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Bifurcation and Stability of Cellular States in Magnetic Fluids

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Prof. Dr. H. Beckert to his 65th birthday

In der vorliegenden Arbeit werden die Untersuchungen des Autors zu Verzweigungs- und Stabilitätsverhältnissen periodischer Gleichgewichtszustände magnetischer Flüssigkeiten in einem vertikalen Magnetfeld auf den Fall endlich tiefer Flüssigkeitsschichten ausgedehnt.

В продолжении исследований автора в работе изучаются устойчивость и бифуркация периодических равновесных состояний магнитной жидкости в вертикальном магнитном поле. Дается распространение на случай жидкости конечной глубины.

This paper continues earlier work by the author concerning bifurcation and stability of periodic equilibrium states of a magnetic fluid subjected to a vertical magnetic field. Here the treatment is extended to cover the case of a fluid of finite depth.

Consider a magnetic fluid in a vertical magnetic field under the influence of gravity and surface tension. This paper continues earlier work [2, 3] by the author on this subject. Here the treatment is extended to cover the case of a fluid of finite depth. Let $-h \leqq z \leqq Z(x, y)$ be a layer of magnetic fluid. Any steady-state equilibrium position of its upper free surface Γ: $z = Z(x, y)$ is characterized by the variational principle $\delta E = 0$, E being the potential energy of the system. Clearly the plane horizontal interface, which may be taken to be the (x, y)-plane, always represents an equilibrium state. As the exterior field $\mathfrak{H}$ increases past a certain critical value H_{cr} this basic solution loses its stability and the system moves into a new nontrivial state.

As in [3] we look for periodic Γ with hexagonal lattice structure Λ. Our approach via Lyapunov-Schmidt procedure is based on analytic expansion of E relative to the Sobolev spaces $\dot{H}_s$ of Λ-periodic functions with mean zero as defined by (1.19). It turns out that (provided $s \geqq 5/2$) the first variation DF of the magnetic energy F acts as an analytic map from $\dot{H}_s$ into $\dot{H}_{s-1}$, this improving a corresponding result of [3]. As an immediate consequence this implies analyticity of DE as a mapping from $\dot{H}_s$ into $\dot{H}_{s-2}$.

An outline of the paper is as follows. In § 1 our objective is to compute the Taylor series of F (resp. E), essentially up to fourth order terms in Γ. Particularly: dim $N \times (D^2E(0)) = 6$ at criticality where N denotes the kernel. In § 2, using the symmetries of E, we solve the branching equations for three types of solutions I, II±. Tested against disturbances in the lattice class Λ the transcritical branch II+ turns out to be stable only. Having (2.15) in mind, this indicates hysteresis at H_{cr} (cf. [5, 7]). The final § 3 is devoted to the proof of Theorem 2.1.

It should be remarked that a further supercritical branch can be determined by means of scaling techniques. Bifurcating solutions to the nonlinear problem (infinite depth) were first constructed formally by Gailitis [5]. For a detailed discussion of bifurcation phenomena in the presence of a symmetry group see the expository paper [7].

§ 1

Consider the upper free surface Γ: $z = Z(x, y)$ separating a layer $-h \leqq z \leqq Z(x, y)$ of an incompressible magnetic fluid of depth $h > 0$ from a vacuum. Subjected to the action of surface tension β, gravity $(0, 0, -g)$ and an exterior vertical magnetic field $\mathfrak{H}$ the plane horizontal interface — say $z = 0$ — always represents an equilibrium state. As $\mathfrak{H}$ increases past a certain critical value H_{cr} this basic solution loses its stability and the system moves into a new nontrivial state.

As in [3] we look for periodic states. Choose dimensionless coordinates $(x_1, x_2, x_3) = (x, y, z)/l$ where $l > 0$ measures the wavelength to be specified later on. If in the (x_1, x_2, x_3) reference-system labelled by

$$\Gamma: x_3 = \zeta(x_1, x_2) = Z(x, y)/l$$

we restrict the interfaces Γ to be Λ-periodic with respect to the hexagonal lattice $\Lambda = \{k_1\omega_1 + k_2\omega_2 : k_1, k_2 \in \mathbf{Z}\}$ generated by $\omega_1 = 2\pi(1, 0)$, $\omega_2 = 2\pi\left(1/2, \sqrt{3}/2\right)$. Let $\mathcal{P}(0, \omega_1, \omega_2, \omega_1 + \omega_2)$ be the fundamental parallelogram of the lattice. On $\mathcal{P}$ we assume the fluid/vacuum to occupy the regions

$$\Omega^{\mathrm{fl}}: (x_1, x_2) \in \mathcal{P}, \qquad -q < x_3 < \zeta(x_1, x_2); \qquad q := h/l > 0 \tag{1.1}$$

resp.

$$\Omega^-: (x_1, x_2) \in \mathcal{P}, x_3 < -q, \qquad \Omega^+: (x_1, x_2) \in \mathcal{P}, x_3 > \zeta(x_1, x_2)$$

(lower/upper vacuum part); let $\Omega = \Omega^- \cup \Omega^+$. If necessary, in the following we shall distinguish the corresponding fields accordingly by indices „fl" ("±"). Let $\zeta = 0$ when $\mathfrak{H} = 0$.

By definition an equilibrium state ζ has to satisfy the variational equaiton $\langle DE(\zeta), h\rangle = 0$ for all admissible variations h where E denotes the energy functional of our system. Considering incompressibility we impose ζ and h to have mean zero:

$$\int_{\mathcal{P}} \zeta \, dx_1 \, dx_2 = 0. \tag{1.2}$$

If the magnetic field $\mathfrak{H} = H \, \nabla\psi$:

$$\begin{aligned} \psi^{\mathrm{fl}}(x, y, z) &= \frac{z}{\mu} + l \, \frac{1 - \mu}{\mu} \, u^{\mathrm{fl}}(x_1, x_2, x_3) \quad \text{on } \Omega^{\mathrm{fl}}, \\ \psi^{\pm}(x, y, z) &= z + l \, \frac{1 - \mu}{\mu} \, u^{\pm}(x_1, x_2, x_3) \quad \text{on } \Omega^{\pm} \end{aligned} \tag{1.3}$$

is permitted to vary in a neighbourhood of $H\left(\nabla(z^{\mathrm{fl}}/\mu), \nabla z^{\pm}\right)$ we get

$$E = \int_{\mathcal{P}} \sqrt{1 + |\nabla\zeta|^2} \, dx_1 \, dx_2 + \frac{q_2^2}{2} \int_{\mathcal{P}} \zeta^2 \, dx_1 \, dx_2 - q_1 q_2 \, \frac{1 + \mu}{\mu} \, F \tag{1.4}$$

for the energy (per unit area) measured in units of β (surface tension), see [3]. Here $F = F(\zeta)$ is defined to be the minimal value to the quadratic variational problem

$$\int_{\Omega} |\nabla u|^2 \, dV + \mu \int_{\Omega_{\mathrm{fl}}} |\nabla u|^2 \, dV \to \min \tag{1.5}$$

($dV = dx_1\,dx_2\,dx_3$) which is to solve subject to boundary and periodicity conditions

$$\begin{aligned} &u = (u^{\mathfrak{n}}, u^{\pm})\ \Lambda\text{-periodic},\\ &u^{+} - u^{\mathfrak{n}} = x_3 + \text{const. on } \Gamma, \qquad u^{\mathfrak{n}} - u^{-} = \text{const. on } x_3 = -q. \end{aligned} \tag{1.6}$$

The dimensionless parameters q_1, q_2 are defined by

$$8\pi\mu(1+\mu)\,q_1 = (\varrho g\beta)^{-1/2}(\mu-1)^2 H^2, \qquad \beta^{1/2}q_2 = l(\varrho g)^{1/2}$$

where $\varrho > 0$ is the density and $\mu > 0$ the magnetic permeability of the fluid ($\mu = 1$ in $\Omega^{\pm}$). Note $q = h/l = h\sqrt{\varrho g}/q_2\sqrt{\beta}$.

