Zeitschrift für Analysis
und ihre Anwendungen
Bd. 4 (5) 1985, S. 385–401

A Class of Nonlinear Singular Integral and Integro-differential Equations with Hilbert Kernel

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Dedicated to Prof. Dr. H. Beckert on the occasion of his 65th birthday

Mit Hilfe einer neuartigen Anwendung des Schauderschen Fixpunktsatzes werden Existenzsätze für eine Klasse von quasilinearen singulären Integrodifferentialgleichungen und eine zugehörige Klasse von nichtlinearen singulären Integralgleichungen bewiesen.

С помощью нового применения принципа Шаудера доказываются теоремы существования для одного класса квазилинейных сингулярных интегро-дифференциальных уравнений и связанного с ним класса нелинейных сингулярных интегральных уравнений.

By means of a novel application of Schauder's fixed point theorem, existence theorems are proved for a class of quasilinear singular integro-differential equations and a related class of nonlinear singular integral equations.

Introduction

Nonlinear singular integral and integro-differential equations with Hilbert or Cauchy kernel have been treated by many authors, cf. POGORZELSKI [12] and the recent monograph [8] by GUSEINOV and MUKHTAROV. But as a rule only existence theorems are given for (in some sense) small nonlinearities. Without smallness assumptions on the data existence assertions were obtained for special classes of such equations by means of the theory of monotone operators in spaces L_p of summable functions [1, 9, 7 (cf. also 8), 2, 13] and recently for Cauchy kernels by means of a nonlocal implicit function theorem in the Sobolev space $W_2{}^1$ and in the space C^1 of continuously differentiable functions [10].

In this paper a class of quasi-linear integro-differential equations with Hilbert kernel and a related class of integral equations are investigated by means of the classical Schauder fixed point theorem in the space C of continuous functions. Reducing the integro-differential equation to an equivalent integral equation of fixed point type, the application of Schauder's theorem yields some general existence theorems for the integro-differential equation under various kinds of assumptions on the data. By differentation a related class of integral equations is reduced to these integro-differential equations. This class of integral equations contains the known Theodorsen integral equation of conformal mapping as a particular case. For its solution in case of a general smooth starlike Jordan curve an existence proof will be given which is independent of the Riemann mapping theorem. For some subclasses of the integral equations the uniqueness of the solution is proved, too.

By means of analogous methods as here the Riemann-Hilbert problem for holomorphic functions and the Poincaré problem for harmonic functions in the unit disk are dealt with in the author's papers [14, 15], respectively.

1. Statement of problems

We look for 2π periodic solutions $\varphi \in W_p^1(-\pi, \pi)$, $p > 1$, of the *quasilinear integro-differential equation*

$$\varphi'(s) + H[M(\cdot, \varphi)\, \varphi'](s) = F(s) + H[N(\cdot, \varphi)](s) \quad \text{for} \quad \text{a.a.} \quad s \in [-\pi, \pi] \tag{1}$$

satisfying the additional condition

$$\varphi(0) = k \tag{2a}$$

or

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s)\, ds = c, \tag{2b}$$

respectively. Here $k, c \in \mathbf{R}$ are given real constants and

$$H[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \cot \frac{\sigma - s}{2}\, d\sigma$$

is the well-known Hilbert transformation with cotangent kernel of a function $\omega \in L_p(-\pi, \pi)$, $p > 1$.

The assumptions on the given functions $M(s, \varphi)$, $N(s, \varphi)$ and $F(s)$ will be specified later. Of course, F has to fulfil the condition

$$\int_{-\pi}^{\pi} F(s)\, ds = 0, \tag{3}$$

which is necessary for the solvability of (1) since $\int_{-\pi}^{\pi} H[\omega](s)\, ds = 0$ for any function $\omega \in L_p(-\pi, \pi)$, $p > 1$.

Further we deal with the *integral equation*

$$\varphi(s) + H[m(\cdot, \varphi)](s) + K[n(\cdot, \varphi)](s) = f(s), \tag{4}$$

where H is the Hilbert transformation and

$$K[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \ln \left(4 \sin^2 \frac{\sigma - s}{2}\right) d\sigma.$$

The transformation K is related to H by the well-known formula

$$\frac{d}{ds} K[\omega](s) = -H[\omega](s) \tag{5a}$$

for any function $\omega \in L_p(-\pi, \pi)$, $p > 1$. Moreover,

$$\frac{d}{ds} H[\omega](s) = H[\omega'](s) \tag{5b}$$

for any function $\omega \in W_p^1(-\pi, \pi)$, $p > 1$. Therefore, differentiating (4) yields the equation (1) with

$$M(s, \varphi) = m_\varphi(s, \varphi), \quad N(s, \varphi) = n(s, \varphi) - m_s(s, \varphi), \quad F(s) = f'(s), \tag{6}$$

if the given functions $m(s, \varphi)$, $n(s, \varphi)$ and $f(s)$ satisfy suitable assumptions. (Precise assumptions on these functions will also be given later. For $f \in W_p^1(-\pi, \pi)$, $p > 1$, the function f' obviously fulfils the necessary solvability condition (3).) Furthermore,

$$\int_{-\pi}^{\pi} K[\omega](s)\, ds = 0$$

for any function $\omega \in L_p(-\pi, \pi)$, $p > 1$. Integrating (4) over $(-\pi, \quad)$ thus leads to the additional condition (2b) with the given constant

$$c = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\, ds. \tag{7}$$

I.e., under suitable assumptions the integral equation (4) is equivalent to the integro-differential equation (1), where M, N, F are given by (6), together with the additional condition (2b), where c is given by (7).

Remark: In case of $n(s, \varphi) \equiv 0$ the equation (4) can also be reduced to the following Riemann-Hilbert problem for the holomorphic function $W(z) = U(z) + iV(z)$ in the unit disk with the boundary values $\varphi(s) = U(e^{is})$:

$$V(e^{is}) = m(s, U(e^{is})) + (Hf)(s)$$

with the additional condition

$$U(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\, ds.$$

Problems of such type are dealt with in our paper [14] by an analogous method as here.

2. Existence theorem

Putting

$$\lambda(s) = M(s, \varphi(s)) \tag{8}$$

and

$$g(s) = F(s) + H[N(\cdot, \varphi)](s), \tag{9}$$

the integro-differential equation (1) takes the form of the linear integral equation (A.1) in the appendix for $v = \varphi'$. Since

$$\int_{-\pi}^{\pi} g(s)\, ds = 0 \tag{10}$$

due to the assumption (3), any solution $v \in L_p(-\pi, \pi)$ to this equation has a vanishing integral over $(-\pi, \pi)$ and therefore represents the derivative φ' of a 2π periodic function $\varphi \in W_p^1(-\pi, \pi)$, $p > 1$.

