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A Class of Nonlinear Singular Integral and Integro-differential Equations with Hilbert Kernel

L. v. WOLFERSDORF

Dedicated to Prof. Dr. H. Beckert on the occasion of his 65th birthday

Mit Hilfe einer neuartigen Anwendung des Schauderschen Fixpunktsatzes werden Existenzsätze für eine Klasse von quasilinearen singulären Integrodifferentialgleichungen und eine zugehörige Klasse von nichtlinearen singulären Integralgleichungen bewiesen.

С помощью нового применения принципа Шаудера доказываются теоремы существования для одного класса квазилинейных сингулярных интегро-дифференциальных уравнений и связанного с ним класса нелинейных сингулярных интегральных уравнений.

By means of a novel application of Schauder's fixed point theorem, existence theorems are proved for a class of quasilinear singular integro-differential equations and a related class of nonlinear singular integral equations.

Introduction

Nonlinear singular integral and integro-differential equations with Hilbert or Cauchy kernel have been treated by many authors, cf. POGORZELSKI [12] and the recent monograph [8] by GUSEINOV and MUKHTAROV. But as a rule only existence theorems are given for (in some sense) small nonlinearities. Without smallness assumptions on the data existence assertions were obtained for special classes of such equations by means of the theory of monotone operators in spaces L_p of summable functions $[1, 9, 7]$ (cf. also 8), 2, 13] and recently for Cauchy kernels by means of a nonlocal implicit function theorem in the Sobolev space W_2 ¹ and in the space C^1 of continuously differentiable functions [10].

In this paper a class of quasi-linear integro-differential equations with Hilbert kernel and a related class of integral equations are investigated by means of the classical Schauder fixed point theorem in the space C of continuous functions. Reducing the integro-differential equation to an equivalent integral equation of fixed point, type, the application of Schauder's theorem yields some general existence theorems for the integro-differential equation under various kinds of assumptions on the data. By differentation a related class of integral equations is reduced to these integro-differential equations. This class of integral equations contains the known Theodorsen integral equation of conformal mapping as a particular case. For its solution in case of a general smooth starlike Jordan curve an existence proof will be given which is independent of the Riemann mapping theorem. For some subclasses of the integral equations the uniqueness of the solution is proved, too.

By means of analogous methods as here the Riemann-Hilbert problem for holomorphic functions and the Poincaré problem for harmonic functions in the unit disk are dealt with in the author's papers [14, 15], respectively.

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1. Statement of problems

We look for 2π periodic solutions φ .
 differential equation We look for 2π periodic solutions $\varphi \in W_p^1(-\pi, \pi)$, $p > 1$, of the *quasilinear integrodition*
 $\varphi'(s) + H[M(\cdot, \varphi) \varphi'](s) = F(s) + H[N(\cdot, \varphi)](s)$ for a.a. $s \in [-\pi, \pi]$

(1) *di//ereniial equation* 3 L. v. WOLFERSDORF

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ferential equation
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 f(8) d8) + $H[M(\cdot, \varphi) \varphi'](s) = F(s) + H[N(\cdot, \varphi)](s)$ for a.a. $s \in [-\pi, \pi]$

(1)

ie additional condition
 $f(\varphi(s)) = k$

(2a)
 $\int_{\pi}^{\pi} \varphi(s) ds =$ We look for 2π periodic solutions $\varphi \in W_p^{-1}(-\pi, \pi)$, $p > 1$, of the quasilinear i
 $\varphi'(s) + H[M(\cdot, \varphi) \varphi'](s) = F(s) + H[N(\cdot, \varphi)](s)$ for a.a. $s \in [-\pi, \pi]$

satisfying the additional condition
 $\varphi(0) = k$

or
 $\frac{1}{2\pi} \int_{-\$

$$
\varphi'(s) + H[M(\cdot,\varphi) \varphi'] (s) = F(s) + H[N(\cdot,\varphi)] (s) \quad \text{for a.a.} \quad s \in [-\pi,\pi]
$$

satisfying-the additional condition

or

V

V

$$
\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\varphi(s)\ ds=c\,,
$$

g the additional condition
\n
$$
\varphi(0) = k
$$
\n
$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s) ds = c,
$$
\n
$$
\text{ely. Here } k, c \in \mathbb{R} \text{ are given real co}
$$
\n
$$
H[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \cot \frac{\sigma - s}{2} d\sigma
$$
\n
$$
\text{cell-known Hilbert' transformation}
$$
\n
$$
[\pi, \pi), p > 1.
$$

is the well-known Hilbert transformation with cotangent kernel of a function or
 $\varphi(0) = k$
 $\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s) ds = c$,
 respectively. Here k, c \in **R** are given real constants and
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is the well-known Hilbert transformation with cotangent ker $\omega \in L_p(-\pi, \pi), p > 1.$
The assumptions on the given functions $M(s, \varphi), N(s, \varphi)$ and $F(s)$ will be specified

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$$
2\pi J
$$

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\n
$$
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$$

\nell-known Hilbert transformation with cotangent kernel of a function
\n $\pi, \pi, \eta, p > 1$.
\nsumptions on the given functions $M(s, \varphi), N(s, \varphi)$ and $F(s)$ will be specified
\ncourse, F has to fulfill the condition
\n
$$
\int_{-\pi}^{\pi} F(s) ds = 0,
$$
\n(3)
\n $\int_{-\pi}^{\pi} F(s) ds = 0,$
\n $\pi, \pi, \eta, p > 1.$
\n π we deal with the integral equation
\n $\varphi(s) + H[m(\cdot, \varphi)](s) + K[n(\cdot, \varphi)](s) = f(s),$
\nis the Hilbert transformation and

which is necessary for the solvability of (1) since $\int H[\omega](s) ds = 0$ for any function *WE LET* $H[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \cos(\sigma) d\sigma$ *

is the well-known Hilbert trans* $\omega \in L_p(-\pi, \pi), p > 1$ *.

The assumptions on the given flater. Of course, <i>F* has to fulfil the *n*
 $\int_{-\pi}^{\pi} F(s) ds = 0$,

which is necessary lity of (1) since $\int_{-\pi}^{\pi} H[\omega](s) ds = 0$ for any function $H[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \cot \frac{\sigma - s}{2} d\sigma$

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 $\omega \in L_p(-\pi, \pi), p > 1$.

Further we deal with the *integral equation*
 $\varphi(s) + H[m(\cdot, \varphi)](s) + K[n(\cdot, \varphi)](s) = f(s)$,

(4)

where *H* is the Hilber

$$
\varphi(s) + H[m(\cdot,\varphi)](s) + K[n(\cdot,\varphi)](s) = f(s), \qquad (4)
$$

later. Of course, F has to fulfill the condition
\n
$$
\int_{-\pi}^{\pi} F(s) ds = 0,
$$
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\nFurther we deal with the *integral equation*
\n $\varphi(s) + H[m(\cdot, \varphi)](s) + K[n(\cdot, \varphi)](s) = f(s),$
\nwhere H is the Hilbert transformation and
\n
$$
K[\omega](s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega(\sigma) \ln \left(4 \sin^2 \frac{\sigma - s}{2}\right) d\sigma.
$$
\nThe transformation K is related to H by the well-known formula
\n
$$
\frac{d}{d\tau} K[\omega](s) = -H[\omega](s)
$$

$$
\frac{d}{ds} K[\omega](s) = -H[\omega](s)
$$
\n(5a)

$$
\omega \in L_p(-\pi, \pi), p > 1.
$$

\n
$$
\omega \in L_p(-\pi, \pi), p > 1.
$$

\nFurther we deal with the *integral equation*
\n
$$
\varphi(s) + H[m(\cdot, \varphi)](s) + K[n(\cdot, \varphi)](s) = f(s),
$$
\nwhere *H* is the Hilbert transformation and
\n
$$
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$$

\nThe transformation *K* is related to *H* by the well-known formula
\n
$$
\frac{d}{ds} K[\omega](s) = -H[\omega](s)
$$
\nfor any function $\omega \in L_p(-\pi, \pi), p > 1$. Moreover,
\n
$$
\frac{d}{ds} H[\omega](s) = H[\omega'(s)]
$$
\nfor any function $\omega \in W_p^{-1}(-\pi, \pi), p > 1$. Therefore, differentiating (4) yields the equation (1) with
\n
$$
M(s, \varphi) = m_{\varphi}(s, \varphi), N(s, \varphi) = n(s, \varphi) - m_{s}(s, \varphi), F(s) = f'(s),
$$
\n(6)

for any function $\omega \in W_p^1(-\pi, \pi)$, $p > 1$. Therefore, differentiating (4) yields the equation (1) with

$$
M(s, \varphi) = m_{\varphi}(s, \varphi), \quad N(s, \varphi) = n(s, \varphi) - m_{s}(s, \varphi), \quad F(s) = f'(s), \tag{6}
$$

$$
\overline{2}
$$

/

if the given functions $m(s, \varphi)$, $n(s, \varphi)$ and $f(s)$ satisfy suitable assumptions. (Precise assumptions on these functions will also be given later. For $f \in W_p^{-1}(-\pi, \pi)$, $p > 1$, the function f' obviously fulfils the necessary solvability condition (3) .) Furthermore,

$$
\int\limits_{-\pi}^{\pi} K[\omega](s)\,ds=0
$$

for any function $\omega \in L_p(-\pi, \pi)$, $p > 1$. Integrating (4) over $(-\pi,)$ thus leads to the additional condition $(2b)$ with the given constant

Integero-differential Equations with Hilbert Kernel	387
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$\int_{-\pi}^{\pi} K[\omega](s) ds = 0$	
for any function $\omega \in L_p(-\pi, \pi)$, $p > 1$. Integrating (4) over $(-\pi, \pi)$ thus leads to the additional condition (2b) with the given constant	
$c = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds$.	
16. under suitable assumptions the integral equation (4) is equivalent to the integro-differential equation (1), where M, N, F are given by (6), together with the additional condition (2b), where c is given by (7).	
Remark: In case of $n(s, \varphi) \equiv 0$ the equation (4) can also be reduced to the following Riemann-Hilbert problem for the holomorphic function $W(z) = U(z) + iV(z)$ in the unit disk with the boundary values $\varphi(s) = U(e^{t_s})$:\n $V(e^{t_s}) = m(s, U(e^{t_s})) + (Hf)(s)$ \n	
with the additional condition	

Le., under suitable assumptions the integral equation (4) is equivalent to the integrodifferential equation (1), where *M, N, F* are given by (6), together with the additional condition (2b), where *c* is given by (7).

Remark: In case of $n(s, \varphi) = 0$ the equation (4) can also be reduced to the following Riemann-Hilbert problem for the holomorphic function $W(z) = U(z)$ of $n(s, \varphi)$ = Hilbert probably Filbert probably Filbert , $U(e^{is})$ + Jinter on $\int_{0}^{\pi} f(s) ds$.

$$
V(e^{is}) = m(s, U(e^{is})) + (Hj) (s)
$$

with the additional condition

$$
U(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds.
$$

Problems of such type are 'dealt with in our paper [14] by an analogous method as here. y = $m(s, U(e^{i\epsilon})) + (Hf)(s)$

tional condition

= $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds$.

such type are dealt with in our paper [14] by an analogous method as

theorem

= $M(s, \varphi(s))$ (8)

= $F(s) + H[N(\cdot, \varphi)](s)$, (9)

ifferential equation (1)

2. Existence theorem

Putting

$$
\lambda(s) = M(s, \varphi(s))
$$

and

$$
g(s) = F(s) + H[N(\cdot, \varphi)](s),
$$

 $U(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds.$

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 $g(s) = F(s) + H[N(\cdot, \varphi)](s),$
 $g(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds.$ the integro-differential equation (1) takes the form of the' linear integral equation (A.1) in the appendix for $v = \varphi'$. Since, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Problems of such type are dealt with in our paper [14] by an analogous method as
\nere.
\n
$$
\begin{aligned}\n\text{L}\text{Existence theorem} \\
\text{Putting} \\
\hat{a}(s) &= M(s, \varphi(s)) \\
\text{and} \\
\hat{g}(s) &= F(s) + H[N(\cdot, \varphi)](s), \\
\text{the integral equation (1) takes the form of the linear integral equation} \\
\text{A.1) in the appendix for } v = \varphi'. \text{ Since}\n\end{aligned}
$$
\n
$$
\int_{-\pi}^{\pi} g(s) \, ds = 0 \tag{10}
$$
\nwe to the assumption (3), any solution $v \in L_p(-\pi, \pi)$ to this equation has a vanishing integral over $(-\pi, \pi)$ and therefore represents the derivative φ' of a 2π periodic

due to the assumption (3), any solution $v \in L_p(-\pi, \pi)$ to this equation has a vanishing integral over $(-\pi, \pi)$ and therefore represents the derivative φ of a 2π periodic function $\varphi \in W_p^{-1}(-\pi, \pi), p > 1$.

