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On the Pendent Liquid Drop[1])

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Dedicated to Herbert Beckert on the occasion of his sixty fifth birthday.

Es wird bewiesen, daß einige mit der Geometrie eines hängenden Tropfen verbundenen Werte, z. B. Durchmesser und Volumen, gleichmäßig unter allen möglichen symmetrischen Tropfen von oben begrenzt sind. Ausführliche Abschätzungen, die auch für diejenige Tropfen gelten, die sich als statisch nicht stabil erweisen, werden gegeben.

Доказывается, что некоторые величины, связанные с геометрией висячей капли, как диаметр и объём, ограничены сверху равномерно относительно всевозможных симметрических капель. Приводятся подробные оценки, имеющие место также для статически неустойчивых капель.

The volume, diameter and other geometrical quantities associated with a pendent liquid drop are shown to be equibounded among all symmetric drops in equilibrium configuration. Explicit bounds are given, and they are shown to be valid also for configurations that are known to be statically unstable.

We consider in this paper a drop of liquid pendent from a homogeneous horizontal plane surface Π and in equilibrium in a uniform gravity field directed downward from Π (see Fig. 1). The liquid will be bounded in part by Π and in part by a free surface S. According to the general theory of capillarity, as initiated by YOUNG [24], LAPLACE [14], and GAUSS [9], the mean curvature H of S satisfies a relation

$$2H = -\varkappa u + \lambda \tag{1}$$

in terms of the distance $(-u)$ to Π. Here $\varkappa$ is the "capillarity constant", $\varkappa > 0$; λ is a Lagrange parameter corresponding to the (prescribed) volume of the drop. The surface S is to meet Π in a prescribed constant angle γ, depending on the materials.

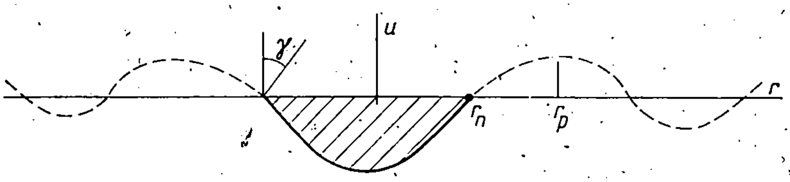

Figure 1: Pendent drop profile and continuation

1. Simple experiments (and everyday experience) suggest that any such drop with sufficiently large volume will be physically unstable. Apparently the first attempts to introduce the volume quantitatively as a criterion for instability were due to HAGEN [10] and to TRAUBE [21]; however, their formal analyses were based on

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erroneous assumptions and led to poor prediction. Lohnstein [15] integrated (1) numerically and determined a family of configurations with increasing drop height, in which the volume achieves a local maximum; he adopted that configuration as criterion for instability. Lohnstein's calculations suggested also that there is an upper bound for the diameter of the wetted disk on Π, at least among stable configurations.