We make the following basic *Assumption* A on the data:

(i) $M(s, \varphi)$ is a continuous function on $[-\pi, \pi] \times \mathbf{R}$ which is 2π periodic in s.

(ii) $N(s, \varphi)$ satisfies the Carathéodory condition on $[-\pi, \pi] \times \mathbf{R}$, i.e. it is measurable in s on $[-\pi, \pi]$ for all $\varphi \in \mathbf{R}$ and continuous in φ on $\mathbf{R}$ for almost all $s \in [-\pi, \pi]$, and it fulfils an estimation of the form

$$|N(s, \varphi)| \leqq N_0(s), \quad N_0 \in L_\varrho(-\pi, \pi), \quad 1 < \varrho < \infty, \tag{11}$$

for φ from bounded intervals of $\mathbf{R}$.

(iii) $F \in L_\varrho(-\pi, \pi)$ with the same exponent ϱ as in (11) and fulfils the condition (3). (If $N(s, \varphi) \equiv 0$ also $\varrho = \infty$ is allowed.)

Then λ is a continuous 2π periodic function and $g \in L_\varrho(-\pi, \pi)$ for any continuous 2π periodic function φ.

The equation (1) is therefore equivalent to the expression (A.8) in the appendix for $v = \varphi'$ with λ, g given by (8), (9). Taking into account the additional condition (2a) or (2b), respectively, we obtain the equivalent *fixed point equation*

$$\varphi = P\varphi \tag{12}$$

for φ, where the operator P is defined for any 2π periodic continuous function ξ by

$$(P\xi)(s) = k + \int_0^s l(\sigma, \xi)\, d\sigma \tag{13}$$

with the kernel

$$l(s, \xi) = D\alpha(s)\, e^{-(H\mu)(s)} + \alpha^2(s)\, g(s) - \alpha(s)\, e^{-(H\mu)(s)} H[\beta g\, e^{H\mu}](s). \tag{14}$$

The constant $D = D[\xi]$ is given by

$$D[\xi] = \tan \bar{\mu} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s)\, g(s)\, e^{(H\mu)(s)}\, ds, \tag{15}$$

where

$$\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s)\, ds,$$

and the functions $g = g(s, \xi)$, $\alpha = \alpha(s, \xi)$, $\beta = \beta(s, \xi)$, $\mu = \mu(s, \xi)$ are defined by

$$g(s, \xi) = F(s) + H[N(\cdot, \xi)](s), \tag{16}$$

$$\mu(s, \xi) = \arctan \lambda(s, \xi), \quad \lambda(s, \xi) = M(s, \xi(s)), \tag{17}$$

$$\alpha(s, \xi) = \cos \mu(s, \xi) = 1/\sqrt{1 + \lambda^2(s, \xi)}, \tag{18a}$$

$$\beta(s, \xi) = \sin \mu(s, \xi) = \lambda(s, \xi)/\sqrt{1 + \lambda^2(s, \xi)}. \tag{18b}$$

In case of (2a) the constant k is prescribed, whereas in case of (2b)

$$k = k[\xi] = c - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^s l(\sigma, \xi)\, d\sigma\, ds. \tag{19}$$

We now estimate the kernel $l(s, \xi)$.

Lemma: *Let p be an arbitrary number satisfying $1 < p < \varrho$ and put $\varkappa = 2p\varrho/[\varrho - p]$. Denote further $\mu_1 = \min \mu(s)$, $\mu_2 = \max \mu(s)$, $s \in [-\pi, \pi]$, and assume the* oscillation of $\mu(s)$

$$2\gamma = \mu_2 - \mu_1 < \pi/\varkappa. \tag{20}$$

Then for the L_p norm of $l(s, \xi)$ the estimation

$$\|l(\cdot, \xi)\|_p = \left(\int_{-\pi}^{\pi} |l(s, \xi)|^p \, ds\right)^{1/p}$$
$$\leq (2\pi)^{2/\varkappa} \|g\|_\varrho \{1 + [A_r + \tan \mu_0] (\cos \varkappa\gamma)^{-2/\varkappa}\} \quad (21)$$

holds, where $\mu_0 = \max[-\mu_1, \mu_2]$ *and* A_r *is the norm of the Hilbert transformation in* $L_r(-\pi, \pi)$, $r = 2p\varrho/[\varrho + p]$;

$$A_r = \begin{cases} \tan(\pi/2r) & \text{if} \quad 1 < r \leq 2 \\ \cot(\pi/2r) & \text{if} \quad 2 \leq r < \infty \end{cases}$$

(*cf.* [11: Kap. IV, § 7]).

Proof: It is

$$\|l(\cdot, \xi)\|_p \leq \|g\|_p + \|e^{-H\mu} H[\beta g \, e^{H\mu}]\|_p + \|e^{-H\mu}\|_p \tan \mu_0 \frac{1}{2\pi} \int_{-\pi}^{\pi} |g| \, |e^{H\mu}| \, ds.$$

Further, since $1/p = 1/\varrho + 2/\varkappa$

$$\|g\|_p \leq (2\pi)^{2/\varkappa} \|g\|_\varrho,$$

since $1/p = 1/r + 1/\varkappa$

$$\|e^{-H\mu}\|_p \leq (2\pi)^{1/r} \|e^{-H\mu}\|_\varkappa$$

and since $1/r = 1/\varkappa + 1/\varrho$

$$\int_{-\pi}^{\pi} |g| \, |e^{H\mu}| \, ds \leq (2\pi)^{1/t} \|e^{H\mu}\|_\varkappa \|g\|_\varrho,$$

where $1/t = 1 - (1/r)$. Finally, according to the well-known Zygmund lemma [16] under the assumption (20) it is

$$\|e^{\pm H\mu}\|_\varkappa \leq (2\pi)^{1/\varkappa} (\cos \varkappa\gamma)^{-1/\varkappa}$$

so that

$$\|e^{-H\mu} H[\beta g \, e^{H\mu}]\|_p \leq \|e^{-H\mu}\|_\varkappa \|H[\beta g \, e^{H\mu}]\|_r$$
$$\leq A_r \|e^{-H\mu}\|_\varkappa \|e^{H\mu}\|_\varkappa \|g\|_\varrho \leq A_r (2\pi)^{2/\varkappa} (\cos \varkappa\gamma)^{-2\varkappa} \|g\|_\varrho.$$

