On Capillary Free Surfaces without Gravity
E. MIERSEMANN CONTROLLANT SET ALL SE
 Dedicated to Professor Herbert Beckert of

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Dedicated to Professor Herbert Beckert on his sixty fifth birthday

Das nichtparametrische Kapillaritätsproblem ohne Schwerkraft läßt sich mittels einer Stromfunktion in ein Dirichiet-Problem für eine quasilineare Gleichung vom gemischten Typ über. fuhren. Wir beweisen ein Maximumprinzip für eine dern Gradienten der Stromfunktion zugeordnete Größe und Regularitätseigenschaften von $C^1(\overline{\Omega})$ -Lösungen in Gebieten Ω mit Ecken.

Непараметрическая задача капиллярности без действия силы тяжести с помощью функции тока может быть сведена к задаче Дирихле для квазилинейного уравнения cмешанного типа. Доказывается принцип максимума для величины, сопоставленной
градиенту функции тока, и свойства регулярности для C¹($\bar{\varOmega}$)-решений в областях Ω с огшее стове una Regularitatseigenschaften von C·(12)-Losungen in Gebieten 12 mit Ecken.
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turen. Wir beweisen ein Maximumppriazip für eine dem Gradienten der

The non-parametric capillary problem in the absence of gravity can be replaced by a Dirichiet problem for a quasilinear equation of mixed type by introducing a stream function. We prove a maximum principle for an expression depending on the gradient of the stream function and, furthermore, regularity properties of $C^1(\overline{\Omega})$ -solutions in-domains Ω with corners. blem in the absence of gravity can be replaced by a Dirichlet
on of mixed type by introducing a stream function. We prove
ression depending on the gradient of the stream function and,
se of $C^1(\overline{\Omega})$ -solutions in domain

1. Introduction

We consider the non-parametric capillary problem in the absence of gravity. One seeks a surface $S: u = u(x)$, of constant mean curvature *H*, defined over a simply connected and bounded base domain $\Omega \subset \mathbb{R}^2$, such that S meets vertical cylinder For a quasilinear equation
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in principle for an expression
in the non-paramy dividend based on the equation
of the boundary of the equa

walls over the boundary $\partial\Omega$ in a prescribed constant angle γ , where $0 \le \gamma \le \frac{\pi}{2}$. This problem leads to the equation

$$
\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{u_{x_i}}{\sqrt{1+|u_x|^2}} = 2H \quad \text{in} \quad \Omega
$$

with the boundary condition

order the non-parametric capillary problem in the absence of gravity. One surface *S*: *u* = *u*(*x*), of constant mean curvature *H*, defined over a simply red and bounded base domain *Ω* ⊂ **R**², such that *S* meets vertical cylinder

\nver the boundary
$$
\partial \Omega
$$
 in a prescribed constant angle γ , where $0 \leq \gamma \leq \frac{\pi}{2}$.

\noblem leads to the equation

\n
$$
\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{u_{x_i}}{\sqrt{1+|u_x|^2}} = 2H
$$
 in *Ω*\nboundary condition

\n
$$
\frac{\partial u}{\partial n}
$$
\nby and Covscus [2] and **FINN** [3]. Here *n* is the exterior unit normal on $\partial \Omega$,

\nby $\frac{\partial \Omega}{\partial n}$, $\frac{|\partial \Omega|}{|\partial \Omega|}$.

see FINN and CONCUS [2] and FINN [3]. Here *n* is the exterior unit normal on, $\partial\Omega$, $|u_x|^2 = u_{x_1}^2 + u_{x_2}^2$ and $2H = \frac{|\partial \Omega|}{|O|}$ $\partial \Omega$, (1.2)

v. Solid Eq. (1.2)

v. Solid Eq. (1.2)

cos y, where $|\Omega|$ is the area of Ω and $|\partial \Omega|$ the walls over the boundary

This problem leads t
 $\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{\partial}{\sqrt{1}}$ with the boundary c
 $\frac{\partial u}{\partial n}$ $\frac{\partial u}{\partial n}$ $\frac{1}{\sqrt{1 + |u_x|^2}}$ see FINN and CONCU

see FINN and CONCU
 $|u_x|^2 = u_x^2 + u_x^2$, and

length of Exist a surrace $S: u = u(x)$, or constant mean curvature H, defined over a simply

innercted and bounded base domain $\Omega \subset \mathbb{R}^2$, such that S meets vertical cylinder

in problem leads to the equation
 $\sum_{i=1}^2 \frac{\partial}{\partial x_i}$

The problem to find explicit geometric criteria for the existence of solutions of (1.1) , (1.2) has been met with only partial success up to the present time, see FINN [3]. It can be replaced by a Dirichlet problem for a quasilinear equation of mixed type by introducing a stream function. We show that the problem of existence of solu-

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tions for this problem can be reduced to a certain boundary gradient estimate. In the last part of this paper we study the behavior of $C^1(\overline{Q})$ -solutions of (1.1), (1.2) over domains with corners. We prove, in particular, that $C^1(\overline{\Omega})$ -solutions belong to the class $C^{1,\lambda}(\overline{\Omega})$, $0 < \lambda < 1$, if $0 < \alpha < \pi$ is satisfied for the interior angles α at the corners. 430 E. MIERSEMANN

