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The Poisson Formula for Euclidean Space Groups and some of its Applications II. The Jacobi Transformation for Flat Manifolds

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Dedicated to Prof. Dr. H. Beckert on the occasion of his 65th birthday

Unter Benutzung der Poissonformel von Teil I dieser Arbeit drücken wir die Spuren der Fundamentallösungen für die Wärmeleitungsgleichung und die Wellengleichung einer kompakten Riemannschen Mannigfaltigkeit mit verschwindender Sektionalkrümmung durch solche Daten aus, die mit ihren geschlossenen geodätischen Linien zusammenhängen.

Используя формулу Пауссона части I этой работы, выражаются следы фундаментальных решений уравнения теплопроводности и волнового уравнения компактного плоского риманова многообразия такими данными, которые связаны с его замкнутыми геодезическими линиями.

Using the Poisson formula of part I of this paper we express the traces of the heat kernel and the wave equation kernel of a compact flat Riemannian manifold by data which are connected with the behaviour of its closed geodesics.

In this second part we consider compact flat Riemannian manifolds (M, g). Using the Poisson formula of Part I we derive a Jacobi transformation formula; it gives a relation between the spectrum of (M, g) and certain data which are connected with the behaviour of the closed geodesic lines of (M, g). In order to explain the meaning of these data we firstly prove a theorem about the closed geodesics on a manifold of the aforesaid type.

Let ω be the set of non-trivial free homotopy classes of closed curves in (M, g).

Theorem 1: *Claim* (a): *To every $\vartheta \in \omega$ there belongs a compact flat manifold $M(\vartheta)$ of dimension $n(\vartheta)$, $1 \leqq n(\vartheta) \leqq n$, and a non-trivial free homotopy class ϑ_0 on $M(\vartheta)$. Through each point $P \in M(\vartheta)$ go closed geodesics belonging to ϑ_0. All of them have equal lengths $l(\vartheta)$; their tangential vectors form $k(\vartheta)$ locally parallel fields on $M(\vartheta)$.*

(It may happen that some of the closed geodesics through P have a self-intersection in P with different tangential vectors in P; it may also happen that each of the $k(\vartheta)$ vectors in P belongs to another closed geodesic of the class ϑ_0.)

Claim (b): *There is an isometric immersion $f_\theta : M(\vartheta) \to M$, such that $f_\theta(M(\vartheta))$ is totally geodesic in (M, g). The closed geodesics of (M, g) belonging to ϑ are exactly the curves $f_\theta(c')$, where c' is any closed geodesic of $M(\vartheta)$ belonging to ϑ_0.*

(It may happen that $f_\theta(M(\vartheta))$ has self-intersections with different tangential planes; double covering is impossible.)

For any $P \in M$ and any closed geodesic c through P we denote by $\Pi_{P,c}$ the linear mapping of $\mathring{M}_P$ onto itself induced by the parallel displacement along c. If $c \in \vartheta$, $P \in f_\theta(M_\theta)$, then $\Pi_{P,c}$ splits into mappings $\Pi_{P,c}^{\text{tang}}$ and $\Pi_{P,c}^{\perp}$, which are tangential and orthogonal to $f_\theta(M(\vartheta))$.

[1]) Part I of this paper was published in this journal in 1 (1982) 1, 13–23.

Claim (c): *We have* $\Pi_{P,c}^{\text{tang}} = \text{Id}$; *further,* $D(\vartheta) := |\text{Det}\,(\Pi_{P,c}^{\perp} - \text{Id})|$ *is different from zero and does not depend on the choice of* $P \in f_\theta(M(\vartheta))$ *and* $c \in \vartheta$.

Let Δ be the Laplace-Beltrami operator and α a covariant harmonic vector on (M, g). The differential operator

$$C^\infty(M) \ni u \mapsto$$

$$L[u] := \Delta u + 4\pi i g^*(\alpha, \nabla u) - 4\pi^2 g^*(\alpha, \alpha)\, u$$

has a self-adjoint extension in $L_2(M)$ with discrete spectrum $\{\lambda_\mathfrak{k}\}_{\mathfrak{k}\in\mathfrak{H}}$, every eigenvalue repeated as often as its multiplicity indicates. Let c be any closed curve in M; then the value

$$p(\vartheta) := 2\pi i \int_c \alpha(dx)$$

depends only on the free homotopy class ϑ of c.

Theorem 2 (*Jacobi transformation formula*): *For* $t \in \mathbf{C}$, $\Re t > 0$, *one has*

$$\sum_{\lambda_\mathfrak{k}\in \operatorname{Spec} L} e^{-\lambda_\mathfrak{k} t} = \frac{\text{vol}\, M}{(4\pi t)^{n/2}} + \sum_{\vartheta\in\omega} \frac{k(\vartheta)\,\text{vol}\, M(\vartheta)}{(4\pi t)^{n(\vartheta)/2}\, D(\vartheta)}\, e^{-l^2(\vartheta)/4t - p(\vartheta)}. \tag{1}$$

Here and in the following "vol" refers to the measure induced by the Riemannian metric g. If M is an n-torus, then (1) coincides with the usual transformation formula for an n-fold thetaseries.

The proofs of the Theorems 1 and 2 are given in § 1 and § 2, respectively. For the general theory of closed geodesics and free homotopy classes see H. BUSEMANN [16], W. KLINGENBERG [23], W. RINOW [25], S. KOBAYASHI and K. NOMIZU [8], D. GROMOLL, W. KLINGENBERG and W. MEYER [19].

The compact flat manifolds of dimension 2 and 3 are classified. (See H. HOPF [22], W. HANTZSCHE and H. WENDT [21], J. WOLF [15]). For the general classification problem see [15]. For the problem of Poisson formulas on Riemannian manifolds see (additionally to the papers quoted in Part I) H. DONELLY [17], P. D. LAX and R. S. PHILLIPS [24]. In particular, we refer the reader to the beautiful report of J. ELSTRODT [18] and its comprehensive bibliography.

§ 1

Let $\mathfrak{V}$ be an n-dimensional vector space over the real field $\mathbf{R}$. We consider $\mathfrak{V}$ also as an affine space, taking the elements of $\mathfrak{V}$ both as vectors and as points. Let $\mathfrak{G}$ be a properly discontinuous group of affine transformations of $\mathfrak{V}$ with compact fundamental domain $\mathcal{F}(\mathfrak{G})$. If $S \in \mathfrak{G}$ is the mapping: $\mathfrak{x} \mapsto \sigma(\mathfrak{x}) + \mathfrak{b}$, where σ is a linear transformation of $\mathfrak{V}$, we write $S = (\sigma, \mathfrak{b})$; the set $\mathfrak{L} := \{\sigma \mid \exists S \in \mathfrak{G} \text{ with } S = (\sigma, \mathfrak{b})\}$ is a finite group of order r, which we call the homogeneous group of $\mathfrak{G}$.