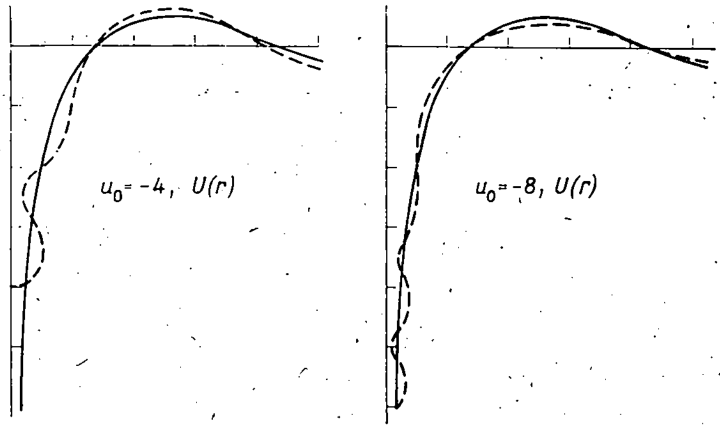

Figure 2. Drop profiles and singulär solution

There is by now a fairly extensive literature on the stability question (cf. [2, 15, 17, 19, 23] and the further references there cited), all of which tends to substantiate Lohnstein's criterion. We remark that all but the last of the cited works develop criteria relative to surfaces that were not yet known to exist. The point is not an idle one, as it cannot be expected that a formal solution will continue to exist when stability fails, and thus the criteria developed could well have been vacuous. The first attempt to obtain a global description of pendent drops seems due to Bashforth and Adams [1], who used the problem to test a new numerical procedure. Lohnstein was aware of the paper but overlooked the section on pendent drops. In a later note [16] he remarks "... entnahm ich kürzlich das genannte Werk der hiesigen Königlichen Bibliothek, und da fand ich dann die erwähnte Tabelle, deren rechtzeitige Benutzung mir viel Arbeit erspart hätte". Kelvin [14] proposed a geometric integration procedure; his result in a specific case suggested the existence of formal solutions of (1) of a sort that had not been physically observed (see Fig. 2). Modern computing methods have facilitated the numerical integration of (1), and many particular cases in agreement with Kelvin's discovery have since been worked out, see, e.g., [5, 11—13, 18]. Hida and Nakanishi [12] were apparently the first to find configurations with a large number of "bulges", and their calculations suggest that the maximal "physical" volume (as defined in § 5) could oscillate to a finite limit as the vertical height increases (cf. the discussion in § 7). Concus and Finn [5] obtained a global existence theorem by characterizing analytically the set of all symmetric "pendent drop" solutions. Wente [22] later proved that every solution is symmetric, so all solutions are included in the family considered.

The family may be obtained by observing that in the symmetric case (1) can be written, after elementary transformation, in the parametric form

$$\frac{du}{ds} = \sin\psi, \quad \frac{dr}{ds} = \cos\psi, \quad \frac{d\psi}{ds} = -u - \frac{1}{r}\sin\psi \tag{2}$$

for u, radial distance r, and inclination angle ψ in terms of arc length s, and by studying the continuation of a (local) solution of (2) with prescribed initial value $u(0) = u_0 < 0$. It is remarkable that in the entire range $0 > u_0 > -\infty$ a solution exists globally and uniquely, and defines a (sectional) curve that extends to infinity without limit sets or double points. With increasing $|u_0|$ the number of bulges increases indefinitely. (Also the value $u_0 = -\infty$ can be considered, in which case a singular solution $U(r)$ is obtained, see [3, 4, 13].) After crossing the r-axis, the curve oscillates about $u = 0$, with peaks of successively decreasing magnitude.

The formal pendent drop solutions are obtained by cutting the curves by horizontal lines; these lines generate the plane Π of support. If Π cuts the surface in more than one circle, and if the configuration is to have physical meaning, then the wetted surface consists only of the disk bounded by the innermost circle, as otherwise the drop would penetrate Π.

The cases $u_0 = -4$ and $u_0 = -8$ are illustrated in Figure 2. Since WENTE [23] has shown that instability must occur before the appearance of a second inflection, it is apparent that the equation for the solution surface is not cognizant of the instability, at least as far as existence and regularity are concerned.

To each section, there corresponds a finite number $\{r_j, u_j\}$ of points at which the section is vertical. The value $2r_v = 2 \max \{r_j\}$ was defined in [5] to be the *diameter* of the drop, and it was shown there that there is an equi-bound for the diameters of all possible drops.

Denote by r_n the coordinate of the first point at which the solution curve crosses the r-axis. It is shown in [5] that r_n plays the role of a dividing point, separating the portion of the curve with the bulges from the portion that oscillates about the r-axis.

Denote by r_p the coordinate of the first peak. For any physical drop the diameter d of the wetted surface on Π satisfies $d \leqq 2 \max \{r_v, r_p\}$.

Let V denote the volume of a (physical) drop, defined as the volume of that component cut off by Π, which meets the u-axis.

