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Weighted Inequalities for Vector-Valued Anisotropic Maximal Functions

V. KOKILASHVILI and J. RÁKOSNÍK

Die anisotrope Maximalfunktion wird hier mit Hilfe einer einparametrigen Familie von Quadern (statt Würfeln) definiert. Für vektorwertige anisotrope Maximalfunktionen werden gewichtete Ungleichungen vom schwachen und starken Typ bewiesen. Die Ungleichung vom . starken Typ wird dann zum Beweis der Stetigkeit des anisotropen Maximaloperators in gewichteten Räumen mit gemischter Norm benutzt.

Анизотропная максимальная функция определяется здесь через однопараметрическое семейство параллелепипедов вместо кубов. Для векторнозначных анизотропных максимальных функций доказываются весовые неравенства слабого и сильного типов. Неравенство сильного типа используется в конце при доказательстве непрерывности анизотропного максимального оператора в весовых пространствах со смешанными нормами.

The anisotropic maximal function is defined by means of one-parametric parallelepipeds instead of cubes. For vector-valued anisotropic maximal functions there are proved weak and strong type weighted inequalities. The strong type inequality is then utilised in the proof of an anisotropic weighted mixed norm maximal inequality.

1. Introduction

1.1. Let \mathbb{R}^n be the Euclidean space of points $x = (x_1, ..., x_n)$, \mathbb{R}_+^n be the set of all points $y = (y_1, \ldots, y_n)$ with $y_i > 0$, $i = 1, \ldots, n$. By a weight function (shortly a weight) we shall mean a non-negative measurable function w: $\mathbb{R}^n \to \mathbb{R}^n$. The weight w generates a measure μ_w given by

$$
\mu_w e = \int\limits_e w(x) \ dx, \qquad e \subset \mathbf{R}^n \text{ measurable.}
$$

 (1.1)

The Lebesgue measure of e will be denoted by $|e|$. For a weight w and $1 \leq p < \infty$ we define the weighted Lebesgue space L_w^p as the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}^n$ with the norm

$$
||f||_{p,w} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x) dx\right)^{1/p} < \infty.
$$

1.2. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a fixed point in \mathbb{R}^n . For $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$ we define

$$
E(x, t) = \left\{ z \in \mathbf{R}^n : |z_i - x_i| \leq \frac{1}{2} t^{\alpha_i}, \quad i = 1, ..., n \right\}
$$

and

$$
E = \{ E(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}^m \}
$$

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1.3. To each measurable function $f: \mathbb{R}^n \to \mathbb{R}^1$ we adjoin the *anisotropic maximal function* 504 • **V.** KORILASHVILI and J. RAKOSNIK
 1.3. To each measurable function $f: \mathbb{R}^n \to \mathbb{R}^1$ we adjoin
 function
 $Mf(x) = \sup_{t>0} |E(x, t)|^{-1} \int_{E(x, t)} |f(z)| dz$

If $\alpha_1 = \cdots = \alpha_n$, then *Mf* is the classical Hardy-Littlewoo

\n- 504. V. KokilashvIII and J. Rárosník
\n- 1.3. To each measurable function
$$
f: \mathbb{R}^n \to \mathbb{R}^1
$$
 we adjoin the *anisotropic maximal function* $Mf(x) = \sup_{t>0} |E(x, t)|^{-1} \int |f(z)| dz.$ (1.2) $E(x, t)$ $E(x, t)$

1.4. Let (Y, S, ν) be a σ -finite measure space and T be a σ -algebra of Lebesgue meas-**1.4.** Let (Y, S, v) be a σ -finite measure space and T be a σ -algebra of Lebesgue measurable sets in \mathbb{R}^n . On the σ -algebra $T \times S$ we define the measure λ as the product of the Lebesgue measure and of the Lebesgue measure and of v. For a λ -measurable function f: $\mathbb{R}^n \times Y \to \mathbb{R}^1$ we define the *vector-valued anisotropic maximal function y* **R**^{*y*} we adjoin the *anisotropic maximal* (1.2)

ardy Littlewood maximal function.
 y and *T* be a *σ*-algebra of Lebesgue meas-
 y define the measure λ as the product of

measurable function *f*: $\mathbb{R}^n \$

$$
M_{(1)}f(x, y) = \sup_{t>0} |E(x,t)|^{-1} \int_{E(x,t)} |f(z, y)| dz, \qquad x \in \mathbb{R}^n, \quad y \in Y. \tag{1.3}
$$

1.5. In [11] (see also [5]) there is given a characterization of those positive functions *^w* • for which the classical Hardy-Littlewood maximal function is bounded operator in L_{w}^{p} . In [7] the well known theorem of Hardy and Littlewood on L^{p} boundedness of maximal functions was generalized for l^{θ} -valued functions in the unweighted case. This result was extended in [2] for functions with values in the spaces $l^{\bar{\theta}}$ and $L^{\bar{\theta}}$ with'mixed norms. For L^{θ} -valued functions the first author obtained in [9] a full description of the weighted Lebesgue spaces in which the Hardy-Littlewood maximal function is a bounded operator. For l^{ϕ} -valued functions a similar result was derived independently in [1]. The weighted weak type inequality for l^{β} valued Hardy-Littlewood maximal functions was established earlier in [8]. with mixed norms. For L^{ϕ} -valued functions the first author obtained in [9] a full
description of the weighted Lebesgue spaces in which the Hardy-Littlewood maximal
function is a bounded operator. For l^{ϕ} -valued f 1.5. In [11] (see also [5]) there is given a characterization control of the classical Hardy-Littlewood maximal function L_w^p . In [7] the well known theorem of Hardy and Littlew maximal functions was generalized for $l^$ 2. Auxiliary notions and assertions

2. μ , E and μ and μ and μ and μ and μ boundedness of maximal functions was generalized for l^6 -valued functions in the unweighted case.

This result was extended **Find Feature was extended on P(2)** or tunctions what vantual μ of the spaces i^{*} and D description of the weighted Lebesgue spaces in which the Hardy-Littlewood maximal function is a bounded operator. For l^* v

In this note there are proved weighted weak and strong type inequalities for a, paper [9]. At the end an application of the strong type inequality to the weighted F-valued anisotropic maxir

[9]. At the end an applic
 c norm maximal inequality d weak sype inequality for $t \rightarrow at$
blished earlier in [8].
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be of the client *1 xype* inequality for l^b -valued Hardy-Little-
1 earlier in [8].
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 ion. The main ideas follow the first author's

the strong type inequality to the weighted
 n.

2.1. *The class* $A_p(x)$ *.* Let E be the set defined in Section 1.2. If $1 < p < \infty$, the class $A_p(\mathbf{E})$ consists of all weights w in \mathbf{R}^n for which there exists such a positive constant c, that for any $E = E(x, t) \in \mathbf{E}$ **1 '** *\p-1* **• ^S** that for any $E = E(x, t) \in E$ $A_n(E)$ consists of all weights w in \mathbb{R}^n for which there exists such a positive constant c. **ary** notion
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 $\frac{1}{2}$ 2.1. The class $A_p(x)$. Let E be the set defined in Section 1.2. If $1 < p < \infty$, the class $A_p(E)$ consists of all weights w in \mathbb{R}^n for which there exists such a positive constant *c*, that for any $E = E(x, t) \in \mathbb{R}$
 DIDS
 E set defined in Section 1.2
 Rⁿ for which there exists $\left(\frac{1}{E}\right)^{n}$
 $\int_{E} w^{-\frac{1}{p-1}}(z) dz$
 $\int_{E}^{p-1} z dz$
 \int_{E}^{p-1}

the class $A_1(E)$ if there ex vector-valued anisotropic maximal function. The main ideas follow the first author

paper [9]. At the end an application of the strong type inequality to the weighte

inxed norm maximal inequality is shown.

