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Some Special Inverse Problems for the Laplace Equation and the Helmholtz Equation

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Die Arbeit beschäftigt sich mit einem speziellen inversen Quellproblem für die Laplace-Gleichung. Eine unbekannte Massenverteilung (Maß), welche auf einem vorgegebenen Gebiet konzentriert ist, ist durch die Randwerte ihres Newtonschen Potentials bzw. durch die Randwerte dessen Gradienten zu bestimmen. In Theorem 1 beweisen wir, daß das zweite Problem auf das erste zurückgeführt werden kann. Es existieren unendlich viele positive Massenverteilungen, die den angegebenen Bedingungen genügen. Deshalb wird auf der Menge aller positiven Massenverteilungen eine Äquivalenzrelation eingeführt. In früheren Arbeiten studierte der erste Autor diese Äquivalenzrelation von einem systematischen Standpunkt aus. Um die unbekannte Massenverteilung eindeutig bestimmen zu können, sind zusätzliche Bedingungen notwendig. Hier studieren wir den Fall einer harmonischen Dichte. Weiter zeigen wir, daß ein Identifikationsproblem für die Helmholtz-Gleichung mit Hilfe unseres speziellen inversen Problems behandelt werden kann.

Статья посвящена специальной обратной проблеме источника, касающейся уравнения Лапласа. Необходимо найти распределение масс, сконцентрированных на заданной области, при помощи краевых значений его Ньютонского потенциала, или краевых значений градиента его Ньютонского потенциала. Теоремой I доказано, что возможно свести вторую задачу к первой. Существуют бесконечно много положительных распределений масс, которые отвечают заданным условиям. Поэтому определяют отношение эквивалентности на множестве всех положительных распределений масс. В прошлых работах первый автор занимался этим отношением эквивалентности с систематической точки зрения. Чтобы однозначно определить неизвестное распределение масс, надо включить дополнительные условия. Здесь мы рассматриваем случай гармонической плотности. Дальше доказываем, что можно исследовать задачу идентификации для уравнения Гельмгольца с помощью нашей специальной обратной проблемы.

The paper deals with a special inverse source problem relative to the Laplace equation. An unknown mass distribution (measure) concentrated on a given domain is to be determined from the boundary values of its Newtonian potential or from the boundary values of its gradient. In Theorem 1 we prove that the second problem can be reduced to the first one. There exist infinitely many positive mass distributions satisfying the condition under consideration. Therefore an equivalence relation on the set of all positive mass distributions is introduced. In earlier papers the first author studied this equivalence relation from a systematic point of view. To uniquely determine the unknown mass distribution additional conditions are necessary. Here we study the case in which the density of a volume distribution is a harmonic function. Further, we prove that an identification problem for the Helmholtz equation can be transformed into the special inverse problem considered.

1. Some remarks on the inverse source problem for the Laplace equation

For the Laplace equation it is well known that the measure μ with support $\operatorname{supp} \mu \subset \bar{\Omega}$, where $\Omega \subsetneq \mathbf{R}^n$ is a bounded domain, cannot be uniquely determinded by the boundary values (see [1–8])

$$G_L\mu|_{\partial\Omega} = g \qquad (1)$$

of its potential $G_L\mu$ or by the boundary values

$$\operatorname{grad} G_L\mu|_{\partial\Omega} = g \tag{2}$$

of the gradient of $G_L\mu$. Here G_L is the Newtonian kernel

$$\begin{aligned} G_L(x, y) &= \frac{1}{2\pi} \log \frac{1}{|x - y|}, \qquad x \neq y,\ n = 2, \\ G_L(x, y) &= \frac{1}{(n-2)\,\omega_n} \frac{1}{|x - y|^{n-2}}, \qquad x \neq y,\ n \geqq 3, \end{aligned} \tag{3}$$

where ω_n is the surface area of a sphere $\partial B(x^0, r)$ with center x^0 and radius $r = 1$. The potential $G_L\mu$ is defined at $x \in \mathbf{R}^n$ by

$$G_L\mu(x) = \int G_L(x, y)\, d\mu(y). \tag{4}$$

Similar facts hold for the heat equation [9] and the potential $G\mu$ with respect to a locally integrable fundamental solution G (see [2, 4, 5, 24]).

