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Interpolation by Special Exponential Polynomials[1])

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Untersucht werden die Verzweigungseigenschaften der nichtlinearen Interpolation durch ein mit einer Exponentialfunktion multipliziertes Polynom.

Изучаются свойства разветвления проблемы нелинейной интерполяции многочленом, умноженным экспоненциальной функцией.

There are investigated the branching properties of the nonlinear interpolation by a polynomial multiplied with an exponential function.

For given $n + 2$ points in the real plane

$$(t_i, z_i), \qquad i = 0, 1, \cdots, n + 1,$$

with $t_0 = 0 < t_1 < t_2 < \ldots < t_{n+1}$ we consider the special exponential polynomials

$$P(t) = e^{xt} \sum_{j=0}^{n} y_j t^j \tag{1}$$

and try to determine the real coefficients $y_1, \ldots, y_n, x$ in such a way that

$$P(t_i) = z_i, \qquad i = 1, \ldots, n + 1. \tag{2}$$

Moreover, we always choose $y_0 = z_0$, so that (2) is also satisfied for $i = 0$.

Approximating values $y_1, \ldots, y_n, x$ for the assumed solution of (2) can be improved by means of Newton's method (cf. H. Schwetlick [2]), calculating the differences to the improved values $y_j + \Delta y_j$ and $x + \Delta x$ from the linear system

$$\sum_{j=1}^{n} t_i^j \Delta y_j + \sum_{j=0}^{n} y_j t_i^{j+1} \Delta x = -\sum_{j=0}^{n} y_j t_i^j, \tag{3}$$

$i = 1, 2, \ldots, n + 1$. The determinant D of this system reads $D = y_n \Delta$, where Δ is Vandermonde's determinant of the $n + 1$ points t_i with $i > 0$ multiplied by $t_1 \ldots t_{n+1}$ (cf. (8) later on). Since Δ is fixed and always positive, $D = 0$ is possible if and only if $y_n = 0$.

In what follows we discuss the situation in the neighbourhood of the branching point $y_n = 0$, where Newton's method usually breaks down numerically. The results are very lucid and may give some qualitative inside in similar nonlinear approximation problems.

We always exclude the trivial case $z_0 = \cdots = z_{n+1} = 0$ with $y_0 = \cdots = y_n = 0$ and an arbitrary x, and without loss of generality we assume $z_0 \geq 0$.

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1. The case $n \leqq 1$

For $n = 0$ equation (2) reads $z_0 e^{xt_1} = z_1$. If $z_0 z_1 > 0$ we have the unique solution

$$x = \frac{1}{t_1} \ln \frac{z_1}{z_0},$$

otherwise there exists no solution.

For $n = 1$ the system (2) reads

$$(z_0 + y_1 t_1) e^{xt_1} = z_1, \qquad (z_0 + y_1 t_2) e^{xt_2} = z_2, \tag{4}$$

which for $z_0 z_1 z_2 \neq 0$ after elimination of x implies the equation

$$(z_0 + y_1 t_1)/z_1 = \left((z_0 + y_1 t_2)/z_2\right)^{t_1/t_2}. \tag{5}$$

At the zero $y_1 = -z_0/t_1$ of the left-hand side the basis at the right-hand side reads

$$\left(z_0 - \frac{z_0}{t_1} t_2\right) \frac{1}{z_2} = \left(1 - \frac{t_2}{t_1}\right) \frac{z_0}{z_2}.$$

In case $z_2 < 0$ this basis is positive and (5) possesses exactly one solution y_1 (cf. Fig. 1 a). The straight lines in Fig. 1 are the graphs of the left-hand side of (5) for diffe-

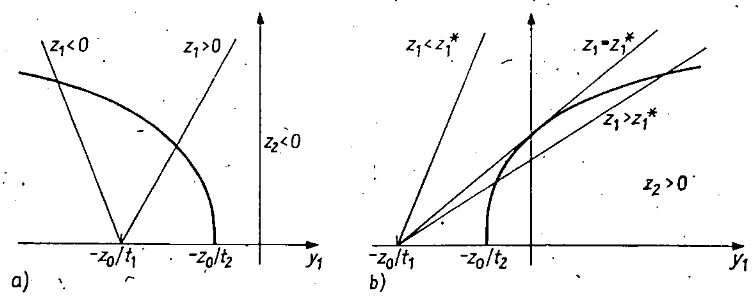

Fig. 1

rent z_1, and the curve is the graph of the right-hand side. In case $z_2 > 0$ let $z_1 = z_1^*$ be the solution of the equation

$$(z_1/z_0)^{t_1} = (z_2/z_0)^{t_2}. \tag{6}$$

Then equation (5) possesses no solutions for $z_1 < z_1^*$, the double solution $y_1 = 0$ for $z_1 = z_1^*$ and exactly two solutions for $z_1 > z_1^*$, one negative and one positive (cf. Fig. 1b). Especially, there is no solution for $z_1 < 0$.