2. **Auxiliary**

c class
$$
A_p(\alpha)
$$
. Let **E** be the set defined in Section 1.2. If $1 < p < \infty$, the class
consists of all weights *w* in **R**^{*n*} for which there exists such a positive constant *c*,
any $E = E(x, t) \in \mathbf{E}$

$$
\left(|E|^{-1} \int_{E} w(z) dz \right) \left(|E|^{-1} \int_{E} w^{-\frac{1}{p-1}}(z) dz \right)^{p-1} \leq c.
$$
 (2.1)
function *w* is said to be of the class $A_1(\mathbf{E})$ if there exists such a constant $c > 0$,
 $M(w) (x) \leq cw(x)$ a.e. in **R**^{*n*}, (2.2)
M is defined by (1.2). Remind two properties of functions from the class

 $\text{stant } c > 0,$
(2.2)

$$
g(x) \leq c w(x) \quad \text{a.e. in } \mathbb{R}^n, \tag{2.2}
$$

where M is defined by (1.2) . Remind, two properties of functions from the class $A_p(E)$: If $w \in A_p(E)$, $1 < p < \infty$, then there exists p_0 such that $1 < p_0 < p$, and $w \in A_{p_0}(E)$; in addition $w \in A_{p_1}(E)$ for arbitrary $p_1 > p$. The second property is a simple corollary of the Hölder inequality, and the first was proved in [10]. There was The function w is said to be of the class $A_1(\mathbf{E})$ if there exists such a c
that
 $M(w)$ $(x) \leq cw(x)$ a.e. in \mathbf{R}^n ,
where M is defined by (1.2). Remind two properties of functions $A_p(\mathbf{E})$: If $w \in A_p(\mathbf{E})$, 1 The function *w* is said to be of the class $A_1(\mathbf{E})$ if there exists such a con

that
 $M(w)$ (x) $\leq cw(x)$ a.e. in \mathbf{R}^n ,

where *M* is defined by (1.2). Remind two properties of functions fro
 $A_p(\mathbf{E})$: If $w \$ *f*); in addition $w \in A_{p_i}(\mathbf{E})$ for arbitrary $p_1 > p$. The second proprollary of the Hölder inequality, and the first was proved in [10]. If the following assertion.
 d the following assertion.
 oosition: Let $1 < p <$ The function w is said to be of the class $A_1(\mathbf{E}$
that $M(w)$ $(x) \leq cw(x)$ a.e. in \mathbf{R}^n ,
where M is defined by (1.2). Remind two pr
 $A_p(\mathbf{E})$: If $w \in A_p(\mathbf{E})$, $1 < p < \infty$, then there $w \in A_p(\mathbf{E})$; in addition $p_1 > p$. The s
first was prover
in Section.

2.2. Proposition: Let $1 < p < \infty$ and **F** *be given in Section* 1.2. Then there exists

$$
\int_{\mathbf{R}^n} [Mf(\alpha)]^p w(x) dx \leq c \int_{\mathbf{R}^n} |f(x)|^p w(x) dx
$$

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2.3. Lemma: For a measurable /unction / we define

Weighted Inequalities for Maximal Functions
\n2.3. Lemma: For a measurable function
$$
f
$$
 we define
\n
$$
\tilde{M}(x) = \sup |E|^{-1} \int_{E} |f(z)| dz, \qquad x \in \mathbb{R}^n,
$$
\nwhere the supremum is taken over all $E \in E$ which contain x .
\nIt holds
\n
$$
Mf(x) \leq \tilde{M}f(x) \leq 2^{|x|/r} Mf(x),
$$
\nwhere $|x| = \alpha_1 + \cdots + \alpha_n$ and
\n
$$
\gamma = \min \alpha_i.
$$
\n(2.5)

It holds.

$$
Mf(x) \leq \tilde{M}f(x) \leq 2^{|a|/r} Mf(x), \qquad (2.4)
$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$
\gamma = \min \alpha_i.
$$

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ma: For a measure $\tilde{M}/(x) = \sup |E|^{-1}$

supremum is taken

s
 $Mf(x) \leq \tilde{M}f(x) \leq$
 $\Rightarrow \alpha_1 + \cdots + \alpha_n$ a
 $\gamma = \min_{i} \alpha_i$.

: The first inequanties Maximal Fund
 \bullet
 \bullet Proof: The first inequality is trivial. Now, suppose that $x \in E(u, t) \in E$. This where the supremum is taken over all $E \in$ E which contain x.

It holds
 $Mf(x) \leq \tilde{M}f(x) \leq 2^{|x|/r} Mf(x)$,

where $|x| = x_1 + \cdots + x_n$ and
 $y = \min x_i$.

Proof: The first inequality is trivial. Now, suppose that $x \in E(u)$,

mean *y* = min α_i .
 *• 1 1 • <i>x***_{***• • <i>x***₁** \leq *****1 <i>1 x***₁** \leq ****} **• • Froof:** The first inequality is trivial. Now, suppose that x *i* means that $|x_i - u_i| \leq \frac{1}{2} t^{a_i}$, $i = 1, 2, ..., n$, and for $y \in E(u, t)$ **w** $|y_i - x_i| \leq |y_i - u_i| + |x_i - u_i| \leq t^{a_i} \leq \frac{1}{2} (t')^{a_i}$, where $t' = 2^{1/t}t$ $2^{|a||y|} Mf(x)$,

and
 $\lim_{a \to a} x \in E(u, t) \in E$
 $\lim_{a \to a} x \in E(u, t) \in E$
 $\lim_{a \to a} |x \to a| \leq t^{\alpha} \leq \frac{1}{2} (t')^{\alpha}$,
 $\lim_{a \to a} |x \to a| \leq t^{\alpha} \leq \frac{1}{2} (t')^{\alpha}$,
 $\lim_{a \to a} |E(x, t)|$. Hence
 $\lim_{b \to a} |E(x, t)|^{-1} \int_{E(x, t)} |f(z)| dz$,
 \lim_{b where the supremum is taken over all $E \in$ which contain x.

It holds $M(x) \leq \tilde{M}f(x) \leq 2^{|x|/p} Mf(x)$,

where $|x| \leq \alpha_1 + \cdots + \alpha_n$ and
 $y = \min \alpha_i$.

Proof: The first inequality is trivial. Now, suppose that $x \in E(u, t)$

me Proof: The first inequality is trivial. Now, suppose that $x \in E(u, t)$

eans that $|x_i - u_i| \leq \frac{1}{2} t^{a_i}$, $i = 1, 2, ..., n$, and for $y \in E(u, t)$ we have
 $|y_i - x_i| \leq |y_i - u_i| + |x_i - u_i| \leq t^{a_i} \leq \frac{1}{2} (t')^{a_i}$,

here $t' = 2^{1/t}t$,

$$
|y_i - x_i| \leq |y_i - u_i| + |x_i - u_i| \leq t^{\alpha} \leq \frac{1}{2} (t^{\alpha})^{\alpha}
$$

where $t' = 2^{1/\gamma}t$, γ defined by (2.5). Consequently $y \in E(x, t')$, i.e. $E(u, t) \subset E(x, t')$,

and, moreover,
$$
|E(x, t')| = (2t)^{|a|} = 2^{|a|} |E(x, t)|
$$
. Hence $|E(u, t)|^{-1} \int |f(z)| dz \leq 2^{|a|/r} |E(x, t')|^{-1} \int |f(z)| dz$ and so the second inequality (2.3) holds