Special measures μ are

$$\mu_\varrho(f) = \int_\Omega f(y)\, \varrho(y)\, dy, \qquad dy \text{ volume element}, \tag{5}$$

$$\mu_\sigma(f) = \int_{\partial\Omega} f(y)\, \sigma(y)\, dS(y), \qquad dS(y) \text{ surface element}. \tag{6}$$

Here f is an element of $C(\bar{\Omega})$, the space of all continuous functions defined on $\bar{\Omega}$ The dual space $C^*(\bar{\Omega})$ consists of all measures ν satisfying $\operatorname{supp} \nu \subset \bar{\Omega}$.

Since μ cannot be uniquely determined by (1) we introduce for $\mu \geqq 0$, $\operatorname{supp} \mu \subset \bar{\Omega}$, the set

$$\mathcal{B}(\mu) = \{\nu \geqq 0 \colon \operatorname{supp} \nu \subset \bar{\Omega},\ G_L\nu(x) = G_L\mu(x),\ x \notin \Omega\}. \tag{7}$$

The set $\mathcal{B}(\mu) \subset C^*(\bar{\Omega})$ is convex and weakly compact [1—8, 24]. Further we introduce the sets

$$\begin{aligned} F^+(\partial\Omega) &= \{\lambda \geqq 0 \colon \operatorname{supp} \lambda \subset \partial\Omega,\ G_L\lambda \text{ continuous}\}, \\ F(\partial\Omega) &= \{\lambda = \lambda_1 - \lambda_2 \colon \lambda_1, \lambda_2 \in F^+(\partial\Omega)\}, \\ D(\partial\Omega) &= \{f \in C(\partial\Omega) \colon f = G_L\lambda,\ \lambda \in F(\partial\Omega)\}. \end{aligned} \tag{8}$$

Special measures $\lambda \in F(\partial\Omega)$ are the measures μ_σ, cf. (6), where $\partial\Omega$ is sufficiently smooth and σ continuous. Integrating the relation in (7) with respect to $\lambda \in F(\partial\Omega)$ we get

$$\int G_L\nu\, d\lambda = \int G_L\mu\, d\lambda, \tag{9}$$

i.e.

$$\int G_L\lambda\, d\nu = \int G_L\lambda\, d\mu. \tag{9'}$$

In applications the relation $\overline{D(\partial\Omega)} = C(\partial\Omega)$ holds which is equivalent to the condition that Green's function is zero on the boundary [2, 24]. Then (7) can be defined as

$$\mathcal{B}(\mu) = \{\nu \geqq 0 \colon \operatorname{supp} \nu \subset \bar{\Omega},\ \int u\, d\nu = \int u\, d\mu,\ u \in H(\bar{\Omega})\}, \tag{7'}$$

where

$$H(\bar{\Omega}) = \{f \in C(\bar{\Omega}) \colon \Delta f = 0 \quad \text{on } \Omega\}. \tag{10}$$

In the case of the boundary values (2) we suppose that the gradient of the potential $G_L\mu$ exists on $\partial\Omega$ and the following conditions are satisfied

$$\frac{\partial}{\partial x_j}\int G_L(x, y)\, d\mu(y) = \int \frac{\partial}{\partial x_j} G_L(x, y)\, d\mu(y), \qquad j = 1, \ldots, n. \tag{11}$$