We make the'following basic *Assumption* A on the data:

(i) $M(s, \varphi)$ is a continuous function on $[-\pi, \pi] \times \mathbf{R}$ which is 2π periodic in s. *(ii)* $N(s, \varphi)$ satisfies the Carathéodory condition on $[-\pi, \pi] \times \mathbf{R}$, i.e. it is measurable in s on $[-\pi, \pi]$ for all $\varphi \in \mathbf{R}$ and continuous in φ on \mathbf{R} for almost all $s \in [-\pi, \pi]$, (i) 4
(ii)
able in
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and it fulfills an estimation of the form

$$
|N(s, \varphi)| \leq N_0(s), \quad N_0 \in L_{\varrho}(-\pi, \pi), \quad 1 < \varrho < \infty,
$$
 (11)
for φ from bounded intervals of **R**.

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I. **v.** WOLFERSDORF
 IN(*s*, φ)| $\leq N_0(\dot{s})$, $N_0 \in L_\varrho(-\pi, \pi)$, $1 < \varrho < \infty$, (11)
 In bounded intervals of **R**.
 $L_\varrho(-\pi, \pi)$ with the same exponent ϱ as in (11) and fulfils the condition
 $(s, \varphi) \equiv 0$ a (iii) $F \in L_{\rho}(-\pi, \pi)$ with the same exponent ρ as in (11) and fulfils the condition (3). (If $N(s, \varphi) \equiv 0$ also $\varrho = \infty$ is allowed.)

Then λ is a continuous 2π periodic function and $g \in L_{\rho}(-\pi, \pi)$ for any continuous 2π periodic function φ .

The equation (1) is therefore equivalent to the expression (A.8) in the appendix for $v = \varphi'$ with λ , g given by (8), (9). Taking into account the additional condition (2a) or (2b), respectively, we obtain the equivalent *fixed point equation q=Pq* $|N(s, \varphi)| \le N_0(\hat{s}), N_0 \in L_{\ell}(-\pi, \pi), 1 < \varrho < \infty,$ (11)
 (Ps) and between the same exponent ϱ as in (11) and fulfils the condition
 $(s, \varphi) \equiv 0$ also $\varrho = \infty$ is allowed.)

is a continuous 2π periodic function and (iii) $F \in L_e(-\pi, \pi)$ with the same exponent ϱ as in (11) and fulfil

(3). (If $N(s, \varphi) \equiv 0$ also $\varrho = \infty$ is allowed.)

Then λ is a continuous 2π periodic function and $g \in L_e(-\pi, \pi)$ for
 2π periodic functio

$$
\varphi = P\varphi \tag{12}
$$

for φ , where the operator P is defined for any 2π periodic continuous function ξ by

2 b), respectively, we obtain the equivalent *fixed point equation*
\n
$$
\varphi = P\varphi
$$
\n(12)
\nhere the operator P is defined for any 2π periodic continuous function ξ by
\n
$$
(P\xi)(s) = k + \int_{0}^{s} l(\sigma, \xi) d\sigma
$$
\n(13)
\nkernel
\n
$$
l(s, \xi) = D\alpha(s) e^{-(H\mu)(s)} + \alpha^2(s) g(s) - \alpha(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}](s).
$$
\n(14)
\nstant $D = D[\xi]$ is given by

$$
l(s,\xi)=D\alpha(s)\,\mathrm{e}^{-(H\mu)(s)}+\alpha^2(s)\,g(s)-\alpha(s)\,\mathrm{e}^{-(H\mu)(s)}H[\beta g\,\mathrm{e}^{H\mu}](s). \qquad (14)
$$

The constant $D = D[\xi]$ is given by

2b), respectively, we obtain the equivalent *fixed point equation*
\n
$$
\varphi = P\varphi
$$
 (12)
\nhere the operator P is defined for any 2π periodic continuous function ξ by
\n $(P\xi)(s) = k + \int_0^s l(\sigma, \xi) d\sigma$ (13)
\nkernel
\n $l(s, \xi) = D\alpha(s) e^{-(H\mu)(s)} + \alpha^2(s) g(s) - \alpha(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}](s)$. (14)
\ntant $D = D[\xi]$ is given by
\n
$$
D[\xi] = \tan \overline{\mu} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s) g(s) e^{(H\mu)(s)} ds,
$$
 (15)
\n
$$
\overline{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s) ds,
$$

\nunctions $g = g(s, \xi), \alpha = \alpha(s, \xi), \beta = \beta(s, \xi), \mu = \mu(s, \xi)$ are defined by
\n $g(s, \xi) = F(s) + H[N(\cdot, \xi)](s),$ (16)
\n $\mu(s, \xi) = \arctan \lambda(s, \xi), \lambda(s, \xi) = M[s, \xi(s)),$ (17)
\n $\alpha(s, \xi) = \cos \mu(s, \xi) = 1/\sqrt{1 + \lambda^2(s, \xi)},$ (18a)

where

$$
\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s) \ ds,
$$

and the functions $g = g(s, \xi), \alpha = \alpha(s, \xi), \beta = \beta(s, \xi), \mu = \mu(s, \xi)$ are defined by

functions
$$
g = g(s, \xi)
$$
, $\alpha = \alpha(s, \xi)$, $\beta = \beta(s, \xi)$, $\mu = \mu(s, \xi)$ are defined by

$$
g(s, \xi) = F(s) + H[N(\cdot, \xi)](s),
$$
(16)

$$
g(s, \xi) = F(s) + H[N(\cdot, \xi)](s),
$$

\n
$$
\mu(s, \xi) = \arctan \lambda(s, \xi), \quad \lambda(s, \xi) = M(s, \xi(s)),
$$
\n(17)

$$
\mu(s,\xi) = \arctan \lambda(s,\xi), \quad \lambda(s,\xi) = M(s,\xi(s)),
$$
\n
$$
\alpha(s,\xi) = \cos \mu(s,\xi) = 1/\sqrt{1 + \lambda^2(s,\xi)},
$$
\n(18a)

$$
\alpha(s,\xi) = \cos \mu(s,\xi) = 1/\sqrt{1 + \lambda^2(s,\xi)},
$$
\n(18a)
\n
$$
\beta(s,\xi) = \sin \mu(s,\xi) = \lambda(s,\xi)/\sqrt{1 + \lambda^2(s,\xi)}.
$$
\n(18b)

In case of $(2a)$ the constant *k* is prescribed, whereas in case of $(2b)$

$$
\bar{\mu} = \frac{1}{2\pi} \int \mu(s) ds,
$$

\nd the functions $g = g(s, \xi)$, $\alpha = \alpha(s, \xi)$, $\beta = \beta(s, \xi)$, $\mu = \mu(s, \xi)$ are defined by
\n
$$
g(s, \xi) = F(s) + H[N(\cdot, \xi)](s), \qquad (16)
$$
\n
$$
\mu(s, \xi) = \arctan \lambda(s, \xi), \quad \lambda(s, \xi) = M(s, \xi(s)), \qquad (17)
$$
\n
$$
\alpha(s, \xi) = \cos \mu(s, \xi) = 1/\sqrt{1 + \lambda^2(s, \xi)}, \qquad (18a)
$$
\n
$$
\beta(s, \xi) = \sin \mu(s, \xi) = \lambda(s, \xi)/\sqrt{1 + \lambda^2(s, \xi)}. \qquad (18b)
$$
\ncase of (2a) the constant k is prescribed, whereas in case of (2b)\n
$$
k = k[\xi] = c - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} \mu(\sigma, \xi) d\sigma ds. \qquad (19)
$$
\nWe now estimate the kernel $l(s, \xi)$.

Lemma: Let p be an arbitrary number satisfying $1 < p < \rho$ and put $x = 2pq$ *Lemma: Let p be an arbitrary number satisfying* $1 < p < \varrho$ *and put* $x = 2pq$ **[** $\varrho - p$ **]. Denote further** $\mu_1 = min \mu(s), \mu_2 = max \mu(s), s \in [-\pi, \pi]$ **, and assume the oscillation of** $\mu(s)$ $\begin{array}{l} \text {stimate } \ \textit{Let p~b} \ \textit{note } \textit{furt.} \ \text{if $\mu(s)$} \ \textit{= μ_2} \end{array}$

$$
2\gamma = \mu_2 - \mu_1 < \pi/\varkappa. \tag{20}
$$

Then for the L_p norm of $l(s, \xi)$ the estimation

$$
||l(\cdot,\xi)||_p = \left(\int_{-\pi}^{\pi} |l(s,\xi)|^p \, ds\right)^{1/p}
$$

$$
\leq (2\pi)^{2/\kappa} ||g||_p \, \{1 + [A_r + \tan \mu_0] \, (\cos \kappa \gamma)^{-2/\kappa}\}
$$
 (21)

holds, where $\mu_0 = \max[-\mu_1, \mu_2]$ and A_r is the norm of the Hilbert transformation in $L_r(-\pi, \pi), r = 2p\varrho/[\varrho + p];$

$$
A_r=\begin{cases}\tan{(\pi/2r)} & if \quad 1
$$

 $(cf. [11: Kap. IV, § 7]).$

Proof: It is

$$
||l(\cdot,\xi)||_p \leq ||g||_p + ||e^{-H\mu}H[\beta g e^{H\mu}]||_p + ||e^{-H\mu}||_p \tan \mu_0 \frac{1}{2\pi} \int |g| |e^{H\mu}| \, ds
$$

Further, since $1/p = 1/\varrho + 2/\varkappa$

$$
|g||_p \leq (2\pi)^{2/\kappa} ||g||_p,
$$

since $1/p = 1/r + 1/x$

$$
||e^{-H\mu}||_p \leq (2\pi)^{1/r} ||e^{-H\mu}||_{\mathbf{x}}
$$

and since $1/r = 1/x + 1/\rho$

 $\int |g| |e^{H\mu}| ds \leq (2\pi)^{1/t} ||e^{H\mu}||_{\kappa} ||g||_{\varrho}$,

where $1/t = 1 - (1/r)$. Finally, according to the well-known Zygmund lemma [16] under the assumption (20) it is

$$
\|\mathrm{e}^{\pm H\mu}\|_{\mathbf{x}}\leq (2\pi)^{1/\mathbf{x}}\ (\cos\,\mathbf{x}\gamma)^{-1}
$$

so that

$$
||e^{-H\mu}H[\beta g e^{H\mu}]||_p \leq ||e^{-H\mu}||_* ||H[\beta g e^{H\mu}]||_r
$$

$$
\leq A_r ||e^{-H\mu}||_* ||e^{H\mu}||_* ||g||_e \leq A_r (2\pi)^{2/\kappa} (\cos \kappa \gamma)^{-2\kappa} ||g||_e.
$$

This together vields (21)

We consider the operator P on the convex compact subset \hat{x} of the space $C[-\pi, \pi]$ of 2π periodic continuous functions defined by

$$
\mathfrak{R} = \{ \xi \in C[-\pi,\pi] : |\xi(s)| \leq R, \quad |\xi(s_1) - \xi(s_2)| \leq R_0 |s_1 - s_2|^{1/q} \},\tag{22}
$$

where q is the exponent conjugate to p and R , R_0 are fixed positive numbers to be specified later. We make the following additional Assumption B :

(i) $M(s, \varphi)$ is a bounded function on $[-\pi, \pi] \times \mathbf{R}$ satisfying the inequality

$$
2\gamma = \mu_2 - \mu_1 < \pi/\kappa \tag{23}
$$

for some p with $1 < p < \varrho$, $\varkappa = 2p\varrho/[\varrho - p]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, $\lambda_1 = \inf M(s, \varphi), \lambda_2 = \sup M(s, \varphi),$ the infimum and supremum are taken over $s \in [-\pi, \pi], \varphi \in \mathbf{R}.$

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(ii) The inequality (11) with a fixed function $N_0 \in L_{\ell}(-\pi, \pi)$ holds uniformly with respect to $\varphi \in \mathbf{R}$. 0 L. v. WOLFERSDORF

(ii) The inequality (11) with a fixed function $N_0 \,\epsilon L_\varrho(-\pi,\pi)$ holds uniform

th respect to $\varphi \in \mathbb{R}$.