2. The associated birthday and the section boundary gradient estimate.

1 last part of this paper we study the behavior of $C^1(\bar{\Omega})$ -solutions of (1.1) , (1.2)

domains with corners. We prove, in pa **f** this paper we study the b
th corners. We prove, in pa
), $0 < \lambda < 1$, if $0 < \alpha < \pi$
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initiating my studies in the
ns and for his constant inter
ciated Dirichlet problem
 $= \frac{u_{x_i}}{\sqrt{1+|u_x|^2}}$

Acknowledgement: I would like to express my gratitude to my teacher Professor Beckert for initiating my studies in the calculus of variations and in partial differen tial equations and for his constant interest in the progress of my work. *Acknowledgement*: I

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equations and for

The associated Dirichnell Price
 $v_i = \frac{u_{x_i}}{\sqrt{1 + |u|}}$

nce $(v_1 - Hx_1)_{x_i} + ($ would like to express my grat

iy studies in the calculus of varial constant interest in the prop

interest in the prop

hlet problem
 $\frac{1}{2}$ $(i = 1, 2)$.
 $\frac{1}{2}$ $- Hx_2x_1 = 0$, it is possible to Acknowledgement: I would like to expre

Beckert for initiating my studies in the calculations and for his constant interest

2. The associated Dirichlet problem

Define
 $v_i = \frac{u_{x_i}}{\sqrt{1+|u_x|^2}}$ $(i = 1, 2)$.

Since $(v_1 - Hx$

Define

$$
v_i = \frac{u_{x_i}}{\sqrt{1+|u_x|^2}} \qquad (i=1, 2).
$$

Since $(v_1 - Hx_1)_{x_1} + (v_2 - Hx_2)_{x_2} = 0$, it is possible to introduce a stream function ψ $\begin{align} \nu_i &= \frac{\nu_i}{\sqrt{1+|u|}} \ \text{since } (\nu_1 - Hx_1)_{x_1} + \nu_2 \ \text{Setting } \nu_{x_1} &= -\nu_2 + \nu_{x_2} &= \nu_1 - \nu_{x_1} \end{align}$ ress of my w

introduce a

introduce a
 $\langle s \rangle$, where s

On its smoc $\begin{aligned}\n\mathbf{v}_i &= \frac{u_{x_i}}{\sqrt{1+|u_x|^2}} \qquad (i=1,2). \\
\text{Since } (\mathbf{v}_1 - H\mathbf{x}_1)_{x_1} + (\mathbf{v}_2 - H\mathbf{x}_2)_{x_2} = 0, \text{ it is possible to introduce a stream function } \mathbf{v} \text{ by setting} \n\end{aligned}$ $\begin{aligned}\n\mathbf{v}_1 &= -\mathbf{v}_2 + H\mathbf{x}_2, \\
\mathbf{v}_2 &= -\mathbf{v}_1 - H\mathbf{x}_1. \\
\text{Suppose the boundary: } \partial\Omega \text{ is given by } \mathbf$

$$
\psi_{x_1} = -\nu_2 + Hx_2, \\
 \psi_{x_1} = \nu_1 - Hx_1.
$$

We assume that the boundary $\partial\Omega$ is piecewise smooth. On its smooth parts we have

$$
y_1 + |u_x|^2
$$

\nSince $(v_1 - Hx_1)x_1 + (v_2 - Hx_2)x_2 = 0$, it is possible to introduce a str
\nby setting
\n
$$
\psi_{x_1} = -v_2 + Hx_2,
$$
\n
$$
\psi_{x_2} = v_1 - Hx_1.
$$
\nSuppose the boundary $\partial\Omega$ is given by $x(s) = (x_1(s), x_2(s))$, where s is
\nWe assume that the boundary $\partial\Omega$ is piecewise smooth. On its smooth
\n
$$
\psi = \psi_x \dot{x}_1 + \psi_x \dot{x}_2 = (-\psi_2 + Hx_2) \dot{x}_1 + (\psi_1 - Hx_1) \dot{x}_2
$$
\n
$$
= \cos \gamma + H(x_2 \dot{x}_1 - \dot{x}_2 x_1).
$$
\nBy integrating from 0 to s we get
\n
$$
\psi(s) = \left(s - |\partial\Omega| \frac{|Q(s)|}{|Q|}\right) \cos \gamma \text{ on } \partial\Omega
$$
\nwhere $Q(s)$ is sketched in Figure 1. We mention that $\psi(|\partial\Omega|) = 0$.

$$
\psi(s) = \left(s - |\partial \Omega| \frac{|\Omega(s)|}{|\Omega|}\right) \cos \gamma \quad \text{on} \quad \partial \Omega
$$
 (2.2)

-

*(*2.1)

 $\frac{1}{\sqrt{2}}$

(2.3)

 $\frac{1}{2}$

where $\Omega(s)$ is sketched in Figure 1. We mention that $\psi(|\partial \Omega|) = 0$.

