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On Duality and the Maximum Principle for Continuous Linear Programming Problems

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In dieser Arbeit wird das Grinoldsche Maximumprinzip für lineare stetige Optimierungsprobleme erweitert auf Fälle, wo die bekannten Dualitätssätze beim Beweis der Existenz einer Optimallösung des Dualproblems in den gebräuchlichen reflexiven L_p-Räumen versagen. Das erweiterte Maximumprinzip wird dann zur Untersuchung eines parabolischen Randsteuerproblems benutzt, bei dem Beschränkungen an die Steuerung und an den Zustand gegeben sind.

В этой работе принцип максимума Гринольда для задач непрерывного линейного программирования расширяется на случай, где известные теоремы двойственности недостаточны для доказательства существования оптимального решения двойственной проблемы в рефлексивных пространствах L_p. После этого расширенный принцип максимума используется для изучения проблемы оптимального управления, описываемой параболическим уравнением, где управление действует на границе области и заданы ограничения на управление и состояние.

In this paper, the Grinold maximum principle for continuous linear programming problems is extended to the case where the known duality theorems do not ensure the existence of an optimal solution of the dual problem in the usual reflexive L_p-spaces. The extended maximum principle is then applied to the investigation of a parabolic boundary control problem with constraints on the control and the state.

1. Introduction

This paper is concerned with applications of duality theorems for continuous linear programs, thus it contributes to a field of optimization theory where many interesting results were found within the last fifteen years. We mention only the basic investigations by Levinson [3], Tyndall [9], and Grinold [1], which have been continued by many others. The reader may find a short bibliography in the author's paper [6]. These investigations were focused on the following pair of linear programs:

Primal problem:

$$\int_0^T a(t)^\top x(t)\,dt = \sup!$$

subject to

$$B(t)\,x(t) \leqq c(t) + \int_0^T K(t,s)\,x(s)\,ds, \qquad x(t) \geqq 0 \tag{1.1}$$

almost everywhere (a.e.) on $[0, T]$, $x(\cdot) \in L_p(0, T; \mathbf{R}^N)$, $p \in [1, \infty)$ (by $^\top$ we shall denote transposition).

Dual problem

$$\int_0^T c(t)^\top v(t)\,dt = \inf!$$

subject to

$$\begin{aligned} B(t)^\top v(t) &\geqq a(t) + \int_0^T K(s,t)^\top v(s)\,ds, \\ v(t) &\geqq \theta \end{aligned} \tag{1.2}$$

a.e. on $[0, T]$, $v(\cdot) \in L_q(0, T; \mathbf{R}^M)$, $1/q + 1/p = 1$.

In this setting, $a(t)$ and $c(t)$ are suitable vector-valued functions, and $B(t)$, $K(t, s)$ are matrix-valued. We will define them in the next section.

Most of the authors supposed $K(t, s)$ to be a continuous Volterra kernel (continuous on $0 \leqq s \leqq t \leqq T$ and vanishing for $s > t$), whereas the more general class (1.1) with (weakly singular) Fredholm kernel was considered by the author [5].

The duality theory refers to the following main questions: Do the primal supremum and the dual infimum coincide? (Without assumptions we know only sup $\leqq$ inf.) Do there exist optimal solutions to one or both of the two dual programs?

In 1970 GRINOLD [2] established his *two-level maximum principle* for continuous linear programs. The first level, being of interest for our paper, is as follows: Suppose that $x^0(t)$ and $v^0(t)$ are optimal for (1.1) and (1.2), respectively. Then

$$\max \left[a(t) + \int_0^T K(s,t)^\top v^0(s)\,ds\right]^\top x$$

subject to

$$\begin{aligned} B(t)\,x &\leqq c(t) + \int_0^T K(t,s)\,x^0(s)\,ds, \\ x &\geqq \theta \end{aligned}$$

($x \in \mathbf{R}^N$) is attained a.e. on $[0, T]$ by $x = x^0(t)$. A similar version holds for the dual problem (1.2). Clearly, the validity of the maximum principle is intimately linked with the existence of a dual optimal solution $v^0(t)$ for (1.2).

