psi \ge 0.
$$
\n
$$
\lim_{u \to +\infty} \psi(u_n) = \inf_{u \in V} \psi(u) = 0.
$$
\n(3.24)
\n
$$
\frac{1}{2} c_2 ||u_n - w(u_n)||^2 \le \psi(u_n) \le 1 \quad \forall n \ge n_0.
$$
\n(3.25)
\ne sequence $\{w(u_n)\}_{n=0}^{+\infty}$ is bounded (cf. Lemma 3.21), the sequence $\{u_n\}_{n=0}^{+\infty}$ has
\ne property. From (3.24) and (3.25) we see that $||u_n - w(u_n)|| \to 0$ for $n \to +\infty$.
\ner, in view of (3.5),
\n
$$
B(u_n, w(u_n), v) = \langle f, v \rangle \quad \forall v \in V.
$$
\n(3.26)
\n
$$
\forall n \in V
$$
\n(3.27)

Then there exists n_0 such that

$$
\frac{1}{2} c_2 \|u_n - w(u_n)\|^2 \leq \psi(u_n) \leq 1 \qquad \forall \ n \geq n_0. \tag{3.25}
$$

Since the sequence ${w(u_n)}_{n=0}^{+\infty}$ is bounded (cf. Lemma 3.21), the sequence ${u_n}_{n=0}^{+\infty}$ has $\frac{1}{2} c_2 ||u_n - w(u_n)||^2 \leq \psi(u_n) \leq 1 \quad \forall n \geq n_0.$
Since the sequence $\{w(u_n)\}_{n=0}^{+\infty}$ is bounded (cf. Lemma 3.21), the sequence the same property. From (3.24) and (3.25) we see that $||u_n - w(u_n)|| \rightarrow$ Further, in view of (3.5), the same property. From (3.24) and (3.25) we see that $||u_n - w(u_n)|| \to 0$ for $n \to +\infty$.

Further, in view of (3.5),

Property. From (3.24) and (3.25) we see that
$$
||u_n - w(u_n)|| \to 0
$$
 for $n \to +\infty$.

\nFor, in view of (3.5),

\n
$$
B(u_n, w(u_n), v) = \langle f, v \rangle \quad \forall v \in V.
$$
\n(3.26)

\ndefine $F_n \in V^*$ by the relation

\n
$$
\langle F_n, v \rangle = B(u_n, u_n - w(u_n), v), \quad v \in V.
$$

\nand (3.1) $|\langle F, v \rangle| \leq c_1 ||u_n - w(u_n)||v||$ and hence, $||F_n||_{V_*} \leq c$, $||u_n - w(u_n)||$.

Let us define $F_n \in V^*$ by the relation

$$
\langle F_n, v \rangle = B(u_n, u_n - w(u_n), v), \qquad v \in V.
$$

Let us define $F_n \in V^*$ by the relation
 $\langle F_n, v \rangle = B(u_n, u_n - w(u_n), v), \quad v \in V$.

By this and (3.1), $|\langle F_n, v \rangle| \leq c_1 ||u_n - w(u_n)|| ||v||$ and hence, $||F_n||_{V^*} \leq c_1 ||u_n - w(u_n)||$
 $\to 0$, if $n \to +\infty$. Now it is evident that (3.23) holds \bl $\rightarrow 0$, if $n \rightarrow +\infty$. Now it is evident that (3.23) holds **I**

Remark 3.27: Now it is already clear why we have chosen in Section 2 the functional ψ as the cost function of our optimal control problem. The *main results* are following:

1. $\psi(u) \geq 0$ for all $u \in V$.

2. inf $\psi(u)=0$.

uEV 3. If we find a minimum point $u \in V$ of ψ , then u is a solution of the transonic flow problem $(2.9, a - b)$.

In other optimal control methods mentioned in Introduction we meet a somewhat different situation: if we minimize the cost function considered (e.g. $||u - w(u)||^2$), then the minimum point need not be a solution of the transonic flow problem, unless we get the zero value of the cost function at this point.. blem (2:9, a-b).

optimal control methods mentioned in Introducti

situation: if we minimize the cost function consi

minimum point need not be a solution of the trans

ne zero value of the cost function at this point.

r

Remark 3.28: In Theorem 3.12 we have shwon that the boundary value problem $(2.9, a - b)$ (which is equivalent to finding a critical point of the functional $C(u)$ $= \Phi(u) - \langle f, u \rangle$ has a unique solution under the condition (3.13). This condition is valid, if

(3.29)

$$
o(s) + 2so'(s) \ge \alpha > 0 \qquad \forall \ s \in [0, +\infty)
$$

with some α . Really, if we denote

for ξ , $\eta \in \mathbf{R}_n$ and $t \in [0, 1]$, then in view of (3.29) and the condition $\varrho' \leq 0$ (under the for ξ , $\eta \in \mathbf{R}_n$ and $t \in [0, 1]$, then in view of (3.29) and the cor
notation $\xi = \xi + t\eta$, we have
 $g'(t) = \varrho(|\xi + t\eta|^2) |\eta|^2 + 2\varrho'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta$

$$
g'(t) = \varrho(|\xi + t\eta|^2) |\eta|^2 + 2\varrho'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2
$$

Transonic Potential
\nand
$$
t \in [0, 1]
$$
, then in view of (3.29) and the condi
\n $\xi + t\eta$), we have
\n
$$
= \varrho(|\xi + t\eta|^2) |\eta|^2 + 2\varrho'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2
$$
\n
$$
= \varrho(|\xi|^2) |\eta|^2 + 2\varrho'(|\xi|^2) \left(\sum_{i=1}^n \xi_i^2 \eta_i^2 + \sum_{\substack{i,j=1 \\ i+j}}^n \xi_i \xi_j \eta_i \eta_j \right)
$$
\n
$$
= \varrho(|\xi|^2) |\eta|^2 + 2\varrho'(|\xi|^2) [|\xi|^2 |\eta|^2 - \sum_{i,j} (\xi_j \eta_j - \xi_j \eta_j)
$$

$$
= \varrho(|\tilde{\xi}|^2) |\eta|^2 + 2\varrho'(|\tilde{\xi}|^2) \left[|\tilde{\xi}|^2 |\eta|^2 - \sum_{i \neq j} (\tilde{\xi}_i \eta_j - \tilde{\xi}_j \eta_i)^2 \right] \geq \alpha |\eta|^2
$$

Hence,

$$
\xi, \eta \in \mathbf{R}_n \text{ and } t \in [0, 1], \text{ then in view of (3.29) and the condition } \varrho'
$$
\n
$$
\text{ation } \xi = \xi + t\eta, \text{ we have}
$$
\n
$$
g'(t) = \varrho(|\xi + t\eta|^2) |\eta|^2 + 2\varrho'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2
$$
\n
$$
= \varrho(|\xi|^2) |\eta|^2 + 2\varrho'(|\xi|^2) \left(\sum_{i=1}^n \xi_i^2 \eta_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n \xi_i \xi_j \eta_i \eta_j \right)
$$
\n
$$
= \varrho(|\xi|^2) |\eta|^2 + 2\varrho'(|\xi|^2) \left[|\xi|^2 |\eta|^2 - \sum_{i \neq j}^n (\xi_i \eta_j - \xi_j \eta_i)^2 \right] \ge
$$
\n
$$
\text{since,}
$$
\n
$$
[\varrho(|\xi + \eta|^2) (\xi + \eta) - \varrho(|\xi|^2) \xi] \eta = g(1) - g(0)
$$
\n
$$
= \int_0^1 g'(t) \, dt \ge \alpha |\eta|^2.
$$
\n
$$
\text{ce}
$$
\n
$$
D\Phi(u, h) = \int_0^1 \varrho(|\nabla u|^2) \nabla u \cdot \nabla h \, dx
$$

Since

$$
D\Phi(u, h) = \int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla h \, dx,
$$

we prove (3.13) (with $c_3 = \alpha$) by putting $\xi = \nabla u(x)$, $\eta = \nabla h(x)$ (for almost every $x \in \Omega$) and by integrating the inequality (3.30) over Ω . (Let us notice that if the con-Since
 $D\Phi(u, h) = \int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla h \, dx,$

we prove (3.13) (with $c_3 = \alpha$) by putting
 $x \in \Omega$) and by integrating the inequality (3

dition $\varrho' \ge 0$ were satisfied, then $g'(t) \ge \varrho$

However from the construc dition $\varrho' \ge 0$ were satisfied, then $g'(t) \ge \varrho(|\xi + t\eta|^2) |\eta|^2 \ge \varrho_\infty |\eta|^2$.

However, from the construction of the function ρ and its extension on the interval (dition $\varrho' \ge 0$ were satisfied, then $g'(t) \ge \varrho(|\xi + t\eta|^2) |\eta|^2 \ge \varrho_\infty |\eta|^2$.)

However, from the construction of the function ϱ and its extension on the interv
 $[0, +\infty)$ it follows that (3.29) is satisfied for $s \$ $[\varrho(|\xi + \eta|^2)(\xi + \eta) - \varrho(|\xi|^2)\xi]\eta = g(1) - g(0)$
 $= \int_{0}^{1} g'(t) dt \ge \alpha |\eta|^2.$

Since
 $D\varPhi(u, h) = \int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla h dx,$

we prove (3.13) (with $c_3 = \alpha$) by putting $\xi = \nabla u(x)$,
 $x \in \Omega$) and by integrating the inequal dition $\varrho' \ge 0$ were satisfied, then $g'(t) \ge \varrho(|\xi + t\eta|^2) |\eta|^2 \ge \varrho_\infty |\eta|^2$.)

However, from the construction of the function ϱ and its extension on the interval
 $[0, +\infty)$ it follows that (3.29) is satisfied for s $\frac{6}{1}$). It means that the condition (3.13) is satisfied on the subset of the space V , formed by the velocity potentials corresponding to strictly subsonic flow fields.. If we are interested in the subsonic flow only, we can apply the secant modulus method directly in the following way. We- choose the can apply the secant modulus method directly in the following way. We choose the constant $s^{**} \in \left(0, \frac{2a_0^2}{\kappa+1}\right)$ and consider the function $\tilde{\rho}: [0, +\infty) \to \mathbb{R}_1$, which is given by the relation $\tilde{\varrho}(s)=\varrho_0\left(1-\frac{\varkappa-1}{2a_0^2}s\right)^{\overline{\varkappa-1}}$ in the interval. $[0,s^{**}]$ and satisfies the conditions (1.19), $(1.21) - (1.23)$ and (3.29) . As an easy exercise the following subsonic theorem can be proved. ing to strictly subsonic flow fields. If we are int
can apply the secant modulus method directly
constant $s^{**} \in \left(0, \frac{2a_0^2}{\kappa+1}\right)$ and consider the f
given by the relation $\bar{\varrho}(s) = \varrho_0 \left(1 - \frac{\kappa - 1}{2a_0^2} s\right)^{\k$

Theorem 3.31 : Let $\varrho := \bar{\varrho}$, where $\bar{\varrho}$ is defined above. Then there exists a unique mini*mum point u of the functional* $C(u) = \Phi(u) - \langle f, u \rangle$. This u is the unique solution of *the problem* (2.9, a-b) *and the secant modulus method converges to it. If* $|\nabla u|^2 \leq s^{**}$ *in Q, then u is the velocity potential of a physically admissible irrotational, subsonic,*

The Proof is an immediate application of Theorem 3.12 I

Let us remark that the solution *u* from the preceding theorem which does not satisfy the condition $|\nabla u|^2 \leq s^{**}$ has no sense from the physical point of view. Similar access to the study of subsonic flows was applied by FEISTAUER in [9-13] with the use of the stream function.