2.4. Lemma [4: *Section 3.6*]: Let $f \in L^1(\mathbb{R}^n)$ and $s > 0$. Then there exist a number 4. Lemma [4: Section 3.0]: Let $f \in L^1(\mathbb{R}^n)$ and $s > 0$. Then there exist a namber $0 \leq 1$ (depending only on α from Section 1.2) and α sequence of non-overlapping arallelepipeds R_j , $j \in \mathbb{N}$ such that
 s *parallelepipeds* R_j , $j \in \mathbb{N}$ such that *sover,* $|\mathbf{E}(x, t')| = (2t)^{|\alpha|} = 2^{|\alpha|} |E(x, t)|$. Hence
 $|E(u, t)|^{-1} \int |f(z)| dz \leq 2^{|\alpha|/r} |\mathbf{E}(x, t')|^{-1} \int |f(z)| dz$,
 $E(x, t)$
 $E(x, t) = \int |f(z)| dz$
 $E(x, t') = \int |f(x)| dz$
 $E(x, t') = \int |f(x)| dz$ *s* defined by (2.3). Consequently $y \in E(x, t)$, i.e. $E(u, t) = E(t, t)$,
 $(x, t')| = (2t)^{|x|} = 2^{|x|} |E(x, t)|$. Hence
 $\int_{E(x, t)} \frac{1}{|f|} f(z) dz \leq 2^{|x|} |E(x, t')|^{-1} \int |f(z)| dz$,
 $E(x, t)$

inequality (2.3) holds \blacksquare

covering lemmas which

$$
s < |R_j|^{-1} \int\limits_{R_j} |f(x)| \, dx \leq (2\alpha_0)^{|a|} \, s, \qquad j \in \mathbb{N},
$$
\n
$$
|f(x)| \leq s \quad \text{a.e. in } \mathbb{R}^n \setminus \bigcup\limits_{j} R_j,
$$
\n
$$
(2.7)
$$

 and *for any* $j \in \mathbb{N}$ there exist U_j , $V_j \in \mathbb{F}$ so that •

i,
\n
$$
|f(x)| \leq s \quad \text{a.e. in } \mathbb{R}^n \setminus \bigcup R_j,
$$
\n
$$
\text{for any } j \in \mathbb{N} \text{ there exist } U_j, V_j \in \mathbb{E} \text{ so that}
$$
\n
$$
V_j \subset R_j \subset U_j \quad \text{and} \quad |U_j| = \alpha_0^{|a|} |V_j|.
$$
\n
$$
\text{Lemma (161): Let } D \text{ be a bounded set in } \mathbb{R}^n \text{ and let for any } x \in D \text{ there be given a}
$$

2.5. Lemma ([6]): Let D be a bounded set in \mathbb{R}^n , and let for any $x \in D$ there be given a *parallelepiped* $R(x)$ with the centre x. Suppose that for-each two points $x_1, x_2 \in D$ the *v*_{*i*} \in *R*_{*i*} \in *D*_{*i*} and $|U_j| = \alpha_0^{|\alpha|} |V_j|$. (2.8)

2.5. Lemma ([6]): Let *D* be a bounded set in **Rⁿ**, and let for any $x \in D$ there be given a parallelepiped $R(x)$ with the centre x . Suppose that fo **2.4.** Lemma [4: Section 3.6]: Let $f \in L^1(\mathbb{R}^n)$ and $s > \alpha_0 \geq 1$ (depending only on α from Section 1.2) and parallelepipeds R_i , $j \in \mathbb{N}$ such that
 $s < |R_j|^{-1} \int |f(x)| dx \leq (2\alpha_0)^{|a|} s$, $j \in \mathbb{N}$,
 $|f(x)| \leq s$ a *contains the other.*
• Then from {R(x): x \in *D} a sequence {R_i} can be selected such that i* $s < |R_j|^{-1} \int_R |f(x)| dx \leq (2\alpha_0)^{|a|} s$, $j \in \mathbb{N}$, (2.6)
 $|f(x)| \leq s$ a.e. in $\mathbb{R}^n \setminus \bigcup R_j$, (2.7)
 or any $j \in \mathbb{N}$ *there exist* $U_j, V_j \in \mathbb{R}$ *so that*
 $V_j \subset R_j \subset U_j$ and $|U_j| = \alpha_0^{|a|} |V_j|$. (2.8)
 *Lemma •• z*₁ *If*(*x*) \leq *s a.e.* in $\mathbb{R}^n \setminus \bigcup R_i$, (2.7)
 and for any j $\in \mathbb{N}$ *there exist* U_j , $V_j \in \mathbb{R}$ *so that* $V_j \subset R_j \subset U_j$ and $|U_j| = \alpha_0^{|\alpha|} |V_j|$. (2.8)

2.5. Lemma ([6]): Let D be a boun *where the number On depends only on the dimension n.*
 and Depending only on the number of parallelepiped $R(x)$ **with the centre x. Suppose that for each two point**

$$
D \subset \bigcup_{j} R_{j}, \qquad (2.9)
$$

$$
\sum_{i} \chi_{R_{j}}(x) \leq \vartheta_{n}, \qquad x \in D, \qquad (2.10)
$$

2.6 In this section we shall prove an anisotropic version *of* the lemma by C. **FEFFEu.** MAN adn E; M. STEIN [7].

(2.5)

Lemma: Let $1 < p < \infty$ and *f*, $g \in L_{loc}(\mathbb{R}^n)$ be non-negative functions. Then Lemma: Let $1 < p < \infty$ and $f, g \in L_{loc}(\mathbb{R}^n)$ be non-negative exists a constant $c > 0$ independent of f and g , such that
 $\int_{\mathbb{R}^n} [Mf(x)]^p g(x) dx \leq c \int_{\mathbb{R}^n} f^p(x) Mg(x) dx$. **EXPRELASHVILI** and **J. F**
 Let $1 < p < \infty$ *ax*

a constant $c > 0$ inc
 $[Mf(x)]^p g(x) dx \leq c$ **•** 506 • *V.* **KOKILASHVILI and J. RAKOSNIK**
 • • <i>• • • <i>• • • <i>s n f s s e n f n g s <i>m n e <i>f m <i>f *****<i>n <i><i>f n <i>n <i><i>f* 506 V. KOKILASHVILI and J. RAKOSNIK

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•
 Lemma: Let $1 < p < \infty$ and $f, g \in L_1$

there exists a constant $c > 0$ independent of
 $\int_{\mathbf{R}^n} [Mf(x)]^p g(x) dx \leq c \int_{\mathbf{R}^n} f^p(x) Mg(x) dx$
 Proof: Let s be an arbitrary positive

$$
\int_{\mathbf{R}^n} [Mf(x)]^p g(x) dx \leq c \int_{\mathbf{R}^n} f^p(x) Mg(x) dx.
$$

Proof: Let s be an arbitrary positive number and $m \in N$. We denote

$$
H_s = \{x \in \mathbb{R}^n \colon Mf(x) > s\} \quad \text{and} \quad H_s^m = H_s \cap \{x \in \mathbb{R}^n \colon |x| \leq m\}.
$$