Although the duality theory has progressed very fast in recent years, it is often the existence of an optimal solution to the dual problem (1.2) which cannot be guaranteed. Regard, as a typical example, the simple problem

$$\begin{aligned} &\int_0^T a(t)\,x(t)\,dt = \sup!, \\ &-c \leqq x(t) - \int_0^t k(t,s)\,x(s)\,ds \leqq c, \qquad 0 \leqq x(t) \leqq 1, \end{aligned} \tag{1.3}$$

a.e. on $[0, T]$, where c, $T > 0$, $a(\cdot) \in C[0, T]$, and a continuous real function $k(t, s)$ on $[0, T] \times [0, T]$ are given, and $x(t)$ is taken from $L_2(0, T)$. This problem fits in (1.1) by $p = 2$, $N = 1$, $M = 3$,

$$B(t) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad K(t,s) = \begin{cases} \begin{pmatrix} k(t,s) \\ -k(t,s) \\ 0 \end{pmatrix}, & s \leqq t \\ \theta, & s > t, \end{cases} \quad c(t) = \begin{pmatrix} c \\ c \\ 1 \end{pmatrix}.$$

The dual problem is therefore

$$\int_0^T [cv_1(t) + cv_2(t) + v_3(t)]\,dt = \inf!,$$
$$v_1(t) - v_2(t) + v_3(t) \geqq a(t) + \int_t^T k(s,t)\big(v_1(s) - v_2(s)\big)\,ds, \tag{1.4}$$
$$v_i(t) \geqq 0, \qquad i = 1, 2, 3,$$

a.e. on $[0, T]$, where $v(t) = \big(v_1(t), v_2(t), v_3(t)\big)^\top$ is taken from $L_2(0, T; \mathbf{R}^3)$. Using the theory of [1] it is easy to prove that an optimal solution of (1.3) exists, and that (1.3), (1.4) admit the same optimal values, i.e. max = inf. It is not clear, however, whether an optimal solution exists in (1.4). The known duality theorems by Grinold, Levinson and Tyndall do not apply, as B, K, and c contain components with different signs.

The difficulties arising from the treatment of problems like (1.1) are caused by the fact that $x(t)$ *occurs under an integral as well as outside the integral* as a „free term", thus the space for defining the inequality constraints of (1.1) must be as large as that for $x(t)$. Consequently, the well-known Slater-conditions cannot be applied to guarantee the solvability of (1.2), if $x(t)$ is defined in the usual L_p*-spaces* with $1 \leqq p < \infty$. Therefore, one could use a *decomposition procedure separating integrals and free terms of* $x(t)$ in order to overcome the obstacles for proving the existence of a dual optimal solution and to establish a satisfactory maximum principle for optimal solutions of (1.1).

We will pursue this idea and its consequences in this paper. Our approach will not lead to entirely new duality results. To a certain extent, our theory is equivalent to the investigation of the problem in the unusual dual space $L_\infty(0, T; \mathbf{R}^M)^*$. However, the decomposition trick enables us to avoid the use of this space completely, thus this idea seems to be interesting in its own right.

As a result, we will obtain a useful generalization of the Grinold maximum principle for problems where the known duality theorems fail to ensure the existence of a dual optimal solution. The maximum principle was successfully applied by the author to the numerical solution of a parabolic boundary control problem (see [7]). In Section 4 we shall investigate analogously a more general *boundary control problem* by the extended maximum principle, in order to characterize optimal controls as precisely as possible.