We introduce in $\mathfrak{V}$ a positive definite scalar product $(\mathfrak{x}, \mathfrak{y}) \mapsto g(\mathfrak{x}, \mathfrak{y})$ which is invariant under the homogeneous group: $g(\sigma(\mathfrak{x}), \sigma(\mathfrak{y})) = g(\mathfrak{x}, \mathfrak{y})$ for every $\sigma \in \mathfrak{L}$. The pair $(\mathfrak{V}, g)$ is a metric vector space or an Euclidean space, for which the elements of $\mathfrak{G}$ are isometries. It is well known that for any given $\mathfrak{G}$ such a g can be found.

Throughout this paper we assume that the elements of $\mathfrak{G}$ act freely on $(\mathfrak{V}, g)$, i.e. they have no fixed points. Then there exists a flat compact Riemannian manifold $M = \mathfrak{V}/\mathfrak{G}$ for which $(\mathfrak{V}, g)$ is the universal covering and whose fundamental group $\pi_1(M)$ is isomorphic to $\mathfrak{G}$. Let

$$\pi : \mathfrak{V} \to M = \mathfrak{V}/\mathfrak{G} \tag{1.1}$$

be the covering map.

In this section we shall study the periodic geodesics on M. For the sake of preciseness we say: Geodesics are always oriented and parametrized by their arc lengths s; the parameter representation is unique up to substitutions $s \mapsto s + a$, $a \in \mathbf{R}$. A periodic geodesic with period $l > 0$ has a representation $\mathbf{R} \ni s \mapsto c(s)$ with $c(s + l) = c(s)$ for every $s \in \mathbf{R}$. A closed geodesic of length l arises from the restriction of the parameter representation of a periodic geodesic with period l to a closed interval of length l.

Definition 1.1: *An oriented straight line $\tilde{c}$ of $(\mathfrak{V}, g)$, parametrized by its arc length $s \mapsto \tilde{c}(s)$, $s \in \mathbf{R}$, is called S-invariant with length $l > 0$, if there is an $S \in \mathfrak{G}$ such that for every $s \in \mathbf{R}$: $S(\tilde{c}(s)) = \tilde{c}(s + l)$.*

The following facts are well known. (See [16, 24].)

(i) If $\tilde{c}$ is S-invariant with length $l > 0$, then $\pi \circ \tilde{c}$ is a periodic geodesic on M with period l.

(ii) Let c be a periodic geodesic on M with period $l > 0$ and let $\tilde{c}$ be any lift of c in $\mathfrak{V}$, then there is exactly one $S \in \mathfrak{G}$, $S \neq \mathrm{Id}$, such that $\tilde{c}$ is S-invariant with length l.

(iii) Let c_1, c_2 be two periodic geodesics on M with periods l_1, l_2, respectively; let $\tilde{c}_1$ be a lift of c_1 in $\mathfrak{V}$ which is S-invariant with length l_1. The closed geodesics $c_{i|[0,l_i]}$, $i = 1, 2$, are free homotopic on M, if and only if there is a lift $\tilde{c}_2$ of c_2 in $\mathfrak{V}$ which is S-invariant with length l_2.

(iv) Let ϑ be a free homotopy class of closed curves in M, which is non-trivial; we denote by ω the set of these classes. There are closed geodesics on M belonging to ϑ, each of them can be extended to a periodic geodesic c on M. The lifts of these c are S-invariant (length = period) for suitable $S \in \mathfrak{G}$. The elements $S \in \mathfrak{G}$ arising in this manner form a conjugacy class θ of $\mathfrak{G}$, $\theta \neq \{\mathrm{Id}\}$.

(v) The set of conjugacy classes of $\mathfrak{G}$ is denoted by Ω. The correspondence between the non-trivial free homotopy classes $\vartheta \in \omega$ and the conjugacy classes $\theta \in \Omega \setminus \{\mathrm{Id}\}$ described under (iv) is bijective.

Lemma 1.2: *Let $S = (\sigma, \mathfrak{b}) \in \mathfrak{G}$, $S \neq \mathrm{Id}$, be given. The S-invariant straight lines of $(\mathfrak{V}, g)$ form a $(n(\sigma) - 1)$ — dimensional family of parallel lines filling out an $n(\sigma)$ — dimensional plane $\mathfrak{e}(S)$; $n(\sigma) \geqq 1$; all of them are S-invariant with the same length $l(S) > 0$. Finally, the isometry S, when restricted to the plane $\mathfrak{e}(S)$, acts as a translation, whose translation vector has the direction of the S-invariant lines and the length $l(S)$.*

Proof: *Let* $s \mapsto \mathfrak{x}(s) = \mathfrak{y} + s\mathfrak{v}$ be the parameter representation of any straight line in $(\mathfrak{V}, g)$, assume $g(\mathfrak{v}, \mathfrak{v}) = 1$. One has $S(\mathfrak{x}(s)) = \sigma(\mathfrak{y}) + \mathfrak{b} + s\sigma(\mathfrak{v})$; therefore, our straight line is S-invariant with length $l > 0$, if and only if

$$\sigma(\mathfrak{v}) = \mathfrak{v}, \qquad \sigma(\mathfrak{y}) + \mathfrak{b} = \mathfrak{y} + l\mathfrak{v}. \tag{1.2}$$

In § 1 of Part I we have shown the direct decomposition

$$\mathfrak{V} = \mathfrak{V}(\sigma) \oplus \mathfrak{V}^{\perp}(\sigma) \tag{1.3}$$

with $\mathfrak{V}(\sigma) = \ker(\sigma - \mathrm{Id})$, $\mathfrak{V}^{\perp}(\sigma) = \mathrm{im}(\sigma - \mathrm{Id})$.