Under the assumptions A and B it is
 $||g||_{\varrho} \leq ||F||_{\varrho} + A_{\varrho} \overline{N}_0 \equiv G < \infty$,

ere \over L. **v.** WOLFERSDORF

e inequality (11) with a fixed function N_0

ect to $\varphi \in \mathbf{R}$.

the assumptions A and B it is
 $||g||_e \leq ||F||_e + A_e \overline{N}_0 \equiv G < \infty$,

o is the norm of N_0 and A_e the norm of

), and by the above (ii) The inequality (11) with a fixed function $N_0 \,\epsilon L_\ell(-\pi,\pi)$ holds unifor
with respect to $\varphi \epsilon$ **R**.
Under the assumptions A and B it is
 $||g||_\ell \leq ||F||_\ell + A_\ell \overline{N}_0 = G < \infty$,
where \overline{N}_0 is the norm of N_0 and $A_\$ I. v. WOLFERSDORF

e: inequality (11) with a fixed function $N_0 \,\epsilon L_\ell(-\pi,\pi)$ holds uniformly

ect to $\varphi \in \mathbb{R}$.

the assumptions A and B it is
 $||g||_\ell \leq ||F||_\ell + A_\ell \overline{N}_0 = G < \infty$, (24)

o is the norm of N_0 and A_ℓ **E**: inequality (11) with a fixed function $N_0 \,\epsilon L_0(-\pi, \pi)$ holds uniformly

ect to $\varphi \in \mathbb{R}$.

the assumptions A and B it is
 $||g||_e \leq ||F||_e + A_e \overline{N}_0 = G < \infty$,

(24)

o is the norm of N_0 and A_ϵ the norm of the

$$
||g||_{\rho} \leq ||F||_{\rho} + A_{\rho} \overline{N}_0 \equiv G < \infty,\tag{24}
$$

where \overline{N}_0 is the norm of N_0 and A_i the norm of the Hilbert transformation in (i) The inequality (11) with a fixed fun
with respect to $\varphi \in \mathbb{R}$.
Under the assumptions A and B it is
 $||g||_e \leq ||F||_e + A_e \overline{N}_0 \equiv G < \infty$,
where \overline{N}_0 is the norm of N_0 and A_e the n
 $L_e(-\pi, \pi)$, and by the abo

$$
||l(\cdot,\xi)||_p \leq (2\pi)^{2/\kappa} G\{1+[A_r+\lambda_0]\left(\cos\kappa\gamma\right)^{-2/\kappa}\} \equiv L
$$
\n(25)

with $\lambda_0 = \max \{-\lambda_1, \lambda_2\}$ for any function $\xi \in C[-\pi, \pi]$. In case of $(2a)$ we take

$$
R_0 = L, \quad R = |k| + (2\pi)^{1/q} L
$$

Then the operator P maps the subset \Re of $C[-\pi, \pi]$ into itself because for ϕ $A_0 = 1$
case of R_0
the of $\ket{\varphi}$ with $\lambda_0 = \max \{-\lambda_1, \lambda_2\}$ for any function

In case of (2a) we take
 $R_0 = L, \quad R = |k| + (2\pi)^{1/q} L$.

Then the operator P maps the subset §

it is
 $|\varphi(s_1) - \varphi(s_2)| \leq \int_{s_1}^{s_1} |l(s, \xi)| ds \leq$

and
 $|\varphi(s)| \leq |\varphi(0)| + |\varphi(s) - \varphi$

$$
\begin{aligned}\n\text{The sum of } & \mathbf{z} \in \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is a } \mathbb{R}^{n} \text{ and } \mathbb{R}^{n} \text{ is
$$

And

$$
|\varphi(s)| \leq |\varphi(0)| + |\varphi(s) - \varphi(0)| \leq |k| + (2\pi)^{1/q} L.
$$

Analogously, in case *of* (2 b) we take

$$
R_0 = L
$$
, $R = |c| + 2(2\pi)^{1/q} L$

$$
R_0 = L, \quad R = |c| + 2(2\pi)^{1/q} L
$$

estimation

$$
|k| \leq |c| + \int_{-\pi}^{\pi} |l(s,\xi)| ds \leq |c| + (2\pi)^{1/q} ||l(\cdot,\xi)||_p \leq |c| + (2\pi)^{1/q} L.
$$

Furthermore, like in our paper [14] one can show that under the assumptions A and B the operator $P: \mathcal{R} \to \mathcal{R}$ is continuous in the norm of $C[-\pi, \pi]$. The Schauder $R_0 = L$, $R = |c| + 2(2\pi)^{1/q} L$

using the estimation
 $|k| \leq |c| + \int_{-\pi}^{\pi} |l(s, \xi)| ds \leq |c| + (2\pi)^{1/q} ||l(\cdot, \xi)||_p \leq |c| + (2\pi)^{1/q} L$.

Furthermore, like in our paper [14] one can show that under the assumptions A

and B the ope fixed point theorem applied to the equation (12) in \Re then yields the existence of a solution $\varphi \in \mathbb{R}$ to the equation (1). Moreover, this solution is absolutely continuous and has a derivative $\varphi'(s) = l(s, \varphi(s)), \varphi' \in L_p(-\pi, \pi)$, i.e. $\varphi \in W_p^{-1}(-\pi, \pi)$. 299

Corollary: **ICO** The inequality (11) with a fixed function $N_0 \in L_4(-\pi, \pi)$ holds uniformly

with respect to $\phi \in \mathbb{R}$.

Uplies $\|\psi\|_{\infty} \leq |\mu^{\nu}\|_{\infty} + A_{\pi}N_0 = G < \infty$,

where N_0 is the norm of the anomore Furthermore, like in our paper [14] one can show that under the assump
and B the operator $P: \mathcal{R} \to \mathcal{R}$ is continuous in the norm of $C[-\pi, \pi]$. The S
fixed point theorem applied to the equation (12) in \mathcal{R} then at under the assumption of $C[-\pi, \pi]$.

then yields the

tion is absolute

.e. $\varphi \in W_p^1(-\pi,$

tegro-differential

ieodory condition

ie differentiable a

lfil the assumption
 π, π).

y nonnegative
 (π/x)). **Example 19.** The point of $P: \mathbb{R} \to \mathbb{R}$ is continuous in the norm of $C[-\pi, \pi]$. The point theorem applied to the equation (12) in \mathbb{R} then yields the exition $\varphi \in \mathbb{R}$ to the equation (1). Moreover, this s

Theorem 1: *Under Assumptions* A *and* B *the integro-differential equation* (1) *possesses a solution* $\varphi \in W_p^1(-\pi, \pi)$.

Corollary: If the function $n(s, \varphi)$ satisfies the Carathéodory condition and an estimation of the form (11), the functions $m(s, \varphi)$ and $f(s)$ are differentiable and the corre-**L** *sponding functions M(s, q), N(s, q), F(s) given by (6) fulfil the assumptions* A *and* B, Theorem 1: Under Assumptions A and B the integro-differential equation (1)
possesses a solution $\varphi \in W_p^{-1}(-\pi, \pi)$.
Corollary: If the function $n(s, \varphi)$ satisfies the Carathéodory condition and an esti-
mation of the for • int theorem applied to the equation (12) in \Re then yields the existence of
on $\varphi \in \Re$ to the equation (1). Moreover, this solution is absolutely continuous
a derivative $\varphi'(s) = l(s, \varphi(s)), \varphi' \in L_p(-\pi, \pi)$, i.e. $\varphi \in W_p$ mation of the form (11), the functions $m(s, \varphi)$ and $f(s)$ are different sponding functions $M(s, \varphi)$, $N(s, \varphi)$, $F(s)$ given by (6) fulfil the as the integral equation (4) possesses a solution $\varphi \in W_p^{-1}(-\pi, \pi)$.

Remar $\begin{array}{l} \textit{main}\ \textit{of} \ \textit{sponding} \ \textit{i} \ \textit{the integral} \ \textit{Remain} \ \textit{bounded} \ \textit{in} \ \textit{seman} \ \textit{bounded} \ \textit{in} \ \textit{semain} \ \textit{in} \ \textit{f} \ \textit{form} \ \textit{L}_{\textit{e}}(\textit{--} \ |\ \textit{in} \ \textit{in}$

$$
\sup M(s,\varphi) < \tan(\pi/\varkappa) \quad \text{(inf } M(s,\varphi) > -\tan(\pi/\varkappa). \tag{26}
$$

• Remark 2: Assumption (ii) of B can be weakened to

$$
|N(s, \varphi)| \le N_1(s) + N_2(s) R^s, \qquad 0 < \delta < 1,
$$
\n(27)

sup $M(s, \varphi) < \tan (\pi/\kappa)$ (inf $M(s, \varphi) > -\tan (\pi/\kappa)$). (26)

Remark 2: Assumption (ii) of B can be weakened to
 $|N(s, \varphi)| \le N_1(s) + N_2(s) R^s$, $0 < \delta < 1$, (27)

for $s \in [-\pi, \pi]$, $|\varphi| \le R$ and any positive R, where N_k , $k = 1, 2$ If K 1: Assumption (1) of B is fulfilled
function $M(s, \varphi)$ satisfying the inequality $M(s, \varphi)$ in $M(s, \varphi)$
 $\forall k$ 2: Assumption (ii) of B can be wear
 $N(s, \varphi)| \leq N_1(s) + N_2(s) R^s$, $0 < \pi, \pi$, $\exists | \varphi| \leq R$ and any positiv $k=1,2$, are

$$
||l(\cdot, \xi)||_p \leq Const_1 + Const_2 R^{\delta}
$$

for $\xi \in \mathbb{R}$ and the operator *P* maps \mathbb{R} into itself if the constants *R*, R_0 are taken sufficiently large. (The same conclusion is valid if (27) is fulfilled only with $\delta = 1$ but the norm of N_2 in $L_0(-\pi, \pi)$ is sufficiently small.)

Remark 3: Since for a continuous function μ the functions $e^{\pm H\mu} \in L_r(-\pi,\pi)$ for any finite $r > 0$, from (12)–(14) it follows that the solution $\varphi \in W_{p}^{-1}(-\pi, \pi)$ for any $1 < p < \varrho$, not only for the value of p determined by the assumption (23). Integro-differible to the operator P maps \hat{x} into ciently large. (The same conclusion is valified norm of N_2 in $L_{\varrho}(-\pi, \pi)$ is sufficiently sn

Remark 3: Since for a continuous further $r > 0$, from (12)–(14) norm of N_2 in $L_e(-1)$

Remark 3: Since

any finite $r > 0$, from
 $1 < p < \varrho$, not only

3. Further existence

We employ a modif

norm of the kernel

replaced by
 $2\gamma = \mu_2$ and the operator P maps \Re into itself if
rge. (The same conclusion is valid if (27
 N_2 in $L_e(-\pi, \pi)$ is sufficiently small.)
rk 3: Since for a continuous function μ
 $e \ r > 0$, from (12)-(14) it follows that
 ϱ *III III III*

We employ a modification of the above method in the case $\rho = 2$ estimating the norm of the kernel $l(s, \xi)$ in another way. In Assumption B the inequality (23) is *B7* μ *B7* μ *B7* μ *B7* μ *B7* μ *B7* μ ² μ _{*B7*} μ ² μ _{*B7*} μ ² μ _{*B7*} μ _{*B7*} **By** a modification of the abov

the kernel $l(s, \xi)$ in another w

by
 $2\gamma = \mu_2 - \mu_1 < \pi/2$.
 $\hat{B}(\cdot, \xi)$ is $\leq E_r$ $||g||$,
 $||l(\cdot, \xi)|| \leq E_r$ $||g||$,
 $||\xi|| = E_r$ $||g||$,
 $||d$ denotes the norm in $L_2(-\pi, \xi)$
 $E_\gamma = B_\gamma + [1 + \$

$$
2\gamma = \mu_2 - \mu_1 < \pi/2. \tag{28}
$$

Then from formula $(A.26)$ of the appendix we have the estimate

$$
l(\cdot,\xi)\| \le E_\gamma \|g\|,\tag{29}
$$

$$
f_{\rm{max}}
$$

replaced by
\n
$$
2\gamma = \mu_2 - \mu_1 < \pi/2.
$$
\nThen from formula (A.26) of the appendix we have the estimate
\n
$$
||l(\cdot, \xi)|| \leq E_r ||g||,
$$
\nwhere $|| \cdot ||$ denotes the norm in $L_2(-\pi, \pi)$. The constant E_r is given by
\n
$$
E_r = B_r + [1 + \lambda_0] (\cos 2\gamma)^{-1},
$$
\nwhere
\n
$$
B_r = 2/[1 - \tan \gamma]
$$
\nand again $\lambda_0 = \max \{-\lambda_1, \lambda_2\}.$
\nNow, under Assumption B it is
\n
$$
||g|| \le ||F|| + \overline{N}_0 = G_0 < \infty,
$$
\nwhere \overline{N}_0 is the L_2 norm of the function N_0 in (11). Hence
\n
$$
||l(\cdot, \xi)|| \leq E_r \cdot G_0 = L_0.
$$
\n(33)

where'

$$
3_{\rm v}=2/[1-\tan\nu]
$$

Now, under Assumption B it is

 $B_{\gamma} = 2/[1 - \tan \gamma]$
and again $\lambda_0 = \max \left[-\lambda_1, \lambda_2 \right]$.
Now, under Assumption B it is
 $||g|| \leq ||F|| + \overline{N}_0 = G_0 < \infty$, (32)
 (33)

where \overline{N}_0 is the L_2 norm of the function N_0 in (11). Hence $||l(\cdot,\xi)|| \leq E_r \cdot G_0 = L_0$.