This together yields (21) ∎

We consider the operator P on the convex compact subset $\mathfrak{K}$ of the space $C[-\pi, \pi]$ of 2π periodic continuous functions defined by

$$\mathfrak{K} = \{\xi \in C[-\pi, \pi] : |\xi(s)| \leq R, \ |\xi(s_1) - \xi(s_2)| \leq R_0 |s_1 - s_2|^{1/q}\}, \quad (22)$$

where q is the exponent conjugate to p and R, R_0 are fixed positive numbers to be specified later. We make the following additional *Assumption B*:

(i) $M(s, \varphi)$ is a bounded function on $[-\pi, \pi] \times \mathbf{R}$ satisfying the inequality

$$2\gamma = \mu_2 - \mu_1 < \pi/\varkappa \quad (23)$$

for some p with $1 < p < \varrho$, $\varkappa = 2p\varrho/[\varrho - p]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, $\lambda_1 = \inf M(s, \varphi)$, $\lambda_2 = \sup M(s, \varphi)$, the infimum and supremum are taken over $s \in [-\pi, \pi]$, $\varphi \in \mathbf{R}$.

(ii) The inequality (11) with a fixed function $N_0 \in L_\varrho(-\pi, \pi)$ holds uniformly with respect to $\varphi \in \mathbf{R}$.

Under the assumptions A and B it is

$$\|g\|_\varrho \leqq \|F\|_\varrho + A_\varrho \overline{N}_0 \equiv G < \infty, \tag{24}$$

where $\overline{N}_0$ is the norm of N_0 and A_ϱ the norm of the Hilbert transformation in $L_\varrho(-\pi, \pi)$, and by the above lemma we have the estimation

$$\|l(\cdot, \xi)\|_p \leqq (2\pi)^{2/\varkappa} G\{1 + [A_r + \lambda_0] (\cos \varkappa\gamma)^{-2/\varkappa}\} \equiv L \tag{25}$$

with $\lambda_0 = \max[-\lambda_1, \lambda_2]$ for any function $\xi \in C[-\pi, \pi]$.

In case of (2a) we take

$$R_0 = L, \quad R = |k| + (2\pi)^{1/q} L.$$

Then the operator P maps the subset $\mathfrak{K}$ of $C[-\pi, \pi]$ into itself because for $\varphi = P\xi$ it is

$$|\varphi(s_1) - \varphi(s_2)| \leqq \int_{s_1}^{s_2} |l(s, \xi)| \, ds \leqq |s_1 - s_2|^{1/q} \|l(\cdot, \xi)\|_p \leqq L|s_1 - s_2|^{1/q}$$

and

$$|\varphi(s)| \leqq |\varphi(0)| + |\varphi(s) - \varphi(0)| \leqq |k| + (2\pi)^{1/q} L.$$

Analogously, in case of (2b) we take

$$R_0 = L, \quad R = |c| + 2(2\pi)^{1/q} L$$

using the estimation

$$|k| \leqq |c| + \int_{-\pi}^{\pi} |l(s, \xi)| \, ds \leqq |c| + (2\pi)^{1/q} \|l(\cdot, \xi)\|_p \leqq |c| + (2\pi)^{1/q} L.$$

Furthermore, like in our paper [14] one can show that under the assumptions A and B the operator $P: \mathfrak{K} \to \mathfrak{K}$ is continuous in the norm of $C[-\pi, \pi]$. The Schauder fixed point theorem applied to the equation (12) in $\mathfrak{K}$ then yields the existence of a solution $\varphi \in \mathfrak{K}$ to the equation (1). Moreover, this solution is absolutely continuous and has a derivative $\varphi'(s) = l(s, \varphi(s))$, $\varphi' \in L_p(-\pi, \pi)$, i.e. $\varphi \in W_p^1(-\pi, \pi)$.

Theorem 1: *Under Assumptions* A *and* B *the integro-differential equation* (1) *possesses a solution* $\varphi \in W_p^1(-\pi, \pi)$.

Corollary: *If the function $n(s, \varphi)$ satisfies the Carathéodory condition and an estimation of the form* (11), *the functions $m(s, \varphi)$ and $f(s)$ are differentiable and the corresponding functions $M(s, \varphi)$, $N(s, \varphi)$, $F(s)$ given by* (6) *fulfil the assumptions* A *and* B, *the integral equation* (4) *possessses a solution* $\varphi \in W_p^1(-\pi, \pi)$.

Remark 1: Assumption (i) of B is fulfilled for any nonnegative (nonpositive) bounded function $M(s, \varphi)$ satisfying the inequality

$$\sup M(s, \varphi) < \tan(\pi/\varkappa) \quad \left(\inf M(s, \varphi) > -\tan(\pi/\varkappa)\right). \tag{26}$$

Remark 2: Assumption (ii) of B can be weakened to

$$|N(s, \varphi)| \leqq N_1(s) + N_2(s) R^\delta, \qquad 0 < \delta < 1, \tag{27}$$

for $s \in [-\pi, \pi]$, $|\varphi| \leqq R$ and any positive R, where N_k, $k = 1, 2$, are fixed functions from $L_\varrho(-\pi, \pi)$. Namely, in this case

$$\|l(\cdot, \xi)\|_p \leqq Const_1 + Const_2 R^\delta$$

for $\xi \in \mathfrak{K}$ and the operator P maps $\mathfrak{K}$ into itself if the constants R, R_0 are taken sufficiently large. (The same conclusion is valid if (27) is fulfilled only with $\delta = 1$ but the norm of N_2 in $L_\varrho(-\pi, \pi)$ is sufficiently small.)

Remark 3: Since for a continuous function μ the functions $e^{\pm H\mu} \in L_r(-\pi, \pi)$ for any finite $r > 0$, from (12)—(14) it follows that the solution $\varphi \in W_p^1(-\pi, \pi)$ for any $1 < p < \varrho$, not only for the value of p determined by the assumption (23).

3. Further existence theorems

We employ a modification of the above method in the case $\varrho = 2$ estimating the norm of the kernel $l(s, \xi)$ in another way. In Assumption B the inequality (23) is replaced by

$$2\gamma = \mu_2 - \mu_1 < \pi/2. \tag{28}$$

Then from formula (A.26) of the appendix we have the estimate

$$\|l(\cdot, \xi)\| \leqq E_\gamma \|g\|, \tag{29}$$

where $\|\cdot\|$ denotes the norm in $L_2(-\pi, \pi)$. The constant E_γ is given by

$$E_\gamma = B_\gamma + [1 + \lambda_0] (\cos 2\gamma)^{-1}, \tag{30}$$

where

$$B_\gamma = 2/[1 - \tan \gamma] \tag{31}$$

and again $\lambda_0 = \max [-\lambda_1, \lambda_2]$.