For each $x \in H_s$ there exists $E_x \in E$ such that

$$
|E_x|^{-1} \int_{E_x} f(z) dz > s. \tag{2.11}
$$

Since the family ${E_x: x \in H_s^m}$ satisfies the assumptions of Lemma 2.5, there is a sequence of non-overlapping sets E_j , from this family, satisfying (2.9) and (2.10) where E_j and H_s^m stand for R_j and D_j , respectively. By the Hölder inequality, and where E_j and H_s^m stand for R_j and D_j , respectively. By the Hölder inequality, and Lemma: Let $1 < p < \infty$ and $f, g \in L_{loc}(\mathbb{R}^n)$ be non-negative funct
there exists a constant $c > 0$ independent of f and g , such that
 $\int_{\mathbb{R}^n} [Mf(x)]^p g(x) dx \leq c \int f^p(x) Mg(x) dx$.
Proof: Let s be an arbitrary positive numb For each $x \in H_s$ there $\exp\{E_x\}$
 $|E_x|^{-1} \int_{E_x} f(z) dz > E_x$.
Since the family $\{E_x : x \in \mathbb{R}\}$
sequence of non-overlapp
where E_j and H_s^m stand
the estimate (2.11) we ha $\begin{array}{l} \text{the}\ \text{fa} \ \text{in} \ \text{of} \ \textit{E}_{j} \ \text{and} \ \text{finite} \ \textit{f}_{i} \ \textit{H}_{i}^{\text{m}} \end{array}$

$$
H_s = \{x \in \mathbb{R}^n : Mf(x) > s\} \text{ and } H_s^m = H_s \cap \{x \in \mathbb{R}^n : |x| \le m\}.
$$

\n $x \in H_s$ there exists $E_s \in E$ such that
\n $|E_x|^{-1} \int f(z) dz > s.$ (2.11)
\ne family $\{E_z : x \in H_s^m\}$ satisfies the assumptions of Lemma 2.5, there is a
\nof non-overlapping sets E, from this family, satisfying (2.9) and (2.10)
\nstate (2.11) we have
\n $\int g(x) dx \le \sum \int g(x) dx (s^{-1} |E_j|^{-1} \int f(z) dz)^p$
\n $H_s^m = \int_{g} g(x) dx = \sum \int g(x) dx (s^{-1} |E_j|^{-1} \int f(z) dz)^p$
\n $\le s^{-p} \sum |E_j|^{-1} \int g(x) dx \int f^p(z) Mg(z) dz$
\n $\le \int |E_j|^{-1} \int g(x) dx$ for all $z \in E_j$, by Lemma 2.3 we get
\n $\left(|E_j|^{-1} \int [Mg(z)]^{-\frac{1}{p-1}} dz\right)^{p-1}$
\n $\left(|E_j|^{-1} \int g(x) dx$ for all $z \in E_j$, by Lemma 2.3 we get
\n $\left(|E_j|^{-1} \int [Mg(z)]^{-\frac{1}{p-1}} dz\right)^{p-1} \le 2^{|a|/p} \left(|E_j|^{-1} \int g(z) dz\right)^{-1}$
\n $\left(|E_j|^{-1} \int_{E_j} [Mg(z)]^{-\frac{1}{p-1}} dz\right)^{p-1} \le 2^{|a|/p} \left(|E_j|^{-1} \int g(z) dz\right)^{-1}$
\n $\left(|E_j|^{-1} \int_{E_j} [Mg(z)]^{-\frac{1}{p-1}} dz\right)^{p-1} \le 2^{|a|/p} \left(|E_j|^{-1} \int_{E_j} g(z) dz\right)^{-1}$
\n $\lim_{n \to \infty} \lim_{n \to \infty} \lim_{n \to \infty} \frac{\ln n}{n}$ (2.12)
\n $\lim_{n \to \infty$

Since $\bar{M}g(z) \geq |E_j|^{-1} \int g(x) dx$ for all $z \in E_j$, by Lemma 2.3 we get

$$
\times \left(|E_j|^{-1} \int\limits_{E_j} \left[Mg(z) \right]^{\frac{1}{p-1}} dz \right)^{p-1}
$$

\n
$$
g(z) \geq |E_j|^{-1} \int\limits_{E_j} g(x) dx \text{ for all } z \in E_j, \text{ by Lemma 2.3 we get}
$$

\n
$$
\left(|E_j|^{-1} \int\limits_{E_j} \left[Mg(z) \right]^{\frac{1}{p-1}} dz \right)^{p-1} \leq 2^{|a|/p} \left(|E_j|^{-1} \int\limits_{E_j} g(z) dz \right)^{-1}
$$

\n
$$
g(z) = 10
$$

and so by (2.10)

$$
\int_{L^m} g(x) dx \leq 2^{|\alpha|/\gamma} \vartheta_n s^{-p} \int_{\mathbf{R}^n} f^p(x) M g(x) dx.
$$
\n(2.12)

Passing to the limit $m \to \infty$ and assuming that the right hand side does not depend on *m* we can write H_s instead of H_s^m in (2.12) and so we obtain the weak type (p, p) inequality for the operator *M* with respect to the measures μ_{Mg} and μ_g . It remains to use the Màrcinkiewicz interpolation theorem (see, e.g. [12]). Since $M_9(z) \le |B_j| = |B_j|$ of z for all $z \in E_j$, by Lemma 2.3 we get
 $\left(|E_j|^{-1} \int_{E_j} [Mg(z)]^{-\frac{1}{p-1}} dz \right)^{p-1} \le 2^{|a|/p} \left(|E_j|^{-1} \int_{E_j} g(z) dz \right)^{-1}$

and so by (2.10)
 $\int_{E_j} g(x) dx \le 2^{|a|/p} \vartheta_n s^{-p} \int_{E_j} f^p(x) Mg(x) dx$.
 \lim assing to the limit $m \to \infty$ and assuming that the right hand side does not depend on
we can write H_s instead of H_s^m in (2.12) and so we obtain the weak type (p, p)
equality for the operator M with respect to the m

3.1. We shall now prove the weak type inequality for the vector-valued anisotropic maximal operator $M_{(1)}$ defined in Section 1.4. This weak type inequality will be then used by the proof of the strong one.

Lemma: If $1 \leq p \leq \vartheta < \infty$, $\vartheta > 1$ and $w \in A_p(\mathbf{E})$, then there exists a positive constant c such that

inequality for the operator
$$
M
$$
 with respect to the measurement
to use the Marcinkiewicz interpolation theorem (see, e.g. [
3. Weak and strong type inequalities
3.1. We shall now prove the weak type inequality for the
maximal operator $M_{(1)}$ defined in Section 1.4. This, weak
tused by the proof of the strong one.
Lemma: If $1 \leq p \leq \vartheta < \infty$, $\vartheta > 1$ and $w \in A_p(\mathbf{E})$, *t*
constant c such that
 $\mu_w \left\{ x \in \mathbf{R}^n : \left(\int_{Y} [M_{(1)}f(x, y)]^{\vartheta} dy \right)^{1/\vartheta} > s \right\}$
 $\leq \frac{1}{\mathbf{R}^n} \left(\int_{Y} |f(x, y)|^{\vartheta} dy \right)^{p/\vartheta} w(x) dx$
for every $s > 0$ and every λ -measurable function f .