To begin, we compute the derivatives of E — at the present stage on a somewhat formal way. Consider, in addition to Γ, a family of neighbouring surfaces Γ_t: $x_3 = \zeta(x_1, x_2) + th(x_1, x_2)$, $\Gamma_0 = \Gamma$. Let $\Omega_t^{\mathfrak{n}}$, $\Omega_t^{\pm}$ be the corresponding family of domains (1.1). Solving (1.5) relative to $\Omega_t^{\mathfrak{n}}$, $\Omega_t^{\pm}$ gives rise to fields $u(t; x_1, x_2, x_3)$. Let a dot denote differentiation with respect to t at $t = 0$: $\dot{u} = \partial u/\partial t\ (0; \cdot, \cdot, \cdot)$. Differentiation of F yields

$$\begin{aligned} \langle DF(\zeta), h\rangle &= \frac{d}{dt} F(\zeta + th)|_{t=0}\\ &= 2\int_{\Omega} \nabla u\,\nabla\dot{u}\,dV + 2\mu\int_{\Omega^{\mathfrak{n}}} \nabla u\,\nabla\dot{u}\,dV\\ &\quad + \int_{\Gamma} (\mu\,|\nabla u^{\mathfrak{n}}|^2 - |\nabla u^{+}|^2)\,h\,dx_1\,dx_2, \end{aligned} \tag{1.7}$$

the last term due to varying the boundary. Note $\Delta u = 0$ in $\Omega^{\mathfrak{n}}$ (resp. $\Omega^{\pm}$) due to (1.5). From (1.6) we get by differentiation

$$\dot{u}^{+} - \dot{u}^{\mathfrak{n}} = (1 - u^{+}_{x_3} - u\,h)^{\mathfrak{n}}_{x_3} + \text{const. on } \Gamma. \tag{1.8}$$

Therefore (1.7) leads to

$$\begin{aligned} \langle DF(\zeta), h\rangle &= \int_{\Gamma} (\mu\,|\nabla u^{\mathfrak{n}}|^2 - |\nabla u^{+}|^2)\,h\,dx_1\,dx_2\\ &\quad + 2\int_{\Gamma} \frac{\partial u^{+}}{\partial n}(u^{+}_{x_3} - u^{\mathfrak{n}}_{x_3} - 1)\,h\,d\Gamma \end{aligned} \tag{1.9}$$

when integrated by parts (note that $u, \dot{u} \in O\left(\exp\left(-2|x_3|/\sqrt{3}\,\right)\right)$). In (1.9) the normal n has to be taken directed to Ω^{+}.

Remark 1.1: Remembering (1.3) we get after retransformation

$$\begin{aligned} \langle DE(\zeta), h\rangle = (\beta l)^{-1}\int_{l\mathscr{P}} \Bigg(&-\beta\,\mathrm{div}\frac{\nabla Z}{\sqrt{1+|\nabla Z|^2}} + \varrho g Z\\ &+ \frac{1-\mu}{8\pi}\left(|\mathfrak{H}_t^{\mathfrak{n}}|^2 + \mu|\mathfrak{H}_n^{\mathfrak{n}}|^2\right)\Bigg)\,h\left(\frac{x}{l}, \frac{y}{l}\right)dx\,dy, \end{aligned}$$

$\mathfrak{H}_t$ (resp. $\mathfrak{H}_n$) being the tangential (resp. normal) component of $\mathfrak{H}$. Because of (1.2) this implies

$$-\beta\,\mathrm{div}\frac{\nabla Z}{\sqrt{1+|\nabla Z|^2}} + \varrho g Z + \frac{1-\mu}{8\pi}\left(|\mathfrak{H}_t^{\mathfrak{n}}|^2 + \mu\,|\mathfrak{H}_n^{\mathfrak{n}}|^2\right) = \text{const.}$$

along an equilibrium interface Γ.

Further differentiation gives

$$D^2F(\zeta)\{h^2\} = \frac{d^2}{dt^2} F(\zeta + th)|_{t=0}$$

$$= 2 \int_\Omega (|\nabla \dot u|^2 + \nabla u \nabla \ddot u)\, dV + 2\mu \int_{\Omega^{\rm fl}} (|\nabla \dot u|^2 + \nabla u \nabla \ddot u)\, dV$$

$$+ 4 \int_\Gamma (\mu \nabla u^{\rm fl} \nabla \dot u^{\rm fl} - \nabla u^+ \nabla \dot u^+)\, h \, dx_1\, dx_2$$

$$+ 2 \int_\Gamma (\mu \nabla u^{\rm fl} \nabla u^{\rm fl}_{x_3} - \nabla u^+ \nabla u^+_{x_3})\, h^2\, dx_1\, dx_2. \tag{1.10}$$

Its value at $\zeta = 0$:

$$D^2F(0)\{h^2\} = 2 \int_\Omega |\nabla \dot u|^2\, dV + 2\mu \int_{\Omega^{\rm fl}} |\nabla \dot u|^2\, dV \tag{1.11}$$

is of particular interest.

For simplicity we adopt the following notation:

$$a(u, v) = \int_\Omega \nabla u \nabla v\, dV + \mu \int_{\Omega^{\rm fl}} \nabla u \nabla v\, dV,$$

$$\dot a(u, v) = \int_\Gamma (\mu \nabla u^{\rm fl} \nabla v^{\rm fl} - \nabla u^+ \nabla v^+)\, h\, dx_1\, dx_2, \tag{1.12}$$

$$\ddot a(u, v) = \int_\Gamma \frac{\partial}{\partial x_3} (\mu \nabla u^{\rm fl} \nabla v^{\rm fl} - \nabla u^+ \nabla v^+)\, h^2\, dx_1\, dx_2.$$

If we keep ζ (hence $\Omega^{\rm fl}$, $\Omega^\pm$) and h both fixed then we have to think of (1.12) as of bilinear forms in u, v. Now (1.10), (1.11) reads

$$\begin{aligned} D^2F(\zeta)\{h^2\} &= 2a(\dot u, \dot u) + 2a(u, \ddot u) + 4\dot a(u, \dot u) + \ddot a(u, u), \\ D^2F(0)\{h^2\} &= 2a(\dot u, \dot u). \end{aligned} \tag{1.13}$$

As above we get by repeated differentiation

$$\begin{aligned} D^3F(0)\{h^3\} &= 6a(\dot u, \ddot u) + 6\dot a(\dot u, \dot u), \\ D^4F(0)\{h^4\} &= 8a(\dot u, u^{(3)}) + 6a(\ddot u, \ddot u) + 24\dot a(\dot u, \ddot u) + 12\ddot a(\dot u, \dot u). \end{aligned} \tag{1.14}$$

We still have to determine the derivatives of u. We start with differentiating the variational equation $a(u, \varphi) = 0$ to (1.5) choosing the test function φ to be sufficiently regular. This yields

$$a(\dot u, \varphi) + \dot a(u, \varphi) = 0 \text{ for all } \varphi\ \Lambda\text{-periodic}.$$

In addition, $\dot u$ has to satisfy (1.6) resp. (1.8). At $\xi = 0$ this particularly reduces to

$$\begin{aligned} &a(\dot u, \varphi) = 0 \quad \text{for all } \varphi\ \Lambda\text{-periodic}; \\ &\dot u^+ - \dot u^{\rm fl} = h + \text{const. along } x_3 = 0, \\ &\dot u^{\rm fl} - \dot u^- = \text{const. along } x_3 = -q. \end{aligned} \tag{1.15}$$

Similarly by repeated differentiation

$$\begin{aligned} &a(\ddot u, \varphi) + 2\dot a(\dot u, \varphi) = 0, \qquad \ddot u\ \Lambda\text{-periodic, for all } \varphi\ \Lambda\text{-periodic}; \\ &\ddot u^+ - \ddot u^{\rm fl} = -2(\dot u_{x_3} - \dot u^{\rm fl}_{x_3})\, h + \text{const. along } x_3 = 0, \\ &\ddot u^{\rm fl} - \ddot u^- = \text{const. along } x_3 = -q \end{aligned} \tag{1.16}$$

at $\zeta = 0$. Integrated by parts this leads to

$$\begin{aligned} &\Delta \dot u^{fl} = 0 \quad \text{on } \Omega^{fl}, \qquad \Delta \dot u^{\pm} = 0 \quad \text{on } \Omega^{\pm};\\ &\dot u^{\pm}_{x_3} - \mu \dot u^{fl}_{x_3} = 0 \quad \text{along } x_3 = 0 \ (x_3 = -q) \end{aligned} \tag{1.17}$$

resp.

$$\begin{aligned} &\Delta \ddot u^{fl} = 0 \quad \text{on } \Omega^{fl}, \qquad \Delta \ddot u^{\pm} = 0 \quad \text{on } \Omega^{\pm};\\ &\ddot u^{+}_{x_3} - \mu \ddot u^{fl}_{x_3} = 2\left(\frac{\partial}{\partial x_1} h(\dot u^{fl+}_{x_1} - \mu \dot u^{fl}_{x_1}) + \frac{\partial}{\partial x_2} h(\dot u^{+}_{x_2} - \mu \dot u^{fl}_{x_2})\right) \quad \text{along} \quad x_3 = 0,\\ &\ddot u^{-}_{x_3} - \mu \ddot u^{fl}_{x_3} = 0 \quad \text{along } x_3 = -q. \end{aligned} \tag{1.18}$$

From now let $\zeta = 0$ be fixed. To solve (1.17) resp. (1.18) expand h in a Fourier series

$$h = \sum_{\omega \in \Lambda'} h_\omega \, e^{i\omega x}, \qquad h_\omega = \bar h_\omega. \tag{1.19}$$

Here $\Lambda' = \{k_1\omega_1' + k_2\omega_2' : k_1, k_2 \in \mathbf{Z}\}$ is the dual lattice to Λ which is generated by $\omega_1' = 2/\sqrt{3}(\sqrt{3}/2, -1/2)$, $\omega_2' = 2/\sqrt{3}\ (0, 1)$ and ωx denotes the scalar product of $\omega \in \Lambda'$ and $x = (x_1, x_2)$. In the following Lemma we consider $\dot u$, $\ddot u$ to be dependent on μ also.