Now, under Assumption B it is

$$\|g\| \leqq \|F\| + \overline{N}_0 \equiv G_0 < \infty, \tag{32}$$

where $\overline{N}_0$ is the L_2 norm of the function N_0 in (11). Hence

$$\|l(\cdot, \xi)\| \leqq E_\gamma \cdot G_0 \equiv L_0. \tag{33}$$

Therefore, we can take $p = \varrho = 2$ and $q = 2$ in the definition (22) of the subset $\mathfrak{K}$ with $R_0 = L_0$ and $R = |k| + \sqrt{2\pi}\, L_0$ or $R = |c| + 2\sqrt{2\pi}\, L_0$ in the case of (2a) or (2b), respectively. Then the operator P again maps $\mathfrak{K}$ into itself.

Besides, the operator P: $\mathfrak{K} \to \mathfrak{K}$ is *continuous* in the norm of $C[-\pi, \pi]$. Namely, let $\xi_n \in \mathfrak{K}$, $n = 1, 2, \ldots$ be a uniformly convergent sequence with the limit function $\xi_0 \in \mathfrak{K}$. We have to show that the functions $P\xi_n$ defined by (12), (13) converge uniformly to $P\xi_0$. For this it is sufficient to prove that the functions $l(s, \xi_n)$ converge weakly in $L_2(-\pi, \pi)$ to $l(s, \xi_0)$. Moreover, in view of the uniform boundedness of the L_2 norms of $l(s, \xi_n)$ by (33) it suffices to show the weak convergence of the functions $l(s, \xi_n)$ to $l(s, \xi_0)$ in the space $L_1(-\pi, \pi)$ only.

Now, obviously, the continuous functions $\lambda_n(s) = \lambda(s, \xi_n)$, $\mu_n(s) = \mu(s, \xi_n)$, $\alpha_n(s) = \alpha(s, \xi_n)$, $\beta_n(s) = \beta(s, \xi_n)$ converge uniformly to $\lambda_0(s) = \lambda(s, \xi_0)$, $\mu_0(s) = \mu(s, \xi_0)$, and so on. Further, the functions $N(s, \xi_n)$ converge strongly in $L_2(-\pi, \pi)$ to $N(s, \xi_0)$ and therefore also the functions $g_n(s) = g(s, \xi_n)$ to $g(s, \xi_0)$. Moreover, due to the assumption (28) and the Zygmund lemma it can be shown like in the corresponding proof in [14] that the functions $\exp [\pm H(\mu_n)]$ converge strongly to $\exp [\pm H(\mu_0)]$ in $L_2(-\pi, \pi)$ and also in $L_{2+\varepsilon}(-\pi, \pi)$ for sufficiently small positive ε.

Therefore, the constants $D[\xi_n]$ defined by (15) converge to $D[\xi_0]$. It remains to prove that the functions

$$\Lambda[\xi_n] \equiv e^{-H\mu_n} H[h_n\, e^{H\mu_n}], \qquad h_n = \beta_n g_n,$$

converge weakly to $\Lambda[\xi_0]$ in $L_1(-\pi, \pi)$. As is shown in Appendix 2 the functions $\Lambda[\xi_n]$ have uniformly bounded L_2 norms under the assumption (28) like the functions $l(s, \xi_n)$ do by (33). Besides, the functions h_n converge strongly in $L_2(-\pi, \pi)$ to $h_0 = \beta(s, \xi_0)\, g(s, \xi_0)$.

Then, for any $\chi \in L_\infty(-\pi, \pi)$ we have

$$\begin{aligned} &|(\Lambda[\xi_n] - \Lambda[\xi_0], \chi)| \\ &\leqq |(\Lambda[\xi_n] - e^{-H\mu_0} H[h_n\, e^{H\mu_n}], \chi)| \\ &\quad + |(e^{-H\mu_0} H[h_n\, e^{H\mu_n}] - \Lambda[\xi_0], \chi)| \\ &\equiv S_1 + S_2, \end{aligned}$$

where $(\cdot, \cdot)$ denotes the usual scalar product.

It is

$$\begin{aligned} S_1 &= |(1 - e^{H\mu_n - H\mu_0}, \Lambda[\xi_n]\, \chi)| \\ &\leqq \|\chi\|_\infty\, \|\Lambda[\xi_n]\|_2\, \|1 - e^{H\mu_n - H\mu_0}\|_2 \\ &\leqq Const\, \|e^{-H\mu_0}(e^{H\mu_0} - e^{H\mu_n})\|_2 \\ &\leqq Const\, \|e^{-H\mu_0}\|_{q'}\, \|e^{H\mu_0} - e^{H\mu_n}\|_{p'} \end{aligned}$$

with some $p' = 2 + \varepsilon > 2$ and $q' = (1/2 - 1/p')^{-1} < \infty$. Taking ε sufficiently small, the functions $\exp[H\mu_n]$ converge strongly to $\exp[H\mu_0]$ in $L_{p'}(-\pi, \pi)$. Also the function $\exp[-H\mu_0]$ is summabel to any power $q' < \infty$ (cf. [6: Kap. IX, § 5]). Therefore, $S_1 \to 0$ as $n \to \infty$.

Finally,

$$\begin{aligned} S_2 &= |(H[h_n\, e^{H\mu_n}] - H[h_0\, e^{H\mu_0}], e^{-H\mu_0}\chi)| \\ &\leqq \|e^{-H\mu_0}\chi\|_{q'}\, A_{r'} \|h_n\, e^{H\mu_n} - h_0\, e^{H\mu_0}\|_{r'}, \end{aligned}$$

where $r' = (1 - 1/q')^{-1} = (1/2 + 1/p')^{-1} > 1$ and $A_{r'}$ is the norm of the Hilbert transformation in $L_{r'}(-\pi, \pi)$. But the functions $h_n \exp[H\mu_n]$ converge strongly to $h_0 \exp[H\mu_0]$ in $L_{r'}(-\pi, \pi)$ since $\exp[H\mu_n]$ converges strongly to $\exp[H\mu_0]$ in $L_{p'}(-\pi, \pi)$ and h_n converges strongly to h_0 in $L_2(-\pi, \pi)$. Hence, also $S_2 \to 0$ as $n \to \infty$.