Special measures μ satisfying (11) are the measures μ_ϱ, $\varrho \in L^\infty(\Omega)$, defined in (5). To get a relation like (9′) we consider measures λ with supp $\lambda \subset \partial\Omega$ generating potentials

$$\frac{\partial}{\partial y_j}\int G_L(x, y)\, d\lambda(x), \qquad j = 1, \ldots, n, \tag{12}$$

which can be continuously extended on $\bar{\Omega} = \Omega \cup \partial\Omega$. Such measures are the harmonic measures $\mu_\xi^{\Omega'}$ relative to $\Omega' = \mathbf{R}^n \setminus \bar{\Omega}$, $\xi \in \Omega'$, where $\mu_\xi^{\Omega'}$ solves the Dirichlet problem at ξ. For the boundary values $u(x) = G_L(x, y)$, $y \in \Omega$, we get

$$\int G_L(x, y)\, d\lambda(x) = \int G_L(x, y)\, d\mu_\xi^{\Omega'}(x) = G_L(\xi, y). \tag{13}$$

In $\mathbf{R}^2$ we have to apply Green's function relative to a ball $\Omega_0 \subset \mathbf{R}^2$, $\bar{\Omega} \subset \Omega_0$. The special measures $\lambda = \mu_\xi^{\Omega'}$ satisfy (12) as far as $\partial\Omega$ consists only of regular boundary points [2, 14, 24, 28].

Theorem 1: *Let $\Omega \subset \mathbf{R}^n$, $n \geqq 3$, be a domain with smooth boundary $\partial\Omega$, ν and μ two positive measures on $\bar{\Omega}$ satisfying* (11) *on a set $M \subset \bar{\Omega}$ containing $\partial\Omega$. From*

$$\operatorname{grad} G_L\nu(x) = \operatorname{grad} G_L\mu(x) \quad \text{for all} \quad x \in \partial\Omega \tag{14}$$

it follows that

$$\int u\, d\nu = \int u\, d\mu \quad \textit{for all} \quad u \in H(\bar{\Omega}). \tag{14′}$$

Proof: Let λ be a measure on $\partial\Omega$ satisfying (12). From

$$\frac{\partial}{\partial x_j} G_L\nu(x) = \frac{\partial}{\partial x_j} G_L\mu(x), \qquad j = 1, \ldots, n, \text{ on } \partial\Omega$$

it follows that

$$\begin{aligned}
\int \left(\frac{\partial}{\partial x_j}\int G_L(x, y)\, d\nu(y)\right) d\lambda(x) &= \int \left(\int \frac{\partial}{\partial x_j} G_L(x, y)\, d\nu(y)\right) d\lambda(x) \\
= \int \left(\int \frac{\partial}{\partial x_j} G_L(x, y)\, d\lambda(x)\right) d\nu(y) &= -\int \left(\int \frac{\partial}{\partial y_j} G_L(x, y)\, d\lambda(x)\right) d\nu(y) \\
= -\int \left(\frac{\partial}{\partial y_j}\int G_L(x, y)\, d\lambda(x)\right) d\nu(y) &= -\int \frac{\partial}{\partial y_j} G_L\lambda(y)\, d\nu(y) \\
= -\int \frac{\partial}{\partial y_j} G_L\lambda(y)\, d\mu(y), &\qquad j = 1, \ldots, n.
\end{aligned} \tag{15}$$

We now consider (15) for $\lambda = \mu_\xi^{\Omega'}$ and obtain

$$-\int \frac{\partial}{\partial y_j} G_L(\xi, y)\, d\nu(y) = -\int \frac{\partial}{\partial y_j} G_L(\xi, y)\, d\mu(y).$$

Since $-\partial G_L(\xi, y)/\partial y_j = \partial G_L(\xi, y)/\partial \xi_j$ we get

$$\frac{\partial}{\partial \xi_j} G_L\nu(\xi) = \frac{\partial}{\partial \xi_j} G_L\mu(\xi), \qquad j = 1, \ldots, n, \quad \xi \in \Omega'.$$