Now, it is easily seen that (1.3) is an orthogonal decomposition. From (1.2) we see that the vector $\mathfrak{b}$ splits as follows

$$\mathfrak{b} = l\mathfrak{v} + (\mathfrak{y} - \sigma(\mathfrak{y})) \tag{1.4}$$

with $\mathfrak{v} \in \mathfrak{V}(\sigma)$, $\mathfrak{y} - \sigma(\mathfrak{y}) \in \mathfrak{V}^\perp(\sigma)$. The vector $\mathfrak{v}$ and the positive number $l > 0$ are uniquely determined by (1.4), whereas $\mathfrak{y}$ is uniquely determined modulo $\mathfrak{V}(\sigma)$. If $\mathfrak{a}_1 = \mathfrak{v}, \mathfrak{a}_2, \ldots, \mathfrak{a}_{n(\sigma)}$ is a basis of $\mathfrak{V}(\sigma)$, then the family of S-invariant straight lines is given by

$$s \mapsto \mathfrak{x}(s) = \mathfrak{y}_0 + \lambda_2\mathfrak{a}_2 + \cdots + \lambda_{n(\sigma)}\mathfrak{a}_{n(\sigma)} + s\mathfrak{v}\,, \tag{1.5}$$

where $\mathfrak{y}_0$ is any fixed vector satisfying (1.4); $\lambda_2, \ldots, \lambda_{n(\sigma)} \in \mathbf{R}$ are the parameters of the family, $n(\sigma) = \dim \mathfrak{V}(\sigma)$. Note that $l = 0$ means $S(\mathfrak{y}) = \mathfrak{y}$, what is excluded. By the same reason $n(\sigma) = 0$ is impossible. From (1.5) the assertions of Lemma 1.2 can be read off. In particular, the plane $\mathfrak{e}(S)$ contains the point $\mathfrak{y}_0$ and is spanned by the vectors $\mathfrak{a}_1 = \mathfrak{v}, \mathfrak{a}_2, \ldots, \mathfrak{a}_{n(\sigma)}$.

Definition 1.3: Using the notations of Lemma 1.2 we define: $\mathfrak{g}(S) := \{T \in \mathfrak{G} \mid T(\mathfrak{e}(S)) = \mathfrak{e}(S)\}$. This is a subgroup of $\mathfrak{G}$. Those elements of $\mathfrak{g}(S)$ which act in $\mathfrak{e}(S)$ as translations form a normal subgroup $\mathfrak{t}(S)$ of $\mathfrak{g}(S)$ with finite factor group $\mathfrak{g}(S)/\mathfrak{t}(S)$; denote $r(S) := \operatorname{ord}(\mathfrak{g}(S)/\mathfrak{t}(S))$. Further, $\mathfrak{g}(S) \cap \mathfrak{T}$ is a subgroup of the group $\mathfrak{t}(S)$ with finite factor group; denote $h(S) = \operatorname{ord}(\mathfrak{t}(S)/\mathfrak{g}(S) \cap \mathfrak{T})$.

In order to verify the correctness of this definition we remark that the elements of $\mathfrak{g}(S)$ act freely in $(\mathfrak{e}(S), g)$ as a properly discontinuous group of isometries. $\mathfrak{T}$ was the subgroup of translations contained in $\mathfrak{G}$, their translation vectors form the lattice Γ. Therefore, $\mathfrak{g}(S) \cap \mathfrak{T}$ contains those translations whose translation vectors belong to $\Gamma(\sigma) = \Gamma \cap \mathfrak{V}(\sigma)$. As we have seen in Part I (Lemma 1.2), $\Gamma(\sigma)$ contains exactly $n(\sigma)$ linearly independent vectors; but $n(\sigma) = \dim \mathfrak{e}(S)$, thus it follows the finiteness of $r(S)$ and $h(S)$. Moreover, we see that $M(S) := \mathfrak{e}(S)/\mathfrak{g}(S)$ is a compact flat manifold of dimension $n(\sigma)$.

Definition 1.4: Let $S = (\sigma, \mathfrak{b}) \in \mathfrak{G}$, $S \neq \mathrm{Id}$, be given and let the vector $\mathfrak{v}$ be determined by the decomposition (1.4). The number of pairwise distinct images of $\mathfrak{v}$ under the action of the elements $\mathfrak{g}(S)$ is denoted by $k(S)$.

We are now able to give the proof of Theorem 1.

Proof of Theorem 1: Ad (a). We consider the manifold $M(S)$ and its universal covering $\mathfrak{e}(S)$. We denote the covering map by $\pi_S\colon \mathfrak{e}(S) \to M(S)$. Further, $S \in \mathfrak{g}(S)$ determines a conjugacy class of $\mathfrak{g}(S)$ and consequently a non-trivial free homotopy class ϑ_0 of $M(S)$. A closed geodesic belongs to ϑ_0 if and only if the associated periodic geodesic has an S-invariant lift in $\mathfrak{e}(S)$. Let Q be any point of $\mathfrak{e}(S)$ and let $\mathfrak{v} = \mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_{k(S)}$ be the $k(S)$ pairwise distinct images of $\mathfrak{v}$ under the group $\mathfrak{g}(S)$. There are $k(S)$ isometries $S_1 = \mathrm{Id}, S_2, \ldots, S_{k(S)} \in \mathfrak{g}(S)$ and points $Q_1 = Q, Q_2, \ldots, Q_{k(S)}$ such that S_i maps the pair $(Q_i, \mathfrak{v})$ on the pair $(Q, \mathfrak{v}_i)$. The closed geodesics of $M(S)$ through $P = \pi_S(Q) \in M(S)$ belonging to ϑ_0 are given by

$$[0, l] \ni s \mapsto \pi_S(Q + s\mathfrak{v}_i)\,, \qquad i = 1, 2, \ldots, k(S)\,.$$

It may happen that these relations do not describe $k(S)$ pairwise distinct closed geodesics. For instance, if there is a value $\bar{s} \in (0, l)$, such that

$$\pi_S(Q + s\mathfrak{v}_i) = \pi_S(Q + [s + \bar{s}]\,\mathfrak{v}_j)\,, \qquad i \neq j\,,$$

then we have the phenomena of self-intersection with different tangential vectors. It is obvious that the images of the vectors $\mathfrak{v}_1, \ldots, \mathfrak{v}_{k(S)}$ under π_S form $k(S)$ locally parallel vector fields on $M(S)$.