We intend to prove that *all four quantities, r_v, r_n, r_p, V are equibounded among all possible drops, without regard to stability considerations*. Specifically, we shall prove:

Theorem 1: *$r_v < \delta$, where $\delta \approx 2.473$ is the unique positive root of*

$$r^3 - 3^{3/2} r - 3^{3/4} = 0. \tag{3}$$

Theorem 2: *$r_n < \mu$, where $\mu \approx 2.888$ is the unique root $r > \delta$ of*

$$\left(1 - \frac{\delta^2}{2}\right) \ln \frac{r}{\delta} + \frac{r^2 - \delta^2}{4} - \frac{1}{2}\left(\delta \ln \frac{r}{\delta}\right)^2 - \frac{1}{\delta^2} = 0. \tag{4}$$

Theorem 3: *Set $\tau = \frac{1}{2}\sqrt{2}\, r_n$. Then*

$$r_p{}^2 < \frac{4 + \tau}{(1 + \tau) \ln\left(1 + \frac{1}{\tau}\right) - 1} \tag{5}$$

and thus $r_p < 5.333$.

Theorem 4: *$V \leqq \pi \sqrt{2}\, r_p{}^2$, and thus $V < 126.4$.*

2. Theorem 1 was already proved in [5]; we include here a proof that has been shortened in some ways. Our discussion depends on the general results of [5], (a) that

the solutions exist and have the appearance indicated in Figures 1, 2, and (b) that every drop has the properties:

(i) all outer vertical points (curvature vectors directed toward the u-axis) lie between the hyperbolae $ru = -1$ and $ru = -2$; the inner vertical points satisfy $-1 < ru < 0$;

(ii) at r_n the sectional curve is locally a graph $u(r)$, and $0 < u'(r_n) < \infty$.

Proof of Theorem 1: For any drop, the maximum diameter occurs at an outer vertical (b, u_b) and is preceded by a segment $u(r)$ of the curve, joining (b, u_b) to a point (a, u_a), which is either an inner vertical or the initial point $(0, u_0)$. On the interval $a < r < b$ we have from (2)

$$(r \sin \psi)_r = -ru. \tag{6}$$

Since $bu_b < -1 < au_a \leq 0$, the curve segment meets the hyperbola $ru = -1$ at a point (c, u_c). Since $\sin \psi_a = 1$ if $a \neq 0$, we have from (6)

$$r \sin \psi - a = -\int_a^r \varrho u \, d\varrho \tag{7}$$

and since $u < 0$ on the segment, there follows $\sin \psi > 0$ and therefore $u' > 0$. Integrating by parts and setting $r = c$, we thus find from (7)

$$c \sin \psi_c > \frac{1}{2}(a + c) \geq \frac{1}{2} c \tag{8}$$

since $au_a > -1$, $cu_c = -1$. Thus, $\sin \psi_c > \frac{1}{2}$. Further, at (c, u_c) the slope of the solution curve cannot exceed that of the hyperbola; we conclude that $c \leq 3^{1/4}$.

We now repeat the procedure, starting the integration at c, to obtain

$$b > c \sin \psi_c + \frac{1}{2}(b - c) + \frac{1}{2}\int_c^b \varrho^2 u'(\varrho) \, d\varrho. \tag{9}$$

In the interval of integration, we have $-2 \leq ru \leq -1$. By (6), $r(\sin \psi)_r = -ru - \sin \psi$; thus there is no inflection on the interval, and $u'(r) > u'(c)$. We conclude from (9), since $\sin \psi_c > \frac{1}{2}$, $c < 3^{1/4}$, that

$$b^3 - 3^{3/2} b - 3^{3/4} < 0. \tag{10}$$

The unique positive root of (3) is easily seen to exceed any value satisfying (10), and the stated bound for r_v follows ∎

3. We proceed to prove Theorem 2. By (ii) above, there must be a segment $u(r) < 0$ of the solution curve, joining an inner vertical (or initial) point (a, u_a) with the crossing point $(r_n, 0)$. We have from (6), if $a \leq \alpha \leq r \leq r_n$,

$$r \sin \psi(r) - \alpha \sin \psi(\alpha) = -\int_\alpha^r \varrho u \, d\varrho. \tag{11}$$