Lemma 1.1: (i) *Let* $\zeta = 0$, *then*

$$\dot u^- = -\frac{2\mu}{(\mu+1)^2} \sum_{\omega \in \Lambda'} \frac{h_\omega \, e^{i\omega x + |\omega| x_3}}{1 - \left(\frac{\mu-1}{\mu+1}\right)^2 e^{-2q|\omega|}},$$

$$\dot u^{fl} = -\frac{1}{\mu+1} \sum_{\omega \in \Lambda'} \frac{h_\omega \, e^{i\omega x}}{1 - \left(\frac{\mu-1}{\mu+1}\right)^2 e^{-2q|\omega|}} \left(e^{|\omega| x_3} + \frac{\mu-1}{\mu+1} e^{-|\omega|(2q+x_3)}\right),$$

$$\dot u^+ = \frac{\mu}{\mu+1} \sum_{\omega \in \Lambda'} \frac{1 - \frac{\mu-1}{\mu+1} e^{-2q|\omega|}}{1 - \left(\frac{\mu-1}{\mu+1}\right)^2 e^{-2q|\omega|}} h_\omega \, e^{i\omega x - |\omega| x_3}.$$

(ii) *If in addition* $\mu = 1$, *then*

$$\ddot u^+(x_1, x_2, 0) = \ddot u^{fl}(x_1, x_2, 0) = \frac{1}{2} A(h^2),$$

$$\ddot u^+_{x_3}(x_1, x_2, 0) = -\ddot u^{fl}_{x_3}(x_1, x_2, 0) = \frac{1}{2} \Delta(h^2)$$

where A denotes the map

$$h \to Ah = \sum_{\omega \in \Lambda'} |\omega| \, h_\omega \, e^{i\omega x}.$$

Proof: (i) is easily verified when inserted in (1.15), (1.17). Let $\mu = 1$, then in view of (i)

$$\dot u^+(x_1, x_2, 0) = -\dot u^{fl}(x_1, x_2, 0) = \frac{h}{2},$$

$$\dot u^+_{x_3}(x_1, x_2, 0) = \dot u^{fl}_{x_3}(x_1, x_2, 0) = -\frac{1}{2} Ah.$$

Consequently (1.16), (1.18) reduces to

$$\begin{aligned}\ddot{u}^+(x_1, x_2, 0) - \ddot{u}^{\mathrm{fl}}(x_1, x_2, 0) &= \text{const.}, \\ \ddot{u}^+_{x_3}(x_1, x_2, 0) - \ddot{u}^{\mathrm{fl}}_{x_3}(x_1, x_2, 0) &= \Delta(h^2).\end{aligned} \tag{1.20}$$

Now consider the harmonic function v on Ω^+ with boundary values h^2 along $x_3 = 0$, and whose Dirichlet integral extended over Ω^+ is finite. Obviously $\ddot{u}^+ = -\frac{1}{2} v_{x_3}$; $\ddot{u}^{\mathrm{fl}}$, $\ddot{u}^-(x_1, x_2, x_3) = -\frac{1}{2} v_{x_3}(x_1, x_2, -x_3)$ represents the desired solution of (1.20). This immediately implies (ii) ∎

Inserting (i) in (1.13) we get after integration by parts

$$\begin{aligned}D^2F(0)\,\{h^2\} &= -2 \int_{\mathcal{P}} \dot{u}^+_{x_3}(x_1, x_2, 0)\, h\, dx_1\, dx_2 \\ &= \frac{2\mu}{\mu+1} |\mathcal{P}| \sum_{\omega\in\Lambda'} \frac{1 - \frac{\mu-1}{\mu+1} e^{-2q|\omega|}}{1 - \left(\frac{\mu-1}{\mu+1}\right)^2 e^{-2q|\omega|}} |\omega|\, |h_\omega|^2,\end{aligned} \tag{1.21}$$

$|\mathcal{P}| = 2\sqrt{3}\,\pi^2$. To stress the dependence on the additional parameters in the following we use the notation $F(\zeta; \mu, q)$, $E(\zeta; \mu, q_1, q_2)$. As above we get from (1.14) by Lemma 1.1

$$\begin{aligned}D^3F(0; \mu, q)\,\{h^3\} &= 6 \int_{\mathcal{P}} \dot{u}^+_{x_3}(\ddot{u}^{\mathrm{fl}} - \ddot{u}^+)|_{x_3=0}\, dx_1\, dx_2 + 6\dot{a}(\dot{u}, \dot{u}) \\ &= \frac{3}{2}(\mu - 1)\left(\left(Ah, hAh - \frac{1}{2} Ah^2\right) - (h^2, \Delta\, e^{-2qA} h)\right) \\ &\quad + O\big((\mu - 1)^2\big),\end{aligned} \tag{1.22}$$

where $(\cdot, \cdot)$ denotes the L_2-scalar product on $\mathcal{P}$ and

$$e^{-2qA} h = \sum_{\omega\in\Lambda'} h_\omega\, e^{-2q|\omega|}\, e^{i\omega x}.$$

We point out that $D^3F(0; 1, q) = 0$.

In order to obtain an analogous expression for $D^4F(0; 1, q)$ we differentiate (1.8) twice in t. Setting $\zeta = 0$, $\mu = 1$, this in view of Lemma 1.1 leads to

$$\begin{aligned}u^{+(3)} - u^{\mathrm{fl}(3)} &= -3(\ddot{u}^+_{x_3} - \ddot{u}^{\mathrm{fl}}_{x_3})\, h - 3(\dot{u}^+ - \dot{u}^{\mathrm{fl}})\, h^2 + \text{const.} \\ &= 3(-h\,\Delta h^2 + h^2\,\Delta h) + \text{const.} = -\Delta h^3 + \text{const.}\end{aligned}$$

along $x_3 = 0$. Now, from (1.14) we get by Lemma 1.1 and the previous formula

$$\begin{aligned}D^4\{F(0; 1, q)\,\{h^4\} &= 8 \int_{\mathcal{P}} \dot{u}^+_{x_3}(u^{\mathrm{fl}(3)} - u^{+(3)})|_{x_3=0}\, dx_1\, dx_2 \\ &\quad + 6 \int_{\mathcal{P}} (\ddot{u}^{\mathrm{fl}}\ddot{u}^{\mathrm{fl}}_{x_3} - \ddot{u}^+\ddot{u}^+_{x_3})|_{x_3=0}\, dx_1\, dx_2 + 24\dot{a}(\dot{u}, \ddot{u}) + 12\ddot{a}(\dot{u}, \dot{u}) \\ &= 4(h, A^3h^3) - 3(h^2, A^3h^2).\end{aligned} \tag{1.23}$$

Stability of the unperturbed state $\zeta = 0$ is determined by the second variation $D^2E(0; \mu, q_1, q_2)$ which we proceed to study. Let

$$Q(\vartheta, \mu, q_1) = \vartheta^2 + 1 - 2q_1\vartheta \frac{1 - \dfrac{\mu - 1}{\mu + 1} e^{-2\vartheta h\sqrt{\varrho g}/\sqrt{\beta}}}{1 - \left(\dfrac{\mu - 1}{\mu + 1}\right)^2 e^{-2\vartheta h\sqrt{\varrho g}/\sqrt{\beta}}},$$

then in view of (1.4) and (1.21)

$$\begin{aligned} D^2E(0; \mu, q_1, q_2) \{h^2\} &= \int_{\mathscr{P}} (|\nabla h|^2 + q_2^2 h^2)\, dx_1\, dx_2 \\ &\quad - q_1 q_2 \frac{1 + \mu}{\mu} D^2F(0; \mu, q_2) \{h^2\} \\ &= q_2^2\, |\mathscr{P}| \sum_{\omega \in \Lambda'} Q\left(\frac{|\omega|}{q_2}, \mu, q_1\right) |h_\omega|^2. \end{aligned} \tag{1.24}$$

Lemma 1.2: *For μ in a neighbourhood of $\mu = 1$ there exist analytic ϑ^{cr}, $q_1^{cr} > 0$ such that for all $\vartheta \geqq 0$*

$$Q(\vartheta, \mu, q_1) > 0 \quad \text{if} \quad 0 \leqq q_1 < q_1^{cr}$$

and $Q(\vartheta^{cr}, \mu, q_1^{cr}) = 0$. Moreover $Q(\vartheta, \mu, q_1^{cr}) > 0$ if $\vartheta \neq \vartheta^{cr}$.

