Zeitschrift für Analysis
und ihre Anwendungen
Bd. 4 (6) 1985, S. 503–511

# Weighted Inequalities for Vector-Valued Anisotropic Maximal Functions

V. Kokilashvili and J. Rákosník

Die anisotrope Maximalfunktion wird hier mit Hilfe einer einparametrigen Familie von Quadern (statt Würfeln) definiert. Für vektorwertige anisotrope Maximalfunktionen werden gewichtete Ungleichungen vom schwachen und starken Typ bewiesen. Die Ungleichung vom starken Typ wird dann zum Beweis der Stetigkeit des anisotropen Maximaloperators in gewichteten Räumen mit gemischter Norm benutzt.

Анизотропная максимальная функция определяется здесь через однопараметрическое семейство параллелепипедов вместо кубов. Для векторнозначных анизотропных максимальных функций доказываются весовые неравенства слабого и сильного типов. Неравенство сильного типа используется в конце при доказательстве непрерывности анизотропного максимального оператора в весовых пространствах со смешанными нормами.

The anisotropic maximal function is defined by means of one-parametric parallelepipeds instead of cubes. For vector-valued anisotropic maximal functions there are proved weak and strong type weighted inequalities. The strong type inequality is then utilised in the proof of an anisotropic weighted mixed norm maximal inequality.

## 1. Introduction

1.1. Let $\mathbf{R}^n$ be the Euclidean space of points $x = (x_1, \ldots, x_n)$, $\mathbf{R}_+^n$ be the set of all points $y = (y_1, \ldots, y_n)$ with $y_i > 0$, $i = 1, \ldots, n$. By a *weight function* (shortly a *weight*) we shall mean a non-negative measurable function $w$: $\mathbf{R}^n \to \mathbf{R}^1$. The weight $w$ generates a measure $\mu_w$ given by

$$\mu_w e = \int_e w(x)\, dx, \qquad e \subset \mathbf{R}^n \text{ measurable}. \tag{1.1}$$

The Lebesgue measure of $e$ will be denoted by $|e|$.
For a weight $w$ and $1 \leqq p < \infty$ we define the weighted Lebesgue space $L_w^p$ as the set of all measurable functions $f: \mathbf{R}^n \to \mathbf{R}^1$ with the norm

$$\|f\|_{p,w} = \Big(\int_{\mathbf{R}^n} |f(x)|^p\, w(x)\, dx\Big)^{1/p} < \infty.$$

1.2. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a fixed point in $\mathbf{R}_+^n$. For $x \in \mathbf{R}^n$ and $t \in \mathbf{R}_+^1$ we define

$$E(x, t) = \left\{z \in \mathbf{R}^n : |z_i - x_i| \leqq \frac{1}{2}\, t^{\alpha_i}, \quad i = 1, \ldots, n\right\}$$

and

$$\mathbf{E} = \{E(x, t) : x \in \mathbf{R}^n, t \in \mathbf{R}_+^1\}.$$

**1.3.** To each measurable function $f\colon \mathbf{R}^n \to \mathbf{R}^1$ we adjoin the *anisotropic maximal function*

$$Mf(x) = \sup_{t>0} |E(x, t)|^{-1} \int_{E(x,t)} |f(z)| \, dz. \tag{1.2}$$

If $\alpha_1 = \cdots = \alpha_n$, then $Mf$ is the classical Hardy-Littlewood maximal function.

**1.4.** Let $(Y, S, \nu)$ be a $\sigma$-finite measure space and $T$ be a $\sigma$-algebra of Lebesgue measurable sets in $\mathbf{R}^n$. On the $\sigma$-algebra $T \times S$ we define the measure $\lambda$ as the product of the Lebesgue measure and of $\nu$. For a $\lambda$-measurable function $f\colon \mathbf{R}^n \times Y \to \mathbf{R}^1$ we define the *vector-valued anisotropic maximal function*

$$M_{(1)}f(x, y) = \sup_{t>0} |E(x, t)|^{-1} \int_{E(x,t)} |f(z, y)| \, dz, \quad x \in \mathbf{R}^n, \quad y \in Y. \tag{1.3}$$

**1.5.** In [11] (see also [5]) there is given a characterization of those positive functions $w$ for which the classical Hardy-Littlewood maximal function is bounded operator in $L_w^p$. In [7] the well known theorem of Hardy and Littlewood on $L^p$ boundedness of maximal functions was generalized for $l^\theta$-valued functions in the unweighted case. This result was extended in [2] for functions with values in the spaces $l^{\bar\theta}$ and $L^{\bar\theta}$ with mixed norms. For $L^\theta$-valued functions the first author obtained in [9] a full description of the weighted Lebesgue spaces in which the Hardy-Littlewood maximal function is a bounded operator. For $l^\theta$-valued functions a similar result was derived independently in [1]. The weighted weak type inequality for $l^\theta$-valued Hardy-Littlewood maximal functions was established earlier in [8].