Setting $\alpha = a$ in (11), we obtain $\sin \psi(r) > ar^{-1} \geq 0$. Another use of (11) now yields

$$\frac{du}{dr} = \tan \psi > \sin \psi > \alpha r^{-1} \sin \psi(\alpha) \tag{12}$$

from which

$$-u(r) > (\alpha \sin \psi(\alpha)) \ln \frac{r_n}{r}. \tag{13}$$

We place (13) into (11) and repeat the above procedure to obtain

$$\frac{r}{\alpha \sin \psi(\alpha)} \frac{du}{dr} > 1 + \frac{1}{2} r^2 \ln \frac{r_n}{r} - \frac{1}{2} \alpha^2 \ln \frac{r_n}{\alpha} + \frac{r^2 - \alpha^2}{4} \tag{14}$$

an integration of which yields

$$\left(1 - \frac{\alpha^2}{2}\right) \ln \frac{r_n}{\alpha} + \frac{r_n^2 - \alpha^2}{4} - \frac{1}{2} \left(\alpha \ln \frac{r_n}{\alpha}\right)^2 < - \frac{u(\alpha)}{\alpha \sin \psi(\alpha)}. \tag{15}$$

The success of the procedure depends on a judicious choice for α. We observe first that an integration by parts in (11) yields $\alpha \sin \psi_\alpha > -\frac{1}{2} \alpha u(\alpha)$, since $a u_a > -1$ by (i) and $\sin \psi_a = 1$ if $a \neq 0$. Thus (15) can be written

$$F(r_n ; \alpha) \equiv \left(1 - \frac{\alpha^2}{2}\right) \ln \frac{r_n}{\alpha} + \frac{r_n^2 - \alpha^2}{4} - \frac{1}{2} \left(\alpha \ln \frac{r_n}{\alpha}\right)^2 - \frac{2}{\alpha^2} < 0. \tag{16}$$

There holds $F_{r_n} > 0$ when $r_n > \alpha$, and (16) thus determines a least upper bound μ_α for r_n. One finds $\frac{d\mu_\alpha}{d\alpha} \gtrless 0$ according as $\alpha \gtrless \alpha_0 \approx 1.4428$; here α_0 is the unique positive solution of

$$\left(1 - \frac{\alpha^2}{2}\right) \frac{\sqrt{4 - \alpha^2}}{\alpha^2} + \frac{\alpha^2}{4} \left[e^{\frac{2}{\alpha^2}\sqrt{4-\alpha^2}} - 1\right] - \frac{4}{\alpha^2} + \frac{1}{2} = 0. \tag{17}$$

If $r_n \leqq \alpha_0$, there is nothing to prove. If $a < \alpha_0 < r_n$, we set $\alpha = \alpha_0$ in (16) to obtain $r_n < 2.806$. If $\alpha_0 < a$, we recall from (i) that $a u_a > -1$, $\sin \psi_a = 1$; thus, (15) can now be written

$$G(r_n ; a) \equiv \left(1 - \frac{a^2}{2}\right) \ln \frac{r_n}{a} + \frac{r_n^2 - a^2}{4} - \frac{1}{2} \left(a \ln \frac{r_n}{a}\right)^2 - \frac{1}{a^2} < 0 \tag{18}$$

and (17) takes the form

$$\left(1 - \frac{a^2}{2}\right) \frac{\sqrt{2 - a^2}}{a^2} + \frac{a^2}{4} \left[e^{\frac{2}{a^2}\sqrt{2-a^2}} - 1\right] - \frac{2}{a^2} + \frac{1}{2} = 0 \tag{19}$$

which has the unique positive solution $a_0 \approx 1.157 < \alpha_0$. Thus, in the equation $G(r_n ; a) = 0$ we will have r_n increasing in a. Since $a \leqq \delta$ (Theorem 1), we obtain from (18) the stated estimate ∎

4. Proof of Theorem 3: We consider the segment of the solution curve joining $(r_n, 0)$ to (r_p, u_p). On that segment we have $u > 0$, $\sin \psi > 0$. Writing $v = \sin \psi$, we have from (6)

$$v_r = -u - v r^{-1} \tag{20}$$

and thus $v_r < -v r^{-1}$, from which

$$r v < r_n v_n \tag{21}$$

on the segment.