^S - • • - -

$$
u_{x_1} = \frac{1}{\sqrt{1 - F}} (v_{x_1} + Hx_1),
$$

$$
u_{x_1} = \frac{1}{\sqrt{1 - F}} (-v_{x_1}/+Hx_2),
$$

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the relation $u_{x_ix_i} = u_{x_ix_i}$ gives us the differential equation (we are using the summation convention)

$$
a_{ij}(p) \; \psi_{x,x_i} = 0 \quad \text{in} \quad \Omega
$$

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the relation $u_{x,x_1} = u_{x,x_1}$ gives us the differential equation (we are using the summa-

tion convention)
 $a_{ij}(p) \psi_{x_ix_j} = 0$ in Ω (2.4)

with the boundary condition (2.2) with the boundary condition (2.2) on $\partial\Omega$, where $p_1 = \psi_{x_1} - H_{x_2}$, $p_2 = \psi_{x_2} + H_{x_1}$ u_{x,x_1} gives us the differential equality
 $= 0$ in Ω

condition (2.2) on $\partial \Omega$, where p_1
 $a_{22} = (1 - p_1^2), a_{12} = a_{21} = p_1 p_2$
 a_{13} is of elliptic type since
 $e^2 \le a_{ij} \xi_i \xi_j \le |\xi|^2$ for all $\xi \in \mathbb{R}^2$.

$$
(1-F)|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^2.
$$

Let $\Omega = B_R(0)$ be a disk with radius R and the center at the coordinate origin. Then we have $\psi(s) = 0$ on $\partial\Omega$. If max $F < 1$ for a solution $\psi \in C^1(\overline{\Omega})$ of the equation (2.4) with the boundary condition $\psi = 0$ on $\partial\Omega$, then the maximum principle implies. $\psi = 0$ in Ω . From (2.3) we obtain immediately $(1 - x)^{|S|} \triangleq a_{iS|S|} \geq |S|$ for all $S \in \mathbb{R}^2$.
 $= B_R(0)$ be a disk with radius R and the center at the coordinate origin. Ther
 $\psi(s) = 0$ on $\partial\Omega$. If max $F < 1$ for a solution $\psi \in C^1(\overline{\Omega})$ of the equation (2.4

$$
u = -\frac{1}{H} \sqrt{1 - H^2 |x|^2} + \text{const.}
$$

•

If $u = -\frac{1}{H} \sqrt{1 - H^2 |x|^2} + \text{const.}$
If $\langle \gamma \rangle \leq \frac{\pi}{2}$ is satisfied, then the assumption max $F < 1$ is fulfilled.

Now we ask for domains with
$$
\psi(s) = 0
$$
 on $\partial\Omega$. The equation (2.2) implies that
\n
$$
x_1 \dot{x}_2 - x_2 \dot{x}_1 = \frac{2 |\Omega|}{|\partial \Omega|}
$$
\n(2.5)

on the smooth parts of $\partial\Omega$. By differentiating we have $x_1\ddot{x}_2 - x_2\ddot{x}_1 = 0$ and, since $x = -\varkappa n$ where \varkappa is the curvature of $\partial\Omega$ at $x(s)$, we can infer that If $\langle y \leq \frac{\pi}{2} \rangle$ is satisfied, then the assumption max $F < 1$ is fu

Now we ask for domains with $\psi(s) = 0$ on $\partial\Omega$. The equation (2.2)
 $x_1 \dot{x}_2 - x_2 \dot{x}_1 = \frac{2 |\Omega|}{|\partial \Omega|}$

on the smooth parts of $\partial\Omega$. By different

 $\alpha \cdot \frac{d |x|^2}{ds} = 0$ by using $n = (\dot{x}_2, -\dot{x}_1).$

Hence, since (2.5) must be satisfied on the smooth parts of $\partial\Omega$, we obtain regular m-goñs and domains which we get from these polygons by rounding off one or some corners by the.incircle with coordinate origin at the point of symmetry. The inequality

 $\max_{\mathbf{x}} F < 1$ is in these cases equivalent to *H* max $|x| < 1$. This means that $\gamma > \frac{\pi}{m}$ f we have $x_1\tilde{x}_2 - x_2\tilde{x}_1 = 0$ and, since

(*s*), we can infer that
 \dot{x}_1).
 about the point of $\partial\Omega$ **, we obtain regular

polygons by rounding off one or some

the point of symmetry. The inequality

nax** $|x| < 1$ **.** must be fulfilled. In fact, this is exactly the corner condition, see FINN [3], $\frac{\alpha}{2} + \gamma$ $> \frac{\pi}{2}$, where α is the interior angle at the corners.

3. A maximum principle for *^F*

Now we prove a maximum principle for *F* by using a method of Bernstein, see GILBARG and TRUDINGER [4: Chapter 14.1].

Theorem: Let $\psi \in C^1(\Omega) \cap C^3(\Omega)$ be a solution of (2.4) and assume that max $F(x, \theta)$ $f(x)$ $<$ 1. Then $\max F = \max F$. $\begin{array}{l} \text{$p\in C^1(\bar{\Omega})$ \cap}\ \emptyset\ \emptyset\ \emptyset\ \text{max}\ F=\begin{bmatrix} \text{max}\ \bar{\Omega} \end{bmatrix}\ \text{max}\ \rho\ \text{res}=\begin{bmatrix} \text{max}\ \text{max}\ \end{bmatrix} \end{array}$

Proof: We derive a differential equation of second order for *F*. The assertion then follows from the classical maximum principle. Set $f_1 = -Hx_2$, $f_2 = Hx_1$ and $F = p_1^2$

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+ p_2^2 , where $p_1 = \psi_{x_1} + f_1$, $p_2 = \psi_{x_2} + f_2$. We have