Proof: The critical values ϑ^{cr}, q_1^{cr} are to be determined from $Q = \partial Q/\partial\vartheta = 0$. Eliminating q_1 leads to

$$4 \frac{\mu - 1}{(\mu + 1)^2} \frac{e^{-2\alpha\vartheta}}{\left(1 - \dfrac{\mu - 1}{\mu + 1} e^{-2\alpha\vartheta}\right)\left(1 - \left(\dfrac{\mu - 1}{\mu + 1}\right)^2 e^{-2\alpha\vartheta}\right)} = \frac{1}{\alpha\vartheta} \frac{\vartheta^2 - 1}{\vartheta^2 + 1}$$

where $\alpha = h\sqrt{\varrho g}/\sqrt{\beta}$. For μ near to 1 this is easily seen to be uniquely solvable for ϑ. Power series expansion shows

$$\begin{aligned} \vartheta^{cr}(\mu) &= 1 + \alpha\, e^{-2\alpha}(\mu - 1) + O((\mu - 1)^2), \\ q_1^{cr}(\mu) &= 1 + \frac{1}{2} e^{-2\alpha}(\mu - 1) + O((\mu - 1)^2) \;\blacksquare \end{aligned} \tag{1.25}$$

Let H_s (s real) be the Sobolev space of Λ-periodic functions (resp. distributions) (1.19) with finite norm

$$\|h\|_s^2 = |h_0|^2 + \sum_{0 \neq \omega \in \Lambda'} |\omega|^{2s} |h_\omega|^2$$

and $\dot{H}_s$ that subspace of functions in H_s satisfying (1.2). Obviously $D^2E(0; \mu, q_1, q_2)$ is continuous on $\dot{H}_1 \times \dot{H}_1$.

If we define the critical "wavelength" to be

$$q_2^{cr}(\mu) = \frac{2}{\sqrt{3}} \vartheta_{cr}^{-1} = \frac{2}{\sqrt{3}} (1 - \alpha\, e^{-2\alpha}(\mu - 1) + O((\mu - 1)^2), \tag{1.26}$$

then Lemma 1.2 implies positivity of $D^2E(0; \mu, q_1, q_2)$ on $\dot{H}_1 \times \dot{H}_1$ as long as $0 \leqq q_1 < q_1^{\mathrm{cr}}$, whereas

$$D^2E(0; \mu, q_1^{\mathrm{cr}}, q_2^{\mathrm{cr}}) \{h^2\} = (q_2^{\mathrm{cr}})^2 \sum_{|\omega|>2/\sqrt{3}} Q\left(\frac{|\omega|}{q_2^{\mathrm{cr}}}; \mu, q_1^{\mathrm{cr}}\right) |h_\omega|^2$$

possesses the six-dimensional kernel

$$N_6 : h = \sum_{|\omega|=2/\sqrt{3}} h_\omega \, e^{i\omega x}, \qquad h_{-\omega} = \bar{h}_\omega .$$

Accordingly $\zeta = 0$ loses its stability as q_1 crosses q_1^{cr}.

§ 2

In this section, assuming $s \geqq 5/2$, we look at E as a functional on the spaces $\dot{H}_s$.

Theorem 2.1: *Assume $s \geqq 5/2$, then* (i) *$F(\zeta; \mu, q)$ as defined by* (1.5), (1.6) *is analytic as a map of a neighbourhood $\mathcal{N}$ of any* $(0; 1, q_0)$ *in $\dot{H}_s \times \mathbf{R}^2$ into* $\mathbf{R}$ *and* (ii) *its derivative DF (with respect to ζ) maps $\mathcal{N}$ into $\dot{H}_{s-1}$ analytically.*

This Theorem is proved in § 3. As an immediate consequence of (ii) and Lemma 1.2 we get

Corollary 2.1: *Let $s \geqq 5/2$.* (i) *$E(\zeta; \mu, (1 + \varepsilon) q_1^{\mathrm{cr}}(\mu), q_2^{\mathrm{cr}}(\mu))$ is analytic from a neighbourhood of $(\zeta; \varepsilon, \mu) = (0; 0, 1)$ in $\dot{H}_s \times \mathbf{R}^2$ into* $\mathbf{R}$. (ii) *$DE(\zeta; \mu, (1 + \varepsilon) q_1^{\mathrm{cr}}(\mu), q_2^{\mathrm{cr}}(\mu))$ considered as a map from $\dot{H}_s \times \mathbf{R}^2$ into $\dot{H}_{s-2}$ is analytic at* $(0; 0, 1)$.

Corollary 2.1 implies by interpolation

Corollary 2.2: *Let $s \geqq 5/2$, then $D^2E(\zeta; \mu, (1 + \varepsilon) q_1^{\mathrm{cr}}(\mu), q_2^{\mathrm{cr}}(\mu))$ — originally considered on $\dot{H}_s \times \dot{H}_s$ — is continuous on $\dot{H}_1 \times \dot{H}_1$. Its continuous extension on $\dot{H}_1 \times \dot{H}_1$ considered as a map from $\dot{H}_s \times \mathbf{R}^2$ into $L(\dot{H}_1, \dot{H}_1; \mathbf{R})$ is analytic at* $(0; 0, 1)$.

Proof: Let

$$E(\zeta; \mu, (1 + \varepsilon) q_1^{\mathrm{cr}}(\mu), q_2^{\mathrm{cr}}(\mu)) = \sum_{i,j,k\geqq 0} \varepsilon^i (\mu - 1)^j E_{ijk+2}(\zeta^{k+2})$$

be the power series expansion of E; E_{ijk+2} denoting certain symmetric and continuous $(k + 2)$-linear forms in $\zeta^{k+2} = (\zeta, \ldots, \zeta) \in \dot{H}_s^{k+2}$. Analyticity of

$$DE(\zeta; \mu, (1 + \varepsilon) q_1^{\mathrm{cr}}, q_2^{\mathrm{cr}}) = \sum_{i,j,k\geqq 0} (k + 2) \, \varepsilon^i (\mu - 1)^j E_{ijk+2}(\zeta^{k+1}, \cdot)$$

as a mapping from $\dot{H}_s \times \mathbf{R}^2$ into $\dot{H}_{s-2}$ — as referred to in Corollary 2.1 — by definition means convergence of

$$\sum_{i,j,k\geqq 0} (k + 2) \, \|E_{ijk+2}\| \, \varepsilon^i (\mu - 1)^j z^{k+1} \tag{2.1}$$

in some neighbourhood of $(0; 0, 1)$ in $\mathbf{R}^3$ where $\|E_{ijk+2}\|$ is defined by

$$\|E_{ijk+2}\| = \sup_{\|\zeta\|_s, \|h\|_{2-s} \leqq 1} |E_{ijk+2}(\zeta^{k+1}, h)| .$$

Since

$$|E_{ijk+2}(\zeta^k, h_1, h_2)| \leqq \frac{(k+1)^{k+1}}{(k+1)!} \|E_{ijk+2}\| \, \|\zeta\|_s{}^k \, \|h_1\|_s \, \|h_2\|_{2-s},$$

(cf. [4]) we get by interpolation

$$|E_{ijk+2}| := \sup_{\|\zeta\|_s, \|h_1\|_1, \|h_2\|_1 \leqq 1} |E_{ijk+2}(\zeta^k, h_1, h_2)| \leqq C \frac{(k+1)^{k+1}}{(k+1)!} \|E_{ijk+2}\| \tag{2.2}$$

where the constant C is independent of i, j, k (see e.g. [6]). From (2.1), (2.2) we deduce the convergence of

$$\sum_{i,j,k \geqq 0} (k+2)(k+1) |E_{ijk+2}| \, \varepsilon^i (\mu - 1)^j z^k$$

in a neighbourhood of $(0; 0, 1) \in \mathbf{R}^3$ and hence the analyticity of

$$D^2E(\zeta;, \mu, (1+\varepsilon) q_1{}^{\mathrm{cr}}; q_2{}^{\mathrm{cr}}) = \sum_{i,j,k \geqq 0} (k+2)(k+1) \varepsilon^i (\mu - 1)^j E_{ijk+2}(\zeta^k, \cdot, \cdot)$$

at $(0; 0, 1) \in \dot{H}_s \times \mathbf{R}^2$ considered as a mapping from $\dot{H}_s \times \mathbf{R}^2$ into $L(\dot{H}_1, \dot{H}_1; \mathbf{R})$ ∎