$$
||l(\cdot,\xi)|| \leq E_{\mathbf{y}} \cdot G_0 = L_0. \tag{33}
$$

E₇ ||g||,

ne norm in $L_2(-\pi, \pi)$.
 $[1 + \lambda_0] (\cos 2\gamma)^{-1}$,
 $-\tan \gamma$]
 $\cdot [-\lambda_1, \lambda_2]$.

mption B it is
 $+\ \overline{N}_0 \equiv G_0 < \infty$,

norm of the function N
 $E_\gamma \cdot G_0 \equiv L_0$.

take $p = \varrho = 2$ and q

i $R = |k| + \sqrt{2\pi} L_0$ or Therefore, we can take $p = \varrho = 2$ and $q = 2$ in the definition (22) of the subset **ft** Therefore, we can take $p = \rho = 2$ and $q = 2$ in the definition (22) of the subset \Re with $R_0 = L_0$ and $R = |k| + \sqrt{2\pi} L_0$ or $R = |c| + 2\sqrt{2\pi} L_0$ in the case of (2a) or \Re (2b), respectively. Then the operator P again maps $\hat{\mathfrak{X}}$ into itself. Therefore, we can take $p = \rho = 2$ and $q = 2$ in the definition (22) of the subset \Re with $R_0 = L_0$ and $R = |k| + \sqrt{2\pi} L_0$ or $R = |c| + 2\sqrt{2\pi} L_0$ in the case of (2a) or (2b), respectively. Then the operator P again ma

Besides, the operator $P: \mathbb{R} \to \mathbb{R}$ is *continuous* in the norm of $C[-\pi, \pi]$. Namely, let $\xi_n \in \mathbb{R}$, $n = 1, 2, \ldots$ be a uniformly convergent sequence with the limit function $\in \mathbb{R}$. We have to show that the functions $P\bar{\xi}_n$ defined by (12), (13) converge uniformly to $P\xi_0$. For this it is sufficient to prove that the functions $l(s, \xi_n)$ converge weakly in $L_2(-\pi, \pi)$ to $l(s, \xi_0)$. Moreover, in view of the uniform boundedness of the L_2 norms of $l(s, \xi_n)$ by (33) it suffices to show the weak convergence of the functions $l(s, \xi_n)$ to $l(s, \xi_0)$ in the space $L_1(-\pi, \pi)$ only. **Example 1.** Then the eperator $P: \mathbb{R} \to \mathbb{R}$ is continuous in the norm of $C[-\pi, \pi]$. Nam
Bet $\xi_n \in \mathbb{R}$, $n = 1, 2, ...$ be a uniformly convergent sequence with the limit func
 $\xi_0 \in \mathbb{R}$. We have to show that the

Now, obviously, the continuous functions $\lambda_n(s) = \lambda(s, \xi_n)$, $\mu_n(s) = \mu(s, \xi_n)$, $\alpha_n(s)$
= $\alpha(s, \xi_n)$, $\beta_n(s) = \beta(s, \xi_n)$ converge uniformly to $\lambda_0(s) = \lambda(s, \xi_0)$, $\mu_0(s) = \mu(s, \xi_0)$, and so on. Further, the functions $N(s, \xi_n)$ converge strongly in $L_2(-\pi, \pi)$ to $N(s, \xi_0)$ and therefore also the functions $g_n(s) = g(s, \xi_n)$ to $g(s, \xi_0)$. Moreover, due to the assumption (28) and the Zygmund lemma it can be shown like in the corresponding proof in [14] that the functions $\exp\left[\pm H(\mu_n)\right]$ converge strongly to $\exp\left[\pm H(\mu_0)\right]$ in $L_2(-\pi, \pi)$ and also in $L_{2+\epsilon}(-\pi, \pi)$ for sufficiently small positive ϵ . weakly in $L_2(-\pi, \pi)$ to $l(s, \xi_0)$. More
 L_2 norms of $l(s, \xi_n)$ by (33) it suffice
 $l(s, \xi_n)$ to $l(s, \xi_0)$ in the space $L_1(-\pi)$

Now, obviously, the continuous
 $= \alpha(s, \xi_n)$, $\beta_n(s) = \beta(s, \xi_n)$ converg

and so on. Fu *e* must be Zygmund lemma it
 i the functions $\exp\left[\pm H(\text{also in } L_{2+\epsilon}(-\pi, \pi) \text{ for sub-}(\text{constants } D[\xi_n])\right]$ defined b notions
 $e^{-H\mu_n}H[h_n e^{H\mu_n}], \qquad h_n =$

Therefore, the constants $D[\xi_n]$ defined by (15) converge to $D[\xi_0]$. It remains to

$$
\Lambda[\xi_n] \equiv e^{-H\mu_n} H[h_n e^{H\mu_n}], \qquad h_n = \beta_n g_n,
$$

 (31)

converge weakly to $A[\xi_0]$ in $L_1(-\pi, \pi)$. As is shown in Appendix 2 the functions $A[\xi_n]$ have uniformly bounded L_2 norms under the assumption (28) like the functions $l(s, \xi_n)$ do by (33). Besides, the functions h_n converge strongly in $L_2(-\pi, \pi)$ to $h_0 = \beta(s, \xi_0) g(s, \xi_0)$. 392 L. v. WOLFERSDORF
 converge weakly to $\Lambda[\xi_0]$ in $L_1(-\pi, \pi)$. A
 $\Lambda[\xi_n]$ have uniformly bounded L_2 norms un
 $l(s, \xi_n)$ do by (33). Besides, the function:
 $h_0 = \beta(s, \xi_0) g(s, \xi_0)$.

Then, for any $\chi \in L_\infty(-\pi,$

\n Inverge weakly to
$$
\Lambda[\xi_0]
$$
 in $L_1(-\pi,\pi)$. As $[\xi_n]$ have uniformly bounded L_2 norms and $[\xi_n]$ do by (33). Besides, the functions $= \beta(s, \xi_0) g(s, \xi_0)$. Then, for any $\chi \in L_{\infty}(-\pi, \pi)$ we have\n
$$
|(\Lambda[\xi_n] - \Lambda[\xi_0], \chi)|
$$
\n
$$
\leq |(\Lambda[\xi_n] - e^{-H\mu_0}H[h_n e^{H\mu_n}], \chi)|
$$
\n
$$
+ |(e^{-H\mu_0}H[h_n e^{H\mu_n}] - \Lambda[\xi_0], \chi)|
$$
\n
$$
= S_1 + S_2,
$$
\n

\n\n here (·, ·) denotes the usual scalar product. It is\n
$$
S_1 = |(1 - e^{H\mu_n - H\mu_0}, \Lambda[\xi_n] \chi)|
$$
\n
$$
\leq ||\chi||_{\infty} ||\Lambda[\xi_n]||_2 ||1 - e^{H\mu_n - H\mu_0}||_2
$$
\n
$$
\leq C_1 \text{ and } C_2 \text{ and } C_3 \text{ are } C_4 \text{ and } C_5 \text{ are } C_6 \text{ and } C_7 \text{ are } C_7 \text{ and } C_8 \text{ are } C_7 \text{ and } C_8 \text{ are } C_8 \text{ and } C_9 \text{ are } C_9 \text{ and } C_9 \text{ are }
$$

where (\cdot, \cdot) denotes the usual scalar product.
It is

 $S_1 = |(1 - e^{H\mu_n - H\mu_0}, A[\xi_n] \gamma)|$ Then, for
 $|\langle A \rangle|$
 \leq
 \equiv
 \equiv
 \equiv
 \equiv
 \equiv
 \equiv
 S_1
 \equiv $\frac{1}{2}$
 $||\chi||_{\infty} ||A[\xi_n]||_2 ||1 - e^{H\mu_n - H\mu_0}||_2$
 Onnet Here *Hunter Hunder Hunder Hunder* $|\langle A[\xi_n] - A[\xi_0], \chi \rangle|$
 $\leq |\langle A[\xi_n] - e^{-H\mu_0} H[h_n e^{H\mu_n}], \chi \rangle|$
 $+ |(e^{-H\mu_0} H[h_n e^{H\mu_n}] - A[\xi_0],$
 $\equiv S_1 + S_2$,

here (\cdot, \cdot) denotes the usual scalar produ

It is
 $S_1 = |(1 - e^{H\mu_n - H\mu_0}, A[\xi_n] \chi)|$
 $\leq ||\chi||_{\infty} ||A[\xi_n]||_2 ||1 - e^{H$ $[\xi_n]$ $- e^{-H\mu_0}H[\n[*\epsilon_n*] - e^{-H\mu_0}H[\nh_0 e^{H\mu} + S_2,\n]$
 Const the usual
 $|(1 - e^{H\mu_n - H\mu_0},\n||\chi||_{\infty} ||A[\xi_n]||_2 ||1$
 Const $||e^{-H\mu_0}||_q$
 $= 2 + \varepsilon > 2$
 Const $||e^{-H\mu_0}||_q$
 $= 2 + \varepsilon > 2$
 Conserverserversions

 $[h_n e^{H\mu_n}], \chi]$

" $] - A[\xi_0], \chi]$

scalar product.
 $A[\xi_n], \chi]$
 $- e^{H\mu_n - H\nu_0}||_2$
 $\|e^{H\mu_0} - e^{H\mu_n}\|_p$.
 $\|e^{H\mu_0} - e^{H\mu_n}\|_p$.

and $q' = (1/2 - 1/p')^{-1} < \infty$. Taking ε sufficient
 $\|$ converge strongly to $\exp[H\mu_0]$ with some $p' = 2 + \varepsilon > 2$ and $q' = (1/2 - 1/p')^{-1} < \infty$. Taking ε sufficiently small, the functions $\exp[H_{\mu_n}]$ converge strongly to $\exp[H_{\mu_0}]$ in $\tilde{L}_{n'}(-\pi, \pi)$. Also the function $\exp[-H\mu_0]$ is summabel to any power $q' < \infty$ (cf. [6: Kap. IX, § 5]).
Therefore, $S_1 \to 0$ as $n \to \infty$. $+ |(e^{-H\mu_1}H[h_n e^{H\mu_n}] - A[\xi_0], \chi)|$
 $\equiv S_1 + S_2,$

where (., .) denotes the usual scalar product.