I,

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\n+
$$
p_2^2
$$
, where $p_1 = \psi_{x_1} + f_1$, $p_2 = \psi_{x_1} + f_2$. We have
\n $F_{x_i} = 2p_r p_r$,
\n $F_{x_ix_j} = 2p_{r,j} p_{r,i} + 2p_r \psi_{,rij}$, (3.1)
\n $p_{l,k} = p_{k,l} + (f_{l,k} - f_{k,l})$, (3.2)
\n(By writing v_i , we mean $\frac{\partial v}{\partial x_i}$.) By differentiating the equation (2.4) with respect
\nto x_k one obtains
\n $a_{ij}(p) \psi_{,kij} + D_{p_i} a_{ij}(p) p_{l,k} \psi_{,ij} = 0$
\nwhere $D_{p_i} = \frac{\partial}{\partial p_i}$. Multiplying by p_k and summing over k, we thus have

$$
a_{ij}(p) \psi_{,kij} + D_{p_i} a_{ij}(p) \ p_{l,k} \psi_{,ij} = 0
$$

where $D_{p_i} = \frac{\partial}{\partial p_i}$. Multiplying by p_k and summing over *k*, we thus have

$$
a_{ij}(p) p_k \psi_{.kij} + D_{p_i} a_{ij}(p) p_{l,k} p_k \psi_{.ij} = 0
$$

or, with (3.2), the, equation

$$
\frac{1}{2} a_{ij}(p) F_{,ij} - a_{ij}(p) p_{r,i} p_{r,j} + D_{p_i} a_{ij}(p) p_{l,k} p_k p_{i,j} = 0.
$$

By using (3.1) and (3.3) we get

where
$$
D_{p_i} = \frac{\partial}{\partial p_i}
$$
. Multiplying by p_k and summing over k, we thus have
\n
$$
a_{ij}(p) p_k \psi_{kij} + D_{p_i} a_{ij}(p) p_{l,k} p_k \psi_{,ij} = 0
$$
\nor, with (3.2), the equation
\n
$$
\frac{1}{2} a_{ij}(p) F_{,ij} - a_{ij}(p) p_{r,i} p_{r,j} + D_{p_i} a_{ij}(p) p_{l,k} p_k \psi_{,ij} = 0.
$$
\nBy using (3.1) and (3.3) we get
\n
$$
\frac{1}{2} a_{ij}(p) F_{,ij} + \frac{1}{2} D_{p_i} a_{ij}(p) \psi_{,ij} F_{,l,i}
$$
\n
$$
= a_{ij}(p) \psi_{,ri} \psi_{,ri} + a_{ij}(p) f_{r,i} f_{r,j} + b_{ij} \psi_{,ij}
$$
\nwhere $b_{11} = 2H p_1 p_2, b_{22} = -2H p_1 p_2, b_{12} = b_{21} = H (p_2^2 - p_1^2)$. Since
\n
$$
\frac{1}{2} F_{z_1} = p_1 \psi_{,11} + p_2 \psi_{,21} + H,
$$
\n(3.4)
\n
$$
\frac{1}{2} F_{z_1} = p_1 (\psi_{,12} - H) + p_2 \psi_{,22},
$$
\n
$$
\frac{1}{2} F_{z_1} = p_1 (\psi_{,12} - H) + p_2 \psi_{,22},
$$
\n
$$
\frac{1}{2} F_{z_1} = p_1 (\psi_{,12} - H)^2.
$$
\nAccording to our hypothesis with respect to F the $\psi_{,ri}$ can be calculated directly from (3.4) and the equation (2.4). This calculation yields
\n
$$
\psi_{,11} = g_i^{-11}(p) F_{,i} - \frac{2H}{F} p_1 p_2,
$$
\n
$$
\psi_{,22} = g_i^{-22}(p) F_{,i} + \frac{H}{F} (p_1^2 - p_2^2),
$$
\n
$$
\psi_{,12} = g_i^{-12}(p) F_{,i} + \frac{H}{F} (p_1^2 - p_2^2).
$$

can be calculated directly from (3.4) and the equation (2.4) . This calculation yields

$$
\frac{1}{2} F_{x_1} = p_1(\psi_{.12} - H) + p_2 \psi_{.22},
$$

\n, we see that $b_{ij}\psi_{.ij} \approx H(p_2F_{,1} - p_1F_{,2}) - 2H^2F$. It is easy to check
\n $\int_{r,j} = 2H^2 - FH^2$. According to our hypothesis with respect to F'th
\ncalculated directly from (3.4) and the equation (2.4). This calculation y
\n $\psi_{.11} = g_i^{11}(p) F_{,i} - \frac{2H}{F} p_1 p_2,$
\n $\psi_{.22} = g_i^{22}(p) F_{,i} + \frac{2H}{F} p_1 p_2,$
\n $\psi_{.12} = g_i^{12}(p) F_{,i} + \frac{H}{F} (p_1^2 - p_2^2)$
\nthe g_i^{lk} are certain regular functions depending on p. Thus we get from

where the g_i^{μ} are certain regular functions depending on p. Thus we get from the above that *F* satisfies the equation

$$
\frac{1}{2} a_{ij}(p) F_{,ij} + b_l F_{,l} = 4H^2(1 - F)
$$

with known functions b_l belonging to $C^1(\Omega)$. The classical maximum principle implies max $F = \max F$, provided $\psi \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ and $\max F < 1$

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Corollary: There exists a solution of (1.1), (1.2), provided $0 < \gamma \leq \frac{\pi}{2}$ and 3Q is ar a circle with respect to the C^2 -norm.

near a circle with respect to the C2-norm. 2- For the proof we observe that a bounded slope condition, cf. GILBARG and TRU-DINGER [4: p. 225], is fulfilled for such domains. Using this condition it is possible to derive the inequality max $F(x, \psi_x(x)) = C < 1$ for a $C^1(\overline{\Omega})$ -solution of (2.4), (2.2) if $\partial\Omega$ is sufficiently near a circle. The constant C'does not depend on ψ . Then, by a standard argument, from the above maximum principle it follows the existence of On Capillary Free Surfaces

Corollary: There exists a solution of (1.1), (1.2), provide

near a circle with respect to the C^2 -norm.