After having determined y_1 we find x form one of the equations (4). In the case $y_1 = 0$ equation (6) guarantees that the given points are lying on the graph of an exponential function $z_0 e^{xt}$ with a certain x. The cases that one or some z_i are vanishing can be considered as limit cases.

2. The general case

For $n \geqq 2$ it is more convenient to eliminate the linear parameters y_j instead of x. For this reason we determine real numbers $\lambda_1, \ldots, \lambda_{n+1}$ from the homogeneous system

$$\sum_{i=1}^{n+1} t_i^j \lambda_i = 0 \tag{7}$$

for $j = 1, 2, \ldots, n$ with the additional condition $\lambda_\nu = 1$ for a certain number ν with $1 \leqq \nu \leqq n+1$. After introducing the notations

$$\Delta_\nu = \begin{vmatrix} t_1 & \ldots t_{\nu-1} & t_{\nu+1} & \ldots t_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ t_1^n & \ldots t_{\nu-1}^n & t_{\nu+1}^n & \ldots t_{n+1}^n \end{vmatrix}, \qquad \Delta = \begin{vmatrix} t_1 & \ldots t_{n+1} \\ \vdots & \vdots \\ t_1^{n+1} & \ldots t_{n+1}^{n+1} \end{vmatrix} \tag{8}$$

we have by Cramer's rule

$$\lambda_i = \frac{(-1)^{i+\nu}}{\Delta_\nu} \begin{vmatrix} t_1 & \ldots t_{i-1} & t_{i+1} & \ldots t_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ t_1^n & \ldots t_{i-1}^n & t_{i+1}^n & \ldots t_{n+1}^n \end{vmatrix}; \tag{9}$$

and the system (2) implies in view of (7) the equation

$$0 = \sum_{i=1}^{n+1} \lambda_i \left(\sum_{j=0}^{n} y_j t_i^j - z_i \, \mathrm{e}^{-x t_i} \right) = \sum_{i=1}^{n+1} (\lambda_i z_0 - \lambda_i z_i \, \mathrm{e}^{-x t_i}). \tag{10}$$

Let $P_\nu(t, x)$ by fixed x be the always existing polynomial (1) satisfying the conditions (2) apart from $i = \nu$, i.e. with $i \neq \nu$, and define $z_\nu(x) = P_\nu(t_\nu, x)$. Then $P_\nu(t, x)$ satisfies all conditions (2), only with $z_\nu(x)$ instead of z_ν.

Theorem: *The polynomial* (1) *with* $y_0 = z_0$ *satisfying the equations* (2) *is supposed to have the coefficients*

$$x = \xi, \qquad y_n = y_{n-1} = \cdots = y_{n-k+1} = 0, \qquad y_{n-k} \neq 0 \tag{11}$$

with $1 \leqq k \leqq n$. *Then the function* $z = z_\nu(x)$ *possesses the properties*

$$z_\nu'(\xi) = \cdots = z_\nu^{(k)}(\xi) = 0, \qquad z_\nu^{(k+1)}(\xi) = (-1)^{\nu+n+k+1} \frac{\Delta}{\Delta_\nu} y_{n-k} \, \mathrm{e}^{\xi t_\nu}. \tag{12}$$

Proof: From (10) we find in view of $\lambda_\nu = 1$

$$z_\nu(x) = \mathrm{e}^{x t_\nu} \left(\sum_{i=1}^{n+1} \lambda_i z_0 - \sum_{i=1}^{n+1}{}' \lambda_i z_i \, \mathrm{e}^{-x t_i} \right), \tag{13}$$

where the dash indicates that the value $i = \nu$ is to drop in the second sum. By differentiation we obtain for $l \geqq 1$

$$z_\nu^{(l)}(x) = \mathrm{e}^{x t_\nu} \sum_{i=1}^{n+1} \lambda_i z_0 t_\nu^{\,l} - \sum_{i=1}^{n+1}{}' \lambda_i z_i (t_\nu - t_i)^l \, \mathrm{e}^{x(t_\nu - t_i)}$$

and according to (1), (2) and (11)

$$z_\nu^{(l)}(\xi) = \mathrm{e}^{\xi t_\nu} \sum_{i=1}^{n+1} \lambda_i \big(t_\nu^{\,l} - (t_\nu - t_i)^l\big) \sum_{j=0}^{n-k} y_j t_i^j,$$

because in view of the factor $(t_\nu - t_i)^l$ we can add here the term with $i = \nu$. Now, from (7) we immediately find the first equations of (11) and from

$$z_\nu^{(k+1)}(\xi) = (-1)^k e^{\xi t_\nu} \sum_{i=1}^{n+1} \lambda_i t_i^{n+1} y_{n-k},$$