In this note there are proved weighted weak and strong type inequalities for a vector-valued anisotropic maximal function. The main ideas follow the first author's paper [9]. At the end an application of the strong type inequality to the weighted mixed norm maximal inequality is shown.

## 2. Auxiliary notions and assertions

**2.1.** *The class $A_p(\alpha)$.* Let $\mathbf{E}$ be the set defined in Section 1.2. If $1 < p < \infty$, the class $A_p(\mathbf{E})$ consists of all weights $w$ in $\mathbf{R}^n$ for which there exists such a positive constant $c$, that for any $E = E(x, t) \in \mathbf{E}$

$$\left(|E|^{-1} \int_E w(z) \, dz\right) \left(|E|^{-1} \int_E w^{-\frac{1}{p-1}}(z) \, dz\right)^{p-1} \leqq c. \tag{2.1}$$

The function $w$ is said to be of the class $A_1(\mathbf{E})$ if there exists such a constant $c > 0$, that

$$M(w)(x) \leqq cw(x) \quad \text{a.e. in } \mathbf{R}^n, \tag{2.2}$$

where $M$ is defined by (1.2). Remind two properties of functions from the class $A_p(\mathbf{E})$: If $w \in A_p(\mathbf{E})$, $1 < p < \infty$, then there exists $p_0$ such that $1 < p_0 < p$, and $w \in A_{p_0}(\mathbf{E})$; in addition $w \in A_{p_1}(\mathbf{E})$ for arbitrary $p_1 > p$. The second property is a simple corollary of the Hölder inequality, and the first was proved in [10]. There was also stated the following assertion.

**2.2.** Proposition: Let $1 < p < \infty$ *and* $\mathbf{E}$ *be given in Section* 1.2. *Then there exists a constant* $c > 0$, *independent of* $f$, *such that*

$$\int_{\mathbf{R}^n} [Mf(x)]^p \, w(x) \, dx \leqq c \int_{\mathbf{R}^n} |f(x)|^p \, w(x) \, dx$$

*if and only if* $w \in A_p(\mathbf{E})$.

**2.3. Lemma:** *For a measurable function $f$ we define*

$$\tilde{M}f(x) = \sup |E|^{-1} \int_E |f(z)|\, dz, \qquad x \in \mathbf{R}^n, \tag{2.3}$$

*where the supremum is taken over all $E \in \mathbf{E}$ which contain $x$.*

*It holds*

$$Mf(x) \leqq \tilde{M}f(x) \leqq 2^{|\alpha|/\gamma} Mf(x), \tag{2.4}$$

*where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and*

$$\gamma = \min_i \alpha_i. \tag{2.5}$$

Proof: The first inequality is trivial. Now, suppose that $x \in E(u, t) \in \mathbf{E}$. This means that $|x_i - u_i| \leqq \frac{1}{2} t^{\alpha_i}$, $i = 1, 2, \ldots, n$, and for $y \in E(u, t)$ we have

$$|y_i - x_i| \leqq |y_i - u_i| + |x_i - u_i| \leqq t^{\alpha_i} \leqq \frac{1}{2} (t')^{\alpha_i},$$

where $t' = 2^{1/\gamma}t$, $\gamma$ defined by (2.5). Consequently $y \in E(x, t')$, i.e. $E(u, t) \subset E(x, t')$, and, moreover, $|E(x, t')| = (2t)^{|\alpha|} = 2^{|\alpha|} |E(x, t)|$. Hence

$$|E(u, t)|^{-1} \int_{E(u,t)} |f(z)\, dz \leqq 2^{|\alpha|/\gamma} |E(x, t')|^{-1} \int_{E(x,t)} |f(z)|\, dz,$$

and so the second inequality (2.3) holds ∎

We recall two covering lemmas which we utilize later.