Now let us go back to the transonic flow problem, when the density ρ has the properties (1.19) —(1.23). Let us prove the second subsonic theorem useful in applications.

(3.30)

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Theorem 3.32: *Let.* $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the functional C or y such

at *that*

\n- M. FESTADER and J. NEČAS
\n- Theorem 3.32: Let
$$
\{u_n\}_{n=0}^{+\infty}
$$
 be a minimizing sequence of the functional C or ψ such that
\n- $|\nabla u_n|^2 \leq s^{**} < \frac{2u_0^2}{z+1}$ in Ω for all $n = 0, 1, \ldots$ (3.33)
\n- en $u_n \to u$ in V and u is the unique solution of the problem (2.9, a - b) in the set $|v \in V : |\nabla v|^2 \leq s^{**}$ in Ω of strictly subsonic velocity potentials.
\n

Then $u_n \to u$ in V and u is the unique solution of the problem (2.9, a-b) in the set

Proof: First, let us assume that ${u_n}_{n=0}^{\infty}$ is a minimizing sequence of ψ . In view Theorem 3.22, ${u_n}_{n=0}^{\infty}$ is bounded. Hence, we can assume that $u_n \rightharpoonup u$ (weakly), if $\rightarrow +\infty$. By (3.23),
 $B(u_n, u_n, u_n - u) - B(u$ of Theorem 3.22, $\{u_n\}_{n=0}^{+\infty}$ is bounded. Hence, we can assume that $u_n \rightharpoonup u$ (weakly), if $n \rightarrow +\infty$. By (3.23), *u,* 2.32 : Let $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the functional C or ψ such $2 \leq s^{**} < \frac{2u_0^2}{z+1}$ in Ω for all $n = 0, 1, ...$ (3.33)
 $\int \{uv_n\}^2 \leq s^{**}$ in Ω of strictly subsonic velocity potentia

$$
B(u_n, u_n, u_n - u) - B(u, u, u_n - u)
$$

= $\langle f, u_n - u \rangle + \langle F_n, u_n - u \rangle - B(u, u, u_n - u) \to 0$ (3.34)

for $n \to +\infty$, since $||F_n||_{V^{\bullet}} \to 0$, $\{u_n - u\}_{n=0}^{+\infty}$ is a bounded sequence and the mapping *"v* $\in V \rightarrow B(u, u, v)$ " is a continuous linear functional on *V*. Further, if we realize that the set \mathfrak{S} is closed and convex, which implies that \mathfrak{S} is weakly closed, we con-
clude that $u \in \mathfrak{S}$.
The function ϱ satisfies the condition
 $\varrho(s) + 2s\varrho'(s) \ge \alpha > 0 \quad \forall s \in [0, s^{**}].$ (3.35) clude that $u \in \mathfrak{S}$. *as the unique solution of the problem*
in Ω of strictly subsonic velocity poter
sume that $\{u_n\}_{n=0}^{+\infty}$ is a minimizing s
is bounded. Hence, we can assume the
 $-B(u, u, u_n - u)$
 $\langle F_n, u_n - u \rangle - B(u, u, u_n - u) \to 0$
 $\rightarrow 0, \{u_n - u$

The function ρ satisfies the condition

$$
\varrho(s) + 2s\varrho'(s) \ge \alpha > 0 \qquad \forall \ s \in [0, s^{**}]. \tag{3.35}
$$

This implies the existence of such $c_3 > 0$ that

$$
= \langle f, u_n - u \rangle + \langle F_n, u_n - u \rangle - B(u, u, u_n - u) \to 0 \qquad (3.34)
$$

for $n \to +\infty$, since $||F_n||_{V^*} \to 0$, $\{u_n - u\}_{n=0}^{+\infty}$ is a bounded sequence and the mapping
" $v \in V \to B(u, u, v)$ " is a continuous linear functional on V. Further, if we realize
that the set \mathfrak{E} is closed and convex, which implies that \mathfrak{E} is weakly closed, we con-
clude that $u \in \mathfrak{E}$.
The function ϱ satisfies the condition
 $\varrho(s) + 2s\varrho'(s) \ge \alpha > 0 \qquad \forall s \in [0, s^{**}]$.
This implies the existence of such $c_3 > 0$ that
 $c_3 ||u_n - u||^2$
 $\leq \int_{\Omega} [\varrho(|\nabla u_n|^2) \nabla u_n - \varrho(|\nabla u|^2) \nabla u] \cdot \nabla(u_n - u) dx$
 $= B(u_n, u_n, u_n - u) - B(u, u, u_n - u)$.
(3.36)
Now, by using (3.34), we see that $u_n \to u$ (strongly) in V, if $n \to +\infty$. From this,
(3.7) and $\psi(u_n) \to 0$, if $n \to +\infty$, we see that $w(u_n) \to u$. Since $w(u)$ is continuous
(see Lemma 3.21), we have $u = w(u)$, which means that u is a solution of the problem
(2.9, a - b).
If $\{u_n\}_{n=0}^{\infty}$ is a minimizing sequence of the functional C, then we can use the known
results from the convex analysis (cf. e.g. [15] or [33]). Again, we can assume that
 $u_n \to u$ (weakly), if $n \to +\infty$. By (3.35),
 $DC(u, u - v) - DC(v, u - v) \ge c_3 ||u - v||^2$, $u, v \in \mathfrak{E}$, (3.37)
with $c_3 > 0$ independent of u, v. This implies that C is weakly lower semicontinuous
in \mathfrak{E}

Now, by using (3:34), we see that $u_n \to u$ (strongly) in *V*, if $n \to +\infty$. From this, (3.7) and $\psi(u_n) \to 0$, if $n \to +\infty$, we see that $w(u_n) \to u$. Since $w(u)$ is continuous (see Lemma 3.21), we have $u = w(u)$, which means that u is a solution of the problem (2.9, a-b).

If $\{u_n\}_{n=0}^{\infty}$ is a minimizing se (see Lemma 3.21), we have $u = w(u)$, which means that u is a solution of the problem $(2.9, a - b).$ Now, by using (3.34), we see that $u_n \to u$ (strongly) in v , if $n \to +\infty$. From this, (3.7) and $\psi(u_n) \to 0$, if $n \to +\infty$, we see that $w(u_n) \to u$. Since $w(u)$ is continuous (see Lemma 3.21), we have $u = w(u)$, which means th $\leq \int_{\Omega} [Q(|\nabla u_n|^2) \nabla u_n - \varrho(|\nabla u|^2) \nabla u] \cdot \nabla (u_n)$
 $= B(u_n, u_n, u_n - u) - B(u, u, u_n - u).$
Now, by using (3.34), we see that $u_n \to u$ (stron (3.7) and $\psi(u_n) \to 0$, if $n \to +\infty$, we see that w
(see Lemma 3.21), we have $u = w(u)$,

If ${u_n}_{n=0}^{\infty}$ is a minimizing sequence of the functional *C*, then we can use the known results from the convex analysis (cf. e.g. [15] or [33]). Again, *we* can assume that minimizing sequence of the

convex analysis (cf. e.g.,
 $\text{if } n \to +\frac{1}{y} \infty$. By (3.35),
 $-\nu$) $-DC(v, u - v) \ge$

condent of u, v. This impl

inf $C(n) = \inf_{n \in V} C(n) = \lim_{n \to +} C(n)$

an value theorem, and theorem.

$$
DC(u, u - v) - DC(v, u - v) \ge c_3 ||u - v||^2, \qquad u, v \in \mathfrak{S}, \qquad (3.37)
$$

$$
C(u) = \inf_{n \in \mathcal{C}} C(n) = \inf_{n \in V} C(n) = \lim_{n \to +\infty} C(u_n).
$$

If we use the mean value theorem, and the relation $DC(u, u_n - u) = 0$, then we get

$$
0 \leq \frac{c_3}{2} ||u_n - u||^2
$$

\n
$$
\leq \int_{0}^{1} [DC(u + t(u_n - u), u_n - u) - DC(u, u_n - u)] dt
$$

\n
$$
= C(u_n) - C(u) \to 0,
$$

if $n \to +\infty$ and thus, $u_n \to u$ (strongly).

S
S

Finally, if $u_1, u_2 \in \mathfrak{S}$ are two subsonic solutions of the problem (2.9, a-b), then on the basis of the relation $DC(u, h) = B(u, u, h) - \langle f, h \rangle$ and (3.37) we get the inequality **• Finally, if** $u_1, u_2 \in \mathfrak{S}$ **are two subsonic solutions of the problem (2 on the basis of the relation** $DC(u, h) = B(u, u, h) - \langle f, h \rangle$ **and (3.37) quality** $c_3 ||u_1 - u_2||^2 \leq B(u_1, u_1, u_1 - u_2) - B(u_2, u_2, u_1 - u_2) = 0$ **.
Hence, u** \mathfrak{S} are two subsonic solutions of the problem (2.9, a-b), then
elation $DC(u, h) = B(u, u, h) - \langle f, h \rangle$ and (3.37) we get the ine-
 $d \leq B(u_1, u_1, u_1 - u_2) - B(u_2, u_2, u_1 - u_2) = 0$.

at a a i a i existence of a solut

$$
c_3\|u_1-u_2\|^2\leq B(u_1,u_1,u_1-u_2)-B(u_2,u_2,u_1-u_2)=0.
$$

Hence, $u_1 = u_2$

4. Differantials of $w(u)$ and the existence of a solution to the transonic flow problem

Theorem 4.1: The mapping " $u \in V \to w(u) \in V$ ", defined by (3.5), has the derivative

under a posteriori estimates on a minimizing sequence
\neorem 4.1: The mapping "
$$
u \in V \to w(u) \in V
$$
", defined by (3.5), has the derivative
\n
$$
Dw(u, h) = \frac{d}{dt} w(u + th)|_{t=0}
$$
\n(4.2)

at every point $u \in V$ *and in every direction* $h \in V \cap W^{1,\infty}(\Omega)$. The mapping " $h \to Dw$ $X(u, h)$ " is linear for each $u \in V$ and uniformly bounded with respect to $u \in V$. For *every* $h \in V \cap W^{1,\infty}(\Omega)$ " $u \to Dw(u, h)$ " is continuous mapping of V into V: Hence, $Dw(w, \cdot)$ is the Gâteaux differential of $w(u)$. \times (u, h) " is linear for each $u \in V$ and uniformly bounded with respect to $u \in V$. For
every $h \in V \cap W^{1,\infty}(\Omega)$ " $u \to Dw(u, h)$ " is continuous mapping of V into V. Hence,
 $Dw(w, \cdot)$ is the Gâteaux differential of $w(u)$.
Pro