This proves the continuity of the operator $P\colon \mathfrak{K} \to \mathfrak{K}$ in $C[-\pi, \pi]$.

Applying again the Schauder fixed point theorem to the equation (12) in $\mathfrak{K}$, we obtain

Theorem 2: *Under Assumptions* A *and* B *with* $\varrho = 2$ *and the inequality* (28) *instead of* (23) *the integro-differential equation* (1) *possesses a solution* $\varphi \in W_2^1(-\pi, \pi)$.

Remark 1: Assumption (i) of B with (28) is fulfilled for any nonnegative (non-positive) bounded function $M(s, \varphi)$.

Remark 2: The remark 2 to Theorem 1 also holds for Theorem 2. If further $M(s, \varphi)$ is a nonnegative function satisfying the inequality

$$M(s, \varphi) \leqq M_1 + M_2 R^\omega, \qquad 0 < \omega < 1, \tag{34}$$

for $s \in [-\pi, \pi]$, $|\varphi| \leqq R$ and any positive R, we have the same estimation for the corresponding quantities λ_0 and $\tan 2\gamma$ in (30) with respect to $|\varphi| \leqq R$. Then the constant E_γ grows not stronger than the function $R^{2\omega}$ as R goes to infinity. Therefore, a solution to equation (1) exists if $\omega < 1/2$ in case the assumption (11) for $N(s, \varphi)$ holds uniformly with respect to $\varphi \in \mathbf{R}$ or if $\delta + 2\omega < 1$ in case the function $N(s, \varphi)$ satisfies (27).

Finally, we deal with the case of a strictly positive or strictly negative function $M(s, \varphi)$:

$$M(s, \varphi) \geqq M_0 > 0 \quad \text{or} \quad M(s, \varphi) \leqq -M_0 < 0 \tag{35}$$

for $s \in [-\pi, \pi]$ and $\varphi \in \mathbf{R}$. The function $v = l(s, \xi)$ is the solution to the equation

$$v(s) + H[\lambda(\cdot, \xi)\, v]\,(s) = g(s, \xi), \tag{36}$$

where $\lambda(s, \xi) = M(s, \xi(s))$ and the function $g(s, \xi)$ given by (16) fulfils the relation

$$\int_{-\pi}^{\pi} g(s, \xi)\, ds = 0$$

in view of the assumption (3). Hence the inequality (A.33) of the appendix yields the estimation

$$\|l(\cdot, \xi)\| \leqq (1/M_0)\, \|g\|. \tag{37}$$

Further, in case of (35) the assumption (28) is fulfilled. Thus the above continuity proof for the mapping P on $\mathfrak{K}$ remains valid.[1] This implies

Theorem 3: *Let $M(s, \varphi)$ be a continuous function on $[-\pi, \pi] \times \mathbf{R}$ which is 2π periodic in s and satisfies an inequality of the form* (35), *and let $N(s, \varphi)$ be a Carathéodory function on $[-\pi, \pi] \times \mathbf{R}$ satisfying an inequality of the form* (27) *with $0 < \delta < 1$, where $\varrho = 2$. Then the integro-differential equation* (1) *has a solution $\varphi \in W_2^1(-\pi, \pi)$ for any $F \in L_2(-\pi, \pi)$ satisfying the relation* (3).

Remark: A solution to equation (1) also exists if the function $M(s, \varphi)$ satisfies an inequality of the form

$$M(s, \varphi) \geqq 1/[M_1 + M_2 R^{\omega_0}], \qquad 0 < \omega_0 < 1, \tag{38}$$

for $s \in [-\pi, \pi]$, $|\varphi| \leqq R$ instead of (35), where $M_1, M_2 > 0$ and $\delta + \omega_0 < 1$.

4. Special case. Uniqueness theorem

A. We consider the following *particular case* of the equation (1):

$$N(s, \varphi) = dM(s, \varphi), \quad d \in \mathbf{R}; \quad F(s) \equiv 0. \tag{39}$$

In this case

$$g(s, \xi) = dH[M(\cdot, \xi)]\,(s) = dH[\lambda]\,(s), \tag{40}$$

and according to formula (A.8$_1$) of the appendix the kernel $l(s, \xi)$ of the integral equation (12) has the simpler form

$$l(s, \xi) = d - d(\cos \bar{\mu})^{-1}\, \alpha(s)\, e^{-(H\mu)(s)}, \tag{41}$$

where

$$\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s, \xi)\, ds.$$

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[1]) From (37) and (14) also the uniform boundedness of the L_2 norms of the functions $\Lambda[\xi_n]$ in the above continuity proof follows.

We only assume that the continuous function $M(s, \varphi)$ is bounded such that the oscillation of the function $\mu = \mu(s, \xi)$ is not greater than $2\gamma = \mu_2 - \mu_1 < \pi$ for any $\xi \in C[-\pi, \pi]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, and $\lambda_1 = \inf M(s, \varphi)$, $\lambda_2 = \sup M(s, \varphi) \times (s, \varphi)$, the infimum and supremum are taken over $s \in [-\pi, \pi]$, $\varphi \in \mathbf{R}$. Applying Zygmund's lemma to (41), we obtain the estimation

$$\|l(\cdot, \xi)\|_p \leqq (2\pi)^{1/p} |d| \left[1 + \sqrt{1 + \lambda_0^2} (\cos p\gamma)^{-1/p}\right] \tag{42}$$

for $1 < p < \pi/2\gamma$, where $\lambda_0 = \max[-\lambda_1, \lambda_2]$. Also the continuity of the corresponding operator P in the space $C[-\pi, \pi]$ can be shown like in [14] or, easier, like in the corresponding proof for Villat's integral equation in the theory of plane cavity flows [3: Chap. VII]. I.e., a solution to equation (1) exists without an additional assumption on the oscillation of $M(s, \varphi)$ of the form (23).

Theorem 4: *Let $M(s, \varphi)$ be a bounded continuous function on $[-\pi, \pi] \times \mathbf{R}$ which is 2π periodic in s. Then the integro-differential equation (1) with (39) possesses a solution $\varphi \in W_p^1(-\pi, \pi)$, $1 < p < \pi/[\mu_2 - \mu_1]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, and $\lambda_1 = \inf M$, $\lambda_2 = \sup M$.*

Remark: As in Remark 3 to Theorem 1, from (41) it follows that indeed the solution $\varphi \in W_p^1(-\pi, \pi)$ for any finite $p > 1$.