In the following let $Lh = -\Delta h + \frac{4}{3} h - \frac{4}{\sqrt{3}} Ah$ denote the linear operator defined by the quadratic form (1.24) at $(\mu, q_1, q_2) = \left(1, q_1{}^{\mathrm{cr}}(1), q_2{}^{\mathrm{cr}}(1)\right) = \left(1, 1, \frac{2}{\sqrt{3}}\right)$. Obviously $L \in L(\dot{H}_s, \dot{H}_{s-2})$ for any real $s \geqq 2$, its range in $\dot{H}_{s-2}$ being $\dot{H}_{s-2} \ominus N_6$. Further:

L acts as an isomorphism onto $\dot{H}_{s-2} \ominus N_6$ when restricted to $\dot{H}_s \ominus N_6$. (*)

We are now in position to solve the equilibrium condition

$$\left\langle DE\left(\zeta; \mu, (1+\varepsilon) q_1{}^{\mathrm{cr}}(\mu), q_2{}^{\mathrm{cr}}(\mu)\right), h\right\rangle = 0, \quad \zeta \in \dot{H}_s, \quad \forall\, h \in \dot{H}_s \tag{2.3}$$

near $(\zeta; \varepsilon, \mu) = (0; 0, 1)$ for ζ. According to Corollary 2.1

$$E\left(\zeta; \mu, (1+\varepsilon) q_1{}^{\mathrm{cr}}(\mu), q_2{}^{\mathrm{cr}}(\mu)\right)$$
$$= \frac{1}{2}(L\zeta, \zeta) + \sum_{i+j>0} \varepsilon^i (\mu - 1)^j E_{ij2}(\zeta^2) + \sum_{i,j,k \geqq 0} \varepsilon^i (\mu - 1)^j E_{ijk+3}(\zeta^{k+3}) \tag{2.4}$$

where $E_{003} = E_{103} = 0$ and

$$\sum_{i+j>0} \varepsilon^i (\mu - 1)^j E_{ij2}(\zeta^2)$$
$$= \frac{(q_2{}^{\mathrm{cr}})^2}{2} |\mathscr{P}| \sum_{\omega \in \Lambda'} Q\left(\frac{|\omega|}{q_2{}^{\mathrm{cr}}}, \mu, (1+\varepsilon) q_1{}^{\mathrm{cr}}\right) |\zeta_\omega|^2 - \frac{1}{2}(L\zeta, \zeta),$$

$$E_{013} = \frac{1}{\sqrt{3}} \left(\left(\frac{1}{2} A\zeta^2 - \zeta A\zeta, A\zeta\right) + \left(\zeta^2, \Delta e^{-\sqrt{3}\alpha A}\zeta\right)\right), \tag{2.5}$$

$$E_{004} = \frac{1}{6\sqrt{3}} \left(3(\zeta^2, A^3\zeta^2) - 4(\zeta, A^3\zeta^3)\right) - \frac{1}{8} \int_{\mathscr{P}} |\nabla\zeta|^4 \, dx_1 \, dx_2,$$

cf. (1.22)–(1.26).

Remark 2.1: Note that (i): $E_{ij2}(\zeta, h) = 0$ $(i + j > 0)$ when $\zeta \in \dot{H}_s \ominus N_6$ and $h \in N_6$ and (ii):

$$\sum_{i+j>0} \varepsilon^i(\mu - 1)^j E_{ij2}(\zeta, h) = \frac{(q_2^{cr})^2}{2} Q(\vartheta^{cr}, \mu, (1 + \varepsilon) q_1^{cr}) (\zeta, h)$$

$$= -\frac{2}{\sqrt{3}} (\zeta, h) q_1^{cr} q_2^{cr} \varepsilon \frac{1 - \frac{\mu - 1}{\mu + 1} e^{-2\alpha\vartheta^{cr}}}{1 - \left(\frac{\mu - 1}{\mu + 1}\right)^2 e^{-2\alpha\vartheta^{cr}}}$$

if $\zeta, h \in N_6$ as a consequence of (2.5) and Lemma 1.2.

Let denote

$$E^{red} = \sum_{i+j>0} \varepsilon^i(\mu - 1)^j E_{ij2}(\zeta^2) + (\mu - 1) E_{013}(\zeta^3) + E_{004}(\zeta^4)$$

and E^{res} the higher order terms $(i + j + k \geqq 2)$ in (2.4):

$$E(\zeta; \mu, (1 + \varepsilon) q_1^{cr}, q_2^{cr}) = \frac{1}{2} (L\zeta, \zeta) + E^{red} + E^{res}.$$

Setting $\zeta = \zeta_1 + \zeta_2$ $(\zeta_1 \in N_6, \zeta_2 \in \dot{H}_s \ominus N_6)$ then (2.3) will be equivalent to

$$(L\zeta_2, h) = -\langle (DE^{red} + DE^{res}) (\zeta_1 + \zeta_2; \varepsilon, \mu), h\rangle \quad \forall\, h \in \dot{H}_s \ominus N_6, \quad (2.6)$$

$$0 = \langle (DE^{red} + DE^{res}) (\zeta_1 + \zeta_2; \varepsilon, \mu), h\rangle \quad \forall\, h \in N_6. \quad (2.7)$$

Because of Corollary 2.1 the linear functional on the right-hand side actually belongs to $\dot{H}_{s-2}$. Thus, according to (*) equation (2.6) can be solved for ζ_2 via the contraction mapping theorem:

$$\zeta_2 = \sum_{i,j\geqq 0, k\geqq 1} \varepsilon^i(\mu - 1)^j Z_{ijk}(\zeta_1^k). \quad (2.8)$$

where $Z_{ij1} = Z_{002} = 0$ $(i, j \geqq 0)$ by comparison of coefficients (cf. Remark 2.1). Substituting (2.8) into (2.7) we get the bifurcation equations

$$\langle DE^{red}(\zeta_1; \varepsilon, \mu), h\rangle + \text{h.o.t.} = 0 \qquad \text{for all } h \in N_6, \quad (2.9)$$

the higher order terms (h.o.t.) being of order $\varepsilon^i(\mu - 1)^j \|\zeta_1\|^{k+2}$ $(i + j + k \geqq 2)$. Introducing on N_6 the real valued Fourier-coefficients $\zeta_{\omega_1'} = a_1 - ib_1$, $\zeta_{\omega_2'} = a_2 - ib_2$, $\zeta_{-\omega_1'-\omega_2'} = a_3 - ib_3$ and setting

$$\sigma_2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2,$$

$$\sigma_3 = a_1a_2a_3 - a_1b_2b_3 - a_2b_3b_1 - a_3b_1b_2,$$

$$\sigma_4^{(1)} = (a_1^2 + b_1^2)^2 + (a_2^2 + b_2^2)^2 + (a_3^2 + b_3^2)^2,$$

$$\sigma_4^{(2)} = (a_1^2 + b_1^2)(a_2^2 + b_2^2) + (a_2^2 + b_2^2)(a_3^2 + b_3^2) + (a_3^2 + b_3^2)(a_1^2 + b_1^2)$$

it is easy to check that for $\zeta_1 \in N_6$

$$E^{red} = |\mathcal{P}| \left\{-\frac{8}{3}(1 + O(\mu - 1))\,\varepsilon\sigma_2 - \frac{8}{\sqrt{3}}(1 + 2e^{-2\alpha})(\mu - 1)\,\sigma_3 + \frac{20}{9}\sigma_4^{(1)} + \frac{8}{3}\left(4\sqrt{3} - 5\right)\sigma_4^{(2)}\right\}. \quad (2.10)$$

We now establish the existence of two types of solutions for which (2.9) reduces to a scalar equation. Considering the translational invariance of the energy functional we may assume $b_1 = b_2 = 0$ without loss in generality:

(I): $a_2 = a_3 = b_3 = 0$, then (2.9) reduces to

$$\frac{\partial}{\partial a_1} E^{\text{red}}(\zeta_1; \varepsilon, \mu) + \text{h.o.t.} = 0, \qquad \zeta_1 = 2a_1 \cos \omega_1' x, \tag{2.11}$$