 $\omega_t = (w_t - w_0)/t$ and $w = w_0 = w(u)$. It holds

r a posteriori estimates on a minimizing sequence
\nm 4.1: The mapping "
$$
u \in V \rightarrow w(u) \in V
$$
", defined by (3.5), has the derivative
\n
$$
Dw(u, h) = \frac{d}{dt} w(u + th)|_{t=0}
$$
\n(4.2)
\npoint $u \in V$ and in every direction $h \in V \cap W^{1,\infty}(\Omega)$. The mapping " $h \rightarrow Dw$
\n" is linear for each $u \in V$ and uniformly bounded with respect to $u \in V$. For
\n $V \cap W^{1,\infty}(\Omega)$ " $u \rightarrow Dw(u, h)$ " is continuous mapping of V into V. Hence,
\nis the Gateaux differential of $w(u)$.
\n f : For $u \in V$ and $h \in V \cap W^{1,\infty}(\Omega)$ we denote $u_t = u + th$, $w_t = w(u_t)$,
\n $t = w_0$)/t and $w = w_0 = w(u)$. It holds
\n
$$
0 = \frac{1}{t} [B(u_t, w_t, v) - B(u_0, w_0, v)]
$$
\n
$$
= \frac{1}{t} \int {\left\{ [Q(|\nabla u|^2) - Q(|\nabla u_0|^2)] \nabla w_t \right\}} + Q(|\nabla u_0|^2) |\nabla w_t - \nabla w_0| \right\} \cdot \nabla v \, dx,
$$
\n
$$
= -2 \int_{\Omega} \int_0^1 g'(|\nabla u_{tt}|^2) (\nabla u_{tt} \cdot \nabla h) \, d\tau \Big) (\nabla w_t \cdot \nabla v) \, dx.
$$
\n
$$
= -2 \int_{\Omega} \int_0^1 g'(|\nabla u_{tt}|^2) (\nabla u_{tt} \cdot \nabla h) \, d\tau \Big) (\nabla w_t \cdot \nabla v) \, dx.
$$
\n
$$
= -2 \int_{\Omega} g'(|\nabla u|^2) \nabla Dw(u, h) \cdot \nabla v \, dx
$$
\n
$$
= -2 \int_{\Omega} g'(|\nabla u|^2) (\nabla u \cdot \nabla h) (\nabla w \cdot \nabla v) \, dx, v \in V.
$$
\n(4.5)
\n
$$

$$

If we use the mean value theorem, then

$$
t \int_{\Omega} \left(|\nabla u_0|^2 \right) \left(\nabla w_t - \nabla w_0 \right) \cdot \nabla v \, dx, \qquad v \in V. \tag{4.3}
$$
\nIf we use the mean value theorem, then\n
$$
\int_{\Omega} \varrho(|\nabla u|^2) \nabla \omega_t \cdot \nabla v \, dx
$$
\n
$$
= -2 \int_{\Omega} \left(\int_{0}^{1} \varrho'(|\nabla u_{tt}|^2) \left(\nabla u_{tt} \cdot \nabla h \right) \, d\tau \right) \left(\nabla w_t \cdot \nabla v \right) \, dx. \tag{4.4}
$$
\nFrom this, for $t \to 0$, we get\n
$$
\int_{\Omega} \varrho(|\nabla u|^2) \nabla Dw(u, h) \cdot \nabla v \, dx
$$
\n
$$
= -2 \int_{\Omega} \varrho'(|\nabla u|^2) \left(\nabla u \cdot \nabla h \right) \left(\nabla w \cdot \nabla v \right) \, dx, \quad v \in V. \tag{4.5}
$$
\nIn view of the assumption $h \in V \cap W^{1,\infty}(\Omega)$ and the properties of ϱ , the right-hand side in (4.5) (considered as a function of $v \in V$) defines a continuous linear functional

From this, for $t \to 0$, we get

$$
\begin{aligned}\n\delta &= -2 \int\limits_{\Omega} \left(\int\limits_{0}^{1} \varrho'(|\nabla u_{tt}|^2) \left(\nabla u_{tt} \cdot \nabla h \right) d\tau \right) (\nabla w_t \cdot \nabla v) d\mathbf{x}.\n\end{aligned}
$$
\nFrom this, for $t \to 0$, we get

\n
$$
\int\limits_{\Omega} \varrho(|\nabla u|^2) \nabla Dw(u, h) \cdot \nabla v d\mathbf{x}.
$$
\n
$$
= -2 \int\limits_{\Omega} \varrho'(|\nabla u|^2) \left(\nabla u \cdot \nabla h \right) \left(\nabla w \cdot \nabla v \right) d\mathbf{x}, \quad v \in V.
$$
\n(4.5)

\nIn view of the assumption $h \in V \cap W^{1,\infty}(\Omega)$ and the properties of ϱ , the right-hand side in (4.5) (considered as a function of $v \in V$) defines a continuous linear functional $F(u, w, h)$ on V . Hence, (4.5) which can be written in the form

\n
$$
B(u, Dw(u, h), v) = \langle F(u, w, h), v \rangle, \qquad v \in V,
$$
\n(4.6)

\nhas a unique solution $Dw(u, h) \in V$. From the continuous dependence of w on u , the continuity of F with respect to u, w, h and the results from [27: Ch. 3, § 6] all remaining assertions of Theorem 4.1 follow 1

In view of the assumption $h \in V \cap W^{1,\infty}(\Omega)$ and the properties of ρ , the right-hand side in (4.5) (considered as a function of $v \in V$) defines a continuous linear functional $F(u, w, h)$ on *V*. Hence, (4.5) which can be written in the form

$$
B(u, Dw(u, h), v) = \langle F(\dot{u}, w, h), v \rangle, \qquad v \in V, \tag{4.6}
$$

'a

0

has a unique solution $Dw(u, h) \in V$. From the continuous dependence of w on u, the continuity of *F* with respect to *u*, *w*, *h* and the results from [27: Ch. 3, §. 6] all remain-
ing assertions of Theorem 4.1 follow \blacksquare

F.

<u>¹</sup> *¹ ¹*** ***¹ ¹*** ***¹ ¹* ****</u>

Remark 4.7: Similarly, supposing that ρ'' is continuous in $[0, +\infty)$ and that the estimate $|p''(s)(1 + s)| \leq c_5$ = const holds in $[0, +\infty)$, we can prove the existence of the second differential $D^2w(u, h, k)$ for $u \in V$ and $h, k \in V \cap W^{1,\infty}(\Omega)$ and the relation $(w = w(u))$

I

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\nrk 4.7: Similarly, supposing that
$$
\varrho''
$$
 is continuous in $[0, +\infty)$ and that the
\n $|\varrho''(s) (1 + s)| \leq c_s = \text{const holds in } [0, +\infty)$, we can prove the existence
\necond differential $D^2w(u, h, k)$ for $u \in V$ and $h, k \in V \cap W^{1,\infty}(\Omega)$ and the
\n $[w = w(u)]$
\n $\int \varrho(|\nabla u|^2) \nabla D^2w(u, h, k) \cdot \nabla v \, dx$
\n $= -2 \int \varrho'(|\nabla u|^2) (\nabla u \cdot \nabla k) (\nabla Dw(u, h) \cdot \nabla v) \, dx$
\n $-4 \int \varrho''(|\nabla u|^2) (\nabla u \cdot \nabla h) (\nabla u \cdot \nabla k) (\nabla w \cdot \nabla v) \, dx$
\n $-2 \int \varrho''(|\nabla u|^2) (\nabla h \cdot \nabla k) (\nabla w \cdot \nabla v) \, dx$
\n $-2 \int \varrho'(|\nabla u|^2) (\nabla u \cdot \nabla h) (\nabla Dw(u, k) \cdot \nabla v) \, dx$ (4.8)
\nna 4.9: The functional w is differentiable in the space V with respect to any

Lemma 4.9 : The functional ψ is differentiable in the space V with respect to any *direction* $h \in V \cap W^{1,\infty}(\Omega)$. The differential $D\psi(u, h)$ has the form

$$
-2 \int_{\Omega} e''(|\nabla u|^2) (\nabla h \cdot \nabla k) (\nabla w \cdot \nabla v) dx
$$

\n
$$
-2 \int_{\Omega} e'(|\nabla u|^2) (\nabla u \cdot \nabla h) (\nabla Dw(u, k) \cdot \nabla v) dx
$$
 (4.8)
\nna 4.9. The functional ψ is differentiable in the space V with respect to any $h \in V \cap W^{1,\infty}(\Omega)$. The differential $D\psi(u, h)$ has the form
\n
$$
D\psi(u, h) = \int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla h dx
$$

\n
$$
- \int_{\partial\Omega} gh ds - \int_{\Omega} \varrho(|\nabla w(u)|^2) \nabla w(u) \cdot \nabla Dw(u, h) dx
$$

\n
$$
+ \int_{\partial\Omega} gDw(u, h) ds, \qquad u \in V, h \in V \cap W^{1,\infty}(\Omega).
$$

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\n12(11)
\n13(12)
\n14(13)
\n15(14)
\n16(15)
\n17(17)
\n18(19)
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\n18(19)
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\n10(11)
\n11(12)
\n12(13)
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\n12(13)
\n13(14)
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\n16(17)
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The Proof follows from the definition (2.13) of ψ and from Lemma 3.21 \blacksquare The following theorem is devoted to the uniqueness of the solution to the problem $(2.9, a - b)$. The Pro
 File follo
 (2.9, a—b). $(u, h) ds$, u

e definition (2

voted to the u

dition
 $Dw(v, u - v)$
 (1.24) , i.e.
 $\frac{6a_0^2}{+1}$,

has at most one em is devoted to the uniqueness of t
 t the condition
 $-v$) $-Dw(v, u - v)$ $ds \le 0$
 ttislying (1.24), i.e.
 $\le s_1 < \frac{6a_0^2}{z+1}$,

, a -b) has at most one solution in th

Theorem *4.11: Let the condition*

$$
\int_{\partial \Omega} g[Dw(u, u - v) - Dw(v, u - v)] ds \leq 0 \tag{4.12}
$$

hold for all $u, v \in V$ *satisfying* (1.24) , i.e.

Green 4.11: Let the condition

\n
$$
\int_{\partial \Omega} g[Dw(u, u - v) - Dw(v, u - v)] ds \leq 0
$$
\n(4.12)

\nall $u, v \in V$ satisfying (1.24), i.e.

\n
$$
|\nabla u|^2, |\nabla v|^2 \leq s_1 < \frac{6a_0^2}{\pi + 1}.
$$
\nwe problem (2.9, a - b) has at most one solution in the class of velocity potentials satisfying (4.13).

\nof: Let u, v be two such solutions. Then $\psi(u) = \psi(v) = \inf_{\xi \in V} \psi(\xi) = 0$ and thus, of Lemma 4.9,

\n
$$
0 = D\psi(u, u - v)
$$
\n
$$
\int_{\xi}^{2\pi} f(x, v) \, dx + \int_{\xi}^{2\pi} f(x, v) \, dv \leq \int_{\xi}^{2\pi} f(x, v) \, dv
$$

Then the problem $(2.9, a - b)$ *has at most one solution in the class of velocity potentials* $u \in V$ *satisfying* (4.13).

problem (2.9, a-b) has at most one solution in the class of velocity potentials $\in V$ satisfying (4.13).
Proof: Let *u*, *v* be two such solutions. Then $\psi(u) = \psi(v) = \inf_{\xi \in V} \psi(\xi) = 0$ and thus, view of Lemma 4.9,

Theorem 4.11: Let the condition
\n
$$
\int_{\partial \Omega} [Dw(u, u - v) - Dw(v, u - v)] ds \leq 0
$$
\n(4.12
\nhold for all $u, v \in V$ satisfying (1.24), i.e.
\n
$$
|\nabla u|^2, |\nabla v|^2 \leq s_1 < \frac{6a_0^2}{\kappa + 1}.
$$
\nThen the problem (2.9, a - b) has at most one solution in the class of velocity potential
\n $u \in V$ satisfying (4.13).
\nProof: Let u, v be two such solutions. Then $\psi(u) = \psi(v) = \inf_{\xi \in V} \psi(\xi) = 0$ and thus
\nin view of Lemma 4.9,
\n
$$
0 = D\psi(u, u - v)
$$
\n
$$
= \int_{\Omega} \varrho(|\nabla w|^2) \nabla u \cdot \nabla (u - v) dx - \int_{\partial \Omega} g(u - v) ds
$$
\n
$$
- \int_{\Omega} \varrho(|\nabla w(u)|^2) \nabla w(u) \cdot \nabla Dw(u, u - v) dx + \int_{\partial \Omega} gDw(u, u - v) ds
$$

and

Transonic Potential Flow Problems 321
\n
$$
0 = D\psi(v, u - v) = \int_{\Omega} \varrho(|\nabla v|^2) \nabla v \cdot \nabla (u - v) dx - \int_{\partial \Omega} g(u - v) ds
$$
\n
$$
- \int_{\Omega} \varrho(|\nabla w(v)|^2) \nabla w(v) \cdot \nabla Dw(v, u - v) dx + \int_{\partial \Omega} gDw(v, u - v) ds. \quad (4.15)
$$
\n
$$
\text{ations } u, v \text{ as the solutions of the problem (2.9, a–b) satisfy the relations}
$$
\n
$$
\int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla (u - v) dx = \int g(u - v) ds.
$$