For the proof we observe that a bounded slope condition

DINGER [4: p. 225], is fulfi

Remark: We may weaken the assumption of the above theorem to $\psi \in C^1(\overline{\Omega})$ only and max $F < 1$. This follows from the weak formulation of the equation (2.4), see sufficiently near a circle. The

argument, from the abov

ion of (2.4) , (2.2)

mark: We may weaken the as

ax $F < 1$. This follows from
 $\frac{5}{6}$

i, since $\psi \in C^{\infty}(\Omega)$ is a consection Part 4, since $\psi \in C^{\infty}(\Omega)$ is a consequence of the regularity theory for quasilinear elliptic equations of second order.

4. Domains with corners

L. SIMON [7] has shown that if $\frac{\alpha}{2} + \gamma > \frac{\pi}{2}$ and $\alpha < \pi$, then a solution of (1.1), (1.2) is differentiable up' to the corner. Here α denotes the interior angle at this corner. From Part 2 it follows that (1.1), (1.2) can be replaced by the Dirichlet problem corner. From Part 2 it follows that (1.1) , (1.2) can be replaced by the Dirichlet problem **f***a*. Evantains what corners is the corner of (1.1) ,
 f and $\alpha < \pi$, then a solution of (1.1),

(1.2) is differentiable up to the corner. Here α denotes the interior angle at this corner. From Part 2 it follows t emptic emptic

4. Doma

L. SIMON

(1.2) is corner. F

corner. F

where

(1.2) is differentiable up' to the corner. Here
$$
\alpha
$$
 denotes the interior angle at this
corner. From Part 2 it follows that (1.1), (1.2) can be replaced by the Dirichlet problem

$$
\int_a a_i(x, \psi_x) v_{x_i} dx = 0 \quad \text{for all } v \in C_0^{\infty}(\Omega),
$$

$$
\psi = \psi(s) \quad \text{on } \partial\Omega
$$

 \rm{where}

- -.

 $\frac{1}{2}$

from Part 2 it follows that (1.1), (1.2) can be
\n
$$
\int_{\Omega} a_i(x, \psi_x) \, v_{x_i} \, dx = 0 \quad \text{for all } v \in C_0^{\infty}.
$$
\n
$$
\psi = \psi(s) \quad \text{on } \partial \Omega
$$
\n
$$
a_1(x, \psi_x) = \frac{1}{\sqrt{1 - F}} (\psi_{x_1} - Hx_2),
$$
\n
$$
a_2(x, \psi_x) = \frac{1}{\sqrt{1 - F}} (\psi_{x_1} + Hx_1)
$$
\ngiven by (2.2).

and ψ is given by (2.2).

Let the origin be a corner of Ω . We assume that $\overline{\partial_1 \Omega}$, $\overline{\partial_2 \Omega}$ are in C^2 and that $\partial_1 \Omega$ has positive and $\partial_2\Omega$ negative slope at the origin, see Figure 2.

-Suppose $u \in C^1(\overline{\Omega})$ is a solution of (1.1), (1.2) and $\alpha < \pi$. This implies that $\frac{\alpha}{2} + \gamma$ $> \frac{\pi}{2}$ must be satisfied. The inequality can be verified by an easy calculation by using (1.2) and the fact that (1.1) is elliptic at the corner. The assumption $u \in C^1(\overline{\Omega})$

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also implies that (4.1) is an elliptic problem since the associated ψ satisfies $\psi \in C^1(\Omega)$ $F < 1$ in \overline{Q} and the equation (4.1).

Theorem: Let $u \in C^1(\bar{\Omega})$ \cap $C^2(\Omega)$ be a solution of (1.1), (1.2). Under the condition **434** E. MIERSEMANN

also implies that (4.1) is an elliptic problem since the associated ψ satisfies $\psi \in C^1(\bar{\Omega})$,
 $F < 1$ in $\bar{\Omega}$ and the equation (4.1).

Theorem: Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of (1.1)

 $\begin{array}{ccc} & & 43 \ & \text{all:} & \ F & & \ 0 & & \ \text{for} & \end{array}$ Proof: Results of this type were proved by MIERSEMANN [5, 6]. Some details differ from [5, 6]. The first distinction is that the vector field $(a_1(x, q), a_2(x, q))$, $q = \psi_x$, is strongly monotone with respect to q for $|\gamma| < 1$ only where again $p_1 = \psi_x$, $-Hx_2$, $F < 1$ in $\overline{\Omega}$ and the equation (4.1).

Theorem: Let $u \in C^1(\overline{\Omega})$ o $C^2(\Omega)$ be a solution of (1.1), (1.2). Under the condition
 $0 < \alpha < \pi$ for all interior angles at the corners we have $u \in C^{1,2}(\overline{\Omega})$ o $H^2(\Omega)$, $p_2 = \psi_x + Hx_1$. The second one is the dependence on x. It is not hard to prove that, the results of [5] stay true for strongly monotone C^1 -vector fields depending on x. We omit the proof of this fact.