(II): $a_1 = a_2 = a_3$, $b_3 = 0$, then (2.9) reduces to

$$\begin{aligned} &\frac{\partial}{\partial a_1} E^{\text{red}}(\zeta_1; \varepsilon, \mu) + \text{h.o.t.} = 0, \\ &\zeta_1 = 2a_1(\cos \omega_1' x + \cos \omega_2' x + \cos (\omega_1' + \omega_2') x). \end{aligned} \tag{2.12}$$

Remark 2.2: As a consequence of the invariance of E under the group of rigid motions every solution of (2.11) (resp. (2.12)) satisfies the complete equilibrium conditions. For, let T_τ be the representation of the translational group defined on $\dot{H}_s$ as usual. Then

$$\left\langle DE(\zeta), \frac{\partial \zeta}{\partial x} \right\rangle = \left\langle DE(\zeta), \frac{\partial \zeta}{\partial y} \right\rangle = 0 \qquad \text{for all } \zeta \in \dot{H}_s,$$

$$\langle DE(\zeta), h \rangle = \langle DE(T_\tau \zeta), T_\tau h \rangle \qquad \text{for all } \zeta, h \in \dot{H}_s,$$

hence $\zeta_2(T_\tau \zeta_1) = T_\tau \zeta_2(\zeta_1)$ in (2.8). In particular, if ζ_1 is a solution of (2.11) then

$$\begin{aligned} \langle DE(\zeta_1 + \zeta_2), 2a_1 \sin \omega_1' x \rangle &= - \left\langle DE(\zeta_1 + \zeta_2), \frac{\partial \zeta_1}{\partial x} \right\rangle \\ &= - \left\langle DE(\zeta_1 + \zeta_2), \frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial x} \right\rangle = 0 \end{aligned}$$

in virtue of $\frac{\partial \zeta_2}{\partial x} \in \dot{H}_{s-1} \ominus \underset{\sim}{N}_6$. Further, if $\tau: x \to x + \frac{\omega_2}{2}$

$$\langle DE(\zeta_1 + \zeta_2), e^{i\omega_2' x} \rangle = \langle DE(\zeta_1 + \zeta_2), T_\tau e^{i\omega_2' x} \rangle = -\langle DE(\zeta_1 + \zeta_2), e^{i\omega_2' x} \rangle,$$

whence $\langle DE(\zeta_1 + \zeta_2), e^{i\omega_2' x} \rangle = 0$ as desired. Similarly $\langle DE(\zeta_1 + \zeta_2), e^{-i(\omega_1' + \omega_2')x} \rangle = 0$ Similar considerations apply to solutions of (2.12).

Returning to (2.11), (2.12) and applying the Weierstrass Preparation Theorem we arrive at the "reduced" bifurcation equation

$$\frac{\partial}{\partial a_1} E^{\text{red}}(\zeta_1; \varepsilon, \mu) = 0. \tag{2.13}$$

Solving (2.13) we get in view of (2.10)

$$\text{(I)}: \; a_1^{\pm} \approx \pm \sqrt{\frac{3}{5}\, \varepsilon} \quad (\varepsilon \geqq 0), \tag{2.14}$$

$$\begin{aligned} \text{(II)}: \; a_1^{\pm} \approx\; & 0.078(1 + 2e^{-2\alpha})(\mu - 1) \\ & \pm \{0.078^2(1 + 2e^{-2\alpha})^2 (\mu - 1)^2 + 0.181\varepsilon\}^{1/2} \end{aligned} \tag{2.15}$$

in cases (I), (II) respectively. In fact both solutions (2.14) coincide under translation $x \to x + \frac{\omega_1}{2}$.

Remark 2.3: In (2.14), (2.15) those terms from (2.13) have been dropped which carry no information about the actual solution. Observe that (2.15) is sufficient as an approximation to (2.13) only in a restricted neighbourhood $|\varepsilon| < \varepsilon_0(\mu - 1)^2$ of (0, 1). Further: Suitable rescaling of (2.11) resp. (2.12) via the Implicit Function Theorem leads to power series expansions

$$\text{(I)}: \quad a_1^{\pm} = \pm\sqrt{\varepsilon}\left\{\sqrt{\frac{3}{5}} + \sum_{i+j>0} a_{ij}\varepsilon^i(\mu - 1)^j\right\},$$

$$\text{(II)}: \quad a_1^{+} = (\mu - 1)\left\{0.156(1 + 2e^{-2\alpha}) + \sum_{i+j>0} a_{ij}^{+}\left(\frac{\varepsilon}{(\mu - 1)^2}\right)^i (\mu - 1)^j\right\}$$

$$a_1^{-} = (\mu - 1)\left\{-\frac{2}{\sqrt{3}}(1 + 2e^{-2\alpha})^{-1}\frac{\varepsilon}{(\mu - 1)^2} + \sum_{i+j>0} a_{ij}^{-}\left(\frac{\varepsilon}{(\mu - 1)^2}\right)^i (\mu - 1)^j\right\}.$$

We conclude this section with a stability result. By definition stability of a solution $\zeta \in \dot{H}_s$ means

$$D^2E(\zeta)\,\{h^2\} > 0 \qquad \text{for all } h \in \dot{H}_s,\ h \neq 0. \tag{2.16}$$

Considering the translational invariance of E which implies

$$D^2E(\zeta)\,\{\zeta_x, h\} = D^2E(\zeta)\,\{\zeta_y, h\} = 0 \qquad \text{for all } h \in \dot{H}_s$$

we have to impose some additional constraint, e.g. $h \perp \dfrac{\partial\zeta_1}{\partial x}, \dfrac{\partial\zeta_1}{\partial y}$ in (2.16).

Theorem 2.2: *Solutions of type* I ($\varepsilon \geqq 0$) *and* II$^-$ *lead to unstable equilibria, whereas the branch* II$^+$ *is stable in the sense above.*

Proof: We first study the branch II$^+$. Inspecting the expansion of the second variation D^2E along our solution $\zeta = \zeta_1 + \zeta_2$ ($\zeta_1 \in N_6$, $\zeta_2 \in \dot{H}_s \ominus N_6$) yields

$$\begin{aligned} &D^2E\big(\zeta, \mu, (1 + \varepsilon)\, q_1^{cr}(\mu), q_2^{cr}(\mu)\big)\,\{h^2\} = (Lh, h) \\ &+ 6(\mu - 1)\, E_{013}(\zeta_1, h^2) + 12E_{004}(\zeta_1^2, h^2) + \text{h.o.t.}, \end{aligned} \tag{2.17}$$

the higher order terms being of order $\left(\dfrac{\varepsilon}{(\mu - 1)^2}\right)^i (\mu - 1)^{2+j}$, $i + j > 0$. Let $h = h_1 + h_2$ ($h_1 \in N_6$, $h_2 \in \dot{H}_s \ominus N_6$), then

$$(Lh, h) \geqq c\|h_2\|_1^2,\ c > 0,$$

$$|(\mu - 1)\, E_{013}(\zeta_1, h_1, h_2)|,\ |E_{004}(\zeta_1^2, h_1, h_2)|$$

$$\leqq C(\mu - 1)^2\, \|h_1\|_1\, \|h_2\|_1 \leqq \frac{C}{2}\left(|\mu - 1|^3\, \|h_1\|^2 + |\mu - 1|\, \|h_2\|_1^2\right).$$

(cf. Corollary 2.2). Thus, for (ε, μ) in a sufficiently small neighbourhood $|\varepsilon| \leqq \varepsilon_0 \times (\mu - 1)^2$ of (0, 1), positivity of (2.17) is implied by that of

$$6(\mu - 1)\, E_{013}(\zeta_1, h_1^2) + 12E_{004}(\zeta_1^2, h_1^2) \tag{2.18}$$

this being true even if we replace ζ_1 by its first approximation $0{,}312(1 + 2e^{-2\alpha}) \times (\mu - 1)\left(\cos\omega_1' x + \cos\omega_2' x + \cos(\omega_1' + \omega_2')\, x\right)$. Since the eigenvalues

$$(\lambda_1, \lambda_{2,3}, \lambda_4, \lambda_{5,6}) = |\mathcal{P}|\,(0{,}72,\ 1{,}37,\ 2{,}17,\ 0)\,(1 + 2e^{-2\alpha})^2\,(\mu - 1)^2$$

of the so modified form (2.18) are positive (with the exception of $\lambda_{5,6}$) the branch II$^+$ is stable on both sides of criticality.