**2.4. Lemma** [4: Section 3.6]: *Let $f \in L^1(\mathbf{R}^n)$ and $s > 0$. Then there exist a number $\alpha_0 \geqq 1$ (depending only on $\alpha$ from Section 1.2) and a sequence of non-overlapping parallelepipeds $R_j$, $j \in \mathbf{N}$ such that*

$$s < |R_j|^{-1} \int_{R_j} |f(x)|\, dx \leqq (2\alpha_0)^{|\alpha|} s, \qquad j \in \mathbf{N}, \tag{2.6}$$

$$|f(x)| \leqq s \quad \text{a.e. in } \mathbf{R}^n \setminus \bigcup_j R_j, \tag{2.7}$$

*and for any $j \in \mathbf{N}$ there exist $U_j$, $V_j \in \mathbf{E}$ so that*

$$V_j \subset R_j \subset U_j \quad \text{and} \quad |U_j| = \alpha_0^{|\alpha|} |V_j|. \tag{2.8}$$

**2.5. Lemma** ([6]): *Let $D$ be a bounded set in $\mathbf{R}^n$, and let for any $x \in D$ there be given a parallelepiped $R(x)$ with the centre $x$. Suppose that for each two points $x_1$, $x_2 \in D$ the parallelepipeds $R(x_1)$ and $R(x_2)$ are comparable, i.e. one of $R(x_1)$ and $(x_1 - x_2) + R(x_2)$ contains the other.*

*Then from $\{R(x): x \in D\}$ a sequence $\{R_j\}$ can be selected such that*

$$D \subset \bigcup_j R_j, \tag{2.9}$$

$$\sum_j \chi_{R_j}(x) \leqq \vartheta_n, \qquad x \in D, \tag{2.10}$$

*where the number $\vartheta_n$ depends only on the dimension $n$.*

**2.6.** In this section we shall prove an anisotropic version of the lemma by C. Fefferman adn E. M. Stein [7].

Lemma: *Let* $1 < p < \infty$ *and* $f, g \in L_{\mathrm{loc}}(\mathbf{R}^n)$ be non-negative functions. Then there exists a constant $c > 0$ independent of $f$ and $g$, such that

$$\int_{\mathbf{R}^n} [Mf(x)]^p g(x)\, dx \leqq c \int_{\mathbf{R}^n} f^p(x)\, Mg(x)\, dx.$$

Proof: Let $s$ be an arbitrary positive number and $m \in \mathbf{N}$. We denote

$$H_s = \{x \in \mathbf{R}^n: Mf(x) > s\} \quad \text{and} \quad H_s^m = H_s \cap \{x \in \mathbf{R}^n: |x| \leqq m\}.$$

For each $x \in H_s$ there exists $E_x \in \mathbf{E}$ such that

$$|E_x|^{-1} \int_{E_x} f(z)\, dz > s. \tag{2.11}$$

Since the family $\{E_x: x \in H_s^m\}$ satisfies the assumptions of Lemma 2.5, there is a sequence of non-overlapping sets $E_j$ from this family, satisfying (2.9) and (2.10) where $E_j$ and $H_s^m$ stand for $R_j$ and $D$, respectively. By the Hölder inequality, and the estimate (2.11) we have

$$\begin{aligned}\int_{H_s^m} g(x)\, dx &\leqq \sum_j \int_{E_j} g(x)\, dx \Big(s^{-1} |E_j|^{-1} \int_{E_j} f(z)\, dz\Big)^p \\ &\leqq s^{-p} \sum_j |E_j|^{-1} \int_{E_j} g(x)\, dx \int_{E_j} f^p(z)\, Mg(z)\, dz \\ &\quad \times \Big(|E_j|^{-1} \int_{E_j} [Mg(z)]^{-\frac{1}{p-1}}\, dz\Big)^{p-1}.\end{aligned}$$

Since $\tilde{M}g(z) \geqq |E_j|^{-1} \int_{E_j} g(x)\, dx$ for all $z \in E_j$, by Lemma 2.3 we get

$$\Big(|E_j|^{-1} \int_{E_j} [Mg(z)]^{-\frac{1}{p-1}}\, dz\Big)^{p-1} \leqq 2^{|\alpha|/\gamma} \Big(|E_j|^{-1} \int_{E_j} g(z)\, dz\Big)^{-1}$$

and so by (2.10)

$$\int_{H_s^m} g(x)\, dx \leqq 2^{|\alpha|/\gamma} \vartheta_n s^{-p} \int_{\mathbf{R}^n} f^p(x)\, Mg(x)\, dx. \tag{2.12}$$

Passing to the limit $m \to \infty$ and assuming that the right hand side does not depend on $m$ we can write $H_s$ instead of $H_s^m$ in (2.12) and so we obtain the weak type $(p, p)$ inequality for the operator $M$ with respect to the measures $\mu_{Mg}$ and $\mu_g$. It remains to use the Marcinkiewicz interpolation theorem (see, e.g. [12]).