2. The maximum principle

At first, we introduce some notations: If X is a Banach space, then we shall denote by $L_p(0, T; X)$, $C[0, T; X]$, or $NBV[0, T; X]$ the spaces of functions on $[0, T]$ with values in X which are p-times integrable, continuous, or of bounded variation and vanishing at $t = 0$ (normalization condition), respectively. $\mathbf{R}^N$ is the Euclidean N-dimensional space (column-vectors) and $\mathbf{R}^{M\times N}$ that of real $M \times N$-matrices. By $\|\cdot\|_p$ we shall indicate the norm of $L_p(0, T)$. If $f \in X^*$, the dual space to X, then we shall write $\langle f, x\rangle$ for the value of f applied to $x \in X$. All other notations will become clear from the context.

In order to complete the definition of the primal problem (1.1), we introduce integers $N \geqq 1$, $M \geqq 1$, real numbers $T > 0$, $p \in [1, \infty)$, and define q by $1/p + 1/q = 1$. We suppose that $a(\cdot) \in L_q(0, T; \mathbf{R}^N)$, $c(\cdot) \in L_\infty(0, T; \mathbf{R}^M)$, and $B(\cdot) \in L_\infty \times (0, T; \mathbf{R}^{M\times N})$ are given. Moreover, we suppose that a measurable function $K(t, s)$:

$[0, T] \times [0, T] \to \mathbf{R}^{M\times N}$ is given such that the integral operator

$$x(\cdot) \mapsto \int_0^T K(\cdot, s)\, x(s)\, ds$$

is a continuous mapping from $L_p(0, T; \mathbf{R}^N)$ into $C[0, T; \mathbf{R}^M]$, and that its adjoint operator, being a mapping from $NBV[0, T; \mathbf{R}^M]$ into $L_q(0, T; \mathbf{R}^N)$, has the form

$$y(\cdot) \mapsto \int_0^T K(s, \cdot)^\top\, dy(s).$$

Here we used the notation

$$\int_0^T z(t)^\top\, dy(t) := \sum_{i=1}^{M} \int_0^T z_i(t)\, dy_i(t).$$

This property holds, if $K(t, s)$ is continuous on $[0, T] \times [0, T]$ or continuous on $0 \leqq s \leqq t \leqq T$ and vanishing on $0 \leqq t < s \leqq T$ (Volterra kernel). Further kernels are discussed in [5]. Now the primal problem is well defined. In our approach, the dual problem will admit another form than (1.2) (see Section 3).

Our investigations will be based on the following two assumptions:

(A1) If $z(\cdot) \in L_\infty(0, T; \mathbf{R}^M)$ is given, then any solution $x(\cdot) \in L_p(0, T; \mathbf{R}^N)$ of

$$B(t)\, x(t) \leqq z(t), \qquad x(t) \geqq \theta \quad \text{a.e. on } [0, T],$$

is bounded and measurable on $[0, T]$.

(A2) (*Slater-condition*): There are $\delta > 0$ and $\bar{x}(t) \geqq \theta$ from $L_\infty(0, T; \mathbf{R}^N)$ such that the strong inequality

$$B(t)\, \bar{x}(t) \leqq c(t) - \Delta_M + \int_0^T K(t, s)\, \bar{x}(s)\, ds$$

holds a.e. on $[0, T]$, where Δ_M is the M-vector with all entries equal to δ.

It should be mentioned that (A2) implies even the existence of $\delta > 0$ and $\bar{x}(t)$, which additionally satisfies $\bar{x}(t) \geqq \Delta_N$ (take $\bar{x}(t) := \bar{x}(t) + \varepsilon \Delta_N$ with ε sufficiently small). Note that in the example (1.3) these assumptions are met!

For proving the duality theorem we shall apply the following statement, which is adopted from [8], formulated for a linear constraint.