These relations are equivalent to

$$\operatorname{grad} G_L(\mu - \nu) = 0 \quad \text{outside } \bar{\Omega}.$$

Therefore $G_L(\mu - \nu)$ is constant outside $\bar{\Omega}$. Since $G_L(\mu - \nu)(x) \to 0$ for $|x| \to \infty$, $n \geqq 3$, we obtain $G_L(\mu - \nu)(x) = 0$ outside $\bar{\Omega}$. The boundary $\partial\Omega$ is smooth. Therefore $G_L(\mu - \nu)(x) = 0$ on $\partial\Omega$ (see [2, 24]). Integrating this equation relative to $\lambda \in F(\partial\Omega)$ we get

$$\int G_L\mu \, d\lambda = \int G_L\nu \, d\lambda \quad \text{or} \quad \int G_L\lambda \, d\mu = \int G_L\lambda \, d\nu.$$

From $\overline{D(\partial\Omega)} = C(\partial\Omega)$ it follows that

$$\int u \, d\nu = \int u \, d\mu \quad \text{for all} \quad u \in H(\bar{\Omega}) \; \blacksquare \tag{16}$$

In $\mathbf{R}^2$ the same result holds with respect to Green's function G_L^{Ω} instead of G_L. For $\mu \geqq 0$ satisfying (11) on the set M we introduce instead of (7′) the set

$$\mathscr{B}_M(\mu) = \{\nu \in \mathscr{B}(\mu) : \nu \text{ satisfies (11) on } M\}. \tag{17}$$

Remark 1: $\mathscr{B}_M(\mu)$ is a convex set, and $\mathscr{B}_M(\mu) \subset \mathscr{B}(\mu) \subset C^*(\bar{\Omega})$.

At the beginning of our article we noted that the set $\mathscr{B}(\mu)$ usually contains more than one element. The same fact holds for the set $\mathscr{B}_M(\mu)$. The mathematicians C. NEUMANN (1909) and G. HERGLOTZ (1914) (see [16, 17]) studied the following inverse problem in $\mathbf{R}^2$. Let $\Omega \subset \mathbf{R}^2$ be a domain bounded by an algebraic curve, μ_{ϱ^*} the measure defined in (5), where $\varrho^*(y) = 1$ on $\bar{\Omega}$. Then there exists a measure $\nu_0 \in \mathscr{B}(\mu_{\varrho^*})$ concentrated on a curve joining the foci of the algebraic curve $\partial\Omega$ [16]. In the case of an ellipse the measure ν_0 is concentrated on the line segment joining the two foci. Similar facts hold in $\mathbf{R}^3$ for rotating bodies. In his paper G. HERGLOTZ [16] proved the first uniqueness theorems in $\mathbf{R}^3$ for the inverse source problem of the Laplace equation. Further uniqueness theorems for the inverse source problem can be found in [2, 4, 10, 13, 21, 22, 27, 28].

To apply functional analysis we introduce the solution of the Dirichlet problem as a projection A on $C(\bar{\Omega})$ and use its dual transformation (projection) A^* on $C^*(\bar{\Omega})$. The transformation A^* contains the physical information and plays a fundamental role in our considerations.

For simplicity let $\partial\Omega$ be sufficiently smooth. For an arbitrary domain the solution of the Dirichlet problem can be found in [15, 24, 28], the latest results in [28]. Let $f \in C(\bar{\Omega})$ and

$$Af(z) = \int_{\partial\Omega} f(y) \, P(z, y) \, dS(y) = \int f(y) \, d\mu_z^{\Omega}(y), \qquad z \in \Omega,$$

$$Af(y) = f(y), \qquad y \in \partial\Omega, \tag{18}$$

where

$$d\mu_z^{\Omega}(y) = P(z, y) \, dS(y) = -\frac{\partial G_L^{\Omega}(z, y)}{\partial n_y} \, dS(y), \tag{18'}$$