It remains to prove that the construction of the manifold $M(S)$ is independent from the choice of S within its conjugacy class θ belonging to a given ϑ. If

$S' = TST^{-1}$, $T \in \mathfrak{G}$, then a simple calculation shows

$$\mathfrak{e}(S') = T\mathfrak{e}(S), \quad \mathfrak{g}(S') = T\mathfrak{g}(S)\,T^{-1}. \tag{1.6}$$

There exists a well-defined isometry φ of $M(S)$ onto $M(S')$ such that $\pi_{S'} = \varphi \circ \pi_S \circ T^{-1}$. The isometry φ maps the periodic geodesics of $M(S)$ with S-invariant lifts on the periodic geodesics of $M(S')$ with S'-invariant lifts and the same period. Further, one has $k(S) = k(S')$. We can now identify the manifolds $M(S)$ and $M(S')$ (according to the isometry φ) and it is therefore legitimate to write $M(\vartheta)$, $l(\vartheta)$, $k(\vartheta)$ and $n(\vartheta)$ instead of $M(S)$, $l(S)$, $k(S)$, $n(\sigma)$, respectively.

Ad (b). Define the map $f_S: M(S) \to M$ by $f_S = \pi \circ \pi_S^{-1}$. This map is locally isometric, because this is true for π and π_S. Because $\pi_S^{-1}(M(S)) = \mathfrak{e}(S) \subsetneqq \mathfrak{V}$ is a plane, $f_S(M(S))$ is totally geodesic in M. The correspondence of the closed geodesics under f_S which belong to ϑ_0 in $M(S)$ and to ϑ in M is now clear.

If we choose instead of S a conjugate element $S' = TST^{-1} \in \mathfrak{G}$, then we find at once $f_{S'} = f_S \circ \varphi^{-1}$ and it is legitimate to write $f_\vartheta: M(\vartheta) \to M$. If there are two different points P_1, P_2 of $M(\vartheta)$ with $f_\vartheta(P_1) = f_\vartheta(P_2)$, then there are two points Q_1, Q_2 of $\mathfrak{e}(S)$, which are $\mathfrak{G}$-equivalent, but not $\mathfrak{g}(S)$-equivalent. In $f_\vartheta(P_1) = f_\vartheta(P_2)$ we have self-intersection of $M(\vartheta)$ with different tangential spaces, but no double covering.

Ad (c). Let $[0, l(\vartheta)] \ni s \mapsto c(s)$ be any closed geodesic in M belonging to $\vartheta \in \omega$ with $P = c(0)$. Let $\bar{c}$ be any S-invariant lift of c in $\mathfrak{V}$ with length $l(\vartheta)$. Assume: $S = (\sigma, \mathfrak{b})$. Denote by π_1, π_2 the differential of π at $\bar{c}(0)$, $\bar{c}(l(\vartheta))$, respectively. Then it is easily seen that

$$\Pi_{P,c} \circ \pi_1 = \pi_2, \quad \pi_2 \circ \sigma = \pi_1.$$

From these relations it follows that

$$\pi_1^{-1} \circ \Pi_{P,c} \circ \pi_1 = \sigma^{-1}.$$

But σ^{-1} when restricted to $\mathfrak{V}(\sigma) = \mathfrak{V}(\sigma^{-1}) = \ker(\sigma - \mathrm{Id})$ equals the identity; on the other hand $\pi_1(\mathfrak{V}(\sigma))$ is the subspace of M_P tangential to $f_\vartheta(M(\vartheta))$ and therefore $\Pi_{P,c}^{\mathrm{tang}} = \mathrm{Id}$. If $\tilde{\sigma}$ is the restriction of σ^{-1} to $\mathfrak{V}^\perp(\sigma) = \mathrm{im}(\sigma - \mathrm{Id})$, then $\tilde{\sigma}$ cannot have the eigenvalue 1 and $\tilde{\sigma} - \mathrm{Id}$ is one-to-one on $\mathfrak{V}^\perp(\sigma)$. Therefore: $\mathrm{Det}(\Pi_{P,c}^\perp - \mathrm{Id}) = \mathrm{Det}(\tilde{\sigma} - \mathrm{Id}) \neq 0$. If $\bar{c}'$ is another lift of c, the element S must be replaced by a conjugate element $S' = TST^{-1}$, one has $S' = (\sigma', \mathfrak{b}')$, $\mathfrak{V}^\perp(\sigma') = T\mathfrak{V}^\perp(\sigma)$ and $\mathrm{Det}(\tilde{\sigma}' - \mathrm{Id}) = \mathrm{Det}(\tilde{\sigma} - \mathrm{Id})$. This determinant depends therefore only on the conjugacy class θ or the free homotopy class ϑ ∎

For later use we prove the following two lemmas.

Lemma 1.5: *Let $S \in \mathfrak{G}$, $S \neq \mathrm{Id}$, be given, assume $S = (\sigma, \mathfrak{b}) \in \theta$. Between the numbers $h(S)$, $t(S)$, $r(S)$* (Definition 1.3 and 1.4) *the following equation is valid*

$$rk(S) = m(\theta)\, h(S)\, r(S) \tag{1.7}$$

($m(\theta)$ was the number of $\mathfrak{T}$-conjugacy classes contained in θ).

Proof: The isometry S acts in $\mathfrak{e}(S)$ as translation with translation vector $l\mathfrak{v}$. Any isometry $T = (\tau, \mathfrak{c})$ of an Euclidean space is commutable with a translation if and only if τ does not change the translation vector. Thus, if $\mathfrak{N}(S)$ denotes the normalizer of S in $\mathfrak{g}(S)$, we have $\mathfrak{N}(S) \cap \mathfrak{t}(S) = \mathfrak{t}(S)$ and

$$k(S) = \mathrm{ord}\,(\mathfrak{g}(S)/\mathfrak{t}(S)) : \mathrm{ord}(\mathfrak{N}(S)/\mathfrak{t}(S)). \tag{1.8}$$