It is
 $S_1 = |(1 - e^{H\mu_n - H\mu_1}, A[\xi_n], \chi)|$
 $\leq ||\chi||_{\infty} ||A[\xi_n]||_2 ||1 - e^{H\mu_n - H\mu_1}||_2$
 \leq *Const* $||e^{-H\mu_1}(e^{H\mu_0} - e^{H\mu_n$ $\equiv S_1 + S_2,$

here (\cdot, \cdot) denotes th

It is
 $S_1 = |(1 - \epsilon)$
 $\leq |x||_{\infty}$ $||x||_{\infty}$
 \leq Const $||x||_{\infty}$
 \leq Const $||x||_{\infty}$

th some $p' = 2 +$

aall, the functions

e function $\exp[-L]$

erefore, $S_1 \to 0$ as

Fin $\begin{aligned} |(1 - e^{H\mu_n - H\mu_0}, A[\xi_n]]\chi)| \ |x||_{\infty} ||A[\xi_n]||_2 ||1 - e^{H\mu_n - H\mu_0}||_2 \ \text{Const} ||e^{-H\mu_0} (e^{H\mu_0} - e^{H\mu_n})||_2 \ \text{Const} ||e^{-H\mu_0} ||e^{H\mu_0} - e^{H\mu_n}||_p \cdot \ = 2 + \varepsilon > 2 \ \text{and} \ q' = (1/2 - 1/p')^{-1} < \infty \ \text{rips} \ \text{exp} \ [H\mu_n] \ \text{converge strongly to } \ \text{exp} \ [1 - H\mu$

S
S

$$
S_2 = |(H[h_n e^{H\mu_n}] - H[h_0 e^{H\mu_0}], e^{-H\mu_0} \chi)|
$$

\n
$$
\leq ||e^{-H\mu_0} \chi||_{q'} A_r ||h_n e^{H\mu_n} - h_0 e^{H\mu_0}||_{r'},
$$

where $r' = (1 - 1/q')^{-1} = (1/2 + 1/p')^{-1} > 1$ and $A_{r'}$ is the norm of the Hilbert transformation in $\bar{L}_r(-\pi, \pi)$. But the functions $h_n \exp[H_{\mu_n}]$ converge strongly to *h*₀ exp $[H\mu_0]$ in $L_r(-\pi,\pi)$ since exp $[H\mu_n]$ converges strongly to exp $[H\mu_0]$ in $L_r(-\pi,\pi)$ since exp $[H\mu_n]$ converges strongly to exp $[H\mu_0]$ in if in $L_r(-\pi, \pi)$ since $\exp [H\mu_n]$ converges strongly to $\exp [H\mu_0]$ in and h_n converges strongly to h_0 in $L_2(-\pi, \pi)$. Hence, also $S_2 \rightarrow 0$ as $n \to \infty$.
This proves the continuity of the operator $P: \mathbb{R} \to \mathbb{R}$ in $C[-\pi, \pi]$. $\leq ||e^{-H\mu} \chi||_q$, $A_r ||h_n e^{H\mu_n} - h_0 e^{H\mu_0}||_r$,

here $r' = (1 - 1/q')^{-1} = (1/2 + 1/p')^{-1} > 1$ and $A_{r'}$ is the norm

ansformation in $L_r(-\pi, \pi)$. But the functions $h_n \exp[H\mu_n]$ converges
 $\exp[H\mu_0]$ in $L_{r'}(-\pi, \pi)$ since $\exp[H\$ small, the functions $\exp [H\mu_n]$ converge strongly to exp

the function $\exp [-H\mu_0]$ is summabel to any power q' <

Therefore, $S_1 \rightarrow 0$ as $n \rightarrow \infty$.

Finally,
 $S_2 = |\langle H[h_n e^{H\mu_n}] - H[h_0 e^{H\mu_0}], e^{-H\mu_0} \chi \rangle|$
 $\leq ||e^{-H\mu_1} \chi||$ *M_W*] in $L_r(-\pi, \pi)$ since exp [*H<sub>W₁*] converges strongly to exp [*H<sub>W₁*] in $L_r(-\pi, \pi)$ since exp [*H_{W_n*]} converges strongly to exp [*H<sub>W₁*] in $L_r(-\pi, \pi)$ and h_n converges strongly to h_0 in $L_2(-\pi, \pi)$

Applying again the Schauder fixed point theorem to the equation (12) in \Re , we

Theorem 2: *Under Assumptions A and B with* $\rho = 2$ *and the inequality (28) instead of* (23) *the integro-differential equation* (1) *possesses a solution* $\varphi \in W_2^1(-\pi, \pi)$.

Remark 1: Assumption (i) of B with (28) is [fulfilled for any nonnegative (nonpositive) bounded function $M(s, \varphi)$.

Remark 2: The remark 2 to Theorem 1 also holds for Theorem 2. If further $M(s, \varphi)$ is a nonnegative function satisfying the inequality

$$
M(s,\varphi)\leq M_1+M_2R^{\omega},\qquad 0<\omega<1,\tag{34}
$$

for $s \in [-\pi, \pi], |\varphi| \leq R$ and any positive *R*, we have the same energy insteads of (23) the integro-differential equation (1) possesses a solution $\varphi \in W_2^1(-\pi, \pi)$.

Remark 1: Assumption (i) of B with (28) is fulfilled for $s \in [-\pi, \pi]$, $|\varphi| \leq R$ and any positive R, we have the same estimation for the corresponding quantities λ_0 and $\tan 2\gamma$ in (30) with respect to $|\varphi| \leq R$. Then the constant E_y grows not stronger than the function R^{2w} as R goes to infinity. Therefore, a solution to equation (1) exists if $\omega < 1/2$ in case the assumption (11) for $N(s, \varphi)$ holds uniformly with respect to $\varphi \in \mathbb{R}$ or if $\delta + 2\omega < 1$ in case the function $N(s, \varphi)$ satisfies (27).

Integro-differential Equations with Hilbert Kernel 393
 M(s, φ) $\geq M_0 > 0$ or $M(s, \varphi) \leq -M_0 < 0$ (35)
 $-\pi, \pi$] and $\varphi \in \mathbb{R}$. The function $v = l(s, \xi)$ is the solution to the equation.
 $v(s) + H[\lambda(\cdot, \xi) v](s) = g(s, \xi$ Integro-differential Equations with Hilbert
 n, we deal with the case of a strictly positive or strictly nearly
 $M(s, \varphi) \ge M_0 > 0$ or $M(s, \varphi) \le -M_0 < 0$
 $-\pi, \pi$] and $\varphi \in \mathbb{R}$. The function $v = l(s, \xi)$ is the soluti Finally, we deal with the case of a strictly positive or strictly negative function $M(x, y)$. $M(s, \varphi)$:

$$
M(s,\varphi)\geq M_0>0 \quad \text{or} \quad M(s,\varphi)\leq -M_0<0 \qquad (35)
$$

for $s \in [-\pi, \pi]$ and $\varphi \in \mathbb{R}$. The function $v = l(s, \xi)$ is the solution to the equation.

$$
v(s) + H[\lambda(\cdot,\xi) v](s) = g(s,\xi), \qquad (36)
$$

where $\lambda(s, \xi) = M(s, \xi(s))$ and the function $g(s, \xi)$ given by (16) fulfils the relation

$$
\int g(s,\,\xi)\,ds=0
$$

in view of the assumption (3). Hence the inequality $(A.33)$ of the appendix yields the estimation

7, 7, 1 and $\varphi \in \mathbf{R}$. The function $v = l(s, \xi)$ is the solution to the equation.
 $I(s) + H[\lambda(\cdot, \xi) v](s) = g(s, \xi),$ (36)
 $I(s) = M(s, \xi(s))$ and the function $g(s, \xi)$ given by (16) fulfils the relation
 $\int_{-\pi}^{\pi} g(s, \xi) ds = 0$
 \there Further, in case of (35) the assumption (28) is fulfilled. Thus the above continuity proof for the mapping P on $\hat{\mathfrak{X}}$ remains valid.¹ This implies

Theorem 3: Let $M(s, \varphi)$ be a continuous function on $[-\pi, \pi] \times \mathbf{R}$ which is 2π *periodic in s. and satisfies an inequality of the form (35), and let N(s,* φ *) be a Carathéodory function on* $[-\pi, \pi] \times \mathbf{R}$ *satisfying an inequality of the form (27) with* $0 < \delta < 1$, *where* $\rho = 2$. Then the integro-differential equation (1) has a solution $\varphi \in W_2^1(-\pi, \pi)$ *for any* $F \in L_2(-\pi, \pi)$ *satisfying the, relation* (3). **•** estimation $|\mathcal{U}(\cdot, \xi)| \leq (1/M_0) ||g||$.
 • (37)
 • *• Further, in case of (35) the assumption (28) is fulfilled. Thus the above continuity
 • Proof for the mapping <i>P* on \Re remains valid ¹. This implie proot for the mapping P on \Re remains valid.¹ This implies

Theorem 3: Let $M(s, \varphi)$ be a continuous function on $[-\pi, \pi] \times \mathbf{R}$ which
 periodic in s and satisfies an inequality of the form (35), and let $N(s, \varphi)$ *Now to Reflere the integro-differential equation* (1) has a solution $\varphi \in W_2^{-1}(-\pi, \pi)$
 $P \in L_2(-\pi, \pi)$ satisfying the relation (3).
 $R: A$ solution to equation (1) also exists if the function $M(s, \varphi)$ satisfies an

Remark: A solution to equation (1) also exists if the function $M(s, \varphi)$ satisfies an inequality of the form for any $F \in L_2(-\pi, \pi)$ satisfying the r

Remark: A solution to equation (

inequality of the form
 $M(s, \varphi) \ge 1/[M_1 + M_2 R^{\omega_o}]$,

for $s \in [-\pi, \pi]$, $|\varphi| \le R$ instead of (3.

4. Special case. Uniqueness theorem

A. We cons

$$
M(s,\,\varphi) \ge 1/[M_1 + M_2 R^{\omega_0}], \qquad 0 < \omega_0 < 1,\tag{38}
$$

inequality of the form
 $M(s, \varphi) \geq 1/[M_1 + M_2 R^{\omega} \cdot], \qquad 0 < \omega_0 < 1$,

for $s \in [-\pi, \pi]$, $|\varphi| \leq R$ instead of (35), where $M_1, M_2 > 0$ and $\delta + \omega_0 < 1$.

A. We consider the following *particular case* of the equation (1):

$$
N(s, \varphi) = dM(s, \varphi), \quad d \in \mathbf{R}; \quad F(s) \equiv 0. \tag{39}
$$

$$
g(s,\xi) = dH[M(\cdot,\xi)](s) = dH[\lambda](s),\tag{4}
$$

S .,

g(*s, e*) = $1/[M_1 + M_2R^{\omega_1}]$, $0 < \omega_0 < 1$, (38)
 g(s, e) = $1/[M_1 + M_2R^{\omega_1}]$, $0 < \omega_0 < 1$, (38)
 $-\pi$, π , $|\varphi| \leq R$ instead of (35), where M_1 , $M_2 > 0$ and $\delta + \omega_0 < 1$.
 age. **Uniqueness theorem**
 g(and according to formula $(A.8_1)$ of the appendix the kernel $l(s, \xi)$ of the integral equation (12) has the simpler form
 $l(s, \xi) = d - d(\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)},$ (41)

where $M(s, \varphi) \ge 1/[M_1 + M_2 R^{\omega_s}], \qquad 0 < \omega_0 < 1,$

for $s \in [-\pi, \pi], |\varphi| \le R$ instead of (35), where $M_1, M_2 >$

4. Special case. Uniqueness theorem

A. We consider the following particular case of the equat
 $N(s, \varphi) = dM(s, \varphi), \quad d \in \$ *l*(*s,* π *)* $|\varphi| \geq R$ *instead of (30), where* $M_1, M_2 > 0$ *and* $\theta + \omega_0 < 1$ *.
 l case. Uniqueness theorem

onsider the following particular case of the equation (1):
* $N(s, \varphi) = dM(s, \varphi), \quad d \in \mathbb{R}; \quad F(s) \equiv 0.$ *(39)

see
* dix the kernel $l(s, \xi)$
of the L_2 norms of the

$$
l(s,\xi) = d - d(\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}
$$

consider the following particular c
\n
$$
N(s, \varphi) = dM(s, \varphi), \quad d \in \mathbb{R};
$$
\ncase
\n
$$
g(s, \xi) = dH[M(\cdot, \xi)](s) = dH[\lambda]
$$
\ncording to formula (A.8₁) of the
\n
$$
I(2)
$$
 has the simpler form
\n
$$
l(s, \xi) = d - d(\cos \overline{\mu})^{-1} \alpha(s) e^{-\mu}
$$
\n
$$
\overline{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s, \xi) ds.
$$
\n(37) and (14) also the uniform bound
\nnowec continuity proof follows.

¹) From (37) and (14) also the uniform boundedness of the L_2 norms of the functions $\Lambda[\xi_n]$ in the above continuity proof follows.