In case of the integral equation (4), where (6) holds, the assumptions (39) write

$$n(s, \varphi) = m_s(s, \varphi) + dm_\varphi(s, \varphi), \qquad f(s) = Const. \tag{43}$$

E.g., this is fulfilled with $d = -1$ if $n(s, \varphi) = f(s) \equiv 0$ and

$$m(s, \varphi) = Q(s + \varphi), \tag{44}$$

where Q is a 2π periodic continuously differentiable function. This case embraces the well-known *Theodorsen integral equation of conformal mapping* (cf. [5: Kap. II]) for which

$$Q(s) = \ln \varrho(s), \tag{45}$$

where $\varrho = \varrho(s)$ is the representation in polar-coordinates of the starlike Jordan curve J to be mapped onto the unit disk. More precisely, for the Theodorsen equation

$$l(s, \xi) = \cos \mu \cdot e^{-(H\mu)(s)} - 1, \tag{46}$$

where $\mu = \mu(s + \xi)$ and

$$\mu(s) = \arctan \frac{\varrho'(s)}{\varrho(s)} \tag{47}$$

is the angle between the outer normal to J and the radius vector in the point $(s, \varrho(s))$ of J. The operator equation (12) with (46) is an integrated form of Friberg's integro-differential equation (cf. [5: Kap. II, § 4.4]).

Theorem 4 thus yields the existence of a solution to the Theodorsen equation for a smooth starlike Jordan curve with continuous function ϱ'.

B. Finally, we state some simple *uniqueness theorems* for continuous solutions of the integral equation (4). In the following we assume that $m(s, \varphi)$, $n(s, \varphi)$ are continuous functions which possess continuous derivatives $m_\varphi(s, \varphi)$ and $n_\varphi(s, \varphi)$.

Let $\varphi_k(s)$, $k = 1, 2$, be two continuous solutions of (4). Then the difference function $\Phi = \varphi_1 - \varphi_2$ satisfies the equation

$$\Phi(s) + H[A(\cdot) \Phi](s) + K[B(\cdot) \Phi](s) = 0 \tag{48}$$

with the continuous functions

$$A(s) = \int_0^1 m_\varphi\big(s, \varphi_2(s) + t[\varphi_1(s) - \varphi_2(s)]\big)\,dt,$$

$$B(s) = \int_0^1 n_\varphi\big(s, \varphi_2(s) + t[\varphi_1(s) - \varphi_2(s)]\big)\,dt.$$

We consider *three cases.*

In *case* I: $n(s, \varphi) \equiv 0$ it is also $B(s) \equiv 0$ and the equation

$$\Phi(s) + H[A(\cdot)\,\Phi]\,(s) = 0 \tag{49}$$

has only the trivial continuous solution $\Phi(s) \equiv 0$ as it follows from Appendix 1.

In *case* II: $n(s, \varphi) = n_0\varphi$, $n_0 \in \mathbf{R}$, it is $B(s) = n_0$ and we have the equation

$$\Phi(s) + H[A(\cdot)\,\Phi]\,(s) + n_0K[\Phi]\,(s) = 0. \tag{50}$$

We multiply (50) by $H[\Phi]$ and integrate it over $(-\pi, \pi)$. It is

$$\int_{-\pi}^{\pi} \Phi H[\Phi]\,ds = \int_{-\pi}^{\pi} K[\Phi]\,H[\Phi]\,ds = 0$$

(recall that $H[\Phi] = -(d/ds)\,K[\Phi]$) and

$$\int_{-\pi}^{\pi} H[\Phi]\cdot H[A\Phi]\,ds = \int_{-\pi}^{\pi} \Phi\cdot A\Phi\,ds$$

on account of the relation $\int_{-\pi}^{\pi} \Phi(s)\,ds = 0$ following from (50) by integrating it over $(-\pi, \pi)$. Therefore, we obtain $\int_{-\pi}^{\pi} A(s)\,\Phi^2(s)\,ds = 0$ which implies $\Phi(s) \equiv 0$ if $A(s)$ has constant sign on $(-\pi, \pi)$. I.e., the solution to the equation (4) is uniquely determined if either

$$m_\varphi(s, \varphi) > 0 \quad \text{or} \quad m_\varphi(s, \varphi) < 0 \tag{51}$$

for almost all $s \in [-\pi, \pi]$ and all $\varphi \in \mathbf{R}$.

In *case* III: $n(s, \varphi) = \nu m(s, \varphi)$, $\nu \in \mathbf{R}$, we multiply the corresponding equation

$$\Phi(s) + H[A(\cdot)\,\Phi]\,(s) + \nu K[A(\cdot)\,\Phi]\,(s) = 0 \tag{52}$$

by $A\Phi$ and integrate it over $(-\pi, \pi)$. This gives

$$\int_{-\pi}^{\pi} A(s)\,\Phi^2(s)\,ds + \nu\int_{-\pi}^{\pi} A\Phi\cdot K[A\Phi]\,ds = 0.$$

Now $\int_{-\pi}^{\pi} \chi K[\chi]\,ds \leqq 0$ for any continuous function χ and therefore $\nu\int_{-\pi}^{\pi} A(s)\,\Phi^2(s)\,ds \geqq 0$. This implies $\Phi(s) \equiv 0$ if $\nu A(s)$ has negative sign on $(-\pi, \pi)$. I.e., the solution to the equation (4) is uniquely determined if

$$\operatorname{sign}\nu\cdot m_\varphi(s, \varphi) < 0 \tag{53}$$

for almost all $s \in [-\pi, \pi]$ and all $\varphi \in \mathbf{R}$.

Theorem 5: *The integral equation* (4) *with continuous coefficient* $m(s, \varphi)$ *possessing a continuous derivative* $m_\varphi(s, \varphi)$ *has at most one continuous solution in the cases* I—III *if in case* II *and* III, *respectively, the condition* (51) *and* (53) *is fulfilled.*

Appendix

1. *Solution of the equation*

$$v(s) + H[\lambda(\cdot)\, v](s) = g(s) \quad \text{for a.a. } s \in [-\pi, \pi]. \tag{A.1}$$