## 3. Weak and strong type inequalities

3.1. We shall now prove the weak type inequality for the vector-valued anisotropic maximal operator $M_{(1)}$ defined in Section 1.4. This weak type inequality will be then used by the proof of the strong one.

Lemma: *If* $1 \leqq p \leqq \vartheta < \infty$, $\vartheta > 1$ *and* $w \in A_p(\mathbf{E})$, *then there exists a positive constant c such that*

$$\begin{aligned}&\mu_w \Big\{x \in \mathbf{R}^n: \Big(\int_Y [M_{(1)}f(x, y)]^\vartheta\, d\nu\Big)^{1/\vartheta} > s\Big\} \\ &\leqq c s^{-p} \int_{\mathbf{R}^n} \Big(\int_Y |f(x, y)|^\vartheta\, d\nu\Big)^{p/\vartheta} w(x)\, dx\end{aligned} \tag{3.1}$$

*for every* $s > 0$ *and every* $\lambda$*-measurable function* $f$.

Proof: Let $s > 0$ and a $\lambda$-measurable function $f: \mathbf{R}^n \times Y \to \mathbf{R}^1$ be given (for the measure $\lambda$ see Section 1.4). Denote

$$N_\vartheta(f)(x) = \left(\int_Y |f(x, y)|^\vartheta \, d\nu\right)^{1/\vartheta} = \|f(x, \cdot)\|_{L_\vartheta(Y)}.$$

By Lemma 2.4 there exist non-overlapping parallelepipeds $R_j$ and parallelepipeds $U_j, V_j \in \mathbf{E}$ such that relations (2.8) hold and, moreover,

$$s < |R_j|^{-1} \int_{R_j} N_\vartheta(f)(x) \, dx \leqq (2\alpha_0)^{|\alpha|} s, \qquad j \in \mathbf{N}, \tag{3.2}$$

$$N_\vartheta(f)(x) \leqq s \quad \text{a.e. in} \quad \mathbf{R}^n \setminus \bigcup_j R_j. \tag{3.3}$$

Put $R = \bigcup_j R_j$, $R' = \mathbf{R}^n \setminus R$, and for $(x, y) \in \mathbf{R}^n \times Y$

$$\psi(x, y) = f(x, y) \chi_{R'}(x), \qquad \varphi(x, y) = f(x, y) - \psi(x, y).$$

One can easily observe that

$$\mu_w\{x \in \mathbf{R}^n : N_\vartheta(M_{(1)}f)(x) > s\} \leqq \mu_w Q_1 + \mu_w Q_2, \tag{3.4}$$

where

$$Q_1 = \left\{x \in \mathbf{R}^n : N_\vartheta(M_{(1)}\varphi)(x) > \frac{s}{2}\right\},$$

$$Q_2 = \left\{x \in \mathbf{R}^n : N_\vartheta(M_{(1)}\psi)(x) > \frac{s}{2}\right\}.$$

Since $w \in A_p(\mathbf{E})$ and $\vartheta \geqq p$, it is $w \in A_\vartheta(\mathbf{E})$, and the Chebyshev inequality, the Fubini theorem and proposition in Section 2.2 yield

$$\mu_w Q_1 \leqq \left(\frac{s}{2}\right)^{-\vartheta} \int_Y \int_{\mathbf{R}^n} [M_{(1)}\varphi(x, y)]^\vartheta \, d\mu_w \, d\nu$$

$$\leqq c_1 s^{-\vartheta} \int_{\mathbf{R}^n} N_\vartheta{}^\vartheta(\varphi)(x) \, d\mu_w \leqq c_1 s^{-\vartheta} \int_{R'} N_\vartheta{}^\vartheta(f)(x) \, d\mu_w.$$

According to (3.3) we conclude that

$$\mu_w Q_1 \leqq c_1 s^{-p} \int_{\mathbf{R}^n} N_\vartheta{}^p(f)(x) \, d\mu_w. \tag{3.5}$$

Now, we shall estimate the second summand of (3.4). Let us introduce the step-function

$$\bar{f}(x, y) = \begin{cases} |R_j|^{-1} \int_{R_j} |f(z, y)| \, dz, & x \in R_j, \\ 0, & x \in R'. \end{cases}$$

With each $U_j = E(x^{(j)}, t_j)$ we associate the parallelepiped $\tilde{U}_j = E(x^{(j)}, \tilde{t}_j)$, where $\tilde{t}_j = 3^{1/\gamma} t_j$, $\gamma$ given by (2.5). We denote $U = \bigcup_j \tilde{U}_j$ and $U' = \mathbf{R}^n \setminus U$.