Similarly

$$D^2E(\zeta; \mu, (1+\varepsilon) q_1^{cr}, q_2^{cr}) \{h^2\} = (Lh, h)$$

$$+ 2\varepsilon E_{102}(h^2) + 6(\mu - 1) E_{013}(\zeta_1, h^2) + \text{h.o.t.}$$

along the branch II$^-$, the higher order terms now being of order $\left(\frac{\varepsilon}{(\mu-1)^2}\right)^{1+i} \times (\mu - 1)^{2+j}$, $i + j > 0$. Thus, on a neighbourhood of $(0, 1)$ as above, its sign is determined by that of

$$2\varepsilon E_{102}(h_1^2) + 6(\mu - 1) E_{013}(\zeta_1, h_1^2). \tag{2.19}$$

Replacing ζ_1 in (2.19) by

$$-\frac{4}{\sqrt{3}} (1 + 2e^{-2\alpha})^{-1} \varepsilon(\mu - 1)^{-1} \left(\cos \omega_1' x + \cos \omega_2' x + \cos (\omega_1' + \omega_2') x\right)$$

we get

$$(\lambda_1, \lambda_{2,3}, \lambda_4, \lambda_{5,6}) = \frac{16}{3} |\mathcal{P}| (1, -2, -3, 0) \varepsilon$$

for the corresponding eigenvalues. Hence the solution II$^-$ turns out to be unstable always.

Concerning the branch I one finds

$$D^2E\left(\zeta; \mu, (1+\varepsilon) q_1^{cr}, q_2^{cr}\right) \{h^2\} = (Lh, h) + 2\varepsilon E_{012}(h^2)$$

$$+ 6(\mu - 1) E_{013}(\zeta_1, h^2) + 12 E_{004}(\zeta_1^2, h^2) + \text{h.o.t.}$$

with h.o.t. of order $\sqrt{\varepsilon}^{1+i}(\mu - 1)^j$, $i + j > 1$. Thus its stability is determined by the eigenvalues of $2\varepsilon E_{012}(h_1^2) + 6(\mu - 1) E_{013}(\zeta_1, h_1^2) + 12 E_{004}(\zeta_1^2, h_1^2)$. After replacing ζ_1 by (2.14) we get for their values

$$(\lambda_1, \lambda_{2,3}, \lambda_{4,5}, \lambda_6) = |\mathcal{P}| \big(10{,}66,\ 0{,}83\varepsilon \pm 3{,}57\,(1 + 2e^{-2\alpha}) \sqrt{\varepsilon}\,(\mu - 1),$$

$$0{,}83\varepsilon \mp 3{,}57(1 + 2e^{-2\alpha}) \sqrt{\varepsilon}\,(\mu - 1), 0\big),$$

showing the (supercritical) solution I to be unstable ∎

Remark 2.4: Concerning the value of E at a solution II$^+$ we get

$$E(\zeta; \mu, (1+\varepsilon) q_1^{cr}, q_2^{cr}) = (\mu - 1) E_{013}(\zeta_1^3) + E_{004}(\zeta_1^4) + \text{h.o.t.}$$

$$= -0{,}004\, |\mathcal{P}| (1 + 2e^{-2\alpha})^4 (\mu - 1)^4 + \text{h.o.t.} < 0$$

with higher order terms $\left(\frac{\varepsilon}{(\mu-1)^2}\right)^i (\mu - 1)^{4+j}$, $i + j > 0$.

§ 3

We now pass to the proof of Theorem 2.1. We adopt the following notation: For any open interval I on the z-axis let $\|u\|_I$ denote the norm of u in $L_2(I)$. Let

$$W_{m,I} = \left\{v \in L_2(I, H_m) : v^{(m)} = \frac{\partial^m v}{\partial z^m} \in L_2(I, H_0)\right\}$$

be the Sobolev space of Λ-periodic functions

$$v(x, y, z) = \sum_{\omega \in \Lambda'} v_\omega(z)\, e^{i\omega x}$$

with distributional derivatives up to order m in $L_2(\mathcal{P} \times I)$, the derivatives up to order $m - 1$ being Λ-periodic again. Choose

$$\|v\|_{m,I}^2 = \|v_0\|_I^2 + \sum_{\omega \in \Lambda'} (|\omega|^{2m} \|v_\omega\|_I^2 + \|v_\omega^{(m)}\|_I^2)$$

to be the norm in $W_{m,I}$. Similarly, for $m \geqq 1$, let

$$V_{m,I} = \{v \in \mathcal{D}'(I, H_m) : v', v^{(m)} \in L_2(I, H_0)\}$$

normed by

$$|v|_{m,I}^2 = \|v_0'\|_I^2 + \sum_{\omega \in \Lambda'} (|\omega|^{2m} \|v_\omega\|_I^2 + \|v_\omega^{(m)}\|_I^2). \tag{3.1}$$

In the following for $I^- = (-\infty, -q_0)$, $I^{\text{fl}} = (-q_0, 0)$, $I^+ = (0, +\infty)$ we shall consider the various spaces $L_2(I^-) \times L_2(I^{\text{fl}}) \times L_2(I^+)$, $W_m = W_{m,I^-} \times W_{m,I^{\text{fl}}} \times W_{m,I^+}$, $V_m = V_{m,I^-} \times V_{m,I^{\text{fl}}} \times V_{m,I^+}$ the corresponding norms of which we denote by $\|\cdot\|$, $\|\cdot\|_m$, $|\cdot|_m$ resp. Further let $\mathcal{S}^\pm = \mathcal{P} \times I^\pm$, $\mathcal{S}^{\text{fl}} = \mathcal{P} \times I^{\text{fl}}$. Accordingly, we write $v = (v^-, v^{\text{fl}}, v^+)$ for a function belonging to W_m (resp. V_m).

Lemma 3.1: *Let $\mu > 0$ and $\mathfrak{F} \in (W_m)^3$; then the unique $v \in V_1$ which satisfies $v^+(x, y, 0) - v^{\text{fl}}(x, y, 0) = \text{const}$, $v^{\text{fl}}(x, y, -q_0) - v^-(x, y, -q_0) = \text{const}$. and*

$$\int_{\mathcal{S}^- \cup \mathcal{S}^+} \nabla v\, \nabla \bar{\varphi}\, dV + \mu \int_{\mathcal{S}^{\text{fl}}} \nabla v\, \nabla \bar{\varphi}\, dV = \int_{\mathcal{S}^- \cup \mathcal{S}^+} \mathfrak{f}\, \nabla \bar{\varphi}\, dV + \int_{\mathcal{S}^{\text{fl}}} \mathfrak{f}\, \nabla \bar{\varphi}\, dV \tag{3.2}$$

for every $\varphi \in V_1$ (satisfying the homogeneous jump conditions) belongs to V_{m+1}. Moreover

$$|v|_{m+1} \leqq C \|\mathfrak{f}\|_m \tag{3.3}$$

with a constant C independent of $\mathfrak{f}$.

Proof: For convenience we assume $\mu = 1$. Getting the estimate (3.3) for general $\mu > 0$ requires minor modifications only. Let $\mathfrak{f} = (f_1, f_2, f_3)$ and

$$f_j = \sum_{\omega \in \Lambda'} f_{j,\omega}(z)\, e^{i\omega x} \qquad (j = 1, 2, 3)$$

its Fourier expansion (notice: $f_j = (f_j^-, f_j^{\text{fl}}, f_j^+)$). We set $\mathfrak{f}_\omega = (f_{1,\omega}, f_{2,\omega}, f_{3,\omega})$ and $\omega f_\omega = \omega(f_{1,\omega}, f_{2,\omega})$. Obviously the Fourier coefficients $v_\omega(z)$ of our solution have to satisfy the variational equations

$$\int_{I^- \cup I^+} (v_\omega' \bar{\varphi}_\omega' + |\omega|^2 v_\omega \bar{\varphi}_\omega)\, dz + \int_{I^{\text{fl}}} (v_\omega' \bar{\varphi}_\omega' + |\omega|^2 v_\omega \bar{\varphi}_\omega)\, dz$$
$$= \int_{I^- \cup I^+} (f_{3,\omega} \bar{\varphi}_\omega' - i\omega f_\omega \bar{\varphi}_\omega)\, dz + \int_{I^{\text{fl}}} (f_{3,\omega} \bar{\varphi}_\omega' - i\omega f_\omega \bar{\varphi}_\omega)\, dz \tag{3.4}$$

subject to the jump conditions $v_0^+(0) - v_0^{\text{fl}}(0) = \text{const.}$, $v_0^{\text{fl}}(-q_0) - v_0^-(-q_0) = \text{const.}$, resp. $v_\omega^+(0) - v_\omega^{\text{fl}}(0) = 0$, $v_\omega^{\text{fl}}(-q_0) - v_\omega^-(-q_0) = 0$ if $\omega \neq 0$. Choosing the test function φ_ω in (3.4) to be v_ω and applying Schwarz's inequality we get

$$\|v_\omega'\|^2 + |\omega|^2 \|v_\omega\|^2 \leqq \|\mathfrak{f}_\omega\| (\|v_\omega'\| + |\omega|\, \|v_\omega\|),$$

whence

$$\|v_\omega'\|^2 + |\omega|^2 \|v_\omega\|^2 \leqq 2\, \|\mathfrak{f}_\omega\|^2. \tag{3.5}$$