The functions u, v as the solutions of the problem $(2.9, a - b)$ satisfy the relations

Transonic Potential Flow Problems 321
\nand
\n
$$
0 = D\psi(v, u - v) = \int_{\Omega} \varrho(|\nabla v|^2) \nabla v \cdot \nabla(u - v) dx - \int_{\partial \Omega} g(u - v) ds
$$
\n
$$
- \int_{\Omega} \varrho(|\nabla w(v)|^2) \nabla w(v) \cdot \nabla Dw(v, u - v) dx + \int_{\partial \Omega} gDw(v, u - v) ds. \quad (4.15)
$$
\nThe functions u, v as the solutions of the problem (2.9, a - b) satisfy the relations
\n(a)
$$
\int_{\Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla(u - v) dx = \int_{\partial \Omega} g(u - v) ds,
$$
\n(b)
$$
\int_{\Omega} \varrho(|\nabla v|^2) \nabla v \cdot \nabla(u - v) dx = \int_{\partial \Omega} g(u - v) ds.
$$
\nIf we subtract (4.15) from (4.14) and take into account (4.5), (4.12), (4.16, a - b) and
\nthe fact that $u = w(u), v = w(v), w$ get
\n
$$
0 = \int_{\partial \Omega} g[Dw(u, u - v) - Dw(v, u - v)] dx
$$
\n
$$
= \int_{\partial \Omega} \varrho(|\nabla u|^2) \nabla u \cdot \nabla Dw(u, u - v) dx
$$

If we subtract (4.15) from (4.14) and take into account (4.5), (4.12), (4.16, a-b) and the fact that $u = w(u)$, $v = w(v)$, we get

$$
-\int_{0}^{1} \rho(|\nabla w(v)|^{2}) \nabla w(v) \cdot \nabla Dw(v, u - v) dx + \int_{\partial D}^{1} \rho Dw(v, u - v) ds. (4.15)
$$

ctions u, v as the solutions of the problem (2.9, a - b) satisfy the relations

$$
\int_{\Omega} \rho(|\nabla v|^{2}) \nabla u \cdot \nabla(u - v) dx = \int_{\partial D} g(u - v) ds.
$$

(4.16)
attract (4.15) from (4.14) and take into account (4.5), (4.12), (4.16, a - b) and
that $u = w(u), v = w(v), w$ get

$$
0 = \int_{\partial D} [Dw(u, u - v) - Dw(v, u - v)] ds
$$

$$
-\int_{\partial D} \rho(|\nabla u|^{2}) \nabla u \cdot \nabla Dw(u, u - v)] dx
$$

$$
+\int_{\Omega} \rho(|\nabla v|^{2}) \nabla v \cdot \nabla Dw(v, u - v)] dx
$$

$$
= 2 \int_{\Omega} \rho'(|\nabla u|^{2}) |\nabla u|^{2} (\nabla u \cdot \nabla(u - v)) dx
$$

$$
= 2 \int_{\Omega} \rho'(|\nabla v|^{2}) |\nabla v|^{2} (\nabla v \cdot \nabla(u - v)) dx.
$$

$$
0 \ge \int_{\Omega} |[p(|\nabla u|^{2}) + \lambda \rho'(|\nabla u|^{2}) | \nabla u|^{2}] \nabla u.
$$

$$
= [e(|\nabla v|^{2}) + \lambda \rho'(|\nabla u|^{2}) | \nabla u|^{2}] \nabla v.
$$

$$
= [e(|\nabla v|^{2}) + \lambda \rho'(|\nabla v|^{2}) | \nabla v|^{2}] \nabla v.
$$

$$
= \int_{\Omega} [h(|\nabla u|^{2}) \nabla u - h(|\nabla v|^{2}) \nabla v] \cdot \nabla(u - v) dx
$$

$$
= \int_{\Omega} [h(|\nabla u|^{2}) \nabla u - h(|\nabla v|^{2}) \nabla v] \cdot \nabla(u - v) dx.
$$
<math display="block</math>

Now, let us multiply (4.17) by $\lambda/2 < 0$ and add the difference (4.16, a) - (4.16, b)

$$
0 \geq \int_{\Omega} \left\{ \left[\nabla (|\nabla u|^2) + \lambda \varrho' (|\nabla u|^2) |\nabla u|^2 \right] \nabla u \right\} - \left[\varrho (|\nabla v|^2) + \lambda \varrho' (|\nabla v|^2) |\nabla v|^2 \right] \nabla v \right\} \cdot \nabla (u - v) dx = \int_{\Omega} \left[h (|\nabla u|^2) \nabla u - h (|\nabla v|^2) \nabla v \right] \cdot \nabla (u - v) dx,
$$
(4.18)

where $h(s) = \rho(s) + \lambda s \rho'(s)$. We have

$$
h(s) + 2sh'(s) = \varrho(s) + (3\lambda + 2) s\varrho'(s) + 2\lambda s^2 \varrho''(s)
$$

= $\varrho(s) + 2s\varrho'(s) - \lambda[-3s\varrho'(s) - 2s^2\varrho''(s)]$

After some calculation we find out that $-3s\varrho'(s) - 2s^2\varrho''(s) \ge \alpha_1 s$ for $s \in [0, s_1]$ $0 \geq \int_{\Omega} {\mathbb{E}(\nabla u|^2) + \lambda \varrho'(\nabla u)}$
 $- [\varrho(|\nabla v|^2) + \lambda \varrho'(|\nabla v|)$
 $= \int_{\Omega} [h(|\nabla u|^2) \nabla u - h(|\nabla u|)$

where $h(s) = \varrho(s) + \lambda s \varrho'(s)$. We have $h(s) + 2sh'(s) = \varrho(s) + (3$
 $= \varrho(s) + 2s$.

After some calculation we find on wi 2a $g(s) + (3\lambda + 2)$
 $g(s) + 2sg'(s) - s$

we find out th
 $s \leq s^{**} < \frac{2a}{s + 1}$
 $f s \geq s^{**}$ and λ we have $\rho(s) + 2s\varrho'(s) \ge \alpha > 0$ and $\frac{0^2}{s}$ we have $\rho(s) + 2s\varrho'(s) \ge \alpha > 0$ and $h(s) + 2sh'(s) = \varrho(s) + (3\lambda + 2) s\varrho'(s) + 2\lambda s^2\varrho''(s)$
 $= \varrho(s) + 2s\varrho'(s) - \lambda[-3s\varrho'(s) - 2s^2\varrho''(s)].$
After some calculation we find out that $-3s\varrho'(s) - 2s^2\varrho''(s) \ge \alpha_1 s$ for $s \in [0, s_1]$
with $\alpha_1 > 0$. For $0 \le s \le s^{**} < \frac{2a$ thus, $h(s) + 2sh'(s) \ge \alpha$. If $s \ge s^{**}$ and λ is close to $-\infty$, then $h(s) + 2sh'(s) \ge \alpha > 0$.
Hence, we see that there exist $\lambda < 0$ and $\beta > 0$ such that $\begin{aligned}\n&= |\ell(|\nabla u|^2) + \lambda \ell(|\nabla v|^2) |\nabla v|^2 |\nabla v \rangle \cdot \nabla (u - v) \, dx, \\
&= \int_{\Omega} [h(|\nabla u|^2) \nabla u - h(|\nabla v|^2) \nabla v] \cdot \nabla (u - v) \, dx, \\
&= \ell(s) + 2s\ell'(s). \text{ We have} \\
&= \ell(s) + 2s\ell'(s) - 2[-3s\ell'(s)] \cdot \nabla (s) - 2s^2\ell''(s)]. \\
&= \ell(s) + 2s\ell'(s) - 2[-3s\ell'(s) - 2s$

$$
h(s) + 2sh'(s) \geq \beta > 0 \qquad \forall \ s \in [0, s_1]. \tag{4.19}
$$

Now we proceed similarly, as in Remark 3.28. For $t \in [0, 1]$, ξ , $\xi \in \mathbb{R}_n$, $|\xi|^2$, $|\xi|^2 \leq$ $h(s) + 2sh'(s) \ge \beta > 0$ \forall
Now we proceed similarly as in Rem
and $\eta = \xi - \xi$ we put

thus,
$$
n(s) + 2sn(s) \leq \alpha
$$
. If $s \geq s^{**}$ and λ is close to $-\infty$, then h . Hence, we see that there exist $\lambda < 0$ and $\beta > 0$ such that $h(s) + 2sh'(s) \geq \beta > 0$ $\forall s \in [0, s_1]$. Now we proceed similarly as in Remark 3.28. For $t \in [0, 1]$, ξ and $\eta = \xi - \xi$ we put\n
$$
g(t) = h(|\xi + t\eta|^2) \left(\xi + t\eta\right) \cdot \eta
$$
\n
$$
= \left[\varrho(|\xi + t\eta|^2) + \lambda \varrho'(|\xi + t\eta|^2) \left|\xi + t\eta|^2\right| (\xi + t\eta) \cdot \eta\right].
$$
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\nThen, in view of (4.19) and the evident inequality
$$
h \ge \rho_{\infty}
$$
, we have
\n
$$
g'(t) = h(|\xi + t\eta|^2) |\eta|^2 + 2h'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2 \ge \beta_1 |\eta|^2,
$$
\n
$$
\beta_1 = \min (\beta, \rho_{\infty}) > 0
$$
\nand

$$
\beta_1 = \min (\beta, \varrho_\infty) > 0
$$

and

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\nview of (4.19) and the evident inequality
$$
h \ge \rho_{\infty}
$$
, we have
\n
$$
g'(t) = h(|\xi + t\eta|^2) |\eta|^2 + 2h'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2 \ge \beta_1 |\eta|^2,
$$
\n
$$
\beta_1 = \min (\beta, \rho_{\infty}) > 0
$$
\n
$$
g(1) - g(0) = \int_0^1 g'(t) dt \ge \beta_1 |\eta|^2.
$$
\n
$$
\text{if we denote } \xi = \nabla v(x), \xi = \nabla u(x) \text{ (for almost every } x \in \Omega \text{), then by (4.20)}
$$
\nfind out that

Finally, if we denote $\xi = \nabla v(x)$, $\xi = \nabla u(x)$ (for almost every $x \in \Omega$), then by (4.20) we easily find out that

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\nThen, in view of (4.19) and the evident inequality
$$
h \ge \rho_{\infty}
$$
, we have
\n
$$
g'(t) = h(|\xi + t\eta|^2) |\eta|^2 + 2h'(|\xi + t\eta|^2) [(\xi + t\eta) \cdot \eta]^2 \ge \beta_1 |\eta|^2,
$$
\n
$$
\beta_1 = \min (\beta, \rho_{\infty}) > 0
$$
\nand
\n
$$
g(1) - g(0) = \int_0^1 g'(t) dt \ge \beta_1 |\eta|^2.
$$
\n(4.20)
\nFinally, if we denote $\xi = \nabla v(x)$, $\xi = \nabla u(x)$ (for almost every $x \in \Omega$), then by (4.20)
\nwe easily find out that
\n
$$
\int_{\Omega} [h(|\nabla u|^2) \nabla u - h(|\nabla v|^2) \nabla v] \cdot \nabla (u - v) dx
$$
\n
$$
\ge \beta_1 \int_{\Omega} |\nabla (u - v)|^2 dx = \beta_1 ||u - v||^2.
$$
\n(4.21)
\nIn virtue of the inequality (4.18), $||u - v||^2 = 0$ which yields $u = v$