We shall show that

$$M_{(1)} \psi(x, y) \leqq d^n M_{(1)} \bar{f}(x, y), \qquad (x, y) \in U' \times Y, \tag{3.6}$$

where $d = 3^{|\alpha|/\gamma}$. Let $x \in U'$ and $t \in \mathbf{R}_+{}^1$. Evidently $x \in E(x, t) \setminus U$. Put $S = \{j \in \mathbf{N} : \Omega_j \cap E(x, t) \neq \emptyset\}$. For $j \in S$ we have

$$|x_i{}^{(j)} - x_i| \leqq \frac{1}{2} (t_j{}^{\alpha_i} + t^{\alpha_i}), \qquad i = 1, \ldots, n. \tag{3.7}$$

Since $x \notin U$, there exists $k$, $1 \leqq k \leqq n$, such that

$$|x_k^{(j)} - x_k| > \frac{1}{2} \bar{t}_j^{\alpha_k}. \tag{3.8}$$

From (3.7) and (3.8) we get $t^{\alpha_k} > t_j^{\alpha_k}$, i.e.

$$t_j < t, \quad j \in S. \tag{3.9}$$

Hence, if $z \in U_j$, $j \in S$, then by (3.7) and (3.9)

$$|z_i - x_i| \leqq |z_i - x_i^{(j)}| + |x_i^{(j)} - x_i|$$
$$\leqq \frac{1}{2} (t_j^{\alpha_i} + t_j^{\alpha_i} + t^{\alpha_i}) < \frac{3}{2} t^{\alpha_i} \leqq \frac{1}{2} \bar{t}^{\alpha_i}$$

and so $U_j \subset E(x, \bar{t})$ for each $j \in S$. On the other hand obviously $3^n \leqq |E(x, \bar{t})| / |E(x, t)| \leqq d^n$ for every $x \in \mathbf{R}^n$ and $t \in \mathbf{R}_+^1$. Thus for $x \in U'$, $y \in Y$ and $t \in R_+^1$ we have

$$|E(x, t)^{-1} \int_{E(x,t)} |\psi(z, y)| \, dz = \sum_{j\in S} |E(x, t)^{-1} \int_{E(x,t)\cap R_j} |\psi(z, y)| \, dz$$
$$\leqq \sum_{j\in S} |E(x, t)|^{-1} \int_{R_j} |f(z, y)| \, dz = \sum_{j\in S} |E(x, t)|^{-1} \int_{R_j} \tilde{f}(z, y) \, dz$$
$$\leqq d^n |E(x, \bar{t})|^{-1} \int_{E(x,\bar{t})} \tilde{f}(z, y) \, dz \leqq d^n M_{(1)} \tilde{f}(x, y),$$

which yields (3.6).

Because of (3.6) we can write

$$Q_2 \subset U \cup Q_3, \tag{3.10}$$

where $Q_3 = \left\{x \in U' : N_\theta(M_{(1)}\tilde{f})(x) > d^{-n} \frac{s}{2}\right\}$. So, instead of $\mu_w Q_2$ it suffices to estimate $\mu_w U$ and $\mu_w Q_3$.

By the generalized Minkowski inequality and (3.2) for $x \in R_j$

$$\left(\int_Y \tilde{f}_\theta(x, y) \, d\nu\right)^{1/\theta} \leqq |R_j|^{-1} \int_{R_j} \left(\int_Y |f(z, y)|^\theta \, d\nu\right)^{1/\theta} dz$$
$$= |R_j|^{-1} \int_{R_j} N_\theta(f)(z) \, dz \leqq (2\alpha_0)^{|\alpha|} s,$$

and so

$$N_\theta{}^\theta(\tilde{f})(x) \leqq (2\alpha_0)^{|\alpha|\theta} s^\theta, \quad x \in R_j. \tag{3.11}$$