Let λ be a 2π periodic continuous function and g a function summable to a power greater than one. We look for a solution v of (A.1) summable to a power greater than one. Substituting

$$w = \lambda v, \qquad v = w/\lambda = g - H[w], \tag{A.2}$$

we obtain the equation

$$w + \lambda H[w] = \lambda g \tag{A.3}$$

for w. This equation has the unique solution (cf. [4: Chap. IV, § 31][2])

$$w(s) = D\beta(s)\, e^{-(H\mu)(s)} + \alpha(s)\, \beta(s)\, g(s) - \beta(s)\, e^{-(H\mu)(s)} H[\beta g\, e^{H\mu}](s), \tag{A.4}$$

where

$$\mu(s) = \operatorname{arc\,tan} \lambda(s), \tag{A.5}$$

$$\alpha(s) = \cos \mu(s) = \frac{1}{\sqrt{1 + \lambda^2(s)}}, \qquad \beta(s) = \sin \mu(s) = \frac{\lambda(s)}{\sqrt{1 + \lambda^2(s)}} \tag{A.6}$$

and the constant D is given by

$$D = \tan \bar{\mu} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s)\, g(s)\, e^{(H\mu)(s)}\, ds \tag{A.7}$$

with the mean value

$$\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s)\, ds.$$

Here it has been used that

$$\tan \bar{\mu} = \int_{-\pi}^{\pi} e^{-H\mu} \sin \mu\, ds \Big/ \int_{-\pi}^{\pi} e^{-H\mu} \cos \mu\, ds$$

is equal to the expression $-\operatorname{Im} \Psi(0)/\operatorname{Re} \Psi(0)$ of the holomorphic function $\Psi(z) = e^{-i(S\mu)(z)}$ in the unit disk, where S is the Schwarz integral.

Therefore, the equation (A.1) has the unique solution

$$v(s) = D\alpha(s)\, e^{-(H\mu)(s)} + \alpha^2(s)\, g(s) - \alpha(s)\, e^{-(H\mu)(s)} H[\beta g\, e^{H\mu}](s). \tag{A.8}$$

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[2]) In formula (31.19) of [4] the sign + has to be changed into −.

The solution (A.4) of the equation (A.3) is derived in [4] for Hölder continuous functions λ and g. But according to the general theory of singular integral equations with continuous coefficients in spaces of summable functions (cf. [11: Kap. III]) it also holds true under the above more general assumptions about λ and g. Also the derivation in [4] can be directly performed under these general assumptions utilizing the solvability of the Dirichlet problem in the Hardy classes with an exponent greater than one (cf. [6: Kap. IX, §§ 4, 5]).

We still simplify the expression (A.8) for v in the *particular cases* $g = 1$ and $g = H\lambda$. By means of the holomorphic function $\Phi(z) = e^{i(S\mu)(z)}$ in the unit disk possessing the boundary values $\Phi(e^{is}) = (\alpha + i\beta)\, e^{H\mu}$ one obtains the relations

$$H[\beta\, e^{H\mu}] = \alpha\, e^{H\mu} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha\, e^{H\mu}\, ds = \alpha\, e^{H\mu} - \cos \bar{\mu}$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \beta\, e^{H\mu}\, ds = \sin \bar{\mu}.$$

Therefore, the solution (A.8) for $g = 1$ takes the form

$$v_0(s) = (\cos \bar{\mu})^{-1}\, \alpha(s)\, e^{-(H\mu)(s)}. \tag{A.8$_0$}$$

Further, the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as can be seen from the equation (A.1). I.e.,

$$v_1(s) = 1 - (\cos \bar{\mu})^{-1}\, \alpha(s)\, e^{-(H\mu)(s)}. \tag{A.8$_1$}$$

2. *Estimations in the* L_2 *norm.*

Let be $g \in L_2(-\pi, \pi)$ and $v, w \in L_2(-\pi, \pi)$. Then, from the second formula in (A.2) the estimation $\|v - g\| \leqq \|w\|$ follows, where $\|\cdot\|$ denotes the norm in $L_2(-\pi, \pi)$. On account of the formulas (A.4) and (A.6) this means that

$$\|\alpha\Psi + \beta\chi\|^2 \leqq \|\alpha\chi - \beta\Psi\|^2 \tag{A.9}$$

for the functions

$$\Psi(s) = e^{-(H\mu)(s)} H[\beta g\, e^{H\mu}]\,(s) - D\, e^{-(H\mu)(s)}, \tag{A.10a}$$

$$\chi(s) = \beta(s)\, g(s). \tag{A.10b}$$

The inequality (A.9) writes

$$\int_{-\pi}^{\pi} A(\Psi, \chi)\,(s)\, ds \leqq 0, \tag{A.11}$$

where

$$\begin{aligned} A(\Psi, \chi) &= (\alpha^2 - \beta^2)\, \Psi^2 + 4\alpha\beta\Psi\chi - (\alpha^2 - \beta^2)\, \chi^2 \\ &\geqq (\alpha^2 - \beta^2 - 2q\alpha^2)\, \Psi^2 - \left(\alpha^2 - \beta^2 + \frac{2}{q}\, \beta^2\right) \chi^2 \\ &= [2(1 - q) \cos^2 \mu - 1]\, \Psi^2 - \left[\left(\frac{2}{q} - 1\right) - 2\left(\frac{1}{q} - 1\right) \cos^2 \mu\right] \chi^2 \end{aligned}$$

for an arbitrary positive constant q.

We now assume that

$$|\mu(s)| \leqq \gamma < \pi/4, \qquad s \in [-\pi, \pi], \tag{A.12}$$

with a constant γ. Then $\cos^2 \mu \geqq \cos^2 \gamma$ and

$$A(\Psi, \chi) \geqq [2(1-q)\cos^2\gamma - 1]\,\Psi^2 - \left[\left(\frac{2}{q} - 1\right) - 2\left(\frac{1}{q} - 1\right)\cos^2\gamma\right]\chi^2$$

if $0 < q < 1$. In particular, choosing

$$q = \frac{1-\tan\gamma}{1+\tan\gamma}\tan\gamma,$$

we obtain

$$A(\Psi, \chi) \geqq K_1 \Psi^2 - K_2 \chi^2 \tag{A.13}$$

with the constants

$$K_1 = \frac{1-\tan\gamma}{1+\tan\gamma}, \qquad K_2 = \frac{1+\tan\gamma}{1-\tan\gamma}. \tag{A.14}$$

From (A.11) and (A.13) with (A.14) the estimation

$$\|\Psi\| \leqq C_\gamma \|\chi\| \tag{A.15}$$

with the constant

$$C_\gamma = \sqrt{\frac{K_2}{K_1}} = \frac{1+\tan\gamma}{1-\tan\gamma} = \frac{\cos 2\gamma}{1-\sin 2\gamma} \tag{A.16}$$

follows. Under the assumption (A.12) the solution v of (A.1) then satisfies the inequality $\|v\| \leqq \|\Psi\| + \|g\| \leqq C_\gamma \|\chi\| + \|g\|$ or because of $\|\chi\| \leqq \sin\gamma \cdot \|g\|$

$$\|v\| \leqq (1 + \sin\gamma \cdot C_\gamma)\,\|g\| \leqq \left(1 + \frac{1}{2}\sqrt{2}C_\gamma\right)\|g\|. \tag{A.17}$$

Obviously, the inequality (A.17) holds true for all functions $g \in L_2(-\pi, \pi)$ if the assumption (A.12) is fulfilled.