In virtue of the inequality (4.18), $||u - v||^2 = 0$ which yields $u = v$ **I**

Theorem 4.11 indicates that the behaviour of the term $\int gDw(u,\,h)\;ds$ will probably play an important role in the study of the transonic flow problem and it will be necessary,to control this term in a suitable way. Let us consider a minimizing sequence Theory
play and
necessaries $\{u_n\}_{n=0}^{+\infty}$ ${u_n}_{n=0}^{+\infty}$ of the functional ψ . Then $\begin{align*}\n\text{rate of a} \\
\text{energy} \\
\text{max of} \\
\frac{1}{20} \\
\text{of} \\
\text{mean} \\
\text{$ cates that the behaviour of the term $\int_{Q} gDw(u, h) ds$ will probably

role in the study of the transonic flow problem and it will b

this term in a suitable way. Let us consider a minimizing sequence

nal ψ . Then
 \sup_{ν dicates that the behaviour of the term
dicates that the behaviour of the term
of this term in a suitable way. Let us
dional ψ . Then
sup $|D\psi(u_n, h)| = 0$.
 $V_0W_1 \otimes (\Omega)^{\leq 1}$
a 4.9, and the relation
 $V_1 \otimes W_2 \otimes W_3 \otimes W_$

$$
\lim_{n\to+\infty}\sup_{\|h\|_{V\cap W^{1,\infty}(\Omega)}\leq 1} |D\psi(u_n,h)|=0.
$$

From this, Lemma 4.9, an4 the relation

$$
\int_{\Omega} \varrho(|\nabla u_n|^2) \nabla u_n \cdot \nabla h \, dx - \int_{\partial \Omega} gh \, ds \to 0 \quad \text{for} \quad n \to +\infty
$$

(which follows from Theorem 3.22), we can see that if. $h \in V \cap W^{1,\infty}(\Omega)$ and $\lim_{n \to +\infty} \frac{1}{\|h\|}$
From this, Lemm
 $\int_{\Omega} \varrho(|\nabla u_n|)$
(which follows fi
 $\|\hbar\|_{V \cap W^{1,\infty}(\Omega)} \leq 1,$
 $0 = \lim_{n \to +\infty}$

Theorem 4.11 indicates that the behaviour of the term
$$
\int_{Q} \mathcal{Y}Du(u, h)
$$
 as with probability
\nplay an important role in the study of the transonic flow problem and it will be
\n necessary to control this term in a suitable way. Let us consider a minimizing sequence
\n $\{u_n\}_{n=0}^{\infty}$ of the functional ψ . Then
\n
$$
\lim_{n \to +\infty} \sup |D\psi(u_n, h)| = 0.
$$
\nFrom this, Lemma 4.9, and the relation
\n
$$
\int_{Q} \varrho(|\nabla u_n|^2) \nabla u_n \cdot \nabla h \, dx - \int_{\partial} gh \, ds \to 0 \text{ for } n \to +\infty
$$
\n(which follows from Theorem 3.22), we can see that if $h \in V \cap W^{1,\infty}(\Omega)$ and
\n
$$
||h||_{V \cap W^{1,\infty}(\Omega)} \leq 1
$$
, then
\n
$$
0 = \lim_{n \to +\infty} D\psi(u_n, h)
$$
\n
$$
= \lim_{n \to +\infty} \left[-\int_{\Omega} \varrho(|\nabla w(u_n)|^2) \nabla w(u_n) \cdot \nabla Dw(u_n, h) \, dx + \int_{\partial\Omega} gDw(u_n, h) \, ds \right].
$$
\nSince $u_n - w(u_n) \to 0$ for $n \to +\infty$ (again by Theorem 3.22), $w(u_n)$ can be approximated by u_n for large *n*. This, (4.5) (where we put $u = v = u_n$) and (4.22) imply that
\nthe term $\int gDw(u_n, h) ds$ can be approximated by
\n
$$
= 2 \int_{\Omega} \varrho'(|\nabla u_n|^2) |\nabla u_n|^2 \nabla u_n \cdot \nabla h \, dx.
$$

mated by u_n for large *n*. This, (4.5) (where we put $u = v = u_n$) and (4.22) imply that the term $\int gDw(u_n, h) ds$ can be approximated by Since $u_n - w(u_n) \to 0$ for $n \to +\infty$ (again by Theorem 3.22), $w(u_n)$ can
mated by u_n for large *n*. This, (4.5) (where we put $u = v = u_n$) and (4.22
the term $\int gDw(u_n, h) ds$ can be approximated by
 $-2 \int_{\Omega} e'(|\nabla u_n|^2) |\nabla u_n|^2 \n$

$$
-2\int\limits_{O}\left[e\left(\vert \nabla u_n\vert^2\right)\vert \nabla u_n\vert^2 \nabla u_n\cdot \dot{\nabla} h\ dx\right].
$$

Therefore, in the following we shall use the condition (1.17) to 'control, the term' *f* $gDw(u, h)$ *ds*. It is possible to show that this condition has a relation to upwinding

the density (used e.g. in [2, 5, 7]).

 $\frac{1}{2}$

Now we shall prove the theorem on the solvability of the transonic flow problem. The entropy condition (1.17) plays the fundamental role **in** the proof because of its

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Theorem 4.23 *(1st fundamental): Let* $\{u_n\}_{n=0}^{+\infty}$ *be a minimizing sequence of the func-*Theorem 4.23 (1st fundamental): Let $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the functional ψ , satisfying a posteriori the condition (4.13) and the entropy condition (1.17) Theorem 4.23 (1st fundamental): Let $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the functional ψ , satisfying a posteriori the condition (4.13) and the entropy condition (1.17) with some $M \in \mathbb{R}_1$. Let $u_n \to u$ (wea *V*, if $n \to +\infty$ and u is a solution of the problem (2.9, a-b), satisfying the condition *(4.13).* mal ψ , satisfying a posteriori the condition (4.13) and the entropy condit
th some $M \in \mathbf{R}_1$. Let $u_n \to u$ (weakly) in *V*, if $n \to +\infty$. Then $\tilde{u}_n \to u$ (st
if $n \to +\infty$ and *u* is a solution of the problem (2.9, Transonic Potential Flow Problems 323

Theorem 4.23 (1st fundamental): Let $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the func-

tional ψ , satisfying a posteriori the condition (4.13) and the entropy condition $(1$ Theorem 4.23 (1st fundamental): Let $\{u_n\}_{n=0}^{+\infty}$ be a minitional ψ , satisfying a posteriori the condition (4.13) and t
with some $M \in \mathbf{R}_1$. Let $u_n \to u$ (weakly) in V , if $n \to +\infty$
 V , if $n \to +\infty$ and u is **Example 1.23** (Ist fundamental): Let $\{u_n\}_{n=0}^{+\infty}$ be a minimizing sequence of the function satisfying a posteriori the condition (4.13) and the entropy condition (1.17) $M \in \mathbf{R}_1$. Let $u_n \to u$ (weakly) in V, if n

$$
\langle G_n, h \rangle = \int_{\Omega} \varrho'(|\nabla u_n|^2) |\nabla u_n|^2 \nabla u_n \cdot \nabla h \, dx, \qquad h \in V. \tag{4.24}
$$

$$
H_n(h) = M \int_a h \, dx - G_n(h), \qquad H(h) = M \int_a h \, dx - G(h). \qquad (4.25)
$$

From the condition (1.17) it follows that $H_n(h) \geq 0$, $H(h) \geq 0$ for $h \geq 0$. If we use, the result of MURAT [26], we get $H_n \to H$ in $(W_0^{1,p}(\Omega))^*$ for each $p > 2$. Hence, the result of MURAT [26], we get $H_n \to H \text{ in } (\overline{W_0}^1, p(\Omega))^*$ for each $p > 2$. Hence, $G_n \to G$ in $(W_0^{1,p}(\Omega))^*$. From the resul
the resul
 $G_n \rightarrow G$ is
Now, I
can write *h* are restriction of $\langle G_n, h \rangle$ or $\langle G, h \rangle$ to $W_0^{1,2}(\Omega)$ by $G_n(h)$ or $G(h)$, respectively,
 $H_n(h) = M \int_h h \, dx - G_n(h)$, $H(h) = M \int_h h \, dx - G(h)$. (4.25)

a condition (1.17) it follows that $H_n(h) \ge 0$, $H(h) \ge 0$ for $h \ge 0$. If

Now, let $h \in V \cap W^{1,\infty}(\Omega)$. Following AGMON, Douglis and NIRENBERG [1], we can write

$$
h=h^1+h^2.
$$

where $\Delta h^1 = 0$ and $h^2 \in W_0^{1,2}(\Omega)$. (4.26) is an orthogonal decomposition of the space $W^{1,2}(\Omega)$ into harmonic functions and functions with zero traces. By MEYERS [25]; there exists $p_1 > 2$ such that the mapping " $h \rightarrow h^{i}$ " $(i = 1, 2)$ is continuous from *Where* $\triangle h^1 = 0$ and $h^2 \in W_0^{1,2}(\Omega)$. (4.26) is an orthogonal decomposition of the space $W^{1,2}(\Omega)$ into harmonic functions and functions with zero traces. By MEYERS [25];
there exists $p_1 > 2$ such that the mapping $W^{1,2}(\Omega)$ into harmonic functions and functions with zero traces. By MEYERS [25];
there exists $p_1 > 2$ such that the mapping " $h \to h^{i}$ " $(i = 1, 2)$ is continuous from
 $W^{1,p_1}(\Omega)$ into $W^{1,p_1}(\Omega)$. Further, since the compact, we can assert that the set $\mathfrak{A} = \{h^1 : ||h||_{V \cap W^{1,\infty}(\Omega)} \leq 1\}$ is compact in $W^{1,p_1}(\Omega)$.
From this and the weak convergence of G_n to G it follows that of Murax [26], we get $H_n \to H$ in $(W_0^{1,p}(Q))^*$ for each $p > 2$. Hence $(W_0^{1,p}(Q))^*$.
 $W_0^{1,p}(Q)^*$.
 $h \in V \cap W^{1,\infty}(\Omega)$. Following AGMON, Douglars and NIRENBERG [1], we
 $= h^1 + h^2$, (4.26)
 $= 0$ and $h^2 \in W_0^{1,2}(\Omega)$. (4. Into narmonic functions and functions with zero traces. By MEYERS [
sts $p_1 > 2$ such that the mapping " $h \rightarrow h^{i}$ " $(i = 1, 2)$ is continuous fi
into $W^{1,p_1}(\Omega)$. Further, since the imbedding $W^{1,\infty}(\partial\Omega) \subset W^{1-\frac{1}{p_1},p_1$ $\begin{align} \n\text{where} \quad W^{1,2}(\text{there} \quad W^{1,2}) \ \text{there} \quad W^{1,p_1} \ \text{com} \quad \text{From} \quad \text{From} \quad \text{and} \quad W^{1,2} \ \text{in} \quad (W^{1,2}(\text{or} \quad \text{in} \quad W^{1,2}_{1}) \ \text{This,} \quad \text{Sim} \quad \text{This,} \quad \text{Sim} \quad \text{In} \$