As we have seen above it is for any $T \in \mathfrak{G}$:

$$\mathfrak{e}(TST^{-1}) = T\mathfrak{e}(S). \tag{1.6}$$

From this relation one finds at once that $\mathfrak{N}(S)$ is also the normalizer of S in $\mathfrak{G}$. Further, one has $\mathfrak{N}(S) \cap \mathfrak{T} = \mathfrak{g}(S) \cap \mathfrak{T}$. This can be seen as follows: if $T \in \mathfrak{g}(S) \cap \mathfrak{T}$, then T is a translation, whose translation vector is in $\mathfrak{e}(S)$; S acts also as translation in $\mathfrak{e}(S)$; the action of T and S on $\mathfrak{e}(S)$ is therefore commutable: $T \in \mathfrak{N}(S)$. Consequently, $\mathfrak{g}(S) \cap \mathfrak{T} \subseteqq \mathfrak{N}(S) \cap \mathfrak{T}$. The inverse inclusion is trivial. From the normal chain

$$\mathfrak{N}(S) \supseteqq \mathfrak{t}(S) \supseteqq \mathfrak{N}(S) \cap \mathfrak{T}$$

we find

$$\begin{aligned}&\operatorname{ord}\big(\mathfrak{N}(S)/\mathfrak{t}(S)\big)\\ &= \operatorname{ord}\big(\mathfrak{N}(S)/\mathfrak{N}(S) \cap \mathfrak{T}\big) : \operatorname{ord}\big(\mathfrak{t}(S)/\mathfrak{N}(S) \cap \mathfrak{T}\big)\\ &= \operatorname{ord}\big(\mathfrak{N}(S) \cdot \mathfrak{T}/\mathfrak{T}\big) : \operatorname{ord}\big(\mathfrak{t}(S)/\mathfrak{g}(S) \cap \mathfrak{T}\big).\end{aligned}$$

According to Remark 3.2 of Part I we have

$$\operatorname{ord}\big(\mathfrak{N}(S) \cdot \mathfrak{T}/\mathfrak{T}\big) = r/m(\theta)$$

and therefore with Definition 1.3:

$$\operatorname{ord}\big(\mathfrak{N}(S)/\mathfrak{t}(S)\big) = r/m(\theta)\; h(S). \tag{1.9}$$

From (1.8) and (1.9) the assertion follows ∎

Lemma 1.6: *For any $S \in \mathfrak{G}$, $S \neq \mathrm{Id}$. one has*

$$\operatorname{vol}\big(M(S)\big) = \operatorname{vol} \mathcal{F}\big(\Gamma(\sigma)\big)/h(S)\, r(S). \tag{1.10}$$

($\mathcal{F}\big(\Gamma(\sigma)\big)$ is any fundamental domain of the lattice $\Gamma(\sigma)$.)

Proof: If $\mathcal{F}_1$ is any fundamental domain of $\mathfrak{e}(S)$ with respect to the translation group $\mathfrak{t}(S)$ we have

$$\operatorname{vol} M(S) = \operatorname{vol} \mathcal{F}_1/r(S). \tag{1.11}$$

As we have seen above, the lattice $\Gamma(\sigma)$ corresponds to the translation group $\mathfrak{T} \cap \mathfrak{g}(S) \subseteqq \mathfrak{t}(S)$; (see the text following Definition 1.5). From

$$h(S) := \operatorname{ord}\big(\mathfrak{t}(S)/\mathfrak{T} \cap \mathfrak{g}(S)\big)$$

it follows at once that

$$\operatorname{vol} \mathcal{F}_1 = \operatorname{vol} \mathcal{F}\big(\Gamma(\sigma)\big)/h(S). \tag{1.12}$$

The formulas (1.11), (1.12) contain the assertion ∎

§ 2

In Section § 2 of Part I we have constructed a complete orthonormal system $\{\psi_{\mathfrak{k}}\}$ in the Hilbert space $L_2(\mathfrak{G})$ of quadratically integrable $\mathfrak{G}$-automorphic functions over $\mathfrak{V}$. The identification of the elements of $L_2(\mathfrak{G})$ with those of $L_2(M)$, $M = \mathfrak{V}/\mathfrak{G}$, is obvious. The construction of the $\psi_{\mathfrak{k}}$ goes as follows. We consider the dual lattice $\Gamma^* \subset \mathfrak{V}^*$ of $\Gamma \subset \mathfrak{V}$; let $\mathfrak{k} = \{\mathfrak{u}_1, \ldots, \mathfrak{u}_l\}$ be any class of pairwise equivalent principal vectors $\mathfrak{u}_i \in \Gamma^*$. (See Definition 2.3 of Part I.) To $\mathfrak{k}$ there belongs the function

$$\mathfrak{V} \ni \mathfrak{x} \mapsto \psi_{\mathfrak{k}}(\mathfrak{x}) := \frac{1}{\sqrt{l}} \sum_{j=1}^{l} \exp\{2\pi i[\langle \mathfrak{u}_1, \mathfrak{a}_j\rangle + \langle \mathfrak{u}_j, \mathfrak{x}\rangle]\}, \tag{2.1}$$

with certain vectors $\mathfrak{a}_j \in \mathfrak{V}$. The vectors $\mathfrak{u}_j$, $j = 1, \ldots, l$, permute under the action of the homogeneous group $\mathfrak{L}$; therefore, they have the equal lengths $g^*(\mathfrak{u}_1, \mathfrak{u}_1) = g^*(\mathfrak{u}_j, \mathfrak{u}_j)$, $j = 1, \ldots, l$.

Let Δ be the Laplace-Beltrami operator and α a covariant harmonic vector on (M, g). We consider the elliptic differential operator

$$C^\infty(M) \ni u \mapsto L[u] := \Delta u + 4\pi i g^*(\alpha, \nabla u) - 4\pi^2 g^*(\alpha, \alpha)\, u.$$

To the vector field α on M there corresponds a constant vector $\tilde{\mathfrak{a}} \in \mathfrak{V}^*$, such that $\sigma^T(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$ for every $\sigma \in \mathfrak{L}$. If we identify functions on M with $\mathfrak{G}$-automorphic functions on $\mathfrak{V}$, then we can write

$$L[u] = \Delta u + 4\pi i g^*(\tilde{\mathfrak{a}}, \nabla u) - 4\pi^2 g^*(\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}})\, u.$$

It is easily seen that

$$L[\psi_{\mathfrak{k}}] + \lambda_{\mathfrak{k}} \psi_{\mathfrak{k}} = 0, \quad \lambda_{\mathfrak{k}} = 4\pi^2 g^*(\mathfrak{u}_1 + \tilde{\mathfrak{a}}, \mathfrak{u}_1 + \tilde{\mathfrak{a}}). \tag{2.2}$$

The set of all classes $\mathfrak{k}$ was denoted by $\mathfrak{H}$ (Part I). The equation (2.2) shows that $\{\psi_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathfrak{H}}$ is a system of eigenfunctions of L with eigenvalues $\{\lambda_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathfrak{H}}$. Because $\{\psi_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathfrak{H}}$ is complete, the set $\{\lambda_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathfrak{H}}$ represents the whole spectrum of the self-adjoint extension $\bar{L}$ of L:

$$\{\lambda_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathfrak{H}} = \operatorname{Spec} \bar{L}. \tag{2.3}$$

After these preparations we are able to prove our second theorem.