G_L^{Ω} is Green's function of Ω, n_y the exterior normal at $y \in \partial\Omega$. If $f \in H(\bar{\Omega})$ then $Af(z) = f(z)$. Further, let

$$(f, \mu) = \int f \, d\mu, \qquad f \in C(\bar{\Omega}), \quad \mu \in C^*(\bar{\Omega}).$$

The adjoint mapping $A^*: C^*(\bar{\Omega}) \mapsto C^*(\bar{\Omega})$ is defined by

$$(Af, \mu) = (f, A^*\mu), \tag{19}$$

where the so-called swept-out measure $A^*\mu$ is concentrated on $\partial\Omega$. For $A^*\mu$ one often writes $\Pi\mu$ (see [24]). Let now $\mu \geqq 0$, $\operatorname{supp} \mu \subset \partial\Omega$. Then the relation

$$\mathcal{B}(\mu) \subset (A^*)^{-1}(\mu) \tag{20}$$

holds. In the case of a smooth boundary and a positive measure ν from (18) and (19) it follows that

$$dA^*\nu(y) = \left(\int P(z, y)\, d\nu(z)\right) dS(y). \tag{21}$$

If $\nu = \delta_z$, $z \in \Omega$, we obtain $dA^*\delta_z(y) = d\mu_z^{\Omega}(y)$, the harmonic measure. The special measues μ_{ϱ^*}, $\varrho^* = 1$ on $\bar{\Omega}$, studied by C. Neumann and G. Herglotz satisfy $\Delta\varrho^* = 0$ in Ω. Further, the boundaries $\partial\Omega$ are analytic curves. It is very useful to consider more general measures μ_ϱ defined in (5) satisfying $\varrho \in H(\bar{\Omega})$, and to study the sets $\mathcal{B}(\mu_\varrho)$. Such special measures play an important role in applications [10, 18, 22, 25, 26]. From (18), (19) and (21) it follows that

$$\begin{aligned}(f, A^*\mu_\varrho) &= \int_{\partial\Omega} f(y) \left(\int_\Omega P(z, y)\, \varrho(z)\, dz\right) dS(y) \\ &= \int_{\partial\Omega} f(y) \left(\int_\Omega P(z, y) \left(\int_{\partial\Omega} P(z, t)\, \varrho_0(t)\, dS(t)\right) dz\right) dS(y) \\ &= \int_{\partial\Omega} f(y) \left(\int_{\partial\Omega} \varrho_0(t) \left(\int_\Omega P(z, y)\, P(z, t)\, dz\right) dS(t)\right) dS(y) \\ &= \int_{\partial\Omega} \left(\int_{\partial\Omega} Q(y, t)\, \varrho_0(t)\, dS(t)\right) f(y)\, dS(y),\end{aligned} \tag{22}$$

where

$$Q(y, t) = \int_\Omega P(z, y)\, P(z, t)\, dz; \qquad y, t \in \partial\Omega. \tag{23}$$

Let

$$\sigma_0(y) = \int_{\partial\Omega} Q(y, t)\, \varrho_0(t)\, dS(t). \tag{24}$$

We have proven the following theorem.

Theorem 2: *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$ and P the kernel defined in* (18). *Further, let ϱ be a harmonic function from $H(\bar{\Omega})$. Then the surface density σ_0 of the swept-out measure $A^*\mu_\varrho = \mu_{\sigma_0}$ is given by* (23), (24), *where ϱ_0 is the restriction of ϱ on $\partial\Omega$.*

2. Some remarks on identification problems

In 1967 and 1970 J. Bony (see [14: p. 96]) proved the following theorem. Let

$$Lu = \sum_{|\alpha| \leqq 2} a_\alpha D^\alpha u \tag{25}$$

be a linear elliptic differential operator of second order, where the coefficients a_α defined on an open set $\Omega \subset \mathbf{R}^n$ are continuous and the coefficient of u is negative. Further, let Mu be a second linear elliptic differential operator with continuous coefficients defined on Ω. If for all u satisfying $Lu = 0$ it follows that $Lu = Mu = 0$, then the relation

$$Lu = h_0 Mu \tag{26}$$

holds, where h_0 is a continuous function.