By a similar way as we obtained (3.5) we derive from Proposition 2.2 and from (3.11) the estimate

$$\mu_w Q_3 \leqq c_2 s^{-\theta} \int_R N_\theta{}^\theta(\tilde{f})(x) \, d\mu_w \leqq c_3 \mu_w R \leqq c_3 \mu_w U. \tag{3.12}$$

Thus, it remains to estimate the measure of $U$. At first, suppose that $p = 1$. From (3.2), (2.8) and (2.4) we obtain

$$\mu_w U \leqq s^{-1} \sum_j \int_{\tilde{U}_j} w(x) \, dx \, |R_j|^{-1} \int_{R_j} N_\theta(f)(z) \, dz$$
$$\leqq d^n \alpha_0{}^{|\alpha|} s^{-1} \sum_j |\tilde{U}_j|^{-1} \int_{\tilde{U}_j} w(x) \, dx \int_{R_j} N_\theta(f)(z) \, dz$$
$$\leqq d^n \alpha_0{}^{|\alpha|} 2^{|\alpha|/\gamma} \sum_j \int_{R_j} N_\theta(f)(z) \, Mw(z) \, dz$$

and using the condition (2.2) we conclude that

$$\mu_w U \leqq c_4 s^{-1} \int_{\mathbf{R}^n} N_\vartheta(f)(z)\, w(z)\, dz. \tag{3.13}$$

Finally, if $1 < p < \infty$, by the use of (3.2), (2.8) and of Hölder's inequality we get

$$\mu_w U \leqq s^{-p} \sum_j \int_{\tilde{U}_j} w(x)\, dx\, |R_j|^{-p} \left( \int_{R_j} N_\vartheta(f)(z)\, dz \right)^p$$

$$\leqq d^{np} \alpha_0^{|\alpha| p} s^{-p} \sum_j |\tilde{U}_j|^{-p} \int_{\tilde{U}_j} w(x)\, dx \int_{R_j} N_\vartheta{}^p(f)(z)\, w(z)\, dz$$

$$\times \left( \int_{\tilde{U}_j} w^{-\frac{1}{p-1}}(z)\, dz \right)^{p-1}.$$

Since $\tilde{U}_j \in \mathbf{E}$, the condition (2.1) yields

$$\mu_w U \leqq c_5 s^{-p} \int_{\mathbf{R}^n} N_\vartheta{}^p(f)(z)\, dz. \tag{3.14}$$

The inequality (3.1) now follows from (3.4), (3.5), (3.10), (3.12), (3.13) and (3.14) ∎

**3.2. Theorem:** *Let $1 < p, \vartheta < \infty$. There exists a constant $c > 0$ such that the inequality*

$$\int_{\mathbf{R}^n} \left( \int_Y [M_{(1)} f(x, y)]^\vartheta\, d\nu \right)^{p/\vartheta} w(x)\, dx \leqq c \int_{\mathbf{R}^n} \left( \int_Y |f(x, y)|^\vartheta\, d\nu \right)^{p/\vartheta} w(x)\, dx \tag{3.15}$$

*holds for every $\lambda$-measurable function $f$ if and only if $w \in A_p(\mathbf{E})$.*

**3.3. Remark:** Let us consider the isotropic case ($\alpha_1 = \cdots = \alpha_n$). In this case Theorem 3.2 was proved by the first author [9]. Particularly, if the measure $\nu$ is concentrated in the natural numbers, an analogous result was obtained independently by K. F. Andersen and R. T. John [1]. In the unweighted case $(w(x) = 1)$ (3.5) is a special case of the maximal inequality proved by R. J. Bagby [2].

**3.4. Proof of Theorem 3.2:** At first we suppose that $1 < p < \vartheta < \infty$. Let $p_0, p_1$ be such that $p_0 < p < p_1 < \vartheta$ and $w \in A_{p_0}(\mathbf{E}) \cap A_{p_1}(\mathbf{E})$ (see Section 2.1). According to Lemma 3.2 there exists $c_1 > 0$ such that

$$\mu_w\{x \in \mathbf{R}^n : N_\vartheta(M_{(1)} f)(x) > s\} \leqq c_1 s^{-p_i} \int_{\mathbf{R}^n} N_\vartheta{}^{p_i}(f)(x)\, w(x)\, dx, \qquad i = 0, 1$$

for every $\lambda$-measurable function $f$. By the use of Marcinkiewicz's interpolation theorem we obtain (3.15).