Proof of Theorem 2: The proof is an application of our Poisson formula (the Theorem in § 3, Part I) to the following function f. Let t be any complex number with $\Re e\, t > 0$; we put

$$\mathfrak{V} \ni \mathfrak{x} \mapsto f(\mathfrak{x}) := \exp\{-g(\mathfrak{x}, \mathfrak{x})/4t - i\langle \mathfrak{x}, 2\pi\tilde{\mathfrak{a}}\rangle\}. \tag{2.4}$$

Obviously, f is an element of the Schwartz space $\mathfrak{S}(\mathfrak{V})$; further, one has for every $\mathfrak{x} \in \mathfrak{V}$ and $\sigma \in \mathfrak{L}$: $f(\sigma(\mathfrak{x})) = f(\mathfrak{x})$.

We need the Fourier transform $\hat{f}$ of f, performed with that Lebesgue measure μ of $\mathfrak{V}$ for which a fundamental domain $\mathcal{F}(\mathfrak{T})$ for the translation group $\mathfrak{T} \subseteq \mathfrak{G}$ has the measure 1. The invariant Lebesgue measure v associated to the metric g differs from μ by a factor $\operatorname{vol}(\mathcal{F}(\mathfrak{T}))$.

Thus we obtain:

$$\begin{aligned}\mathfrak{V}^* \ni \mathfrak{u} \mapsto \hat{f}(\mathfrak{u}) &= \int_{\mathfrak{V}} \exp\{-i\langle \mathfrak{u} + 2\pi\tilde{\mathfrak{a}}, \mathfrak{z}\rangle - g(\mathfrak{z}, \mathfrak{z})/4t\}\, d\mu(\mathfrak{z}) \\ &= (4\pi t)^{n/2} \left(1/\operatorname{vol}\mathcal{F}(\mathfrak{T})\right) \exp\{-t g^*(\mathfrak{u} + 2\pi\tilde{\mathfrak{a}}, \mathfrak{u} + 2\pi\tilde{\mathfrak{a}})\}.\end{aligned} \tag{2.5}$$

From (2.2) and (2.5) it follows that

$$\sum_{\mathfrak{k}\in\mathfrak{H}} \hat{f}(2\pi\mathfrak{k}) = (4\pi t)^{n/2} \left(1/\operatorname{vol}\mathcal{F}(\mathfrak{T})\right) \sum_{\lambda_{\mathfrak{k}}\in\operatorname{Spec} L} e^{-\lambda_{\mathfrak{k}} t}. \tag{2.6}$$

(Note: $\bar{f}(2\pi\mathfrak{k}) := \bar{f}(2\pi\mathfrak{u}_1) = \cdots = \bar{f}(2\pi\mathfrak{u}_l)$.) This is the left hand side of the above quoted Poisson formula. The right hand side has the shape:

$$\frac{1}{r} \sum_{\theta \in \Omega} m(\theta)\, I_\theta(f). \tag{2.7}$$

In order to evaluate $I_\theta(f)$ we choose any element $S = (\sigma, \mathfrak{b})$ of the conjugacy class θ. According to the equation (3.13) of Part I we have

$$I_\theta(f) = \big(1/e(\sigma)\big) \int_{\mathfrak{B}^\perp(\sigma)} f(\mathfrak{z} + \mathfrak{b})\, d\mu_\sigma^\perp(\mathfrak{z}). \tag{2.8}$$

If $\theta = \{\mathrm{Id}\}$, then $\mathfrak{B}^\perp(\sigma) = \{0\}$, $\mathfrak{b} = 0$, $e(\sigma) = 1$, and $m(\theta) = 1$. Consequently, we have

$$m(\theta)\, I_\theta(f) = f(0) = 1. \tag{2.9}$$

Now we assume $\theta \neq \{\mathrm{Id}\}$. We use the decomposition (1.4)

$$\mathfrak{b} = l\mathfrak{v} + \big(\mathfrak{y} - \sigma(\mathfrak{y})\big),$$

with

$$\mathfrak{v} \in \mathfrak{B}(\sigma), \quad \mathfrak{y} - \sigma(\mathfrak{y}) \in \mathfrak{B}^\perp(\sigma), \quad g(\mathfrak{v}, \mathfrak{v}) = 1, \quad l = l(S).$$

This enables us to write

$$I_\theta(f) = \big(1/e(\sigma)\big) \int_{\mathfrak{B}^\perp(\sigma)} f(\mathfrak{z} + l\mathfrak{v})\, d\mu_\sigma^\perp(\mathfrak{z}).$$

Here the vectors $\mathfrak{z} \in \mathfrak{B}^\perp(\sigma)$ and the unit vector $\mathfrak{v}$ as well as the vector $\tilde{\mathfrak{a}}$ are orthogonal. We can therefore write:

$$I_\theta(f) = \big(1/e(\sigma)\big)\, e^{-l^2(S)/4t - 2\pi i \langle l(S)\mathfrak{v}, \tilde{\mathfrak{a}}\rangle} \int_{\mathfrak{B}^\perp(\sigma)} f(\mathfrak{z})\, d\mu_\sigma^\perp(\mathfrak{z}).$$

According to Definition 3.1 of Part I the measure $\mu_\sigma^\perp$ is the Lebesgue measure of the $\big(n - n(\sigma)\big)$-dimensional vector space $\mathfrak{B}^\perp(\sigma)$ normalized in such a manner that any fundamental domain $\mathcal{F}\big(\Gamma^\perp(\sigma)\big)$ of the lattice $\Gamma^\perp(\sigma)$ has the measure 1. If we transist to that Lebesgue measure $v_\sigma^\perp$ which is induced in $\mathfrak{B}^\perp(\sigma)$ by the metric g we must write:

$$\int_{\mathfrak{B}^\perp(\sigma)} f(\mathfrak{z})\, d\mu_\sigma^\perp(\mathfrak{z}) = \big(1/\mathrm{vol}\, \mathcal{F}(\Gamma^\perp(\sigma))\big) \int_{\mathfrak{B}^\perp(\sigma)} f(\mathfrak{z})\, dv_\sigma^\perp(\mathfrak{z}).$$