We now integrate (20) in u along the segment, from 0 to u_p, obtaining

$$\int_0^{u_p} \frac{\sin\psi}{r}\,du - (1 - \cos\psi_n) = -\frac{1}{2} u_p^2$$

from which

$$u_p < \sqrt{2}\sqrt{1 - \cos\psi_n} = \sqrt{2}\,\frac{\sin\psi_n}{\sqrt{1 + \cos\psi_n}} < \sqrt{2}\,v_n. \tag{22}$$

Thus by (6)

$$(rv)_r = -ru > -ru_p > -\sqrt{2}\,rv_n \tag{23}$$

from which

$$v(r) > \frac{r_n v_n}{r}\left(1 + \frac{\sqrt{2}}{2} r_n\right) - \frac{\sqrt{2}}{2} r v_n. \tag{24}$$

Thus, $v(r)$ remains positive at least until the value λr_n, with

$$\lambda^2 = \frac{1}{r_n}\left(\sqrt{2} + r_n\right). \tag{25}$$

We now observe $u'(r) > v(r)$ and integrate the inequality resulting from (24) from r_n to r. We obtain

$$u(r) > r_n v_n \left\{\left(1 + \frac{1}{2}\sqrt{2}\,r_n\right)\ln\frac{r}{r_n} - \frac{\sqrt{2}\,r_n}{4}\left(\frac{r^2}{r_n^2} - 1\right)\right\}. \tag{26}$$

Once the height $u(\lambda r_n)$ is attained, we find from (6) $(rv)_r = -ru < -ru(\lambda r_n)$ and thus, for $\lambda r_n < r < r_p$,

$$rv < \lambda r_n v(\lambda r_n) - u(\lambda r_n)\frac{r^2 - (\lambda r_n)^2}{2}.$$

It follows that the value r_p, at which $v = \sin\psi = 0$, must satisfy

$$r_p^2 < \frac{2\lambda r_n v(\lambda r_n)}{u(\lambda r_n)} + (\lambda r_n)^2. \tag{27}$$

In (27) we estimate the denominator from (26), and the numerator by placing (26) into (6) and integrating from r_n to λr_n. Using (25), we are led to the stated inequality (5) ∎

5. **Proof of Theorem 4**: The volume of any physical drop cannot exceed the volume V_p of the drop obtained by taking the supporting plane Π through (r_p, u_p). We have, in terms of arc length s measured from the vertex,

$$V_p = \pi r_p^2 u_p - 2\pi\int_0^{s_p} \varrho u \frac{d\varrho}{ds}\,ds = \pi r_p^2 u_p + 2\pi\int_0^{s_p} \frac{d}{ds}(\varrho\sin\psi)\,ds = \pi r_p^2 u_p$$

by the governing equations (2), since $\sin\psi = 0$ at $s = 0$, s_p. According to (22), $u_p < \sqrt{2}$, and the theorem follows ∎

6. The estimates we have given are in nondimensional form. To obtain the corresponding dimensional results for a given physical configuration, we need only multiply each spatial coordinate that appears in any relation by $\sqrt{\varkappa}$. Thus, for a water drop hanging from a glass plate in vacuo in the earth's gravitational field, for which situation one has $\gamma \approx 0$, $\varkappa \approx 29$, we find $r_v < 0.46$ cm, $r_n < 0.54$ cm, $r_p < 0.99$ cm, $V < 0.81$ cm³. These values, although they are within the range of reality, are certainly not sharp. The results do provide, however, general qualitative information that could not be obtained from the methods that were previously available. We remark that as $u_0 \to 0$, r_n and r_p become the first positive zeros of the Bessel functions $J_0(r)$ and $J_1(r)$; thus $r_n \approx 2{,}405$, $r_p \approx 3.832$. In the dimensional case just considered, $r_n \approx 0.447$, $r_p \approx 0.712$.