We only assume that the continuous function $M(s, \varphi)$ is bounded such that the oscillation of the function $\mu = \mu(s, \xi)$ is not greater than $2\gamma = \mu_2 - \mu_1 < \pi$ for any $\zeta \in C[-\pi, \pi]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, and $\lambda_1 = \inf M(s, \varphi)$, $\lambda_2 = \sup M(s, \varphi)$ \times (s, φ), the infimum and supremum are taken over $s \in [-\pi, \pi]$, $\varphi \in \mathbb{R}$. Applying Zygmund's lemma to (41), we obtain the estimation I. v. WOLFERSDORF

Illustration $M(s, \varphi)$ is bounded in the function $\mu = \mu(s, \xi)$ is not greater than $2\gamma = \mu_2 - \pi, \pi$], where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, and $\lambda_1 = \inf M(s, \varphi)$, λ
 λ , π], where $\mu_k = \arctan \lambda_k$, $k = 1,$

$$
||l(\cdot,\xi)||_p \le (2\pi)^{1/p} |d| \left[1 + \sqrt{1 + \lambda_0^2} \left(\cos p\gamma \right)^{-1/p} \right] \tag{42}
$$

for $1 < p < \pi/2\gamma$, where $\lambda_0 = \max[-\lambda_1, \lambda_2]$. Also the continuity of the corresponding operator *P* in the space $C[-\pi, \pi]$ can be shown like in [14] or, easier, like in the corre-
sponding proof for Villat's integral equation in the theory of plane cavity flows [3:
Chap. VII]. I.e., a solution to equation Chap. VII]. I.e., a solution to equation (i) exists without an additional assumption on the oscillation of $M(s, \varphi)$ of the form (23). Also the continuity of the corresponding

vn like in [14] or, easier, like in the corre-

in the theory of plane cavity flows [3:

) exists without an additional assump-

1 (23).
 ntinuous function on $[-\pi, \pi] \times \mathbf{R}$

sponding proof for Villat's integral equation in the theory of plane cavity flows [3: *Chap. VII].* I.e., a solution to equation (1) exists without an additional assumption on the oscillation of $M(s, \varphi)$ of the form (23) Theorem 4: Let $M(s, \varphi)$ be a bounded continuous function on $[-\pi, \pi] \times \mathbf{R}$ which *is* 2π periodic in s. Then the integro-differential equation (1) with (39) possesses a solu*tion* $\varphi \in W_p^{-1}(-\pi, \pi)$, $1 < p < \pi/[\mu_2 - \mu_1]$, where $\mu_k = \arctan \lambda_k$, $k = 1, 2$, and $\lambda_1 = \inf M$, $\lambda_2 = \sup M$. *m*(*s, q) em a* solution to equation (1) exists without an additional assump-
 mem 4: Let $M(s, \varphi)$ *be a bounded continuous function on* $[-\pi, \pi] \times \mathbf{R}$ *which*
 *wh*¹(*x- m, Then the integro-differential eq*

Remark: As in Remark *3* to Theorem 1, from *(41)* it follows that indeed the solution $\varphi \in W_p^1(-\pi, \pi)$ for any finite $p > 1$. *n* **h**: As in Remark 3 to Theorem 1, $W_p^{-1}(-\pi, \pi)$ for any finite $p > 1$.
 of the integral equation (4), where
 $n(s, \varphi) \doteq m_s(s, \varphi) + dm_\varphi(s, \varphi)$,
 i is fulfilled with $d = -1$ if $n(s, \varphi)$.

- In case of the integral equation (4), where *(6)* holds, the assumptions *(39)* write

$$
n(s,\varphi) \doteq m_s(s,\varphi) + dm_\varphi(s,\varphi), \qquad f(s) = Const. \qquad (43)
$$

E.g., this is fulfilled with $d = -1$ if $n(s, \varphi) = f(s) \equiv 0$ and

$$
m(s,\varphi)=Q(s+\varphi),\qquad \qquad (44)
$$

where Q is a 2π periodic continuously differentiable function. This case embraces the well-known *Theodorsen iategral equation of conformal mapping* (cf. *[5:* Kap. II]) for which $\vec{B} = \inf_{\mathbf{M}} \vec{M}, \lambda_2 = \sup \vec{M}.$

Remark: As in Remark 3 to Ti

n $\varphi \in W_p^1(-\pi, \pi)$ for any finite

In case of the integral equation
 $n(s, \varphi) = m_s(s, \varphi) + dm_\varphi$

g., this is fulfilled with $d = -1$
 $m(s, \varphi) = Q(s + \varphi),$

here In case of the integral equation (4), where (6) holds, the assumptions (39) write
 $n(s, \varphi) = m_s(s, \varphi) + dm_{\varphi}(s, \varphi)$, $f(s) = Const$. (43)
 E.g., this is fulfilled with $d = -1$ if $n(s, \varphi) = f(s) \equiv 0$ and
 $m(s, \varphi) = Q(s + \varphi)$, (44)

w $n(s, \varphi) = m_s(s, \varphi) + dm_\varphi(s, \varphi),$ $f(s) = Const.$
 E.g., this is fulfilled with $d = -1$ if $n(s, \varphi) = f(s) \equiv 0$ and
 $m(s, \varphi) = Q(s + \varphi),$

where Q is a 2π periodic continuously differentiable function. This case

well-known *Theodor* E.g., this is fulfilled with $d = -1$ if $n(s, \varphi) = f(s) \equiv 0$ and
 $m(s, \varphi) = Q(s + \varphi)$, (44)

where Q is a 2*x* periodic continuously differentiable function. This case embraces the

well-known *Theodorsen integral equation of c*

$$
Q(s) = \ln \varrho(s),
$$

where $\rho = \varrho(s)$ is the representation in polar-coordinates of the starlike Jordan curve *J* to be mapped onto the unit disk. More precisely, for the Theodorsen equation

$$
l(s,\xi) = \cos \mu \cdot e^{-(H\mu)(s)} - 1, \tag{46}
$$

/

$$
\mu(s) = \arctan \frac{\varrho'(s)}{\varrho(s)}
$$

^{$\varrho(s)$}

is the angle between the outer normal to *J* and the radius vector in the point $(s, \varrho(s))$

of *J*. The operator equation (cf. [5: Kap. II, § 4.4]).

Theorem 4 thus yields the existence of a solution to the Theo of *J.* The operator equation *(12)* with *(46)* is an integrated form of Friberg's integrodifferential equation $(cf. [5: Kap. II, § 4.4]).$ is the angle between the outer normal to *J* and the radius vector in the point $(s, \varrho(s))$
of *J*. The operator equation (cf. [5: Kap. II, § 4.4]).
Theorem 4 thus yields the existence of a solution to the Theodorsen equati $\mu(s) = \arctan \frac{q'(s)}{q(s)}$ (47)

is the angle between the outer normal to *J* and the radius vector in the point $(s, \varrho(s))$

of *J*. The operator equation (12) with (46) is an integrated form of Friberg's integro-

different

Theorem 4 thus yields the existence of a solution to the Theodorsen equation for

B. Finally; we state some simple *uniqueness theorems* for continuous solutions of the integral equation (4). In the following we assume that $m(s, \varphi)$, $n(s, \varphi)$ are continuous functions which possess continuous derivatives $m_{\varphi}(s, \varphi)$ and $n_{\varphi}(s, \varphi)$.

Let $\varphi_k(s)$, $k = 1, 2$, be two continuous solutions of (4). Then the difference function $\Phi = \varphi_1 - \varphi_2$ satisfies the equation

$$
\Phi(s) + H[A(\cdot) \Phi](s) + K[B(\cdot) \Phi](s) = 0
$$

(45)

the contin¹ with the continuous functions

Integro-differential Equations with Hilbert Kernel
\nthe continuous functions
\n
$$
A(s) = \int_0^1 m_\varphi(s, \varphi_2(s) + t[\varphi_1(s) - \varphi_2(s)]) dt,
$$
\n
$$
B(s) = \int_0^1 n_\varphi(s, \varphi_2(s) + t[\varphi_1(s) - \varphi_2(s)]) dt.
$$
\nWe consider three cases.
\nIn case I: $n(s, \varphi) \equiv 0$ it is also $B(s) \equiv 0$ and the equation
\n
$$
\Phi(s) + H[A(\cdot) \Phi](s) = 0
$$
\nso only the trivial continuous solution $\Phi(s) \equiv 0$ as it follows from Appendix 1.
\nIn case II: $n(s, \varphi) = n_0\varphi$, $n_0 \in \mathbb{R}$, it is $B(s) = n_0$ and we have the equation
\n
$$
\Phi(s) + H[A(\cdot) \Phi](s) + n_0 K[\Phi](s) = 0.
$$
\n(50)
\ne multiply (50) by $H[\Phi]$ and integrate it over $(-\pi, \pi)$. It is

$$
B(s) = \int_{0}^{t} n_{\varphi}(s, \varphi_{2}(s) + t[\varphi_{1}(s) - \varphi_{2}(s)]) dt.
$$

We consider three cases.
\nIn case I:
$$
n(s, \varphi) \equiv 0
$$
 it is also $B(s) \equiv 0$ and the equation
\n
$$
\Phi(s) + H[A(\cdot) \Phi](s) = 0
$$
\nhas only the trivial continuous solution $\Phi(s) \equiv 0$ as it follows from Appendix 1.

In *case* II: $n(s, \varphi) = n_0 \varphi$, $n_0 \in \mathbb{R}$, it is $B(s) = n_0$ and we have the equation

$$
\Phi(s) + H[A(\cdot) \Phi](s) + n_0 K[\Phi](s) = 0.
$$

We multiply (50) by $H[\Phi]$ and integrate it over $(-\pi, \pi)$. It is

$$
\Phi(s) + H[A(\cdot) \Phi](s) = 0
$$
\nthe trivial continuous solution $\Phi(s) \equiv 0$ as it follows from Appendix 1.
\nwe II: $n(s, \varphi) = n_0 \varphi, n_0 \in \mathbb{R}$, it is $B(s) = n_0$ and we have the equation
\n
$$
\Phi(s) + H[A(\cdot) \Phi](s) + n_0 K[\Phi](s) = 0.
$$
\n
$$
\text{tiny } \int \Phi H[\Phi] \text{ and integrate it over } (-\pi, \pi). \text{ It is}
$$
\n
$$
\int_{-\pi}^{\pi} \Phi H[\Phi] \, ds = \int_{-\pi}^{\pi} K[\Phi] H[\Phi] \, ds = 0
$$
\n
$$
\text{hat } H[\Phi] = -(d/ds) K[\Phi]) \text{ and}
$$
\n
$$
\int_{-\pi}^{\pi} H[\Phi] \cdot H[A\Phi] \, ds = \int_{-\pi}^{\pi} \Phi \cdot A\Phi \, ds
$$
\n
$$
\int_{-\pi}^{\pi} H[\Phi] \cdot H[A\Phi] \, ds = 0 \text{ following from (50) by integrating it of}
$$

(recall that $H[\Phi] = -(d/ds) K[\Phi]$) and

$$
\int\limits_n^n H[\Phi] \cdot H[A\Phi] \, ds = \int\limits_{-n}^n \Phi \cdot A\Phi \, ds
$$

on account of the relation $\int \Phi(s) ds = 0$ following from (50) by integrating it over $(-\pi, \pi)$. Therefore, we obtain $\int A(s) \Phi^2(s) ds = 0$ which implies $\Phi(s) \equiv 0$ if $A(s)$ has constant sign on $(-\pi, \pi)$. I.e., the solution to the equation (4) is uniquely deter-(recall that $H[\Phi] = -\frac{\pi}{2} K[\Phi] H[\Phi] ds$
 $\int_{-\pi}^{\pi} \Phi H[\Phi] ds = \int_{-\pi}^{\pi} K[\Phi] H[\Phi] ds$

(recall that $H[\Phi] = -\frac{d}{ds} K[\Phi]$) and
 $\int_{-\pi}^{\pi} H[\Phi] \cdot H[A\Phi] ds = \int_{-\pi}^{\pi} \Phi \cdot A\Phi d\Phi$

on account of the relation $\int_{-\pi}^{\pi} \Phi(s) ds = 0$ f
 $(-\pi$ (recall that $H[\Phi] = -(d/ds) K[\Phi]$) and
 $\cdot \cdot$
 $\int_{-\pi}^{\pi} H[\Phi] \cdot H[A\Phi] ds = \int_{-\pi}^{\pi} \Phi \cdot A\Phi ds$

on account of the relation $\int_{-\pi}^{\pi} \Phi(s) ds = 0$ following from (50) by integrating it over
 $(-\pi, \pi)$. Therefore, we obtain $\int_{-\pi}^$ on account of the relation $\int_{-\pi}^{\pi} \Phi(s) ds = 0$ following from (50
 $(-\pi, \pi)$. Therefore, we obtain $\int_{-\pi}^{\pi} A(s) \Phi^2(s) ds = 0$ which is

has constant sign on $(-\pi, \pi)$. I.e., the solution to the equation

mined if either
 $(-\pi, \pi)$. Therefore, we obtain $\int_{-\pi}^{\pi} A(s) \Phi^2(s) ds = 0$ which implies $\Phi(s) \equiv 0$ if
has constant sign on $(-\pi, \pi)$. I.e., the solution to the equation (4) is uniquely d
mined if either
 $m_{\varphi}(s, \varphi) > 0$ or $m_{\varphi}(s, \varphi)$ $\int_{-\pi}^{\pi} H[\Phi] \cdot H[A\Phi] ds = \int_{-\pi}^{\pi} \Phi \cdot A\Phi ds$

nt of the relation $\int_{-\pi}^{\pi} \Phi(s) ds = 0$ following from (50) by integrating it over