The lattice $\Gamma_e^\perp(\sigma) = (\sigma - \mathrm{Id})(\Gamma)$ was a sublattice of $\Gamma^\perp(\sigma) = \Gamma \cap \mathfrak{B}^\perp(\sigma)$ and the latter decomposes in exactly $e(\sigma)$ cosets modulo $\Gamma_e^\perp(\sigma)$. (See Definition 1.3 of Part I.) Thus we have

$$e(\sigma)\, \mathrm{vol}\, \mathcal{F}\big(\Gamma^\perp(\sigma)\big) = \mathrm{vol}\, \mathcal{F}\big(\Gamma_e^\perp(\sigma)\big).$$

Now we obtain

$$I_\theta(f) = (4\pi t)^{(n - n(\sigma))/2} \big(\mathrm{vol}\, \mathcal{F}(\Gamma_e^\perp(\sigma))\big)^{-1} e^{-l^2(S)/4t - 2\pi i \langle l(S)\mathfrak{v}, \tilde{\mathfrak{a}}\rangle}.$$

If ϑ is the free homotopy class of closed curves on M corresponding to the conjugacy class $\theta \neq \{\mathrm{Id}\}$, we can write $l(S) = l(\vartheta)$, $n(\sigma) = n(\vartheta)$. Further, it is easily seen that

$$2\pi i l(S) \langle \mathfrak{v}, \tilde{\mathfrak{a}}\rangle = 2\pi i \int_c \alpha(dx) = p(\vartheta),$$

where c is a closed geodesic having an S-invariant lift, i.e. belongs to ϑ. Finally, we can use the following lemma which we shall prove at the end of this section.

Lemma 2.1: *In the just used notations one has*

$$\frac{m(\theta)}{r} \cdot \frac{\operatorname{vol} \mathcal{F}(\Gamma)}{\operatorname{vol} \mathcal{F}(\Gamma_e^\perp(\sigma))} = \frac{k(\vartheta) \operatorname{vol} M(\vartheta)}{D(\vartheta)}.$$

From the Poisson formula

$$\sum_{\mathfrak{k} \in \mathfrak{H}} \hat{f}(2\pi \mathfrak{k}) = \frac{1}{r} \sum_{\theta \in \Omega} m(\theta) \, I_\theta(f)$$

we obtain finally

$$\sum_{\lambda_\mathfrak{k} \in \operatorname{Spec} L} e^{-\lambda_\mathfrak{k} t} = \frac{\operatorname{vol} M}{(4\pi t)^{n/2}} + \sum_{\vartheta \in \omega} \frac{k(\vartheta) \operatorname{vol} M(\vartheta)}{(4\pi t)^{n(\vartheta)/2} D(\vartheta)} e^{-l^2(\vartheta)/4t - p(\vartheta)}.$$

This formula is exactly the desired result of Theorem 2 ∎

Proof of Lemma 2.1: Let $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ be any $\mathbf{Z}$-basis of Γ, such that $\mathfrak{x}_1, \ldots, \mathfrak{x}_{n(\sigma)}$ is a $\mathbf{Z}$-basis of $\Gamma(\sigma)$. Further, let $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$ be any orthonormal $\mathbf{R}$-basis of $\mathfrak{V}$, such that $\mathfrak{y}_1, \ldots, \mathfrak{y}_{n(\sigma)}$ is an $\mathbf{R}$-basis of $\mathfrak{V}(\sigma)$. The matrix $\mathfrak{X}$ whose entries are the coordinates of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ with respect to $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$ has the form

$$\mathfrak{X} = \begin{pmatrix} \mathfrak{X}_1 & * \\ 0 & \mathfrak{X}_2 \end{pmatrix},$$

where $\mathfrak{X}_1$, $\mathfrak{X}_2$ are matrices of type $\big(n(\sigma), n(\sigma)\big)$, $\big(n - n(\sigma), n - n(\sigma)\big)$.
We have

$$\operatorname{vol} \mathcal{F}(\Gamma) = |\operatorname{Det} \mathfrak{X}_1 \cdot \operatorname{Det} \mathfrak{X}_2|, \quad \operatorname{vol} \mathcal{F}\big(\Gamma(\sigma)\big) = |\operatorname{Det} \mathfrak{X}_1|.$$

Let $\hat{\mathfrak{X}}$ be the matrix whose entries are the coordinates of $(\sigma - \mathrm{Id})(\mathfrak{x}_1), \ldots, (\sigma - \mathrm{Id})(\mathfrak{x}_n)$ with respect to $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$. Then we find

$$\hat{\mathfrak{X}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix} \begin{pmatrix} \mathfrak{X}_1 & * \\ 0 & \mathfrak{X}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A}\mathfrak{X}_2 \end{pmatrix}.$$

Here $\mathfrak{A}$ is the matrix of the restriction of $\sigma - \mathrm{Id}$ on $\mathfrak{V}^\perp(\sigma)$ taken with respect to the orthonormal basis $\mathfrak{y}_{n(\sigma)+1}, \ldots, \mathfrak{y}_n$. Further, $\mathfrak{A}\mathfrak{X}_2$ represent the coordinates of a $\mathbf{Z}$-basis of $\Gamma_e^\perp(\sigma) = (\sigma - \mathrm{Id})(\Gamma)$ with respect to $\mathfrak{y}_{n(\sigma)+1}, \ldots, \mathfrak{y}_n$. Consequently, we have

$$|\operatorname{Det}(\mathfrak{A}\mathfrak{X}_2)| = \operatorname{vol} \mathcal{F}\big(\Gamma_e^\perp(\sigma)\big) = |\operatorname{Det} \mathfrak{A}| \, |\operatorname{Det} \mathfrak{X}_2|$$

$$= \frac{|\operatorname{Det} \mathfrak{A}| \operatorname{vol} \mathcal{F}(\Gamma)}{\operatorname{vol} \mathcal{F}(\Gamma(\sigma))}. \tag{2.10}$$

If $\tilde{\sigma}$ denotes the restriction of σ^{-1} to $\mathfrak{V}^\perp(\sigma)$, then we have $D(\vartheta) = |\operatorname{Det}(\tilde{\sigma} - \mathrm{Id})|$, (compare the proof of Theorem 1 (c)). Therefore, we find $|\operatorname{Det} \mathfrak{A}| = D(\vartheta)$, and from (2.10) it follows that

$$\frac{\operatorname{vol} \mathcal{F}(\Gamma)}{\operatorname{vol} \mathcal{F}(\Gamma_e^\perp(\sigma))} = \frac{\operatorname{vol} \mathcal{F}(\Gamma(\sigma))}{D(\vartheta)}$$

From Lemma 1.5 and 1.6 it follows that

$$\frac{m(\theta)}{r} \cdot \frac{\operatorname{vol} \mathcal{F}(\Gamma)}{\operatorname{vol} \mathcal{F}(\Gamma_e^\perp(\sigma))} = \frac{k(S) \operatorname{vol} M(S)}{D(\vartheta)}.$$

But $M(S)$ and $k(S)$ depend only on the conjugacy class θ or equivalently on ϑ ∎

As a counterpart of the Jacobi transformation formula (1) we shall give a corresponding cos-formula.