It is sufficient to prove the regularity properties near the corners. Set $\Omega_{\ell} = \Omega$ $n B_e(0)$ where $B_e(0)$ is a disk with radius ρ and the center at the origin. Assuming $\overline{\partial_i\Omega} \in C^2$, there exists a continuation $\Phi \in C^2(\overline{\Omega_0})$ of ψ , provided $0 < \alpha < \pi$ and $\rho > 0$ is sufficiently small. The proof of the Theorem is almost the same as in [5] \blacksquare **Let us that for the proof of this fact.**
 Let us sufficient to prove the regularity properties near the corners.
 $B_e(0)$ where $B_e(0)$ is a disk with radius ϱ and the center at the origin
 $\overline{2} \in C^2$, there exi

Since (2.4) is of mixed type, the barrier construction of $[5\colon {\rm p.\ 61}]$ must be modified. Let *E* be the set of all functions $\varphi \in C^1(\overline{\Omega_o})$ such that $F(x, \varphi_x) < 1$ in $\overline{\Omega_e}$. Clearly, *E* is a convex set. *faultionally small. The proof of the set of all functions* $\varphi \in C^1(\overline{\Omega_e})$ *is of mixed type, the barrier con the set of all functions* $\varphi \in C^1(\overline{\Omega_e})$ *is ex set.

a: Assume that for* ψ *,* $\varphi \in E$ *we have \int_a^b a_i(x,*

$$
\text{Lemma:} \ A \text{ssume that for } \psi, \ \varphi \in E \text{ we have } \psi \leq \varphi \text{ on } \partial \Omega_e \text{ and}
$$
\n
$$
\int_a a_i(x, \psi_x) \ v_{x_i} \, dx \leq \int_a a_i(x, \varphi_x) \ v_{x_i} \, dx
$$
\n
$$
\text{for all } v \in C_0^{-1}(\Omega_e) \text{ with } v \geq 0 \text{ in } \Omega_e. \text{ Then one has } \psi \leq \varphi \text{ in } \Omega_e.
$$

Proof: This comparison principle follows immediately by using the convexity of *E* and the fact that for all $\xi \in \mathbb{R}^2$ we have

$$
(1 - F)^{-1/2} |\xi|^2 \leq a_{ij}\xi_i\xi_j \leq (1 - F)^{-3/2} |\xi|^2,
$$

where in this part we put $a_{ij} = \frac{\partial a_i(x, q)}{\partial x}$ with $q_i =$ Here φ is a function belonging to E Proof: This comparison principle follows immediately by

E and the fact that for all $\xi \in \mathbb{R}^2$ we have
 $(1 - F)^{-1/2} |\xi|^2 \leq a_{ij}\xi_i\xi_j \leq (1 - F)^{-3/2} |\xi|^2$,

ere in this part we put $a_{ij} = \frac{\partial a_i(x, q)}{\partial q_j}$ with $q_i = \varphi_{x_i}$ ison principle follows immediate

or all $\xi \in \mathbb{R}^2$ we have
 $\frac{d}{dx} \leq a_{ij} \xi_i \xi_j \leq (1 - F)^{-3/2} |\xi|^2$,

ut $a_{ij} = \frac{\partial a_i(x, q)}{\partial q_j}$ with $q_i = \varphi_{x_i}$,

onging to $E \parallel$

follows that there exists at most
 $\langle 1, \rangle \cap C^2(\overline$

From the Lemma it follows that there exists at most one solution $\psi \in C^1(\overline{\Omega})$ of (4.1) such that $F(x, \psi_x) < 1$.

1) such that $F(x, \psi_x) < 1$.
A function $\varphi \in C^1(\overline{\Omega_e}) \cap C^2(\overline{\Omega_e} \setminus \{0\})$ is said to be a *barrier* of ψ at $x = 0$ if the following conditions are fulfilled:

$$
(i) \ \ \varphi \in E \ ,
$$

(ii) $\varphi \geq \psi$ on $\partial \Omega_{\varrho}$,
(iii) $a_{ij}(x, \varphi_x) \varphi_{x_i x_j} \leq 0$ in Ω_{ϱ} .

From the Lemma we obtain $\varphi \geq \psi$ in $\overline{\Omega} \varrho$ for a barrier φ . Put

$$
\varphi = \varPhi(0) + \varPhi_{x_i}(0) x_i + A r^{1+\mu} \sin (1+\tau) \varTheta
$$

where (r, Θ) denote polar coordinates in the plane and A, μ and τ positive constants. We will show that one can choose these constants in such a way that φ satisfies (i) to Here φ is a function belonging to E **i**

From the Lemma it follows that there exists at most one solution $\psi \in C^1(\overline{\Omega}$

(4.1) such that $F(x, \psi_x) < 1$.

A function $\varphi \in C^1(\overline{\Omega}_e) \cap C^2(\overline{\Omega}_e \setminus \{0\})$ is said to (x) denote polar coordinates in the plane and A, μ and τ
bow that one can choose these constants in such a way there $\rho \in C^2(\overline{\Omega \varrho})$ we have
 $(x) = \Phi(0) + \Phi_{x_i}(0) x_i + h(x)$ where $|h(x)| \leq c_1 |x|^2$. (4.1) such that $F(x, \psi_x) < 1$.