If the solutions u of $Lu = 0$ are not known on a ball $B(x^0, r) \subset \Omega$ then the coefficients a_α cannot be uniquely determined. In $\mathbf{R}^1$ we consider the following counterexample. Let x and 1 be a fundamental system of $Lu = u''$. Further, let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R}^1)$ be two functions satisfying supp $\varphi_j \subset B(x^0, r)$. Now we consider the two fundamental systems

$$u_1 = x + \varphi_1, \qquad u_2 = 1 + \varphi_1, \tag{27}$$

$$v_1 = x + \varphi_2, \qquad v_2 = 1 + \varphi_2. \tag{28}$$

Using the Wronskian we can construct the linear differential operator

$$L_1u = u''(1 + g_2(x)) + u'g_1(x) + ug_0(x) \tag{29}$$

and

$$L_2u = u''(1 + h_2(x)) + u'h_1(x) + uh_0(x) \tag{30}$$

having u_1, u_2 resp. v_1, v_2 as fundamental system. Here the g_j and h_j are zero outside $B(x^0, r)$. For $x \notin B(x^0, r)$ L_1u and L_2u have the form

$$L_1u = L_2u = u''. \tag{31}$$

There exist infinitely many different differential operators producing on $\mathbf{R}^1 \setminus B(x^0, r)$ the same differential operator. The same construction holds in $\mathbf{R}^n$ relative to the Laplace operator and solutions depending only on r. To construct counterexamples for more general differential operators the methods of J. Bony can be applied.

In applications these results are important. The solutions u of $Lu = 0$ are not known on the whole domain $\Omega \subset \mathbf{R}^n$. Without additional information coming from the applied problem the coefficients a_α cannot be uniquely determined.

3. A special identification problem for the Helmholtz equation

We shall see that an identification problem for the Helmholtz equation is governed by the same kernel Q defined in (23). This kernel was introduced in [3] the first time. Let

$$\Delta u + q(x)\, u = 0 \tag{32}$$

be the Helmholtz equation, where u satisfies on $\partial\Omega$ the boundary condition $u|_{\partial\Omega} = h$. Further, let $u = u_1 + u_2$, where $\Delta u_1 = 0$, $u_1|_{\partial\Omega} = h$. Then $u_2|_{\partial\Omega} = 0$ and

$$\Delta u_2 + q(x)\, u_1 + q(x)\, u_2 = 0. \tag{33}$$

Following V. G. ROMANOV [23] we consider the so-called linearized problem neglecting the term $q(x)\, u_2$

$$\Delta u_2 + q(x)\, u_1 = 0, \qquad u_2|_{\partial\Omega} = 0. \tag{34}$$

In this case $q(x)\, u_2$ must be sufficiently small. If $u_1 = 1$ the solution of (34) is given by

$$u_2(x) = \int_\Omega G_L{}^\Omega(z, x)\, q(z)\, dz, \tag{35}$$

where $G_L{}^\Omega$ is the Green function of Ω. To uniquely determine q additional conditions are necessary. In the following let q be a harmonic function, i.e. $\Delta q = 0$ in Ω. Using (18) and Fubini's theorem we get

$$u_2(x) = \int_{\partial\Omega} \Big(\int_\Omega G_L{}^\Omega(z, x)\, P(z, t)\, dz\Big)\, q_0(t)\, dS(t). \tag{36}$$

To determine q_0 we can use the following condition

$$-\left.\frac{\partial \tilde{u}_2}{\partial n}\right|_{\partial\Omega} = g. \tag{37}$$

From (35), (36) and (37) it follows that

$$g(y) = \int_{\partial\Omega} Q(y, t)\, q_0(t)\, dS(t), \tag{38}$$

where Q was defined in (23). Equations (24) and (38) have the form

$$b(y) = Tk(y) = \int_{\partial\Omega} Q(y, t)\, k(t)\, dS(t). \tag{39}$$

At the end of this paper we shall write down the concrete formula of T and T^{-1} for a ball $B(x^0, r) \subset \mathbf{R}^2$. The complete study of this transformation, especially its inverse transformation $k = T^{-1}b$, can be found in [19, 20].