Further, let $1 < \vartheta < p < \infty$. According to Section 2.1 there exists $p_0$, $1 < p_0 < p$ such that $w \in A_{p_0}(\mathbf{E})$. Choose $\vartheta_0$, $1 < \vartheta_0 < p p_0^{-1}$. We can consider two cases: $1 < \vartheta \leqq \vartheta_0$ and $\vartheta_0 < \vartheta < p$.

If $1 < \vartheta \leqq \vartheta_0$, then $w \in A_{p/\vartheta}(\mathbf{E})$. We have

$$\left\{ \int_{\mathbf{R}^n} \left( \int_Y [M_{(1)} f(x, y)]^\vartheta\, d\nu \right)^{p/\vartheta} d\mu_w \right\}^{\vartheta/p} = \sup \int_{\mathbf{R}^n} \left( \int_Y [M_{(1)} f(x, y)]^\vartheta\, d\nu \right) h(x)\, dx, \tag{3.16}$$

where the supremum is taken over all non-negative functions $h: \mathbf{R}^n \to \mathbf{R}^1$ for which

$$\int_{\mathbf{R}^n} [h(x)]^{\frac{p}{p-\vartheta}} [w(x)]^{-\frac{\vartheta}{p-\vartheta}}\, dx \leqq 1.$$

Using the Fubini theorem and Lemma 2.6 we can write

$$\int_{\mathbf{R}^n} \left( \int_Y [M_{(1)}f(x, y)]^\vartheta \, d\nu \right) h(x) \, dx = \int_Y \left( \int_{\mathbf{R}^n} [M_{(1)}f(x, y)]^\vartheta \, h(x) \, dx \right) d\nu$$
$$\leqq c_2 \int_{\mathbf{R}^n} \left( \int_Y |f(x, y)^\vartheta \, d\nu \right) Mh(x) \, dx.$$

Hence, by the Hölder inequality we obtain

$$\int_{\mathbf{R}^n} \left( \int_Y [M_{(1)}f(x, y)]^\vartheta \, d\nu \right) h(x) \, dx$$
$$\leqq c_2 \left( \int_{\mathbf{R}^n} \left( \int_Y |f(x, y)|^\vartheta \, d\nu \right)^{p/\vartheta} w(x) \, dx \right)^{\vartheta/p} \left( \int_{\mathbf{R}^n} [Mh(x)]^{\frac{p}{p-\vartheta}} [w(x)]^{-\frac{\vartheta}{p-\vartheta}} \, dx \right)^{\frac{p-\vartheta}{p}}. \tag{3.17}$$

Since $w \in A_{p/\vartheta}(\mathbf{E})$, it is $w^{-\frac{\vartheta}{p-\vartheta}} \in A_{\frac{p}{p-\vartheta}}(\mathbf{E})$, and applying Lemma 2.6 to the second integral on the right hand side of (3.17) we get

$$\int_{\mathbf{R}^n} \left( \int_Y [M_{(1)}f(x, y)]^\vartheta \, d\nu \right) h(x) \, dx \leqq c_3 \left( \int_{\mathbf{R}^n} \left( \int_Y |f(x, y)|^\vartheta \, d\nu \right)^{p/\vartheta} w(x) \, dx \right)^{\vartheta/p}. \tag{3.18}$$

The inequality (3.15) now follows from (3.16) and (3.18).

Since, among other, we have just proved the inequality (3.15) for $\vartheta = \vartheta_0$ and the inequality (3.15) with $\vartheta = p$ is a simple consequence of the Fubini theorem and of Proposition 2.2, we can use the interpolation theorem for spaces with mixed norms (see, e.g. [3]). By this way we obtain the inequality (3.15) for $\vartheta_0 < \vartheta \leqq p$.

Thus we proved, that the condition $w \in A_p(\mathbf{E})$ is sufficient for (3.15).

Vice versa, if we consider functions $f$ of the form $f(x, y) = \varphi(x) \, \psi(y)$, where the functions $\varphi$, $\psi$ satisfy the conditions

$$\int_Y |\psi(y)|^\vartheta \, d\nu > 0, \qquad \int_{\mathbf{R}^n} |\varphi(x)|^p \, w(x) \, dx < \infty,$$

then the necessity of the condition $w \in A_p(\mathbf{E})$ follows from the corresponding assertion of Proposition 2.2 ∎

## 4. Maximal inequality with mixed norms

4.1. We shall use Theorem 3.2 for the proof of an inequality for the anisotropic maximal function in spaces with mixed norms.