A function $\varphi \in C^1(\overline{\Omega_e}) \cap C^2(\overline{\Omega_e} \setminus \{0\})$ is said to be a *barrier* of ψ at $x = 0$ if the following conditions are fulfilled:

(i) $\varphi \in K$,

(ii) $\varphi \in \mathbb{F}$,

(ii) $\varphi \in \mathbb{F}$ We will show that one can choose these constant

(iii), provided $\varrho > 0$ is small enough. Since $\Phi \in$
 $\Phi(x) = \Phi(0) + \Phi_{x_i}(0) x_i + h(x)$ where

According to our assumption one has
 $\psi(x) = \Phi(0) + \Phi_{x_i}(0) x_i + R(x)$ where

with a fu

$$
\Phi(x) = \Phi(0) + \Phi_{x}(0) x_i + h(x)
$$
 where $|h(x)| \leq c_1 |x|^2$.

$$
\psi(x) = \Phi(0) + \Phi_{x_i}(0) x_i + R(x) \quad \text{where} \quad |R(x)| \leq c(\varrho) \varrho \quad \text{in } \Omega_{\varrho}
$$

We shall prove that

$$
\varphi \geq \psi \quad \text{on} \quad \Omega \cap \partial B_{\rho}(0) \tag{4}
$$

On Capillary Free Surfaces without Gravity 435

ove that
 ψ on $\Omega \cap \partial B_{\rho}(0)$ (4.2)
 $A \varrho^{1+\mu} \sin (1 + \tau) \Theta \ge c(\varrho) \varrho$ is fulfilled, then we have the inequality $\varphi \geq \psi$ on $\Omega \cap \partial B_{\varrho}(0)$
is satisfied. If $A \varrho^{1+\mu} \sin(1+\tau) \vartheta \geq c(\varrho) \varrho$ is fulfilled, then we have the inequality (4.2). Set

$$
c_2 = \min_{(\tau,\theta)\in\bar{D}_{\varrho}} \sin{(1+\tau)\theta}.
$$

(b) On Capillary Free Surfaces without Gravity 435
 (d) (4.2)
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 (Provided $\tau_0 > 0$ and $\rho > 0$ are sufficiently small, we see that $c_2 > 0$ for all $0 < \tau$ We shall prove that
 $\varphi \geq \psi$ on $\Omega \cap \partial B_{\theta}(0)$

is satisfied. If $A\varrho^{1+\mu} \sin(1 + \tau)$ (4.2). Set
 $c_2 = \min_{(\tau,\theta) \in \overline{\Omega}_{\theta}} \sin(1 + \tau) \Theta$.

Provided $\tau_0 > 0$ and $\varrho > 0$ are s
 $\leq \tau_0$. Put $c_0(\varrho) = \max \{c(\varrho), \varrho^{\theta}\}$
 for a δ with $0 < \delta < 1$. If we choose $A =$ $c_2 = \min_{(r,\theta)\in\bar{G}_{\theta}} \sin(1+\tau) \Theta.$

Provided $\tau_0 > 0$ and $\varrho > 0$ are sufficientl
 $\leq \tau_0$. Put $c_0(\varrho) = \max \{c(\varrho), \varrho^{\delta}\}$ for a $\delta \times \varrho^{-\mu}$ then the inequality (4.2) is satisfied.

Now we will prove $\varphi \geq \psi$ on $\partial \Omega \cap B_{\rho}$. Again, this inequality is true if we have

$$
c_0(\varrho) \varrho^{-\mu} r^{1+\mu} \geq c_1 r^2 \ (0 < r \leq \varrho) \quad \text{or} \quad c_0(\varrho) \geq c_1 \varrho \, .
$$

 $\geq \tau_0$. Fut $c_0(q) = \max \{c(q), e^s\}$ for a $\times e^{-\mu}$ then the inequality (4.2) is satisfied

Now we will prove $\varphi \geq \psi$ on $\partial\Omega \cap L$
 $c_0(q) e^{-\mu_7 t + \mu} \geq c_1 r^2$ ($0 < r \leq \varrho$)

From the definition of $c_0(q)$ it follows t From the definition of $c_0(q)$ it follows that the previous inequality is satisfied for all $0 < \varrho \leq \varrho_0$, provided ϱ_0 is small enough.

• Now we check the assumption (i). An easy calculation yields

$$
\times e^{-\mu}
$$
 then the inequality (4.2) is satisfied.
\nNow we will prove $\varphi \geq \psi$ on $\partial \Omega \cap B_e$. Again, this in
\n $c_0(\varrho) e^{-\mu_1 t + \mu} \geq c_1 r^2$ ($0 < r \leq \varrho$) or $c_0(\varrho) \geq c_1 \varrho$
\nFrom the definition of $c_0(\varrho)$ it follows that the previous is
\n $0 < \varrho \leq \varrho_0$, provided ϱ_0 is small enough.
\nNow we check the assumption (i). An easy calculation
\n
$$
F(x, \varphi_x) = \left(\varphi_{x_1}(0) - Hx_2 + \frac{c_0(\varrho)}{c_2} \left(\frac{r}{\varrho}\right)^{\mu} g_1\right)^2 + \left(\varphi_{x_1}(0) + Hx_1 + \frac{c_0(\varrho)}{c_2} \left(\frac{r}{\varrho}\right)^{\mu} g_2\right)^2
$$
\nwhere
\n
$$
g_1 = (1 + \mu) \cos \theta \sin (1 + \tau) \theta - (1 + \tau) \sin \theta
$$
\n
$$
g_2 = (1 + \mu) \sin \theta \sin (1 + \tau) \theta + (1 + \tau) \cos \theta
$$
\nConsequently, the function φ satisfies $F(x, \varphi_x) < 1$ in 2
\n $\varrho_0 > 0$.
\nThe inequality (iii) was proved in [5: Lemma 2.2].
\nReplacing A by $-A$ we obtain $|\psi - \varphi| \leq c |x|^{1+\mu}$ in $\overline{\Omega}$
\nget
\n $\psi \in C^{1,1}(\overline{\Omega}_{\varrho}) \cap H^2(\Omega_{\varrho})$ for a λ with $0 < \lambda < 1$,
\nsee [5]: Since u_x is given by (2.3), the same is true for u .
\nRemark: If we assume that $u \in C^{1,\beta}(\overline{\Omega})$, then we can