(8) and (9) we find the last equation of (12) ∎

Since the determinants (8) are always positive we have the following

Corollary: *Under the conditions of the theorem the function $z_\nu(x)$ possesses for $x = \xi$ a turning point if k is an even number and an extremum if k is an odd number. In the last case the extremum is a maximum for $(-1)^{\nu+n+k}\, y_{n-k} > 0$ and a minimum for $(-1)^{\nu+n+k}\, y_{n-k} < 0$.*

Defining a subneighbourhood of a point as an open subset of a neighbourhood of this point with this point as a limit point, the corollary says that in the case of an extremum there exist a subneighbourhood of the given points (t_i, z_i) where the interpolation problem is unsolvable and other subneighbourhoods with even numbers of different solutions. Especially, we have this situation in the usual case $k = 1$. In the case of a turning point there exist in subneighbourhoods odd numbers of solutions, especially, at least one solution. On the boundaries of these subneighbourhoods two or more solutions coincide.

The case (11) arises if already the graph of an exponential polynomial (1) with $n - k$ instead of n goes through the given points. In this case we can drop k equations in (2) and solve the remaining equations, and the numerical difficulties are removed. If the interpolation problem is unsolvable it makes sense to search for the best approximation of the given points (cf. L. COLLATZ and W. KRABS [1]).

3. Numerical results

For the numerical solution of the interpolation problem it is not necessary to use the method (3), because it is easier to determine first x from (13) with $z_\nu(x) = z_\nu$ and afterwards the coefficients y_j from the linear system (2) with $i \neq \nu$. To illustrate the investigated possibilities in the neighbourhood of given points with (11) we consider two examples with $n = 2$ and $t_i = i$.

1°. Let be $z_i = 1 - i/3$, then $x = 0$, $y_1 = -1/3$, $y_2 = 0$. According to (12) the value $z_2 = -1/3$ is a maximum if we fix the other z_i. For some smaller values we have e.g. the following rounded results:

z_2	0.33		0.333		0.3333		0.33333	
x	−0.1054	0.0953	−0.0321	0.0311	−0.0101	0.0100	−0.0032	0.0032
y_2	−0.0370	0.0303	−0.0109	0.0102	−0.0034	0.0033	−0.0012	0.0011
y_1	−0.2222	−0.4242	−0.3007	−0.3640	−0.3232	−0.3432	−0.3302	−0.3365

2°. Let be $z_i = 1$, then $x = y_1 = y_2 = 0$ and the values z_ν are turning points. For fixed $z_0 = z_3 = 1$ we have in the neighbourhood of $z_1 = z_2 = 1$ the rounded results

z_1	1.1	1	0.9	0.9	1		1.1	
z_2	1	0.9	0.9	1	1.1		1.1	
x	0.7403	0.5971	0	−0.5971	−0.7403	0	0.5411	−0.5411
y_2	0.0891	0.0859	0.05	0.5153	0.8211	−0.05	0.0460	0.2335
y_1	−0.5644	−0.5355	−0.15	0.1199	0.2755	0.15	−0.4057	0.6562

where in the case $z_1 = z_2 = 1.1$ we have found three different solutions. In the (z_1, z_2)-plane the points with $y_2 = 0$ are lying on the curve

$$4(z_1^3 + z_2^3) + 1 = 3z_1^2z_2^2 + 6z_1z_2, \tag{14}$$

which under the substitution $z_1 = 1 + (1/\sqrt{2})\ (u - v)$, $z_2 = 1 + (1/\sqrt{2})\ (u + v)$ turns over into

$$v^2 = (3\sqrt{2} + u)^2 - 6 - 12\left(1 + \frac{\sqrt{2}}{3}u\right)^{3/2}. \tag{15}$$

Here we have cancelled the branch with the plus sign at the square root, since it has no meaning for real solutions. For small u the curve (15) is approximately equal to Neil's parabola $v^2 = (\sqrt{2}/18)\ u^3$, cf. Fig. 2. At the inside of the vertex we have three solutions of the interpolation problem, and one at the outside. The dotted curve is the meanigless branch of (14).

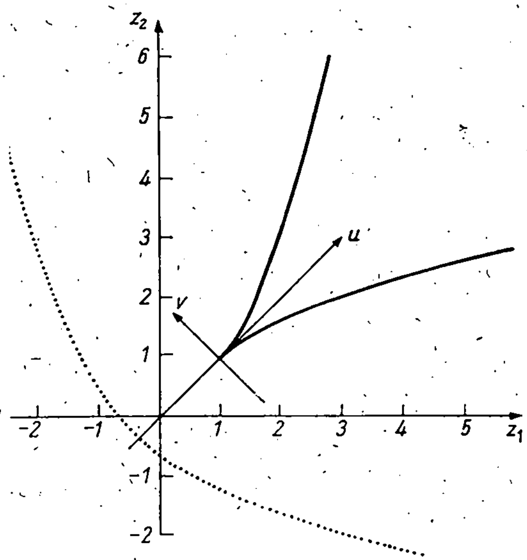

Fig. 2

LITERATURE

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