Therefore, we obtain $\int_{-\pi}^{\pi} A(s) \Phi^2(s) ds = 0$ which implies $\Phi(s) \equiv 0$ if $A(s)$

cant sign on account of the relation $\int_{-\pi}^{\pi} \Phi(s) ds = 0$ following from (50) by i
 $(-\pi, \pi)$. Therefore, we obtain $\int_{-\pi}^{\pi} A(s) \Phi^2(s) ds = 0$ which implies

has constant sign on $(-\pi, \pi)$. I.e., the solution to the equation (4)

mi

$$
\Phi(s) + H[A(\cdot)\Phi](s) + \nu K[A(\cdot)\Phi](s) = 0 \qquad (52)
$$

by $A\Phi$ and integrate it over $(-\pi, \pi)$. This gives

$$
\int\limits_{-\pi}^{\pi} A(s) \, \Phi^2(s) \, ds + \nu \int\limits_{-\pi}^{\pi} A \, \dot{\Phi} \cdot K[A\Phi] \, ds = 0 \, .
$$

Now $\int \chi K[\chi] ds \leq 0$ for any continuous function χ and therefore $\nu \int A(s) \Phi^2(s) ds$ ≥ 0 . This implies $\Phi(s) \equiv 0$ if $vA(s)$ has negative sign on $(-\pi, \pi)$. I.e., the solution to the equation (4) is uniquely determined if by $A\Phi$ and integrate it over $(-\pi, \pi)$. This gives
 $\int_{-\pi}^{\pi} A(s) \Phi^{2}(s) ds + v \int_{-\pi}^{\pi} A\Phi \cdot K[A\Phi] ds = 0$.
 $\int_{-\pi}^{\pi} \chi K[\chi] ds \leq 0$ for any continuous function $\chi \geq 0$. This implies $\Phi(s) \equiv 0$ if $\nu A(s)$ has negative sig 111: $n(s, \varphi) = \nu m(s, \varphi), \nu \in \mathbb{R}$, we multiply the corresponding equation
 $\Phi(s) + H[A(\cdot) \Phi](s) + \nu K[A(\cdot) \Phi](s) = 0$ (52)

nd integrate it over $(-\pi, \pi)$. This gives
 $\int_A A(s) \Phi^2(s) ds + \nu \int_A \Phi \cdot K[A\Phi] ds = 0$.
 π
 $K[\chi] ds \leq 0$ for a for almost all $s \in [-\pi, \pi]$ and all $\varphi \in \mathbb{R}$.
 $\phi(s) + H[A(\cdot) \Phi](s) + \nu K[A(\cdot) \Phi](s) = 0$ (52)

by $A\Phi$ and integrate it over $(-\pi, \pi)$. This gives
 $\int_{-\pi}^{\pi} A(s) \Phi^2(s) ds + \nu \int_{-\pi}^{\pi} A\Phi \cdot K[A\Phi] ds = 0$.

Now $\int_{-\pi}^{\pi} \chi K[\chi] ds \$

$$
\text{sign } \nu \cdot m_{\omega}(s, \varphi) < 0
$$

12

- (50)

Theorem 5: The integral equation (4) with continuous coefficient $m(s, \varphi)$ possessing *a continuous derivative m_v(s,* φ *) has at most one continuous solution in the cases I--III if in case II and III, respectively, the condition (51) and (53) is fulfilled.* Fheorem 5: The integral equation (4) with continuous

continuous derivative $m_{\varphi}(s, \varphi)$ has at most one continuous
 n case II and III, respectively, the condition (51) and
 pendix
 Solution of the equation
 $v(s) +$ $\begin{array}{c}\n \begin{array}{c}\n \text{coefficient m} \\
 \text{is solution in t} \\
 \text{for } n \geq 1\n \end{array}\n \end{array}$ *were 5: The integral equation* (4) *with continuous derivative* $m_{\varphi}(s, \varphi)$ has at most one continuous \cdot **II** and **III**, respectively, the condition (51) and
 $v(s) + H[\lambda(\cdot) v](s) = g(s)$ for a.a. $s \in [-\pi, a \ 2\pi$ periodic *exergion m(s,* φ *) possessing*
solution in the cases I—III
3) *is fulfilled.*
(A.1)
on summable to a power
le to a power greater than
(A.2)
(A.3)

Appendix

1. *Solution of the equation*

$$
\begin{aligned}\n\text{Hilon of the equation} \\
\text{Hilon of the equation} \\
\text{Hilon
$$

Let λ be a 2π periodic continuous function and g a function summable to a power greater than one. We look for a solution *v* of (A.1) summable to a power greater than one. Substituting *won of the equation*
 $v(s) + H[\lambda(\cdot) v](s) = g(s)$ **f**
 a 2π periodic continuous fund than one. We look for a solutification
 $w = \lambda v$, $v = w/\lambda = g -$
 he equation
 $w + \lambda H[w] = \lambda g$

is equation has the unique s
 $w(s) = D\beta(s) e^{-(H\mu$ *H*[λ (·) *v*] (*s*) = *g*(*s*) for a.a. $s \in [-\pi, \pi]$. (A.1)
 *i*riodic continuous function and *g* a function summable to a power
 i. We look for a solution *v* of (A.1) summable to a power greater than
 g
 i $v =$ $v(s) + H[\lambda(\cdot) v](s) = g(s)$ for a.a.

Let λ be a 2π periodic continuous function is

greater than one. We look for a solution v of

one. Substituting
 $w = \lambda v$, $v = w/\lambda = g - H[w]$,

we obtain the equation
 $w + \lambda H[w] = \lambda g$

for $\begin{aligned} &\text{a } 2\pi \text{ periodic continuous function} \ &\text{than one. We look for a solution} \ &\text{stituting} \ &\text{w} = \lambda v, \hspace{5mm} v = w/\lambda = g-H \ &\text{in the equation} \ &\text{w} + \lambda H[w] = \lambda g \ &\text{is equation has the unique solu} \ &\text{w}(s) = D\beta(s) \ \text{e}^{-(H\mu)(s)} + \alpha(s) \ \beta(s) \ &\text{f}(\beta s) = \text{f}^{-(H\mu)(s)} H[\beta g \ \text{e}^{H\mu}] \ &\text{g}(s) = \text{arctan} \ \lambda(s), \ &\text{g}(s$

$$
w = \lambda v, \qquad v = w/\lambda = g - H[w], \tag{A.2}
$$

we obtain the equation

$$
w + \lambda H[w] = \lambda g \tag{A.3}
$$

for *w*. This equation has the unique solution (cf. $[4: Chap. IV, § 31]^2$)

estituting

\n
$$
w = \lambda v, \quad v = w/\lambda = g - H[w],
$$
\n(A.2)

\nn the equation

\n
$$
w + \lambda H[w] = \lambda g
$$
\n(A.3)

\nhis equation has the unique solution (cf. [4: Chap. IV, § 31]²)

\n
$$
w(s) = D\beta(s) e^{-(H\mu)(s)} + \alpha(s) \beta(s) g(s)
$$
\n
$$
-\beta(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}](s),
$$
\n
$$
\mu(s) = \arctan \lambda(s),
$$
\n
$$
\alpha(s) = \cos \mu(s) = \frac{1}{\sqrt{1 + \lambda^2(s)}}, \quad \beta(s) = \sin \mu(s) = \frac{\lambda(s)}{\sqrt{1 + \lambda^2(s)}} \quad (A.6)
$$
\nconstant D is given by

$$
\mu(s) = \arctan \lambda(s), \tag{A.5}
$$

$$
\begin{aligned}\n\omega(s) &= \mathcal{L}\rho(s) \, e^{-(H\mu)(s)} H[\beta g \, e^{H\mu}](s), \\
\mu(s) &= \arctan \lambda(s), \\
\alpha(s) &= \cos \mu(s) = \frac{1}{\sqrt{1 + \lambda^2(s)}}, \qquad \beta(s) = \sin \mu(s) = \frac{\lambda(s)}{\sqrt{1 + \lambda^2(s)}}\n\end{aligned} \tag{A.6}
$$

and the constant D is given by

for *w*. This equation has the unique solution (cf. [4: Chap. IV, § 31]²)
\n
$$
w(s) = D\beta(s) e^{-(H\mu)(s)} + \alpha(s) \beta(s) g(s)
$$
\n
$$
-\beta(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}](s),
$$
\nwhere
\n
$$
\mu(s) = \arctan \lambda(s),
$$
\n
$$
\alpha(s) = \cos \mu(s) = \frac{1}{\sqrt{1 + \lambda^2(s)}},
$$
\n
$$
\beta(s) = \sin \mu(s) = \frac{\lambda(s)}{\sqrt{1 + \lambda^2(s)}}
$$
\n(A.6)
\nand the constant *D* is given by
\n
$$
D = \tan \bar{\mu} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s) g(s) e^{(H\mu)(s)} ds
$$
\nwith the mean value
\n
$$
\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s) ds.
$$
\nHere it has been used that
\n
$$
\tan \bar{\mu} = \int_{-\pi}^{\pi} e^{-H\mu} \sin \mu ds / \int_{-\pi}^{\pi} e^{-H\mu} \cos \mu ds
$$

$$
\bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s) \, ds.
$$

$$
\tan \overline{\mu} = \int_{-\pi}^{\pi} e^{-H\mu} \sin \mu \, ds \bigg/ \int_{-\pi}^{\pi} e^{-H\mu} \cos \mu \, ds \bigg/
$$

is equal to the expression $-\text{Im } \Psi(0)/\text{Re } \Psi(0)$ of the holomorphic function $\Psi(z)$ $= e^{-i(S\mu)(z)}$ in the unit disk, where S is the Schwarz integral. $v(t)$ to the
 $v(s)$ in the
 $v(s) = 0$ *Da*
 Da
 Da
 Da
 Da
 *Da(s) e-(H^p)(8) -2(g) -2(g) -2(g) -2(g) -2(g) -2(g)

<i>Da(s)* e-(Hp)(s) + $\alpha^2(s) g(s) - \alpha(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}] (s).$

(A.8)

19) of [4] the sign + has to be changed into -.

Therefore, the equation (A.1) has the unique solution

$$
v(s) = D\alpha(s) e^{-(H\mu)(s)} + \alpha^2(s) g(s) - \alpha(s) e^{-(H\mu)(s)} H[\beta g e^{H\mu}](s).
$$
 (A.8)

²) In formula (31.19) of [4] the sign $+$ has to be changed into $-$.

The solution $(A.4)$ of the equation $(A.3)$ is derived in [4] for Hölder continuous functions λ and g. But according to the general theory of singular integral equations with continuous coefficients in spaces of summable functions (cf. $[11: Kap. 'III]$). it also holds true under the above more general assumptions about λ and q . Also the derivation in [4] can be directly performed under these general assumptions utilizing the solvability of the Dirichiet problem in the Hardy classes with an exponent greater than one (cf. [6: Kap. IX, \S § 4, 5]). with columbous coefficients in spaces of summable functions (cf. [1]
it also holds true under the above more general assumptions about
the derivation in [4] can be directly performed under these genera
utilizing the solva e direct

e Diricl

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sion (A

+ *i* β) e
 $\frac{1}{2\pi} \int_{-\pi}^{\pi}$

We still simplify the expression (A.8) for *v* in the *particular cases* $q = 1$ *and* $q = H\lambda$ *.* By means of the holomorphic function $\Phi(z) = e^{i(S\mu)(z)}$ in the unit disk possessing the boundary values $\Phi(e^{is}) = (\alpha + i\beta) e^{i\mu}$ one obtains the relations utilizing the solvability of the Dirichlet problem in

nent greater than one (cf. [6: Kap. IX, §§ 4, 5]).