We further derive analogous estimations to (A.15) and (A.17) under the less restrictive *assumption*

$$|\nu(s)| \leqq \gamma < \pi/4, \tag{A.18}$$

where

$$\nu(s) = \mu(s) - \tilde{\mu}, \qquad 2\tilde{\mu} = \max_{s\in[-\pi,\pi]} \mu(s) + \min_{s\in[-\pi,\pi]} \mu(s). \tag{A.19}$$

The inequality (A.18) is fulfilled if

$$2\gamma = \max_{s\in[-\pi,\pi]} \mu(s) - \min_{s\in[-\pi,\pi]} \mu(s) < \pi/2. \tag{A.20}$$

At first we introduce the functions

$$\Psi_0(s) = H[\beta g\, e^{H\mu}](s) - D, \tag{A.21 a}$$

$$\chi_0(s) = \beta(s)\, g(s)\, e^{(H\mu)(s)}, \tag{A.21 b}$$

where $\Psi_0 = H\chi_0 - D$, and the constant

$$D_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_0(s)\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s)\, g(s)\, e^{(H\mu)(s)}\, ds. \tag{A.22}$$

Then the function

$$W_0(z) = e^{-i(S\nu)(z)}[(S\chi_0)(z) + iD_0 \tan \bar{\nu}], \tag{A.23}$$

where

$$\bar{\nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \nu(s)\, ds$$

and S again denotes the Schwarz integral, is holomorphic in the unit disk and satisfies the condition $\operatorname{Im} W_0(0) = 0$.

The function W_0 has the boundary values

$$\operatorname{Re} W_0(e^{is}) = e^{-H\mu}[\cos \nu \cdot \chi_0 - \sin \nu\, (\Psi_0 + D_1)],$$

$$\operatorname{Im} W_0(e^{is}) = -e^{-H\mu}[\cos \nu\, (\Psi_0 + D_1) + \sin \nu \cdot \chi_0],$$

where

$$D_1 = D - D_0 \tan \bar{\nu} = D_0[\tan \bar{\mu} - \tan \bar{\nu}]. \tag{A.24}$$

Therefore, putting $\alpha_0 = \cos \nu(s)$, $\beta_0 = \sin \nu(s)$, we obtain the inequality

$$\|e^{-H\mu}[\alpha_0(\Psi_0 + D_1) + \beta_0\chi_0]\|^2 \leqq \|e^{-H\mu}[\alpha_0\chi_0 - \beta_0(\Psi_0 + D_1)]\|^2,$$

which is analogous to the inequality (A.9).

From this inequality in the same way as above the inequality

$$\|e^{-H\mu}(\Psi_0 + D_1)\| \leqq C_\gamma \|e^{-H\mu}\chi_0\|$$

follows, where the constant C_γ is given by (A.16) again. Due to the formulas (A.10a, b) and (A.21a, b) this means that $\|\Psi + D_1 e^{-H\mu}\| \leqq C_\gamma\|\chi\|$, and on account of (A.24) we obtain $\|\Psi\| \leqq C_\gamma \|\chi\| + |D_0|\; |\tan \bar{\mu} - \tan \bar{\nu}|\; \|e^{-H\mu}\|$. Finally, $|D_0| \leqq (1/2\pi)\, \|g\| \times \|e^{H\mu}\|$, and under the assumption (A.18) $|\tan \bar{\mu} - \tan \bar{\nu}| \leqq |\tan \bar{\mu}| + 1$ and $\|e^{\pm H\mu}\| \leqq \left(\frac{2\pi}{\cos 2\gamma}\right)^{1/2}$ in virtue of the well-known Zygmund lemma [16]. Therefore,

$$\|\Psi\| \leqq C_\gamma \|\chi\| + [1 + |\tan \bar{\mu}|]\, (\cos 2\gamma)^{-1}\, \|g\|. \tag{A.25}$$

From (A.25) the estimation

$$\|v\| \leqq E_\gamma \|g\| \tag{A.26}$$

with the constant

$$E_\gamma = B_\gamma + [1 + |\tan \bar{\mu}|]\, (\cos 2\gamma)^{-1}, \tag{A.27}$$

where

$$B_\gamma = 1 + C_\gamma = 2/[1 - \tan \gamma], \tag{A.28}$$

for the solution v of (A.1) follows. The inequality (A.26) holds for all $g \in L_2(-\pi, \pi)$ if the assumption (A.18) is fulfilled.

If the function λ is strictly positive (or strictly negative) and $\int_{-\pi}^{\pi} g(s)\, ds = 0$ the solution v of (A.1) can be estimated in another way. Namely, applying the Hilbert operator to (A.1), we obtain the equation

$$Hv - \lambda v = Hg + Const \tag{A.29}$$

taking into account that

$$H^2\varphi = -\varphi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(s)\, ds$$

for any function $\varphi \in L_2(-\pi, \pi)$. Further, from (A.1) it follows that

$$\int_{-\pi}^{\pi} v(s)\, ds = \int_{-\pi}^{\pi} g(s)\, ds = 0 \tag{A.30}$$

by assumption. Multiplying (A.29) by v and integrating it over $(-\pi, \pi)$, therefore yields the relation

$$\int_{-\pi}^{\pi} \lambda(s)\, v^2(s)\, ds = -\int_{-\pi}^{\pi} vHg\, ds \tag{A.31}$$

using that $\int_{-\pi}^{\pi} vHv\, ds = 0$.

Let now the *assumption*

$$\lambda(s) \geqq M_0 > 0 \quad \text{or} \quad \lambda(s) \leqq -M_0 < 0 \quad \text{in } [-\pi, \pi] \tag{A.32}$$

be fulfilled, respectively. Then (A.31) implies the estimation

$$M_0 \int_{-\pi}^{\pi} v^2(s)\, ds \leqq \pm \int_{-\pi}^{\pi} \lambda(s)\, v^2(s)\, ds \leqq \|v\|\, \|g\|,$$

since $\|Hg\| = \|g\|$, and we obtain the inequality

$$\|v\| \leqq (1/M_0)\, \|g\| \tag{A.33}$$

for the solution v of (A.1).

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Manuskripteingang: 28. 09. 1983

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