4. Properties of the kernel R

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary and ν a measure on $\bar{\Omega}$. Then the swept-out measure $A^*\nu$ concentrated on $\partial\Omega$ has the form (see (21))

$$dA^*\nu(y) = \sigma(y)\, dS(y) \quad \text{where} \quad \sigma(y) = \int P(z, y)\, d\nu(z).$$

In particular, if the measure ν has the form

$$d\nu_P(z) = P(z, t)\, dz, \qquad t \in \partial\Omega, \tag{40}$$

then the swept-out measure is given by its density

$$Q(y, t) = \int_{\Omega} P(z, y)\, P(z, t)\, dz; \qquad y, t \in \partial\Omega. \tag{41}$$

Furthermore, if Ω is the ball $B(0, r) \subset \mathbf{R}^2$ one can find a measure $\nu_0 \in \mathcal{B}(\nu_P)$ on the line segment joining 0 and t. This procedure is an analytic continuation for the potentials considered.

Theorem 3: *Let $B(0, r_0) \subset \mathbf{R}^2$ be a ball and $t \in \partial B(0, r_0)$ a fixed point. The measure ν_P defined in (40) can be replaced by the measure*

$$d\nu_0(z) = \frac{r_0}{2}\, d\delta_{\overline{0t}}(z) = \frac{1}{2}\, ds$$

concentrated on the line segment $\overline{0t}$ joining the points 0 and t such that $\frac{r_0}{2}\,\delta_{\overline{0t}} \in \mathcal{B}(\nu_P)$, i.e.,

$$Q(y, t) = \frac{r_0}{2} \int P(\xi, y)\, d\delta_{\overline{0t}}(v) = \frac{1}{2} \int_{s=0}^{r_0} P\left(s\,\frac{t}{r_0}, y\right) ds. \tag{42}$$

Proof: In polar coordinates the kernel Q has the form

$$Q(y, t) = \int_{\Omega} P(z, y)\, P(z, t)\, dz = \int_{r=0}^{r_0} \int_{\partial B(0,r)} P(z, y)\, P(z, t)\, dS(z)\, dr, \tag{43}$$

where $z \in \partial B(0, r)$. Further

$$P(z, t) = \frac{1}{2\pi r_0}\,\frac{r_0^2 - |z|^2}{|z - t|^2} = \frac{1}{2\pi r_0}\,\frac{r_0^2 - r^2}{|z - t|^2}. \tag{44}$$

We now fix $r < r_0$ and apply the Kelvin transformation [15] $t \mapsto t^* = \frac{r^2}{r_0^2} t$ relative to the circle $\partial B(0, r)$. The following relation holds for $z \in \partial B(0, r)$

$$\frac{r}{r_0} \frac{|z - t|}{\left|z - \frac{r^2}{r_0^2} t\right|} = 1.$$

Hence it follows that

$$\frac{1}{2\pi r_0} \frac{r_0^2 - r^2}{|t - z|^2} = \frac{r}{r_0} \frac{1}{2\pi r} \frac{r^2 - \left(\frac{r^2}{r_0}\right)^2}{\left|z - \frac{r^2}{r_0^2} t\right|^2}. \tag{45}$$