 (3.1)

Proof: Let $s > 0$ and a λ -measurable function $f: \mathbb{R}^n \times Y \to \mathbb{R}^1$ be given (for the measure *2* see Section 1.4). Denote

$$
N_{\theta}(f)(x) = \left(\int\limits_Y |f(x, y)|^{\theta} dy\right)^{1/\theta} = ||f(x, \cdot)||_{L_{\theta}(Y)}.
$$

By Lemma-2.4 there exist non-overlapping parallelepipeds R_i and parallelepipeds U_j , $V_j \in E$ such that relations (2.8) hold and, moreover, *s < jR1I -1 f ^N⁰ (/) (x) di (2x)1' S,* j € N, (3.2) $N_{\theta}(f) (x) = \left(\int_{I} |f(x, y)|^{\theta} dy\right)^{1/\theta} = ||f(x, \cdot)||_{L_{\theta}(Y)}$.
By Lemma 2.4 there exist non-overlapping parallelepipeds R_i and parallelepi U_j , $V_j \in E$ such that relations (2.8) hold and, moreover,
 $s < |R_j|^{-1} \int_{R_j} N_{\theta}(f) (x) dx$

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\n: Let
$$
s > 0
$$
 and a λ -measurable function $f : \mathbb{R}^n \times Y \to \mathbb{R}^1$ be given (for the
\n λ see Section 1.4). Denote
\n $N_{\theta}(f) (x) = \left(\int_{Y} |f(x, y)|^{\theta} dy\right)^{1/\theta} = ||f(x, \cdot)||_{L_{\theta}(Y)}$.
\n $\max 2.4$ there exist non-overlapping parallelepipeds R_j and parallelepipeds
\nE such that relations (2.8) hold and, moreover,
\n $s < |R_j|^{-1} \int_{R_j} N_{\theta}(f) (x) dx \leq (2\alpha_0)^{|a|} s, \quad j \in \mathbb{N},$
\n $N_{\theta}(f) \cdot (x) \leq s$ a.e. in $\mathbb{R}^n \setminus \bigcup_{j} R_j$.
\n $\bigcup_{j} R_j, R' = \mathbb{R}^n \setminus R_j$ and for $(x, y) \in \mathbb{R}^n \times Y$

$$
N_{\theta}(f),(x) \leq s \quad \text{a.e. in} \quad \mathbf{R}^{n} \setminus \bigcup_{j} R_{j}. \tag{3.3}
$$
\n
$$
\bigcup_{j} R_{j}, \ R' = \mathbf{R}^{n} \setminus R, \text{ and for } (x, y) \in \mathbf{R}^{n} \times Y
$$
\n
$$
\psi(x, y) = f(x, y) \chi_{R'}(x), \qquad \varphi(x, y) = f(x, y) - \psi(x, y).
$$
\neasily observe that

Put $R = \bigcup R_i$, $R' = \mathbb{R}^n \setminus R$, and for $(x, y) \in \mathbb{R}^n \times Y$

$$
\psi(x, y) = f(x, y) \chi_R(x), \qquad \varphi(x, y) = f(x, y) - \psi(x, y).
$$

asily observe that

$$
\mu_w\{x \in \mathbb{R}^n : N_\theta(M_{(1)}f)(x) > s\} \leq \mu_w Q_1 + \mu_w Q_2,
$$
 (3.4)

$$
R_j, R' = \mathbf{R}^n \setminus R, \text{ and for } (x, y) \in \mathbf{R}^n \times Y
$$

\n
$$
(x, y) = f(x, y) \chi_R(x), \qquad \varphi(x, y) = f(x, y) - \psi(x)
$$

\n
$$
\text{d)} \text{where that}
$$

\n
$$
e^{\{x \in \mathbf{R}^n : N_\theta(M_{(1)}f)(x) > s\}} \leq \mu_w Q_1 + \mu_w Q_2,
$$

- -

By Lemma 2.4 there exist non-overlapping parallelepipeds
$$
R_j
$$
 and parallelepipeds
\n U_j , $V_j \in \mathbf{E}$ such that relations (2.8) hold and, moreover,
\n
$$
s < |R_j|^{-1} \int_{R_j} N_{\theta}(f) (x) dx \leq (2\alpha_0)^{|a|} s, \quad j \in \mathbf{N},
$$
\n
$$
N_{\theta}(f) \cdot (x) \leq s \quad \text{a.e. in } \mathbf{R}^n \setminus \bigcup_{j} R_j.
$$
\n(3.3)
\nPut $R = \bigcup_{j} R_j$, $R' = \mathbf{R}^n \setminus R$, and for $(x, y) \in \mathbf{R}^n \times Y$
\n
$$
\psi(x, y) = f(x, y) \chi_R(x), \quad \varphi(x, y) = f(x, y) - \psi(x, y).
$$
\nOne can easily observe that
\n
$$
\mu_w \{x \in \mathbf{R}^n : N_{\theta}(M_{(1)}f) (x) > s\} \leq \mu_w Q_1 + \mu_w Q_2,
$$
\n(3.4)
\nwhere
\n
$$
Q_1 = \left\{ x \in \mathbf{R}^n : N_{\theta}(M_{(1)}\varphi) (x) > \frac{s}{2} \right\},
$$
\n
$$
Q_2 = \left\{ x \in \mathbf{R}^n : N_{\theta}(M_{(1)}\varphi) (x) > \frac{s}{2} \right\}.
$$
\nSince $w \in A_p(\mathbf{E})$ and $\vartheta \geq p$, it is $w \in A_{\theta}(\mathbf{E})$, and the Chebyshev inequality, the Fubini

theorem and proposition in Section *2.2* yield

where
\n
$$
Q_1 = \left\{ x \in \mathbb{R}^n : N_{\theta}(M_{(1)}\varphi) \ (x) > \frac{s}{2} \right\},
$$
\n
$$
Q_2 = \left\{ x \in \mathbb{R}^n : N_{\theta}(M_{(1)}\varphi) \ (x) > \frac{s}{2} \right\}.
$$
\nSince $w \in A_p(\mathbf{E})$ and $\vartheta \ge p$, it is $w \in A_{\theta}(\mathbf{E})$, and the Chebyshev inequality, the Fubini theorem and proposition in Section 2.2 yield
\n
$$
\mu_w Q_1 \le \left(\frac{s}{2}\right)^{-\theta} \int \int \int \int [M_{(1)}\varphi(x, y)]^{\theta} d\mu_w dv
$$
\n
$$
\le c_1 s^{-\theta} \int N_{\theta}^{\theta}(\varphi) \ (x) \ d\mu_w \le c_1 s^{-\theta} \int N_{\theta}^{\theta}(f) \ (x) \ d\mu_w.
$$
\nAccording to (3.3) we conclude that
\n
$$
\mu_w Q_1 \le c_1 s^{-p} \int N_{\theta}^{\theta}(f) \ (x) \ d\mu_w.
$$
\nNow, we shall estimate the second summand of (3.4). Let us introduce the step-function
\n
$$
\tilde{f}(x, y) = \begin{cases} |R_j|^{-1} \int |f(z, y)| dz, & x \in R_j, \\ 0, & x \in R'. \end{cases}
$$
\nWith each $U_j = E(x^{(j)}, t_j)$ we associate the parallelepiped $\tilde{U}_j = E(x^{(j)}, t_j)$, where
\n $I_j = 3^{1/p} t_j, \gamma$ given by (2.5). We denote $U = \bigcup \tilde{U}_j$ and $U' = \mathbf{R}^n > U$.
\nWe shall show that

•
•
•
•

$$
\mu_w Q_1 \leq c_1 s^{-p} \int_{\mathbf{R}^n} N_p p(f) \left(x \right) d\mu_w. \tag{3.5}
$$

Nov, we shall estimate the second summand of (3.4). Let us introduce the step- (3.4) . Let us introduce the

ng to (3.3) we conclude that
\n
$$
\mu_{\nu}Q_{1} \leq c_{1}s^{-p} \int_{\mathbf{R}^{n}} N_{\theta}p(f) (x) d\mu_{\nu}.
$$
\nwe shall estimate the second summand of (3.4
\n
$$
\tilde{f}(x, y) = \begin{cases}\n|R_{j}|^{-1} \int_{R_{j}} |f(z, y)| dz, & x \in R_{j}, \\
0, & x \in R'.\n\end{cases}
$$
\nch $U_{j} = E(x^{(j)}, t_{j})$ we associate the parallele
\n t_{j}, γ given by (2.5). We denote $U = \bigcup_{j} \tilde{U}_{j}$ and
\nall show that
\n
$$
M_{(1)} \psi(x, y) \leq d^{n} M_{(1)} \tilde{f}(x, y), \quad (x, y) \in U' \times
$$
\n
$$
= 3^{|a|/r}.
$$
 Let $x \in U'$ and $t \in \mathbf{R}_{+}^{1}$. Evidently $x \in$
\n $t_{j} \neq 0$. For $j \in S$ we have
\n
$$
|x_{i}^{(j)} - x_{i}| \leq \frac{1}{2} (t_{j}^{\alpha_{i}} + t^{\alpha_{i}}), \qquad i = 1, ..., n.
$$