Let $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ be families of one-parametric parallelepipeds in the spaces $\mathbf{R}^m$, $\mathbf{R}^n$, corresponding to the vectors $\alpha^{(1)} \in \mathbf{R}^m$, $\alpha^{(2)} \in \mathbf{R}^n$, respectively (see Section 1.2). By $\mathbf{E}$ we denote the family of all $E = E^{(1)} \times E^{(2)}$, $E^{(i)} \notin \mathbf{E}^{(i)}$, $i = 1, 2$. Let us introduce the maximal function

$$M^*f(x, y) = \sup |E|^{-1} \int_E f(u, z) \, du \, dz,$$

where the supremum is taken over all $E \in \mathbf{E}$ with the centre at $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$.

4.2. Theorem: *Let $1 < p_1, p_2 < \infty$ and $w_i \in A_{p_i}(\mathbf{E}^{(i)})$, $i = 1, 2$. Then*

$$\int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^m} [M^*f(x, y)]^{p_1} \, w_1(x) \, dx \right)^{p_2/p_1} w_2(y) \, dy$$
$$\leqq c \int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^m} |f(x, y)|^{p_1} \, w_1(x) \, dx \right)^{p_2/p_1} w_2(y) \, dy$$

*with $c > 0$ independent of $f$.*

Proof: Let $M_{(1)}f$ be the maximal function defined by (1.3) and

$$M_{(2)}f(x, y) = \sup |E^{(2)}|^{-1} \int_{E^{(2)}} |f(x, z)\, dz,$$

where the supremum is taken over all $E^{(2)} \in \mathbf{E}^{(2)}$ containing $y$. Then, obviously,

$$M^*f(x, y) \leqq M_{(1)}(M_{(2)}f)(x, y), \quad (x, y) \in \mathbf{R}^m \times \mathbf{R}^n. \tag{4.1}$$

According to (4.1) and Proposition 2.2 we have

$$\int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^m} [M^*f(x, y)]^{p_1} w_1(x)\, dx \right)^{p_2/p_1} w_2(y)\, dy$$
$$\leqq \int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^m} [M_{(1)}(M_{(2)}f)(x, y)]^{p_1} w_1(x)\, dx \right)^{p_2/p_1} w_2(y)\, dy$$
$$\leqq c \int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^m} [M_{(2)}f(x, y)]^{p_1} w_1(x)\, dx \right)^{p_2/p_1} w_2(y)\, dy.$$

It remains to use Theorem 3.2 with $Y = \mathbf{R}^m$ and $\nu = \mu_{w_1}$ ∎

## References


[1] Andersen, K. F., and R. T. John: Weighted inequalities for vector-valued maxima functions and singular integrals. Studia Math. 69 (1980), 19–31.

[2] Bagby, R. J.: An extended inequality for the maximal function. Proc. Amer. Math. Soc. 48 (1975), 419–422.

[3] Benedek, A., and R. Panzone: The spaces $L^p$ with mixed norm. Duke Math. J. 28 (1961), 301–324.

[4] Бесов, О. В., Ильин, В. П., и С. М. Никольский: Интегральные представления функций и теоремы вложения. Москва: Изд-во Наука 1975.

[5] Coifman, R. R., and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. Studia Math. 51 (1974), 241–250.

[6] de Guzmán, M.: A covering lemma with applications to differentiability of measures and singular integral operators. Studia Math. 34 (1970), 299–317.

[7] Fefferman, C., and E. M. Stein: Some maximal inequalities. Amer. J. Math. 93 (1971), 107–115.

[8] Heinig, H. P.: Weighted maximal inequalities for $l^r$-valued functions. Canad. Math. Bull. 19 (1976), 445–453.

[9] Кокилашвили, В. М.: Максимальные неравенства и мультипликаторы в весовых пространствах Лизоркина-Трибеля. Докл. Акад. Наук СССР 239 (1978), 42–45.

[10] Kurtz, D. S.: Weighted norm inequalities for the Hardy-Littlewood maximal function for one parameter rectangles. Studia Math. 53 (1975), 39–54.

[11] Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. 165 (1972), 207–225.

[12] Zygmund, A.: Trigonometric series (Vol. II, Second Ed.). Cambridge: Univ. Press 1959.


Manuskripteingang: 23. 05. 1984

VERFASSER

Prof. Dr. V. Kokilashvili
Mathematical Institute of Georgian Academy of Sciences
USSR-380093 Tbilisi, ul. Ruchadze 1

Dr. J. Rákosník
Matematický ústav ČSAV
ČSSR-11567 Praha 1, Žitná 25