Let, we can assert that the set
$$
\mathfrak{A} = \{h^1 : ||h||_{V \cap W^{1,\infty}(Q)} \leq 1\}
$$
 is compact in $W^{1,p_1}(Q)$.
this and the weak convergence of G_n to G it follows that

$$
\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(Q)} \leq 1}
$$

$$
|\langle G_n - G, h^1 \rangle| = 0.
$$
(4.27)
moving this, we can use a finite ε -network in the set \mathfrak{A} .) Moreover, from $G_n \to G$

$$
\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(Q)} \leq 1} |\langle G_n - G, h^2 \rangle| = 0.
$$

$$
\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(Q)} \leq 1} |\langle G_n - G, h^2 \rangle| = 0.
$$

4.26) and (4.27) imply $\langle G_n \to G$ in $[V \cap W^{1,\infty}(Q)]^*$.
are also

(For proving this, we can use a finite ε -network in the set \mathfrak{A} .) Moreover, from $G_n \to G$ For $\lim_{n \to +\infty} \sup_{\|h\|_{V_0(W^{1,\infty}(\Omega))} \le 1} |\langle G_n - G, h^{\perp} \rangle| = 0.$

(For proving this, we can use a finite ε -network in the set in $(W_1^{(1,p_1(\Omega))})^*$ and the properties of the mapping ${}^n h \to 0$ in $(W_0^{1,p_1}(Q))$ ^{*} and the properties of the mapping $h \to h^{2}$ it follows that also

From this and the weak convergence of G_n to G_j it follows:
 $\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(\Omega)}} |\langle G_n - G, h^1 \rangle| = 0.$
 \therefore
 $\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(\Omega)}} |\langle G_n - G, h^1 \rangle| = 0.$
 $\lim_{n \to +\infty} \sup_{\|h\|_{V \cap W^{1,\infty}(\Omega)}} |\langle G_n - G, h$ Similarly as in the proof of Theorem 4.11, we put $h(s) = \varrho(s) + \lambda s \varrho'(s)$ in $\left[0, \frac{6a_0^3}{s} \right]$. and choose λ close to $-\infty$ so that the condition (4.19) is valid. Then, by (4.24) and Theorem 3.22 for $h_n = u - u_n$, Theorem 3.22 for $h_n = u - u_n$,
 $\int h(|\nabla u_n|^2) \nabla u_n \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \nabla u_n \, dx$
 $\int h(|\nabla u_n|^2) \cdot \n$ begins, we can use a time e-network in the set 21.) More

(a) \ast and the properties of the mapping $\ast h \rightarrow h^{2}$, it foll

im sup $|\langle G_n - G, h^2 \rangle| = 0$.

and (4.27) imply $G_n \rightarrow G$ in $[V \cap W^{1,\infty}(\Omega)]^*$.

as in the proof of The $\varrho'(s)$ in $\left[0, \frac{6a_0^2}{\varkappa + 1}\right]$
 en, by (4.24) and (For provide)

in $(W_{l_0}$

This, (e^{iS_l})

and ch

Theore
 \therefore

21.

$$
(\mathcal{L}^{(1)}(2))^{T}
$$
 and the properties of the mapping " $h \to h^{2}$ " it follows that also
\n
$$
\lim_{n \to +\infty} \sup_{\|h\|_{V\cap W^{1,\infty}(\Omega)}} |\langle G_n - G, h^2 \rangle| = 0.
$$

\n4.26) and (4.27) imply $G_n \to G$ in $[V \cap W^{1,\infty}(\Omega)]^*$.
\nlarly as in the proof of Theorem 4.11, we put $h(s) = \varrho(s) + \lambda s \varrho'(s)$ in $[0, \frac{6\alpha}{\varkappa - 1}]$
\n $\text{cos } \lambda$ close to $-\infty$ so that the condition (4.19) is valid. Then, by (4.24)
\n $\text{cos } \lambda$ of $h_n = u - u_n$,
\n
$$
\int_{\Omega} h(|\nabla u_n|^2) \nabla u_n \cdot \nabla h_n dx
$$

\n
$$
= \int_{\Omega} \varrho(|\nabla u_n|^2) \nabla u_n \cdot \nabla h_n dx + \lambda \int_{\Omega} \varrho'(|\nabla u_n|^2) |\nabla u_n|^2 \nabla u_n \cdot \nabla h_n dx
$$

\n
$$
= \int_{\partial\Omega} gh_n ds + \langle F_n, h_n \rangle + \lambda \langle G_n, h_n \rangle.
$$

\n(4)

 $.28)$

 $21₁$

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324 M. FEISTAUER and J. NECAS
From $h_n \rightharpoonup 0$ (weakly) and the properties of F_n and G_n we get that the expression in (4.28) tends to zero, if $n \rightarrow +\infty$. If we subtract the expression (4.28) tends to zero, if $n \to +\infty$. If we subtract the expression M. FEISTAUER and J. NECAS
 $\rightarrow 0$ (weakly) and the properties of F_n and G

rds to zero, if $n \rightarrow +\infty$. If we subtract the e
 $\int h(|\nabla u|^2) \nabla u \cdot \nabla h_n dx$ ($\rightarrow 0$, if $n \rightarrow +\infty$)
 \approx

(8) and use (4.21), where $v := u_n$ and

$$
\int\limits_{\Omega} h(|\nabla u|^2) \nabla u \cdot \nabla h_n \, dx \qquad (\to 0, \text{ if } n \to +\infty)
$$

from (4.28) and use (4.21), where $v:= u_n$ and $u - v := h_n$, then, we get

from
$$
h_n \rightharpoonup 0
$$
 (weakly) and the properties of F_n and G_n we get that the expression in 1.28) tends to zero, if $n \to +\infty$. If we subtract the expression\n
$$
\int_{\Omega} h(|\nabla u|^2) \nabla u \cdot \nabla h_n \, dx \quad (\to 0, \text{ if } n \to +\infty)
$$
\nfrom (4.28) and use (4.21), where $v := u_n$ and $u - v := h_n$, then we get\n
$$
\beta_1 ||h_n||^2 \leq \int_{\partial\Omega} g h_n \, ds + \langle F_n, h_n \rangle + \lambda \langle G_n, h_n \rangle - \int_{\Omega} h(|\nabla u|^2) \nabla u \cdot \nabla h_n \, dx \to 0.
$$
\n(4.29)\nThis yields $h_n \to 0$ (if $n \to +\infty$). It means that $u_n \to u$ (strongly) and, in view of theorem 3.22 and the continuity of the mapping $w(u)$, we have $u = w(u)$. Hence,

This yields $h_n \to 0$ (if $n \to +\infty$). It means that $u_n \to u$ (strongly) and, in view of Theorem 3.22 and the continuity of the mapping $w(u)$, we have $u = w(u)$. Hence, u is a solution of the problem $(2.9, a - b)$ from (4.28) and use (4.21), where $v := u_n$
 $\beta_1 ||h_n||^2 \leq \int_{\partial \Omega} gh_n ds + \langle F_n, h_n \rangle +$

This yields $h_n \to 0$ (if $n \to +\infty$). It mean

Theorem 3.22 and the continuity of the 1
 u is a solution of the problem (2.9, a-b) I

In th $|u|^2 \leq \int_{\partial\Omega} gh_n ds + \langle F_n, h_n \rangle$
 $\rightarrow 0$ (if $n \rightarrow +\infty$). It is

and the continuity of the

of the problem (2.9, a --1

wing theorem we shall s

17, 18, 32], has similar c

4.30 (2nd fundamental):

and
 $-\alpha_1 \sqrt{s} \leq c < +\infty$,
 u_n and $u - v := h_n$, then, we get
 $+ \lambda \langle G_n, h_n \rangle - \int h(|\nabla u|^2) \nabla u \cdot \nabla h_n dx \to 0.$

(4.29)

cans that $u_n \to u$ (strongly) and, in view of

ne mapping $w(u)$, we have $u = w(u)$. Hence,

(4.29)

how that also the entropy condition

In the following theorem we shall show that also the entropy condition (1.18), used e.g. in [3, 17, 18, 32], has similar compactification properties as (1.17).

Theorem 4.30 (2nd fundamental): Let us assume that o satisfies the conditions

$$
|g(s) - \alpha| \sqrt{s} \leq c < +\infty, \qquad s \in [0, +\infty)^{1}
$$

with constants α , $c > 0$ *and let us consider the boundary condition (2.4). Let* $\{u_n\}_{n=0}^{+\infty}$ *he a* minimizing sequence of the functional ψ , satisfying a posteriori the condition (1.18), and let $u_n \rightharpoonup u$ (weakly) in V, if $n \rightarrow +\infty$. Then $u_n \rightarrow u$ (strongly) in V, u is a solu- $|g(s) - \alpha| \sqrt{s} \leq c < +\infty$, $s \in [0, +\infty)^{1}$ (4.31
 with constants α , $c > 0$ and let us consider the boundary condition (2.4). Let $\{u_n\}_{n=0}^{+\infty}$
 be a minimizing sequence of the functional ψ , satisfying a poste **the** *tion of* $e^{i\theta}$ *<i>tion of the problem (2.9, a—b)**and let us consider the boundary condition a minimizing sequence of the functional* ψ *, satisfying a posterio and let* $u_n \to u$ *(weakly) in* V *, if* $n \to +\infty$ *. Then tion of the problem* $(2.9, a - b)$ *and* $||u_n||_{W^{1,p_1}(\Omega)}$, $||u||_{W^{1,p_2}(\Omega)} \leq \tilde{c}$ *with some* $p_2 > 2$ *and* $\tilde{c} > 0$. In the following theorem we shall show that and e.g. in [3, 17, 18, 32], has similar compactific:

Theorem 4.30 (2nd fundamental): Let us ass

19)-(1.23) and
 $|g(s) - \alpha| \sqrt{s} \le c < +\infty$, $s \in [0, +\infty]$

th constants $\alpha, c > 0$ (1.19)-(1.23) and
 $|g(s) - \alpha| \sqrt{s} \leq c < +\infty$, $s \in [0, +\infty)^1$)

with constants $\alpha, c > 0$ and let us consider the boundary

be a minimizing sequence of the functional ψ , satisfying a p

and let $u_n \to u$ (weakly) in V , if tants $\alpha, c > 0$ and let us consider the boundary condition (2.4). Let $\{u_n\}_{n=0}^{\infty}$
mizing sequence of the functional v , satisfying a posteriori the condition (1.18),
 $\rightarrow u$ (weakly) in V, if $n \rightarrow +\infty$. Then $u_n \rightarrow u$

Proof: By Theorem 3.22,

$$
\int_{\Omega} \varrho(|\nabla u_n|^2) \nabla u_n \cdot \nabla v \, dx = \int_{\partial \Omega} g v \, ds \quad \forall y \in V, \quad v \in V,
$$

where $F_n \to 0$ in V^* . From this we get

$$
\begin{aligned}\n&\rightarrow u \ (weakly) \ in \ V, \ if \ n \rightarrow +\infty. \ Then \ u_n \rightarrow u \ (strongly) \ in \ V, \ u \ is \ a \ solu-\\
\text{problem (2.9, a - b) and } \|u_n\|_{W^{1,p_1}(Q)} \cdot \|u\|_{W^{1,p_2}(Q)} \leq \tilde{c} \ with \ some \ p_2 > 2 \ and \\
\text{By Theorem 3.22,} \\
&\int \varrho(|\nabla u_n|^2) \ \nabla u_n \cdot \nabla v \, dx = \int \varrho v \, ds \quad \text{by } \langle F_n, v \rangle, \quad v \in V, \\
&\rightarrow 0 \text{ in } V^* \text{ From this we get} \\
&\quad \int \nabla u_n \cdot \nabla v \, dx = \int \left(\alpha - \varrho(|\nabla u_n|^2) \right) \nabla u_n \cdot \nabla v \, dx \\
&\quad + \int \varrho v \, ds + \langle F_n, v \rangle, \qquad v \in V'.\n\end{aligned}\n\qquad (4.32)
$$
\n
$$
V \text{ be a solution of the problem} \\
\alpha \int \nabla u_n! \cdot \nabla v \, dx = \int \left(\alpha - \varrho(|\nabla u_n|^2) \nabla u_n \cdot \nabla v \, dx \right. \\
&\quad + \int \varrho v \, ds \qquad \forall v \in V \qquad (4.33)
$$
\n
$$
V \text{ be a solution of the problem} \\
\alpha \int \nabla u_n^2 \cdot \nabla v \, dx = \langle F_n, v \rangle \qquad \forall v \in V. \qquad (4.34)
$$
\n
$$
\alpha \int \nabla u_n^2 \cdot \nabla v \, dx = \langle F_n, v \rangle \qquad \forall v \in V. \qquad (4.35)
$$
\n
$$
\text{a solution is satisfied e.g., if we extend } \varrho \text{ to } [0, +\infty] \text{ in such a way that } \varrho(s) = \text{const.}
$$