With each $U_j = E(x^{(j)}, t_j)$ we associate the parallelepiped $\tilde{U}_j = E(x^{(j)}, \tilde{t}_j)$, where $\tilde{J}_j = 3^{1/2}t_j$, γ given by (2.5). We denote $U = \bigcup_j \tilde{U}_j$ and $U' = \mathbb{R}^n \searrow U$.
We shall show that $Y \t R^n$
 $\leq c_1 s^{-\theta} \int_{\mathbf{R}^n} N_{\theta}(\varphi) (x) d\mu_w \leq c_1 s^{-\theta}$

coording to (3.3) we conclude that
 $\mu_w Q_1 \leq c_1 s^{-p} \int_{\mathbf{R}^n} N_{\theta} p(f) (x) d\mu_w$.

Now, we shall estimate the second summan

notion
 $\tilde{f}(x, y) = \begin{cases} |R_j|^{-1}$ $\tilde{f}(x, y) = \begin{cases} \tilde{\kappa}_j & x \in R'. \\ 0, & x \in R'. \end{cases}$
 $\begin{aligned} \text{ch } U_j &= E(x^{(j)}, t_j) \text{ we associate the parallelepip, } \\ \text{in } Y_j & \text{given by (2.5). We denote } U = \bigcup_i \tilde{U}_j \text{ and } U'. \\ \text{all show that} & \begin{aligned} M_{(1)} \psi(x, y) &\leq d^n M_{(1)} \tilde{f}(x, y), \quad (x, y) \in U' \times Y, \\ \text{in } X & \in U' \text{ and } t \in \mathbb{R$ *fl,flE(x,t)* 0). For Swehave - —. *(tç' + es'), i ='l,°..., n; - (3.7)-*

$$
M_{(1)} \psi(x, y) \leq d^{n} M_{(1)} \tilde{f}(x, y), \quad (x, y) \in U' \times Y, \tag{3.6}
$$

where $d = 3^{|a|/r}$. Let $x \in U'$ and $t \in \mathbb{R}_+$. Evidently $x \in E(x, t) \setminus U$. Put $S = \{j \in \mathbb{N} :$
 $\Omega_j \cap E(x, t) \neq \emptyset\}$. For $j \in S$ we have

$$
|x_i^{(j)} - x_i| \leq \frac{1}{2} (t_j^{\alpha_i} + t^{\alpha_i}), \qquad i = 1, ..., n.
$$
 (3.7)

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Since $x \notin U$, there exists $k, 1 \leq k \leq n$, such that

$$
|x_k^{(j)}-x_k|>\frac{1}{2'}\tilde{t}_j^{\alpha_k}.
$$

From (3.7) and (3.8) we get $t^{\alpha_k} > t_i^{\alpha_k}$, i.e.

$$
t_j < t, \qquad j \in S.
$$

 $|z_i|$

Hence, if $z \in U_j$, $j \in S$, then by (3.7) and (3.9)

$$
- |x_i| \leq |z_i - x_i^{(j)}| + |x_i^{(j)} - x_i|
$$

\n
$$
\leq \frac{1}{2} (t_j^{a_i} + t_j^{a_i} + t^{a_i}) < \frac{3}{2} t^{a_i} \leq
$$

and so $U_i \subset E(x, \bar{t})$ for each $j \in S$. On the other hand obviously $3^n \leq |E(x, \bar{t})|$ $|X| \leq |E(x, t)| \leq d^n$ for every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. Thus for $x \in U'$, $y \in Y$ and $t \in R_+$ ¹ we have

$$
|E(x, t)^{-1} \int_{E(x, t)} |\psi(z, y)| dz = \sum_{j \in S} |E(x, t)^{-1} \int_{E(x, t) \cap R_j} |\psi(z, y)| dz
$$

\n
$$
\leq \sum_{j \in S} |E(x, t)|^{-1} \int_{R_j} |f(z, y)| dz = \sum_{j \in S} |E(x, t)|^{-1} \int_{R_j} \tilde{f}(z, y) dz
$$

\n
$$
\leq d^n |E(x, t)|^{-1} \int_{E(x, t)} \tilde{f}(z, y) dz \leq d^n M_{(1)} \tilde{f}(x, y),
$$

 \tilde{t}^{α_i}

which yields (3.6).

Because of (3.6) we can write

$$
Q_2 \subset U \cup Q_3,
$$

where $Q_3 = \left\{ x \in U' : N_{\theta}(M_{(1)}) \leq d^{-n} \frac{s}{2} \right\}$. So, instead of $\mu_{\theta} Q_2$ it suffices to estimate $\mu_w U$ and $\mu_w Q_3$.

By the generalized Minkowski inequality and (3.2) for $x \in R_i$

$$
\left(\int\limits_{Y} \tilde{f}_{\theta}(x, y) \, dv\right)^{1/\theta} \leq |R_j|^{-1} \int\limits_{R_j} \left(\int\limits_{Y} |f(z, y)|^{\theta} \, dv\right)^{1/\theta} dz
$$
\n
$$
= |R_j|^{-1} \int\limits_{R_j} N_{\theta}(f) \, (z) \, dz \leq (2\alpha_0)^{|\alpha|} \, s \, ,
$$

and so

 $N_{\theta}^{\theta}(\tilde{f})$ $(x) \leq (2\alpha_0)^{|\alpha| \theta} s^{\theta}, \quad x \in R_i$.

By a similar way as we obtained (3.5) we derive from Proposition 2.2 and from (3.11) the estimate

$$
\mu_w Q_3 \leq c_2 s^{-\theta} \int_R N_{\theta}^{\theta}(\tilde{f}) \ (x) \ d\mu_w \leq c_3 \mu_w R \leq c_3 \mu_w U \ . \tag{3.12}
$$

Thus, it remains to estimate the measure of U. At first, suppose that $p = 1$. From (3.2) , (2.8) and (2.4) we obtain

$$
\mu_w U \leq s^{-1} \sum_{j} \int_{\tilde{U}_j} w(x) dx |R_j|^{-1} \int_{R_j} N_{\theta}(f) (z) dz
$$

\n
$$
\leq d^n \alpha_0^{|\alpha|} s^{-1} \sum_{j} |\tilde{U}_j|^{-1} \int_{\tilde{U}_j} w(x) dx \int_{R_j} N_{\theta}(f) (z) dz
$$

\n
$$
\leq d^n \alpha_0^{|\alpha|} 2^{|\alpha|/y} \sum_{j} \int_{R_j} N_{\theta}(f) (z) M w(z) dz
$$

 (3.9)

 (3.10)

 (3.11)

 (3.8)

and using the condition (2.2) we conclude that

Weighted Inequalities for Maximal Functions 509
\ng the condition (2.2) we conclude that
\n
$$
\mu_w U \leq c_4 s^{-1} \int_{\mathbf{R}^n} N_{\theta}(f) (z) w(z) dz.
$$
\n(3.13)
\nif $1 < p < \infty$, by the use of (3.2), (2.8) and of Hölder's inequality we get