Figures 3 and 4 show results of computer calculations of r_n and of V, with increasing $|u_0|$. The appears to be either continued oscillation about, or very slow convergence toward, the values for the singular solution $U(r)$ (see § 7).

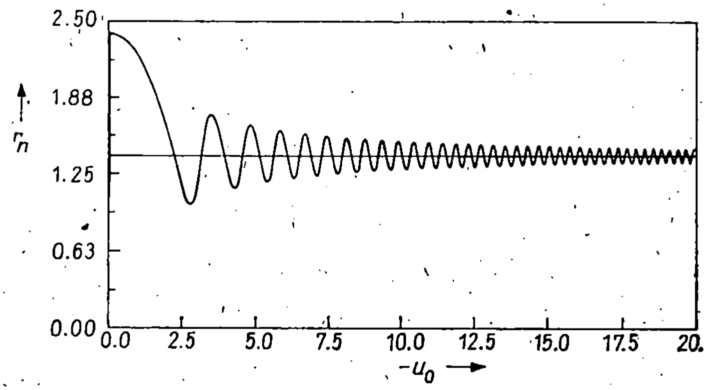

Figure 3. Initial null-point for drop profile and singulär solution

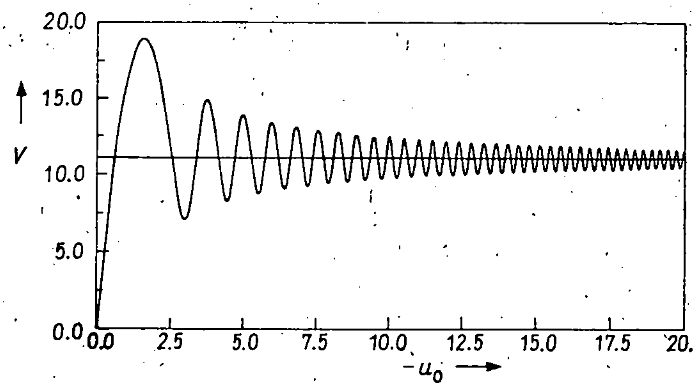

Figure 4. Maximal volume for pendent drop and singulär drop

7. Minor extensions of the estimates given above, in conjunction with estimates already available in [5], suffice to show that from any family of pendent drop solutions, with $u_0 \to -\infty$, there is a subsequence that converges uniformly in compacta, to a limit curve that is again a solution of (2) and has the general appearance of

the "singular solution" shown in Figure 2. The existence of such solution as a graph $U(r)$ was proved in [3]. It can be shown (cf. [5]) that any limit curve obtained by the above procedure is asymptotically close to $U(r)$ for small r, in the sense that both curves are asymptotic to the hyperbola $ru = -1$; however, it is not yet known whether the limit curve is uniquely determined, nor is it known under what circumstances it can be represented as a graph, nor has the uniqueness of $U(r)$ been proved. For partial results and conjectures relating to these questions, see [3—5, 7].

8. The bound we have shown for the volume has an interesting heuristic consequence. Imagine a drop of liquid, situated above and resting on a horizontal homogeneous plane Π in a vertical gravity field. It is known (cf. [6]) that a unique such drop exists for any V. Let us choose V larger than the bound given by Theorem 4. We now rotate Π about an axis through a diameter of the wetted disk, supposing that the adhesion forces suffice to keep the wetted surface unchanged. It can be shown [8] that the contact angle distribution can no longer be constant when Π is tilted, nevertheless it seems reasonable to suppose that, at least for small angles of tilt, a formal solution of the physical equations should continue to exist. If the plane could be rotated through an angle π, we would obtain a pendent drop configuration, which by Wente's theorem [22] must be symmetric and governed by the system (2). But by Theorem 4, no such solution exists. Thus, at some angle φ_0 of tilt, the solution must develop an instability, of a sort that prevents its further continuation (the configuration studied in the present paper could be continued in u_0 regardless of the instability). Some computational evidence of such non-continuability was encountered by Milinazzo (see the Appendix in [8]).

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