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$$
g_1 = (1 + \mu) \cos \Theta \sin (1 + \tau) \Theta - (1 + \tau) \sin \Theta \cos (1 + \tau) \Theta,
$$

 $g_2 = (1+\mu)\sin\Theta\sin(1+\tau)\Theta + (1+\tau)\cos\Theta\cos(1+\tau)\Theta.$

 $g_1 = (1 + \mu) \cos \theta \sin (1 + \tau) \theta - (1 + \tau) \sin \theta \cos (1 + \tau) \theta$,
 $g_2 = (1 + \mu) \sin \theta \sin (1 + \tau) \theta + (1 + \tau) \cos \theta \cos (1 + \tau) \theta$.

Consequently, the function φ satisfies $F(x, \varphi_x) < 1$ in $\overline{\Omega_{\varrho_0}}$ for a sufficiently small $\varrho_0 > 0$. $\varrho_0 > 0.$ msequently, the function φ satisfies $F(x, \varphi)$
 > 0 .

The inequality (iii) was proved in [5: Lemn

Replacing A by $-A$ we obtain $|\psi - \varphi| \le$

t

The inequality (iii) was proved in $[5:$ Lemma 2.2].

Replacing A by $-A$ we obtain $|\psi - \Phi| \le c |x|^{1+\mu}$ in $\overline{\Omega_{\epsilon}}$. From this inequality, we

$$
\psi \in C^{1,\lambda}(\bar{\Omega}_{\varrho}) \cap H^2(\Omega_{\varrho}) \text{ for a } \lambda \text{ with } 0 < \lambda < 1,
$$

see [5]. Since u_x is given by (2.3), the same is true for u .

Remark: If we assume that $u \in C^{1,\beta}(\overline{\Omega})$, then we can control $c(\rho)$ and we get $||u||_{C^{1,1}(\bar{\Omega})} \leq C$ for $0 < \beta < \lambda < 1$, provided we have chosen $\beta > 0$ sufficiently small. The constant *C* depends, for fixed Ω and γ , on $||u||_{C^1,\beta(\overline{\Omega})}$ only.

Once one has obtained $\psi \in C^{1,1}(\overline{\Omega_e})$, it is possible to apply the theory of linear elliptic equations of second order in domains with corners, see AzzAM [1]. The linear transformation which transforms the equation $a_{ij}(0, \psi_z(0)) \psi_{z_i z_j} = 0$ into the Laplace equation transforms γ into a new angle ω . The vector $\psi_z(0)$ can be obtained from (2.1) The inequality (iii) was proved in [5: Lemma 2.2].

Replacing A by $-A$ we obtain $|\psi - \Phi| \le c |x|^{1+\mu}$ in $\overline{\Omega_{\theta}}$. From this inequality, y

get
 $\psi \in C^{1.4}(\overline{\Omega}_{\theta}) \cap H^2(\Omega_{\theta})$ for a λ with $0 < \lambda < 1$,

see [5]: Sinc Remark: If we assume that $u \in C^{1,\beta}(\overline{\Omega})$
 $||u||_{C^{1,\lambda}(\overline{\Omega})} \leq C$ for $0 < \beta < \lambda < 1$, provided λ

The constant C depends, for fixed Ω and γ , θ

Once one has obtained $\psi \in C^{1,\lambda}(\overline{\Omega_e})$, it is

elliptic equ and (1.2). An easy calculation yields for ω , $0 < \omega < \pi$, the equation, $\beta < \lambda < 1$, provided we have chosen $\beta > 0$ sufficien
ds, for fixed Ω and γ , on $||u||_{C^{1,\beta}(\overline{\Omega})}$ only.
ined $\psi \in C^{1,\lambda}(\overline{\Omega_{\varrho}})$, it is possible to apply the theory
econd order in domains with corners, see Azz

$$
\operatorname{tg}\frac{\omega}{2} = \left(1 - \frac{\cos^2\gamma}{\sin^2\frac{\alpha}{2}}\right) \operatorname{tg}\frac{\alpha}{2}.
$$

\-

The angle ω is exactly the angle at the corner of the surface $u(x)$ about the origin. Provided $\overline{\partial_1\Omega}$, $\overline{\partial_2\Omega}$ are sufficiently regular, then we have *E.* MIERSEMANN
 e ω is exactly the *a*
 $\overline{\partial_1 \Omega}$, $\overline{\partial_2 \Omega}$ are suffic
 $u \in C^{m+2.4}(\overline{\Omega_{e_0}})$ The angle ω is exactly

Provided $\overline{\partial_1\Omega}$, $\overline{\partial_2\Omega}$ are
 $u \in C^{m+2,1}(\overline{\Omega_e})$

if $\omega < \frac{\pi}{2+m}$ is satisf

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$$
u\in C^{m+2,\lambda}(\overline{\Omega_{\mathfrak{e}_0}}) \qquad (0<\lambda<1\,;\,m=0,\,1,\,2,\,\ldots)
$$

if $\omega < \frac{\pi}{2+m}$ is satisfied.

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