Weighted Inequalities for Maximal Functions
\nand using the condition (2.2) we conclude that
\n
$$
\mu_w U \leq c_4 s^{-1} \int N_{\theta}(f) (z) w(z) dz.
$$
\n(3.
\nFinally, if $1 < p < \infty$, by the use of (3.2), (2.8) and of Hölder's inequality we get
\n
$$
\mu_w U \leq s^{-p} \sum_{j} \int_{U_j} w(x) dx |R_j|^{-p} \int_{R_j} N_{\theta}(f) (z) dz \rangle^p
$$
\n
$$
\leq d^{np} \alpha_0^{|a|p} s^{-p} \sum_{j} |\tilde{U}_j|^{-p} \int_{U_j} w(x) dx \int N_{\theta}^p(f) (z) w(z) dz
$$
\n
$$
\times \left(\int_{U_j} w^{-\frac{1}{p-1}}(z) dz \right)^{p-1}
$$
\nSince $\tilde{U}_j \in E$, the condition (2.1) yields
\n
$$
\mu_w U \leq c_5 s^{-p} \int N_{\theta}^p(f) (z) dz.
$$
\n(3.
\nThe inequality (3.1) now follows from (3.4), (3.5), (3.10), (3.12), (3.13) and (3.14)
\n3.2. Theorem: Let $1 < p, \theta < \infty$. There exists a constant $c > 0$ such that the inequality
\n
$$
\int_{\mathbb{R}^n} \int_{Y} [M_{(1)}(x, y)]^{\theta} dy)^{p/\theta} w(x) dx \leq c \int_{\mathbb{R}^n} \int_{Y} |f(x, y)|^{\theta} dy)^{p/\theta} w(x) dx.
$$
\n(3.
\nholds for every λ -measurable function f if and only if $w \in A_p(\mathbf{E})$.
\n3.3. Remark: Let us consider the isotropic case $(\alpha, = \dots = \infty)$. In this case Theorem

Since $\tilde{U}_i \in \mathbb{E}$, the condition (2.1) yields

$$
\mu_w U \leq c_5 s^{-p} \int_{\mathbf{R}^n} N_{\theta} p(f) \ (z) \ dz. \tag{3.14}
$$

The inequality (3.1) now follows from (3.4), **(3.5), (3.10), (3.12),** (3.13) and (3.14) I

3.2. Theorem: Let $1 < p, \vartheta < \infty$. There exists a constant $c > 0$ such that the inequality

$$
\mathbf{R}^n
$$

uality (3.1) now follows from (3.4), (3.5), (3.10), (3.12), (3.13) and (3.14) \blacksquare
orem: Let $1 < p, \vartheta < \infty$. There exists a constant $c > 0$ such that the inequality

$$
\iint_{\mathbf{R}^n} \left(\int_{Y} [M_{(1)}\hat{f}(x, y)]^{\theta} dy \right)^{p/\theta} w(x) dx \leq c \int_{\mathbf{R}^n} \left(\int_{Y} |f(x, y)|^{\theta} dy \right)^{p/\theta} w(x) dx \qquad (3.15)
$$

holds for every λ *-measurable function <i>f* if and only if $w \in A_n(E)$.

3.3. Remark: Let us consider the isotropic case $(\alpha_1 = \cdots = \alpha_n)$. In this case Theorem 3.2 was proved by the first author [9]. Particularly, if the measure v is concentrated in the natural numbers, an analogous result was obtained independently by K. F. ANDERSEN and R.T. JOHN [1]. In the unweighted case $(w(x) = 1)$ (3.5) is a special case of the maximal inequality proved by R. J. BAGEY [2]. 1.6 Interpretative (3.1) now follows from (5.4), (5.9), (6.10), (8.2), (3.

3.4. Proof of Theorem 3.2: At first we suppose that $1 < p < \theta < \infty$. Let p_0, p_1 be such that $p_0 < p < p_1 < \vartheta$ and $w \in A_{p_1}(E) \cap A_{p_1}(E)$ (see Section 2.1). According EN and R. T. JOHN [1]. In the unweighted case $(w(x) = 1)$ (3.5) is a spectric in the maximal inequality proved by R. J. BAGBY [2].

of of Theorem 3:2: At first we suppose that $1 < p < \vartheta < \infty$. Let p_0 ,

that $p_0 < p < p_1 < \vartheta$

$$
\mu_w\{x\in {\bf R}^n\colon N_{\theta}(M_{(1)}f)(x)>s\}\leqq c_1s^{-p_i}\int_{\bf R}N_{\theta}^{p_i}(f)(x)\,w(x)\,dx, \qquad i=0,\,1
$$

for every λ -measurable function f . By the use of Marcinkiewicz's interpolation theorem we obtain (3.15).

Further, let $1 < \vartheta < p < \infty$. According to Section 2.1 there exists $p_0, 1 < p_0 < p$ such that $w \in A_{p_0}(E)$. Choose ϑ_0 , $1 < \vartheta_0 < p p_0^{-1}$. We can consider two cases: $1 < \vartheta$ ϑ_0 and $\vartheta_0 < \vartheta < p$.
If $1 < \vartheta \leq \vartheta_0$, then $\begin{array}{l} \text{there,} \ \text{that } i \leq \theta \ < \theta \ \text{if} \ \begin{array}{l} \text{if} \ \text{if} \ \text{if} \ \text{if} \ \end{array} \end{array}$ λ -measurable function *f*. By the use o
tain (3.15).

, let $1 < \vartheta < p < \infty$. According to Se
 $w \in A_{p_0}(\mathbf{E})$. Choose ϑ_0 , $1 < \vartheta_0 < pp_0$
 $\vartheta_0 < \vartheta < p$.
 $\vartheta \leq \vartheta$, then $w \in A_{p/\theta}(\mathbf{E})$. We have
 $\iint_{\mathbf{R}^n}$ $\begin{aligned} \text{tion 2.1 th} \ \text{We can c} \ \int \int_{Y} \Big(\int_{Y} \big[M_{\theta} \Big] \end{aligned}$ (1). According $i = 0, 1$

bolation the

space of $i < n_0$.

cases: 1
 $\left(\frac{1}{3}, 1\right)$
 $h(x) dx$,

(3.1

R¹ for whi dx , $i = 0, 1$

interpolation theo-

ists $p_0, 1 < p_0 < p$

ir two cases: $1 < \vartheta$
 y) $\left| \vartheta \right|$
 $\left| \$

If $1 < \vartheta \leq \vartheta_0$, then $w \in A_{p/\vartheta}(E)$. We have

$$
\left\{\int\limits_{\mathbf{R}^n}\left(\int\limits_Y\left[M_{(1)}f(x,y)\right]^\theta\,dy\right)^{p/\theta}\,d\mu_w\right\}^{\theta/p}=\sup\limits_{\mathbf{R}^n}\int\limits_Y\left(\int\limits_Y\left[M_{(1)}f(x,y)\right]^\theta\,\tilde{d}\nu\right)h(x)\,dx\,,\tag{3.16}
$$

where the supremum is taken over all non-negative functions $h: \mathbb{R}^n \to \mathbb{R}^1$ for which *-*

$$
\int_{\mathbb{R}^n} [h(x)]^{\frac{p}{p-\theta}} [w(x)]^{-\frac{\theta}{p-\theta}} dx \leq 1.
$$

Using the Fubmi theorem and Lemma *2.6* we can write

V. Korkuashvnu and J. Rákosník
\ne Fubini theorem and Lemma 2.6 we can write
\n
$$
\iint_{\mathbf{R}^n} \left(\int_Y \left[M_{(1)} f(x, y) \right]^{\phi} dy \right) h(x) dx = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \left[M_{(1)} f(x, y) \right]^{\phi} h(x) dx \right) dy
$$
\n
$$
\leq c_2 \int_{\mathbf{R}^n} \left(\int_Y |f(x, y)|^{\phi} dy \right) M h(x) dx.
$$
\nby the Hölder inequality we obtain