Let $P^{B(0,r_0)}$ be the Poisson kernel relative to $B(0, r_0)$. Then (45) can be written in the form

$$P^{B(0,r_0)}(z, t) = \frac{r}{r_0} P^{B(0,r)}(t^*, z). \tag{46}$$

Using (46) we obtain

$$\int_{\partial B(0,r)} P^{B(0,r_0)}(z, y)\, P^{B(0,r_0)}(z, t)\, dS(z)$$
$$= \int_{\partial B(0,r)} P^{B(0,r_0)}(z, y) \frac{r}{r_0} P^{B(0,r)}(t^*, z)\, dS(z) = \frac{r}{r_0} P^{B(0,r_0)}(t^*, y). \tag{47}$$

From (43) and (47) it follows that

$$Q(y, t) = \int_{B(0,r_0)} P(z, y)\, P(z, t)\, dz = \int_{r=0}^{r_0} \frac{r}{r_0} P\left(\frac{r^2}{r_0^2} t, y\right) dr$$
$$= \frac{1}{2} \int_{r=0}^{r_0} P\left(\frac{r^2}{r_0^2} t, y\right) \frac{2r}{r_0}\, dr = \frac{1}{2} \int_{s=0}^{r_0} P\left(s \frac{t}{r_0}, y\right) ds$$
$$= \frac{r_0}{2} \int P(\zeta, y)\, d\delta_{\overline{0t}}(\zeta) \;\blacksquare \tag{48}$$

Remark 2: In the case of m points $t_1, \ldots, t_m \in \partial B(0, r_0)$ the measure

$$d\nu_{P_\mu}(z) = \big(a_1 P(z, t_1) + \cdots + a_m P(z, t_m)\big)\, dz, \tag{49}$$

where $a_j \geqq 0$, $a_1 + \cdots + a_m = 1$ and $\mu = a_1\delta_{t_1} + \cdots + a_m\delta_{t_m}$, can be replaced by the measure

$$\nu_0 = \frac{r_0}{2} (a_1 \delta_{\overline{0t_1}} + \cdots + a_m \delta_{\overline{0t_m}}) \in \mathscr{B}(\nu_{P_\mu}). \tag{50}$$

Similar results hold relative to

$$d\nu_{P_\mu}(z) = \int P(z, t)\, d\mu(t), \tag{51}$$

where μ is a positive measure on $\partial B(0, r_0)$ satisfying $\int d\mu = 1$.

Let $y = r_0 e^{i\alpha}$, $t = r_0 e^{i\beta}$. From (42) it follows that

$$Q^*(\alpha, \beta) = Q(y, t) = \frac{1}{2} \int_{r=0}^{r_0} \frac{1}{2\pi r_0} \frac{(r_0^2 - r^2)}{r_0^2 + r^2 - 2r_0 r \cos(\alpha - \beta)}\, dr. \tag{52}$$

Using the abbreviation $l = \alpha - \beta$ we get

$$Q^*(l) = \frac{1}{4\pi r_0} \int\limits_{r=0}^{r_0} \frac{(r_0^2 - r^2)}{r_0^2 + r^2 - 2r_0 r \cos l} \, dr. \tag{53}$$

The transformation defined in (39) has the form

$$b(\alpha) = Tk(\alpha) = r_0 \int\limits_{-\pi}^{+\pi} k(\alpha - l)\, Q^*(l)\, dl. \tag{54}$$

Its inverse T^{-1} is given by [19, 20]

$$k(\beta) = T^{-1}b(\beta) = \frac{2}{r_0} \left[b(\beta) + \frac{1}{\pi} \int\limits_{-\pi}^{+\pi} b''(\beta - l) \log 2 \left| \sin \frac{l}{2} \right| dl \right]. \tag{55}$$

The inverse contains derivations of the second order. The problem to replace a given surface distribution by a volume distribution with harmonic density is ill-posed with respect to the topology of $C(\bar{\Omega})$. It is well known that most problems in applications are ill-posed problems [5—8].

The results of paragraph 4 were proved by the second author, the other ones by the first author.

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