Let $u_n^1 \in V$ be a solution of the problem

$$
\alpha \int_{\Omega} \nabla u_n! \cdot \nabla v \, dx = \int_{\Omega} (\alpha - \varrho (|\nabla u_n|^2) \nabla u_n \cdot \nabla v \, dx + \int_{\partial \Omega} g v \, ds \qquad \forall v \in V
$$
\n(4.33)

and $u_n^2 \in V$ be a solution of the problem

$$
\alpha \int_{\Omega} \nabla u_n^2 \cdot \nabla v \, dx = \langle F_n, v \rangle \qquad \forall \ v \in V. \tag{4.34}
$$

¹) This condition is satisfied e.g., if we extend ϱ to $[0, +\infty]$ in such a way that $\varrho(s) = \text{const}$ for large *s* (cf. Section 1).

 $\frac{1}{2}$ Then $u_n = u_n^1 + u_n^2$. It is evident that $u_n^2 \to 0$ in *V* (since $F_n \to 0$ in *V*)*. Hence,
 $u_n^1 \to u$. Then $u_n = u_n^1 + u_n^2$. It is evident that $u_n^2 \to 0$ in *V* (si
 $u_n^1 \to u$.
 Now, by [25]
 $||u_n^1||_{W^{1,p_1(p)}} \le \tilde{c} = \text{const} \quad \forall n = 0, 1, ...$ ¹

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1 Interval 1, 2 . It is evident that $u_n^2 \to 0$ in V (since $F_n \to 0$ in V^*). Hence,

Now, by [25]
 $||u_n||_{W^{1,p_1(Q)}} \le \tilde{c} = \text{const} \quad \forall n = 0, 1, ...$ (4.35)

th some $p_2 > 2$ an Then $u_n = u_n^{-1} + u_n^{-2}$. It is evident
 $u_n^{-1} \rightharpoonup u$.

Now, by [25]
 $||u_n^{-1}||_{W^{1,p_1}(Q)} \leq \tilde{c}$ = const

with some $p_2 > 2$ and $\tilde{c} > 0$. Since
 $G \in V^*$ by the relations
 $\langle G_n, h \rangle = \int_{\tilde{C}} \nabla u_n \cdot \nabla h \, dx$,

and
 t $u_n^2 \to 0$ in V (since $F_n \to 0$ in V^*). Hence,
 $\forall n = 0, 1, ...$ (4.35)
 $\rightarrow u$, also $||u||_{W^{1,p_{\sharp}}(Q)} \leq \tilde{c}$. If we define G_n and
 $\in V$ (4.36)
 V , (4.37)

ving we shall proceed similarly as in the proof

at all

$$
\|u_n\|_{W^{1,p_1}(\Omega)} \leq \tilde{c} = \text{const} \qquad \forall \ n = 0, 1, \dots \tag{4.35}
$$

Now, by [25]
 $||u_n||_{W^{1,p_i}(\Omega)} \le \tilde{c} = \text{const} \quad \forall n = 0, 1, ...$ (4.35)

with some $p_2 > 2$ and $\tilde{c} > 0$. Since $u_n^{-1} \to u$, also $||u||_{W^{1,p_i}(\Omega)} \le \tilde{c}$. If we define G_n and
 $G \in V^*$ by the relations $G \in V^*$ by the relations $||u_n||_{W^{1,p_{\mathfrak{t}}(\Omega)}} \leq \tilde{c} = \text{const} \qquad \forall \ n = 0, 1, ...$ (4.35)
 $e \ p_2 > 2 \text{ and } \tilde{c} > 0. \text{ Since } u_n^1 \to u, \text{ also } ||u||_{W^{1,p_{\mathfrak{t}}(\Omega)}} \leq \tilde{c}. \text{ If we define } G_n \text{ and }$
 $\forall G_n, h \rangle = \int_{\Omega} \nabla u_n \cdot \nabla h \, dx, \qquad h \in V$ (4.36)

$$
\langle G_n, h \rangle = \int \nabla u_n \cdot \nabla h \, dx, \qquad h \in V \tag{4.36}
$$

by the relations
\n
$$
\langle G_n, h \rangle = \int_{\Omega} \nabla u_n \cdot \nabla h \, dx, \qquad h \in V
$$
\n
$$
\langle G, h \rangle = \int_{\Omega} \nabla u \cdot \nabla h \, dx, \qquad h \in V,
$$
\n(4.36)
\n
$$
\langle G, h \rangle = \int_{\Omega} \nabla u \cdot \nabla h \, dx, \qquad h \in V,
$$
\n(4.37)

respectively, then $G_n \rightharpoonup G$. In the following we shall proceed similarly as in the proof of Theorem 4.23: First, if we realize that all u_n satisfy the condition (1.18), then we see that in view of [26], $G_n \to G$ in $(W_0^{1,p}(\Omega))^*$ for all $p > 2$. Further, let every $h \in V \cap W^{1,p_1}(\Omega)$ be decomposed in the form (4.26) . Using again Meyers' results from [25] and considering p_1 from the proof of Theorem 4.23 such that $2 < p_1 < p_2$. then in view of the compact imbedding of $W^{1-\frac{n}{p_1},p_1}(\partial\Omega)$ into $W^{1-\frac{n}{p_1},p_1}(\partial\Omega)$, we find $h \in V \cap W^{1,p}(\Omega)$ be decomposed in the form [25] and considering p_1 from the proof
then in view of the compact imbedding of
out that the set $\mathfrak{A} = \{h^1 : ||h||_{V \cap W^{1,p}(\Omega)} \le$
same arguments as in the proof of Theoren out that the set $\mathfrak{A} = \{\bar{h}^1 : ||h||_{V \cap W^{1,p_{\mathfrak{r}}(Q)}} \leq 1\}$ is compact in $W^{1,p_1}(Q)$. Now, by the same arguments as in the proof of Theorem 4.23, we conclude that $G_n \to G$ in $[V \cap W^{1,p_2}(Q)]^*$. and $\langle G, h \rangle = \int_{\Omega} \nabla u$
respectively, then G_n -
of Theorem 4.23: First,
see that in view of [2
 $h \in V \cap W^{1,p_1}(\Omega)$ be defrom [25] and consideri
then in view of the co
out that the set $\mathfrak{A} =$
same arguments as in
 $[V$ respectively, then $G_n \rightharpoonup G$. In the following we shall proceed simila
of Theorem 4.23: First, if we realize that all u_n satisfy the conditio
see that in view of [26], $G_n \rightharpoonup G$ in $(W_0^{1,p}(Q))^*$ for all $p > 2$. If
 h Theorem 4.23: First, if we realize that all u_n satisfy the condition (1.18)
 e that in view of [26], $G_n \to G$ in $(W_0^{1,p}(\Omega))^*$ for all $p > 2$. Further,
 $\in V \cap W^{1,p}(\Omega)$ be decomposed in the form (4.26). Using again Mey

Let $h_n = u_n^{-1} - u$. Hence, $h_n \rightharpoonup 0$ (weakly). If we use (4.32), (4.33), (4.36), (4.37), then • edges and the set of t

Let
$$
h_n = u_n^1 - u
$$
. Hence, $h_n \rightharpoonup 0$ (weakly). If we use
\nthen
\n
$$
\alpha ||h_n||^2 = \alpha \int_{\Omega} |\nabla h_n|^2 dx
$$
\n
$$
= \alpha \int_{\Omega} \nabla u_n \cdot \nabla h_n dx \rightharpoonup \alpha \int_{\Omega} \nabla u \cdot \nabla h_n dx + \langle F_n, h_n \rangle
$$
\n
$$
= \alpha \langle G_n - G, h_n \rangle + \langle F_n, h_n \rangle \rightharpoonup 0
$$
\nand,
\nthus, $u_n^1 \rightharpoonup u$ in V. Hence, $u_n \rightharpoonup u$ and u is a solution
\n5. Regularity of the minimizing sequence
\nIn the light of Theorems 4.23 and 4.30 we come to a, nat
\nthe minimizing sequence of the functional ψ can be found

and, thus, $u_n^1 \rightarrow u$ in *V*. Hence, $u_n \rightarrow u$ and *u* is a solution of the problem (2.9, a -b) **I**

In the light of Theorems 4.23 and 4.30 we come to a natural question "How regular the minimizing sequence of the functional ψ can be found?" In this section we shall assume that $\partial\Omega$ is smooth- and consider the Neumann condition (2.4) on $\partial\Omega$. $u_n^1 \rightarrow u$ in *V*. Hence, $u_n \rightarrow u$ and *u* is a solution of the problem (2.9, a-b) **s**
 $u_n^1 \rightarrow u$ in *V*. Hence, $u_n \rightarrow u$ and *u* is a solution of the problem (2.9, a-b) **s**

arity of the minimizing sequence

the of Theorems **IIII** is the minimizing sequence
 $\frac{1}{2}$ and 4.30 we come to a natural question "How regular
 $\frac{1}{2}$ right of Theorems 4.23 and 4.30 we come to a natural question "How regular
 $\frac{1}{2}$ $\frac{1}{2}$ a given and consi

Theorem 5.1 Let the conditions (1.19)—(1.23) *and* (4.31) *he satisfied and let* ${u_n}_{n=0}^{\infty}$ be a minimizing sequence of the functional ψ . Then, for the decomposition u_n $= u_n^1 + u_n^2$ *from the proof of Theorem 4.30, it holds is* and $\partial\Omega$ is smooth and consider the Neuma
 i $\partial\Omega$ is smooth and consider the Neuma
 5.1: Let the conditions $(1.19) - (1.2$:
 minimizing sequence of the functional
 from the proof of Theorem 4.30, it hold

ri

$$
||u_n^2|| \to 0 \quad \text{(in } V),
$$

$$
||u_n^1||_{W^{1,p}(\Omega)} \leqq c(p) < +\infty \qquad \forall \, p < +\infty, \, n = 0, 1, \ldots
$$

and $\{u_n\}_{n=0}^{+\infty}$ is also a minimizing sequence.

Proof: The assertion (5.2) has been already proved in Theorem 4.30. With respect to (4.33), the regularity of $\partial\Omega$ and (4.35), we get (5.3) on the basis of $W^{1,p}(\Omega)$ – estimates of the solution to (4.33) (cf. $[1]$).