Definition 2.2: For an integer $m \geqq 1$ and a real number $a \geqq 0$ we define the distribution $T(m, a) \in \mathcal{D}'(\mathbf{R})$ as follows:

$$\langle T(m, a), \varphi\rangle := (-2\pi)^{(1-m)/2} \Lambda^{(1-m)/2}[\varphi(t) + \varphi(-t)]_{t=a} \tag{2.11}$$

if m odd, and

$$\langle T(m, a), \varphi\rangle = 2(-2\pi)^{-m/2} \int_a^\infty \frac{t}{\sqrt{t^2 - a^2}} \Lambda^{(m/2)}[\varphi(t) + \varphi(-t)]\, dt \tag{2.12}$$

if m even; in both cases $\varphi \in \mathcal{D}(\mathbf{R})$ and $\Lambda := (1/t)\, d/dt$.

We remark that for m even and $a = 0$ we can write

$$\langle T(m, 0), \varphi\rangle = (-2\pi)^{-m/2} \int_{-\infty}^{\infty} \Lambda^{(m/2)}[\varphi(t) + \varphi(-t)]\, dt.$$

Proposition 2.3: *In the sense of distributions over* $\mathbf{R}$ *we have*:

$$2D_n := 2 \sum_{\lambda_{\mathfrak{f}} \in \operatorname{Spec} L} \cos \sqrt{\lambda_{\mathfrak{f}}}\, t = \operatorname{vol} M \cdot T(n, 0) + \sum_{\vartheta \in \omega} \frac{k(\vartheta) \operatorname{vol} M(\vartheta)}{D(\vartheta)} e^{-p(\vartheta)}\, T\big(n(\vartheta), l(\vartheta)\big). \tag{2.13}$$

Remark 2.4: a) One has:

$$\operatorname{sing\,supp} D_n = \{\pm l \in \mathbf{R} \mid \exists\, \vartheta \quad \text{with} \quad l = l(\vartheta)\} \cup \{0\}.$$

b) The distribution D_n is the trace of the fundamental solution of the wave equation over M:

$$\frac{\partial^2 u}{\partial t^2} - L[u] = 0.$$

If n is odd, then *Huygens principle* is valid for this equation. Consequently, one must have for a suitable neighbourhood U of $0 \in \mathbf{R}$:

$$(\operatorname{supp} D_n) \cap U = (\operatorname{sing\,supp} D_n) \cap U = \{0\}.$$

This is in accordance with (2.13). It is obvious that in the case m even such a neighbourhood cannot be found. In this connection the following corollary seems remarkable.

Corollary 2.5: *Let n be an odd number. If M is orientable then*

$$\operatorname{supp} D_n = \operatorname{sing\,supp} D_n.$$

If M is not orientable, then this relation is false.

Proof: Let M be an orientable manifold. For every element $S = (\sigma, \mathfrak{b}) \in \mathfrak{G}$ we must have $\operatorname{Det} \sigma = 1$. Taking into account that n is an odd number we find $n(\sigma)$ odd, i.e. $n(\vartheta)$ odd. The assertion follows from $\operatorname{supp} T\big(n(\vartheta), a\big) = \{a, -a\}$ in that case. On the other hand, if M is not orientable then at least one $n(\vartheta)$ must be an even integer and we have $\operatorname{supp} T\,\big(n(\vartheta), a\big) = [a, \infty) \cup (-\infty, -a]$ ∎

Proof of the Proposition 2.3: Let τ, λ be positive real numbers, λ sufficiently large. Using integration in the complex ω-plane we define

$$A(\tau, \lambda) := \frac{2^{\lambda-1}\pi^{n/2}}{2\pi i} \int_{1-\infty i}^{1+\infty i} e^{\tau^2\omega/4} w^{-(\lambda+1)/2} \sum_{\lambda_\mathfrak{k}\in \operatorname{Spec} L}' e^{-\lambda_\mathfrak{k}/w} dw. \tag{2.14}$$

Well-known integral formulas give

$$A(\tau, \lambda) = \pi^{n/2} \sum_{\lambda_\mathfrak{k}\in \operatorname{Spec} L} \left(2\tau/\sqrt{\lambda_\mathfrak{k}}\right)^{(\lambda-1)/2} J_{(\lambda-1)/2}\left(\sqrt{\lambda_\mathfrak{k}}\, \tau\right). \tag{2.15}$$

Here, J_ν denotes the Bessel function with index ν. If $\lambda > 2n$, then the last series is uniformly convergent in every compact τ-interval. (Compare the analogous calculations in the non-euclidean case [20].) Now, we use our Jacobi formula; we obtain from (2.14):

$$A(\tau, \lambda) = \frac{\operatorname{vol} M \cdot \tau^{\lambda-n-1}}{\Gamma((\lambda - n + 1)/2)} + \sum_{\vartheta\in\omega} \frac{\pi^{(n-n(\vartheta))/2} k(\vartheta) \operatorname{vol} M(\vartheta)}{D(\vartheta)\, \Gamma((\lambda - n(\vartheta) + 1)/2)} e^{-p(\vartheta)} \{\tau^2 - l^2(\vartheta)\}_+^{(\lambda-n(\vartheta)-1/2)}. \tag{2.16}$$

We choose the value $\lambda = 2n + 2$ and complete the definition of $A(\tau, 2n + 2)$ by putting $A(0, 2n + 2) = 0$ and $A(-\tau, 2n + 2) = -A(\tau, 2n + 2)$. We consider $A(\tau, 2n + 2)$ as an odd element of $\mathcal{D}'(\mathbf{R})$, to which we apply the operator

$$2^{-(n+1)}\pi^{(1-n)/2} \frac{d}{dt} A^{(n)}.$$

This can be done term by term in both expressions (2.15) and (2.16) of $A(\tau, 2n + 2)$. The comparison of the arising series gives the desired result ∎

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