This proves (3.3) for $m = 0$. Likewise by differentiating the Euler-Lagrange equations to (3.4) we get

$$-v_\omega^{(m+1)} + |\omega|^2 v_\omega^{(m-1)} = -i\omega f_\omega^{(m-1)} - f_{3,\omega}^{(m)} \qquad (m \geqq 1).$$

Thus

$$\|v_\omega^{(m+1)}\|^2 \leqq 3(|\omega|^4 \|v_\omega^{(m-1)}\|^2 + |\omega|^2 \|f_\omega^{(m-1)}\|^2 + \|f_{3,\omega}^{(m)}\|^2).$$

Applying the well known inequality

$$\|u_\omega^{(m)}\|^2 \leqq \text{const.}\left(\varepsilon^k \|u_\omega^{m+k}\|^2 + \frac{1}{\varepsilon^{m-l}} \|u_\omega^{(l)}\|^2\right), \quad \varepsilon > 0$$

and recalling (3.1), (3.5) gives the desired estimate for $m \geqq 1$ ∎

Proof of Theorem 2.1: The strategy is to transform (1.5) into a variational problem posed on the fixed domain $(\mathscr{S}^\pm, \mathscr{S}^{\text{fl}})$. Note that by interpolation it is sufficient to assume $s = m + 1/2$, $2 \leqq m \in \mathbf{N}$.

Now, for $(\zeta, q) \in \dot{H}_{m+1/2} \times \mathbf{R}$ in a neighbourhood of $(0, -q_0)$ let

$$x_1 = x,\ x_2 = y,\ x_3 = z + w(x, y, z) \tag{3.6}$$

define a diffeomorphism from $\mathscr{S}^-$ (resp. $\mathscr{S}^{\text{fl}}$, $\mathscr{S}^+$) as defined above onto Ω^- (resp. Ω^{fl}, Ω^+), cf. (1.1). As the example

$$w^- = q_0 - q,\ w^+ = \sum_{\omega \neq 0} \zeta^\omega\, e^{i\omega x - |\omega| z},$$

$$w^{\text{fl}} = \frac{q_0 + q}{-q_0} z + \sum_{\omega \neq 0} \zeta_\omega\, e^{i\omega x}\left(\frac{e^{-|\omega|z}}{1 - e^{2|\omega|q_0}} + \frac{e^{|\omega|z}}{1 - e^{-2|\omega|q_0}}\right)$$

shows, it is always possible to require: 1st *the transition function* $w = (w^-, w^{\text{fl}}, w^+)$ *to belong to* V_{m+1} *and* 2$^{\text{nd}}$ *the map*

$$(\zeta, q)\ (\in \dot{H}_{m+1/2} \times \mathbf{R}) \to w \in V_{m+1}$$

to be analytic at $(0, -q_0)$. Let $w = 0$ when $(\zeta, q) = (0, -q_0)$.

According to (3.6) the variational problem (1.5), (1.6) transforms into

$$\int_{\mathscr{S}^- \cup \mathscr{S}^+} \left(|\nabla v|^2 (1 + w_z) - 2 v_z\, \nabla v\, \nabla w + \frac{v_z^2\, |\nabla w|^2}{1 + w_z}\right) dV$$

$$+ \mu \int_{\mathscr{S}^{\text{fl}}} \left(|\nabla v|^2 (1 + w_z) - 2 v_z\, \nabla v\, \nabla w + \frac{v_z^2\, |\nabla w|^2}{1 + w_z}\right) dV \to \min \tag{3.7}$$

$(dV = dx\, dy\, dz)$, which has to be solved for $v(x, y, z)\ \big(= u(x_1, x_2, x_3)\big)$ subject to $v^+(x, y, 0) - v^{\text{fl}}(x, y, 0) = \zeta(x, y) + \text{const}$, $v^{\text{fl}}(x, y, -q_0) - v^-(x, y, -q_0) = \text{const}$. To show, in a first step, the analytic dependence of its solution v on (ζ, μ, q) near $(0, 1, q_0)$ set $v = v_1 + v_2$ where

$$v_1 = \frac{1}{2} \operatorname{sgn} z \sum_{\omega \neq 0} \zeta_\omega\, e^{i\omega x + |\omega||z|}$$

is the solution to

$$\int_{\mathscr{S}^- \cup \mathscr{S}^+} |\nabla v_1|^2\, dV + \int_{\mathscr{S}^{\text{fl}}} |\nabla v_1|^2\, dV \to \min$$

subject to $v_1^+(x, y, 0) - v_1^{\text{fl}}(x, y, 0) = \zeta(x, y) + \text{const.}$, $v_1^{\text{fl}}(x, y, -q_0) - v_1^-(x, y, -q_0) = \text{const}$. Notice $|v_1|_{\widehat{m+1}} = \dfrac{1}{\sqrt{2}} \|\zeta\|_{m+1/2}$. Then, collecting higher order terms $\mathfrak{f} = \mathfrak{f}(\zeta, \mu, q; v_2)$ and denoting $e_z = (0, 0, 1)$:

$$\mathfrak{f}^\pm = -w_z\, \nabla v + v_z\, \nabla w + \left(\nabla v\, \nabla w - \frac{v_z\, |\nabla w|^2}{1 + w_z}\right) e_z,$$

$$\mathfrak{f}^{\text{fl}} = -(\mu - 1)\, \nabla v_1 - \mu w_z\, \nabla v + \mu v_z\, \nabla w + \mu\left(\nabla v\, \nabla w - \frac{v_z\, |\nabla w|^2}{1 + w_z}\right) e_z \tag{3.8}$$

on the right-hand side, v_2 has to satisfy a variational equation subject to homogeneous jump conditions as referred to in Lemma 3.1. If $T \in L(W_m{}^3, V_{m+1})$ denotes the solution map for (3.2) this equation will be equivalent to

$$v_2 = T\{\mathfrak{f}(\zeta, \mu, q; v_2)\}. \tag{3.9}$$

Recall that the spaces W_s form Banach algebras provided that $s > 3/2$ (see [1]). Hence, under the assumption $m \geqq 2$, mapping $\mathfrak{f}$ which transforms $(\zeta, \mu, q; v_2) \in \dot{H}_{m+1/2} \times \mathbf{R}^2 \times V_{m+1}$ according to (3.8) into $\mathfrak{f}(\zeta, \mu, q; v_2) \in W_m$ turns out to be analytic at $(0, 1, q_0, 0)$. Thus we can solve (3.9) via the contraction mapping theorem for $v_2 \in V_{m+1}$ as an analytic function of $(\zeta, \mu, q) \in \dot{H}_{m+1/2} \times \mathbf{R}^2$ near $(0, 1, q_0)$. Obviously this implies $v = v_1 + v_2$ to be analytic too.

We proceed by expanding the minimal value (3.7). Its analytic dependence on $(v, w, \mu) \in V_1 \times V_{m+1} \times \mathbf{R}$ is easily seen by Sobolev's embedding theorem. Replacing v, w by its power series expansions we get analyticity of (3.7) as a function of $(\zeta, \mu, q) \in \dot{H}_{m+1/2} \times \mathbf{R}^2$. This proves part (i) of the theorem.

The remaining part (ii) now follows in a few lines. Observe the earlier formula (1.9) — obtained in § 1 by formal differentiation — actually to be valid in virtue of the present hypothesis. By transforming (1.9) according to (3.5) we get

$$\langle DF(\zeta), h\rangle = \int_{\mathcal{P}} \Phi(\nabla\zeta, \nabla v|_{z=0}|, \nabla w|_{z=0})\, h\, dx\, dy$$

where the integrand is analytic in its arguments. By the trace mapping theorem $\nabla v|_{z=0}$, $\nabla w|_{z=0} \in H_{m-1/2}$. Consequently $\Phi \in H_{m-1/2}$, since the spaces H_s are Banach algebras provided that $s > 1$. This finishes the proof ∎

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