Hence, by the Holder inequality we obtain

We have
$$
\iint_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} \left[M_{(1)} f(x, y) \right]^{\phi} dy \right) h(x) dx = \int_{\mathbf{Y}} \left(\int_{\mathbf{R}^n} \left[M_{(1)} f(x, y) \right]^{\phi} h(x) dx \right) dx
$$

\n
$$
\leq c_2 \int_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} |f(x, y)|^{\phi} dv \right) M h(x) dx.
$$

\nby the Hölder inequality we obtain
\n
$$
\int_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} |M_{(1)} f(x, y)|^{\phi} dv \right) h(x) dx
$$

\n
$$
\leq c_2 \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} |f(x, y)|^{\phi} dv \right)^{p/\phi} w(x) dx \right)^{\phi/p} \left(\int_{\mathbf{R}^n} \left[M h(x) \right]^{\frac{p}{p-\phi}} [w(x)]^{-\frac{\theta}{p-\phi}} dx \right)^{\frac{p-\theta}{p}}.
$$

\n
$$
\leq A_{p/\phi}(\mathbf{E}), \text{ it is } w^{-\frac{\theta}{p-\phi}} \in A_{\frac{p}{p-\phi}}(\mathbf{E}), \text{ and applying Lemma 2.6 to the second}
$$

\non the right hand side of (3.17) we get
\n
$$
\int_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} |M_{(1)} f(x, y)|^{\phi} dv \right) h(x) dx \leq c_3 \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{Y}} |f(x, y)|^{\phi} dv \right)^{p/\phi} w(x) dx \right)^{\phi/p}.
$$

\n
$$
\text{quality (3.15) now follows from (3.16) and (3.18).} \tag{3.18}
$$

Since $w \in A_{p/\theta}(\mathbf{E})$, it is $w^{-\frac{\theta}{p-\theta}} \in A_{\frac{p}{p-\theta}}(\mathbf{E})$, and applying Lemma 2.6 to the second integral on the right hand side of (3.17) we get
 $\iint_{\mathbf{R}^n} \left(\int_{\mathbf{F}} [M_{(1)}f(x, y)]^{\theta} dy \right) h(x) dx \leq c_3 \left(\int_{\mathbf{R}^n$ integral on the right hand side of $\frac{\overline{p-\theta}}{(3.17)}$ we get

$$
\int_{\mathbf{R}^n} \left(\int\limits_Y \left[M_{(1)} f(x, y) \right]^{\theta} dy \right) h(x) dx \leq c_3 \left(\int\limits_{\mathbf{R}^n} \left(\int\limits_Y |f(x, y)|^{\theta} dy \right)^{p/\theta} w(x) dx \right)^{\theta/p}.
$$
\n(3.18)

The inequality *(3.15)* now follows from *(3.16)* and (3.18).

Since, among other, we have just proved the inequality *(3.15)* for $\vartheta = \vartheta_0$ and the inequality (3.15) with $\vartheta = p$ is a simple consequence of the Fubini theorem and of Proposition 2.2, we can use the interpolation theorem for spaces with mixed norms (see, e.g. [3]). By this way we obtain the inequality (3.15) for $\hat{\theta}_0 < \hat{\theta} \leq p$. The inequality (3.15) now follows from (3.16) and (3.18).
Since, among other, we have just proved the inequality (3.15) for $\vartheta = \vartheta$
inequality (3.15) with $\vartheta = p$ is a simple consequence of the Fubini theore
Proposition *•* among other, we have
y (3.15) with $\vartheta = p$ is
on 2.2, we can use the
[3]). By this way we α
re proved, that the con-
ressa, if we consider fu
 φ , ψ satisfy the condi
 $\int_{\mathbf{r}} |\psi(y)|^{\vartheta} d\nu > 0$, $\int_{\mathbf{R}^n}$
n

Thus we proved, that the condition $w \in A_p(E)$ is sufficient for (3.15).

Thus we proved, that the condition $w \in A_p(E)$ is sufficient for (3.15).
Vice versa, if we consider functions f of the form $f(x, y) = \varphi(x) \psi(y)$, where the functions φ , ψ satisfy the conditions. for 2.2, we can use the complete the consider $f(\mathbf{y})$. By this way we consider the consider the consider of φ , ψ satisfy the cond $\int_{\mathbf{r}} |\psi(y)|^{\theta} d\nu > 0$,

$$
\int\limits_Y |\psi(y)|^{\theta} d\nu > 0, \quad \int\limits_{\mathbf{R}^n} |\varphi(x)|^p w(x) dx < \infty,
$$

then the necessity of the condition $w \in A_p(\mathbf{E})$ follows from the corresponding assertion of Proposition 2.2 **■**

4. Maximal inequality with **mixed norms**

4.1.' We shall use Theorem *3.2* for the proof 'of an inequality for the anisotropic maximal function in spaces with mixed norms.

Let $E^{(1)}$, $E^{(2)}$ be families of one-parametric parallelepipeds in the spaces \mathbb{R}^m , \mathbb{R}^n , corresponding to the vectors $\alpha^{(1)} \in \mathbb{R}^m$, $\alpha^{(2)} \in \mathbb{R}^n$, respectively (see Section 1.2). By E we denote the family of all $E = E^{(1)} \times E^{(2)}$, $E^{(i)} \notin E^{(i)}$, $i = 1, 2$. Let us introduce the maximal function *where in spaces with mixed norms.*
 Let $E^{(1)}$, $E^{(2)}$ be families of one-parametric parallelepipeds in the sportesponding to the vectors $\alpha^{(1)} \in \mathbb{R}^m$, $\alpha^{(2)} \in \mathbb{R}^n$, respectively (see By E we denote the f

$$
M^*f(x, y) = \sup |E|^{-1} \int_R f(u, z) du dz,
$$

where the supremum is taken over all $E \in E$ with the centre at $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$.

4.2. Theorem: Let $1 < p_1, p_2 < \infty$ and $w_i \in A_{p_i}(\mathbf{E}^{(i)}), i = 1, 2$. Th

$$
\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} [M^* f(x, y)]^{p_1} w_1(x) dx \right)^{p_1/p_1} w_2(y) dy
$$

\n
$$
\leq c \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} |f(x, y)|^{p_1} w_1(x) dx \right)^{p_1/p_1} w_2(y) dy
$$

Proof: Let $M_{(1)}$ be the maximal function defined by (1.3) and

$$
M_{(2)}f(x, y) = \sup |E^{(2)}|^{-1} \int_{B^{(1)}} |f(x, z) dz,
$$

where the supremum is taken over all $E^{(2)} \in E^{(2)}$ containing y. Then, obviously,

$$
M^*/(x, y) \leq M_{(1)}(M_{(2)}f)(x, y), \qquad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n.
$$

According to (4.1) and Proposition 2.2 we have

$$
\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} [M^* f(x, y)]^{p_1} w_1(x) dx \right)^{p_1/p_1} w_2(y) dy
$$
\n
$$
\leqq \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} [M_{(1)} (M_{(2)} f) (x, y)]^{p_1} w_1(x) dx \right)^{p_1/p_1} w_2(y) dy
$$
\n
$$
\leqq c \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} [M_{(2)} f(x, y)]^{p_1} w_1(x) dx \right)^{p_1/p_1} w_2(y) dy.
$$

It remains to use Theorem 3.2 with $Y = \mathbb{R}^m$ and $\nu = \mu_w$.

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