We still simplify the expression (A.8) for v in the p

By means of the holomorphic function $\Phi(z) = e^{i(S\mu)}$

boun

By means of the holomorphic function
$$
\Phi(z) = e^{i(S\mu)(z)}
$$
 in the unit disk possessing the
\nboundary values $\Phi(e^{is}) = (\alpha + i\beta) e^{H\mu}$ one obtains the relations
\n
$$
H[\beta e^{H\mu}] = \alpha e^{H\mu} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha e^{H\mu} ds = \alpha e^{H\mu} - \cos \overline{\mu}
$$
\nand
\n
$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \beta e^{H\mu} ds = \sin \overline{\mu}.
$$
\nTherefore, the solution (A.8) for $g = 1$ takes the form
\n
$$
v_0(s) = (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}.
$$
\n(A.8₀)
\nFurther, the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as can be seen from the
\nequation (A.1). I.e.,
\n
$$
v_1(s) = 1 - (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}.
$$
\n(A.8₁)
\n2. Estimations in the L₂ norm.

$$
\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\beta e^{H\mu}\,ds=\sin \overline{\mu}\,.
$$

Therefore, the solution (A.8) for $g = 1$ takes the form

$$
v_0(s) = (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}
$$

Further, the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as can be seen from the equation (A.1). I.e., and

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equation

2. Estin

Let, be $\frac{1}{2\pi} \int_{-\pi}^{\pi} \beta e^{H\mu} ds = \sin \overline{\mu}.$

re, the solution (A.8) for $g = 1$ takes the form
 $v_0(s) = (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}.$

the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as
 $v_1(s) = 1 - (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}.$

$$
v_1(s) = 1 - (\cos \bar{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}.
$$
 (A.8₁)

2. Estimations in the L_2 norm.

Let, be $g \in L_2(-\pi, \pi)$ and $v, w \in L_2(-\pi, \pi)$. Then, from the second formula in (A.2)
the estimation $||v - g|| \le ||w||$ follows, where $||\cdot||$ denotes the norm in $L_2(-\pi, \pi)$. On
account of the formulas (A.4) and (A.6) this mea the estimation $||v - g|| \le ||w||$ follows, where $||\cdot||$ denotes the norm in $L_2(-\pi, \pi)$. On account-of the formulas $(A.4)$ and $(A.6)$, this means that. *W₀*(s) = $(\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}$. (A.8₀)

the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as can be seen from the

(A.1). I.e.,
 $v_1(s) = 1 - (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}$. (A.8₁)

tions in the L₂ norm.
 $\in L$ *z*₁(*s*) = 1 - (cos $\overline{\mu}$)⁻¹ α (*s*) e<sup>-(*H* μ)(*s*). (A.8₁)
 Ations in the L₂ *norm*.
 $\in L_2(-\pi, \pi)$ and $v, w \in L_2(-\pi, \pi)$. Then, from the second formula in (A.2)

ations $||v - g|| \le ||w||$ follows, where $||$ Let, be $g \in L_2(-\pi, \pi)$ and $v, w \in L_2(-\pi, \pi)$. Then, from the estimation $||v - g|| \le ||w||$ follows, where $||\cdot||$ denotes to account of the formulas (A.4) and (A.6) this means that $||\alpha\Psi + \beta \chi||^2 \le ||\alpha \chi - \beta \Psi||^2$
for the functio

$$
\|\alpha \Psi + \beta \chi\|^2 \le \|\alpha \chi - \beta \Psi\|^2 \tag{A.9}
$$

for the functions

$$
\Psi(s) = e^{-(H\mu)(s)}H[\beta g e^{H\mu}](s) - D e^{-(H\mu)(s)}, \qquad (A.10a)
$$

$$
\chi(s) = \beta(s) g(s). \tag{A.10b}
$$

The inequality (A.9) writes

$$
\int_{-\pi}^{\pi} A(\Psi, \chi) \, (s) \, ds \leq 0, \tag{A.11}
$$

Therefore, the solution (A.8) for
$$
g = 1
$$
 takes the form
\n $v_0(s) = (\cos \pi)^{-1} \alpha(s) e^{-(H\mu)(s)}$ (A.8₀)
\nFurther, the solution v_1 for $g = H\lambda$ is given by $v_1 = 1 - v_0$ as can be seen from the
\nequation (A.1). I.e.,
\n $v_1(s) = 1 - (\cos \overline{\mu})^{-1} \alpha(s) e^{-(H\mu)(s)}$ (A.8₁)
\n2. Estimations in the L₂ norm.
\n2. Estimations in the L₂ norm.
\n2. Estimations in the L₂ norm.
\n2. the estimation $||v - g|| \leq ||w||$ follows, where $||\cdot||$ denotes the norm in $L_2(-\pi, \pi)$. On
\nthe estimations $|w - g|| \leq ||w||$ follows, where $||\cdot||$ denotes the norm in $L_2(-\pi, \pi)$. On
\naccount of the formulas (A.4) and (A.6). this means that
\n $||\alpha \Psi + \beta \chi||^2 \leq ||\alpha \chi - \beta \Psi||^2$ (A.9)
\nfor the functions
\n $\Psi(s) = e^{-(H\mu)(s)} H[\beta g e^{H\mu}] (s) - D e^{-(H\mu)(s)}$,
\n $\chi(s) = \beta(s) g(s)$.
\n(A.10a)
\n $\chi(s) = \beta(s) g(s)$.
\n4.11)
\nwhere
\n $A(\Psi, \chi) = (\alpha^2 - \beta^2) \Psi^2 + 4\alpha \beta \Psi \chi - (\alpha^2 - \beta^2) \chi^2$
\n $\geq (\alpha^2 - \beta^2 - 2qx^2) \Psi^2 - (\alpha^2 - \beta^2 + \frac{2}{q} \beta^2) \chi^2$
\n $= [2(1 - q) \cos^2 \mu - 1] \Psi^2 - [(\frac{2}{q} - 1) - 2(\frac{1}{q} - 1) \cos^2 \mu] \chi^2$
\nfor an arbitrary positive constant q.

We now assume that

$$
|\mu(s)| \leq \gamma < \pi/4, \quad s \in [-\pi, \pi], \tag{A.12}
$$

with a constant γ . Then $\cos^2 \mu \geq \cos^2 \gamma$ and

$$
A(\Psi,\chi) \geq \left[2(1-q)\cos^2\gamma - 1\right]\Psi^2 - \left[\left(\frac{2}{q}-1\right) - 2\left(\frac{1}{q}-1\right)\cos^2\gamma\right]\chi^2
$$

if $0 < q < 1$. In particular, choosing

$$
q=\frac{1-\tan\gamma}{1+\tan\gamma}\tan\gamma,
$$

we obtain

$$
A(\Psi,\chi) \geq K_1 \Psi^2 - K_2 \chi^2 \tag{A.13}
$$

with the constants

$$
K_1 = \frac{1 - \tan \gamma}{1 + \tan \gamma}, \qquad K_2 = \frac{1 + \tan \gamma}{1 - \tan \gamma}.
$$
 (A.14)

 $(A.15)$

From $(A.11)$ and $(A.13)$ with $(A.14)$ the estimation

$$
\|\Psi\|\leq C, \|\chi\|
$$

with the constant

$$
C_{\gamma} = \sqrt{\frac{K_2}{K_1}} = \frac{1 + \tan \gamma}{1 - \tan \gamma} = \frac{\cos 2\gamma}{1 - \sin 2\gamma}
$$
 (A.16)

follows. Under the assumption (A.12) the solution v of (A.1) then satisfies the inequality $||v|| \le ||\Psi|| + ||g|| \le C_r ||\chi|| + ||g||$ or because of $||\chi|| \le \sin \gamma \cdot ||g||$

$$
\|v\| \le (1 + \sin \gamma \cdot C_{\gamma}) \|g\| \le \left(1 + \frac{1}{2} \sqrt{2} C_{\gamma}\right) \|g\|.
$$
 (A.17)

Obviously, the inequality (A.17) holds true for all functions $g \in L_2(-\pi, \pi)$ if the assumption $(A.12)$ is fulfilled.

We further derive analogous estimations to $(A.15)$ and $(A.17)$ under the less restrictive assumption

$$
|\nu(s)| \leq \gamma < \pi/4 \tag{A.18}
$$

where

$$
\nu(s) = \mu(s) - \tilde{\mu}, \qquad 2\tilde{\mu} = \max_{s \in [-\pi,\pi]} \mu(s) + \min_{s \in [-\pi,\pi]} \mu(s).
$$
 (A.19)

The inequality (A.18) is fulfilled if

$$
2\gamma = \max_{s \in \{-\pi,\pi\}} \mu(s) - \min_{s \in \{-\pi,\pi\}} \mu(s) < \pi/2. \tag{A.20}
$$

At first we introduce the functions

$$
\Psi_0(s) = H[\beta g e^{H\mu}] \langle s \rangle - D, \qquad (A.21a)
$$

$$
\chi_0(s) = \beta(s) g(s) e^{(H\mu)(s)}, \qquad (A.21 b)
$$

where $\Psi_0 = H\chi_0 - D$, and the constant

$$
D_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_0(s) \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s) \, g(s) \, e^{(H\mu)(s)} \, ds. \tag{A.22}
$$

Then the function

$$
W_0(z) = e^{-i(Sv)(z)}[(S_{\chi_0}) (z) + iD_0 \tan \bar{v}],
$$

where

$$
\bar{\nu}=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\nu(s)\,ds
$$

and S again denotes the Schwarz integral, is holomorphic in the unit disk and satisfies the condition Im $W_0(0) = 0$.

The function W_0 has the boundary values

$$
\text{Re }W_0(\mathrm{e}^{\mathcal{U}})=\mathrm{e}^{-H\mu}[\cos \nu\cdot \chi_0-\sin \nu\left(\mathcal{Y}_0+D_1\right)],
$$

Im
$$
W_0(e^{i\theta}) = -e^{-H\mu}[\cos \nu (\Psi_0 + D_1) + \sin \nu \cdot \chi_0],
$$

where

$$
D_1 = D - D_0 \tan \bar{\nu} = D_0 [\tan \bar{\mu} - \tan \bar{\nu}].
$$

Therefore, putting $\alpha_0 = \cos v(s)$, $\beta_0 = \sin v(s)$, we obtain the inequality

$$
|e^{-H\mu}[\alpha_0(\Psi_0+D_1)+\beta_0\chi_0] \|^2 \leq \|e^{-H\mu}[\alpha_0\chi_0-\beta_0(\Psi_0+D_1)]\|^2.
$$

which is analogous to the inequality $(A.9)$.

From this inequality in the same way as above the inequality

$$
||e^{-H\mu}(\Psi_0+D_1)||\leq C_r ||e^{-H\mu}\chi_0||
$$

follows, where the constant C_r , is given by (A.16) again. Due to the formulas (A.10a, b) and (A.21 a, b) this means that $||\Psi| + D_1 e^{-H\mu}|| \leq C_{\gamma} ||\chi||$, and on account of (A.24)
we obtain $||\Psi|| \leq C_{\gamma} ||\chi|| + |D_0|$ [tan $\bar{\mu} - \tan \bar{\nu}| ||e^{-H\mu}||$. Finally, $|D_0| \leq (1/2\pi) ||g||$.
 $\times ||e^{H\mu}||$, and under the assumptio $\left(\frac{2\pi}{\cos 2\nu}\right)^{1/2}$ in virtue of the well-known Zygmund lemma [16]. Therefore, $||\Psi|| \leq C_{\mathbf{v}} ||\chi|| + [1 + |\tan \overline{\mu}|] (\cos 2\gamma)^{-1} ||g||.$ $(A.25)$

From $(A.25)$ the estimation

$$
||v|| \le E_r ||g|| \tag{A.26}
$$

with the constant

$$
E_{\gamma} = B_{\gamma} + [1 + |\tan \bar{\mu}|] (\cos 2\gamma)^{-1}, \tag{A.27}
$$

where

$$
B_{\mathbf{y}} = 1 + C_{\mathbf{y}} = 2/[1 - \tan \gamma], \tag{A.28}
$$

for the solution v of (A.1) follows. The inequality (A.26) holds for all $g \in L_2(-\pi, \pi)$ if the assumption $(A.18)$ is fulfilled.

If the function λ is strictly positive (or strictly negative) and $\int g(s) ds = 0$ the

 $(A.23)$

 $(A.24)$

solution v of (A.1) can be estimated in another way. Namely, applying the Hilbert operator to $(A.1)$, we obtain the equation

$$
Hv - \lambda v = Hg + Const
$$
 (A.29)

taking into account that

$$
H^2\varphi=-\varphi+\frac{1}{2\pi}\int\limits_{-\pi}\varphi(s)\,ds
$$

for any function $\varphi \in L_2(-\pi, \pi)$. Further, from (A.1) it follows that

$$
\int_{-\pi}^{\pi} v(s) \, ds = \int_{-\pi}^{\pi} g(s) \, ds = 0 \tag{A.30}
$$

by assumption. Multiplying (A.29) by v and integrating it over $(-\pi, \pi)$, therefore yields the relation

$$
\int_{-\pi}^{\pi} \lambda(s) \, v^2(s) \, ds = - \int_{-\pi}^{\pi} v H g \, ds \tag{A.31}
$$

 $(A.33)$

using that $\int vHv ds = 0$.

Let now the assumption

 $\lambda(s) \ge M_0 > 0$ or $\lambda(s) \le -M_0 < 0$ in $[-\pi, \pi]$ $(A.32)$

be fulfilled, respectively. Then (A.31) implies the estimation

$$
M_0 \int\limits_{-\pi}^{\pi} v^2(s) \, ds \leqq \pm \int\limits_{-\pi}^{\pi} \lambda(s) \, v^2(s) \, ds \leq ||v|| \, ||g||,
$$

since $||Hg|| = ||g||$, and we obtain the inequality

 $||v|| \leq (1/M_0) ||g||$

for the solution v of $(A.1)$.

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