It remains to show that $\{u_n\}_{n=0}^{+\infty}$ is also a minimizing sequence of ψ . In virtue of the relation $\psi(u) = C(u) - C(w(u))$ and the Lipschitz-continuity of C, it is sufficient *to prove that* $w(u_n) - w(u_n^1) \to 0$ *in <i>V*, if $n \to +\infty$. Let us put $w_n = w(u_n)$, w_n^1 $= w(u_n^1)$. With respect to (3.5) and (3.18) we have to (4.33), the regulari

mates of the solution

It remains to show

the relation $\psi(u) = C$

to prove that $w(u_n)$
 $= w(u_n^1)$. With respe
 $B(u_n, w_n - i)$
 $= \int_{\Omega} e(|\nabla u_n|^2)$
 $= \int_{\Omega} [e(|\nabla u_n|^2)$

If we put $v := w_n - i$

side, t

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\n
$$
\therefore
$$
 The assertion (5.2) has been already proved in Theorem 4.30. With respect
\n, the regularity of $\partial\Omega$ and (4.35), we get (5.3) on the basis of $W^{1,p}(\Omega)$ – esti-
\nthe solution to (4.33) (cf. [1]).
\nnains to show that $\{u_n\}_{n=0}^{1}$ is also a minimizing sequence of ψ . In virtue of
\nion $\psi(u) = C(u) - C(w(u))$ and the Lipschitz continuity of C , it is sufficient
\nthat $w(u_n) - w(u_n) \rightarrow 0$ in V , if $n \rightarrow +\infty$. Let us put $w_n = w(u_n)$, w_n
\n
$$
\therefore
$$
 With respect to (3.5) and (3.18) we have
\n
$$
B(u_n, w_n - w_n^1, v)
$$
\n
$$
= \int_{\Omega} \varrho(|\nabla u_n|^2) (\nabla w_n - \nabla w_n^1) \cdot \nabla v \, dx, \qquad v \in V.
$$
\n
$$
= \int_{\Omega} \varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2)| \nabla w_n^1 \cdot \nabla v \, dx, \qquad v \in V.
$$
\n(5.4)
\n
$$
= \int_{\Omega} \varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2)|^2 |\nabla w_n^1|^2 dx \Big|^{1/2}
$$
\n
$$
\leq \int_{\Omega} \varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2)|^2 |\nabla w_n^1|^2 dx \Big|^{1/2} ||w_n - w_n||.
$$
\n(5.5)

. .-

If we put $v := w_n - w_n^1$, use (3.2) and apply the Cauchy inequality to the right-hand -

$$
= \int_{\Omega} \left[\varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2) \right] \nabla w_n^1 \cdot \nabla v \, dx, \qquad v \in V. \tag{5.4}
$$

If we put $v := w_n - w_n^1$, use (3.2) and apply the Cauchy inequality to the right-hand side, then

$$
c_2 ||w_n - w_n^1||^2
$$

$$
\leq \left\{ \int_{\Omega} \left[\varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2)|^2 |\nabla w_n|^2|^2 dx \right\}^{1/2} ||w_n - w_n||. \tag{5.5}
$$

Hence, it will do to prove that

$$
\int_{\Omega} \left[\varrho(|\nabla u_n|^2) - \varrho(|\nabla u_n|^2)|^2 |\nabla w_n|^2|^2 dx \to 0, \right. \tag{5.6}
$$

if $n \to +\infty$.
We can assume that $\nabla u_n^2(x) \to 0$ almost everywhere in Ω . Using again [25], we get
 $p_3 > 2$ such that

$$
||w(u)||_{W^{1,p_1}(\Omega)} \leq e^* \qquad \forall u \in V.
$$

Now, let us choose an arbitrary $\varepsilon > 0$. Let $\mathfrak{M} \subset \Omega$ be a measurable set. By the
Hölder inequality and (5.7) we get. (5.7)

Hence, it will do to prove that

$$
\int_{\Omega} \left[\varrho (|\nabla u_n|^2)^2 - \varrho (|\nabla u_n|^2)^2 \, |\nabla w_n|^2 \right] dx \to 0,
$$
\n
$$
-\infty.
$$
\n(5.6)

0

0

if $n \to +\infty$.
We can assume that $\nabla u_n^2(x) \to 0$ almost everywhere in *Q*. Using again [25], we get $p_3 > 2$ such that

$$
||w(u)||_{W^{1,p_{u(Q)}}} \leq e^* \qquad \forall \ u \in V. \tag{5.7}
$$

$$
\leq \left\{ \int_{\Omega} \left[\varrho (|\nabla u_n|^2) - \varrho (|\nabla u_n|^2) \right]^2 |\nabla w_n|^2 d x \right\}^{1/2} ||w_n - w_n||. \tag{5.5}
$$
\nHence, it will do to prove that\n
$$
\int_{\Omega} \left[\varrho (|\nabla u_n|^2) - \varrho (|\nabla u_n|^2) \right]^2 |\nabla w_n|^2 d x \to 0,
$$
\nif $n \to +\infty$.\nWe can assume that $\nabla u_n^2(x) \to 0$ almost everywhere in Ω . Using again [25], we get\n
$$
p_3 > 2 \text{ such that } \qquad ||w(u)||_{W^{1,p_4}(\Omega)} \leq e^* \qquad \forall u \in V.
$$
\nNow, let us choose an arbitrary $\varepsilon > 0$. Let $\mathfrak{M} \subset \Omega$ be a measurable set. By the Hölder inequality and (5.7) we get.\n
$$
\left(\int_{\mathfrak{M}} |\nabla w_n|^2 d x \right)^{1/2} \leq \mu_n(\mathfrak{M})^{\frac{1}{2} - \frac{1}{p_1}} \left(\int_{\Omega} |\nabla w_n|^2|^p d x \right)^{\frac{p_1 - 2}{p_2}}.
$$
\n(5.8)

Now, let us choose an arbitrary $\varepsilon > 0$. Let $\mathfrak{M} \subset \Omega$ be a measurable set. By the Hölder inequality and (5.7) we get.
 $\left(\int_{\mathfrak{M}} |\nabla w_n|^2|^2 dx\right)^{1/2} \leq \mu_n(\mathfrak{M})^{\frac{1}{2 - \mu}} \left(\int_{\Omega} |\nabla w_n|^2|^{p_n'} dx\right)^{\frac{1}{p_n}} \leq c^* \$ We denote by μ_n the *n*-dimensional Lebesque measure in \mathbf{R}_n .) We choose $\mathfrak{M} \subset \mathbb{R}$

(We denote by μ_n the *n*-dimensional Lebesque measure in \mathbf{R}_n .) We choose $\mathfrak{M} \subset \mathbb{R}$

with $\mu_n(\mathfrak{M}) < \left(\$ virtue of Jegorov's theorem, $\nabla u_n^2(x) \to 0$ uniformly in $\Omega = \mathfrak{M}$. If $\delta > 0$, let n_0 be such that $|\nabla u_n^2(x)| < \delta$ in $\Omega - \mathfrak{M}$ for all $n \ge n_0$. Since ϱ is Lipschitz-continuous in $[0, +\infty)$ and $u_n = u_n! + u_n^2$, we get (by using the Hölder inequality) $[0, +\infty)$ and $u_n = u_n^1 + u_n^2$, we get (by using the Hölder inequality) in **R**_n.) We choose $\mathfrak{M} \subset \Omega$

21)) in such a way that in
 $\Omega - \mathfrak{M}$. If $\delta > 0$, let n_0 be

2 is Lipschitz-continuous in

ar inequality)
 $n_1^2 dx$
 $\left.\begin{matrix} 1/2 \\ 1/2 \end{matrix}\right|$ **o —**

$$
\begin{array}{ll}\n\left(\frac{1}{m}\right) & \left(\frac{1}{m}\right) \\
\text{denote by } \mu_n \text{ the } n\text{-dimensional Lebesque measure in } \mathbf{R}_n. \right) \text{ We choose } 2 \\
\mu_n(\mathfrak{M}) < \left(\frac{\varepsilon}{4c^*e_0}\right)^{\frac{2p_1}{p_1-2}} (e_0 \text{ is the constant from (1.21)) in such a way} \\
\text{or } \left(\frac{\varepsilon}{4c^*e_0}\right)^{\frac{2p_2}{p_1-2}} (e_0 \text{ is the constant from (1.21)) in such a way} \\
\text{or } \left(\frac{\varepsilon}{4c^*e_0}\right)^{\frac{2p_2}{p_1-2}} < e_0 \text{ is the constant from } \left(1.21\right) \\
\text{or } \left(\frac{\varepsilon}{4c^*e_0}\right)^{\frac{2p_2}{p_1-2}} < e_0 \text{ in } \Omega - \mathfrak{M} \text{ for all } n \geq n_0. \text{ Since } e_0 \text{ is Lipschitz continuity} \\
\text{so) and } u_n = u_n! + u_n^2, \text{ we get } (\text{by using the Hölder inequality}) \\
\left(\frac{\varepsilon}{2\pi}\right)^{\frac{2p_2}{p_1}} < \left(\frac{\varepsilon}{2\pi}\right)^{\frac{2p_2}{p_1}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_1}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_2}} \\
\text{so, so } \left\{\int_{\Omega} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_1}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_2}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_1}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_2}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_1}}\n\right) \\
\text{so, so } \left\{\int_{\Omega} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_1}} \left(\frac{\varepsilon}{\pi}\right)^{\frac{2p_2}{p_2}} < e_0 \text{ for all } n \geq n_0. \text{ Since } e \text{
$$

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\n
$$
\leq \text{const.} \left\{ \delta^2 \left(\int_{\Omega} |\nabla w_n|^2 dx \right)^{1/2} \right\}
$$
\n
$$
+ \delta \left(\int_{\Omega} |\nabla u_n|^2 \frac{2p_1}{p_1 - 2} dx \right)^{\frac{p_2 - 2}{2p_1}} \left(\int_{\Omega} |\nabla w_n|^2|^{p_1} dx \right)^{\frac{1}{p_1}} \right\}
$$
\n
$$
\leq \text{const.} \left\{ \delta^2 ||w_n|| + \delta ||u_n||_{\mathcal{W}^{1, \frac{2p_1}{p_1 - 2}}(\Omega)} \cdot ||w_n||_{\mathcal{W}^{1, \frac{p_1}{p_1 - 2}}(\Omega)} \right\}
$$
\nNow, if we use Lemma 3.21, (5.3) and (5.7), we see that there exists a constant c^* such that
\n
$$
I(\Omega - \mathfrak{M}) \leq c^{**}(\delta^2 + \delta).
$$
\nIf we choose $\delta > 0$ such that $c^{**}(\delta^2 + \delta) < \varepsilon/2$, then in view of (5.8) and (5.9),

Now, if we use Lemma 3.21, (5.3) and (5.7), we see that there exists a constant *^c*

$$
I(\Omega - \mathfrak{M}) \leq c^{**}(\delta^2 + \delta).
$$

If we choose $\delta > 0$ such that $c^{**}(\delta^2 + \delta) < \varepsilon/2$, then in view of (5.8) and (5.9), $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$
I(\Omega - \mathfrak{M}) \leq c^{**}(\delta^2 + \delta).
$$

pose $\delta > 0$ such that $c^{**}(\delta^2 + \delta) < \varepsilon/2$, the

$$
\left\{ \int_{\Omega} [\varrho(|\nabla u_n|^2] - \varrho(|\nabla u_n|^2)]^2 |\nabla w_n|^2|^{2} dx \right\}^{1/2} <
$$

for $n \geq n_0$, which concludes the proof **I**

 $\frac{1}{2}$

Now, if we use Lemma 3.21,

such that
 $I(\Omega - \mathfrak{M}) \leq c^{**}(\delta^2 - \mathfrak{M})$

If we choose $\delta > 0$ such that
 $\left\{\int_{\Omega} [e(|\nabla u_n^1|^2) - e(|\nabla$

for $n \geq n_0$, which concludes t

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