Theorem 2.1: *Let V and Z be real Banach spaces, $C \subseteqq V$ a convex closed set, $f: V \to \mathbf{R}^1$ a continuously Frechét-differentiable functional, and $T: V \to Z$ a linear continuous operator. Suppose that v^0 is optimal for the problem*

$$f(v) = \min!, \qquad Tv = \theta, \qquad v \in C, \tag{2.1}$$

and that the regularity conditions

$$TV = Z \tag{2.2}$$

and

$$T(\bar{v} - v^0) = \theta \quad \text{for some} \quad \bar{v} \in \operatorname{int} C \tag{2.3}$$

are fulfilled. Then there is a Lagrange multiplier $y \in Z^*$ such that

$$\langle f'(v^0), v - v^0 \rangle + \langle y, T(v - v^0) \rangle \geqq 0 \quad \text{for all} \quad v \in C. \tag{2.4}$$

In the theorem, $f'(v^0)$ is the Frechét-derivative of f at v^0. In what follows, we shall assume that an optimal solution $x^0(t)$ for the primal problem (1.1) exists. This holds true, if the feasible set of (1.1) is bounded in the norm of $L_p\,(0, T; \mathbf{R}^N)$.

Now we shall prove a Lagrange multiplier rule, from which all other statements can be easily derived.

Theorem 2.2: *Let $x^0(t)$ be optimal for (1.1), and suppose that the assumptions (A1), (A2) are satisfied. Then there is a vectorvalued function $y(t)$ from NBV $[0, T; \mathbf{R}^M]$ (henceforth called Lagrange multiplier) such that the pair $(x^0(t), z^0(t))$,*

$$z^0(t) := \int_0^T K(t, s)\, x^0(s)\, ds,$$

is a solution to the linear programming problem

$$\int_0^T \left[a(t) + \int_0^T K(s, t)^\top\, dy(s)\right]^\top x(t)\, dt - \int_0^T z(t)^\top\, dy(t) = \max! \tag{2.5}$$

subject to

$$B(t)\, x(t) \leqq c(t) + z(t), \qquad x(t) \geqq \theta \quad \text{a.e. on} \quad [0, T], \tag{2.6}$$

$$x(\cdot) \in L_\infty(0, T; \mathbf{R}^N), \qquad z(\cdot) \in C[0, T; \mathbf{R}^M].$$

Proof: In order to apply Theorem 2.1, we write the primal problem (1.1) in the equivalent, decomposed form

$$\begin{aligned} \int_0^T a(t)^\top x(t)\, dt = \max!, \quad & \int_0^T K(t, s)\, x(s)\, ds - z(t) = \theta \quad \text{on} \quad [0, T], \\ & B(t)\, x(t) - z(t) \leqq c(t) \quad \text{a.e. on} \quad [0, T], \\ & x(t) \geqq \theta. \end{aligned} \tag{2.7}$$

According to (A1) and the assumption on $K(t, s)$ we can assume $x(\cdot) \in L_\infty(0, T; \mathbf{R}^N)$, $z(\cdot) \in C[0, T; \mathbf{R}^M]$. Now we define

$$V := L_\infty(0, T; \mathbf{R}^N) \times Z, \qquad Z := C[0, T; \mathbf{R}^M],$$

denote the elements of V by $v(t) := (x(t), z(t))$, and introduce

$$f(v(\cdot)) := \int_0^T \left(-a(t)\right)^\top x(t)\, dt, \qquad \left(Tv(\cdot)\right)(t) := \int_0^T K(t, s)\, x(s)\, ds - z(t),$$

$$C := \{v(\cdot) \in V \mid B(t)\, x(t) - z(t) \leqq c(t),\ x(t) \geqq \theta \quad \text{a.e. on} \quad [0, T]\}.$$

In this way, the problem (2.7) becomes equivalent to (2.1), and $v^0(t) = (x^0(t), z^0(t))$ solves (2.1). According to (A2), a pair $v(t) := (\bar{x}(t) + \varepsilon\Delta_N, \bar{z}(t))$ with $\bar{z}(t) := \int_0^T K(t, s) \times (\bar{x}(s) + \varepsilon\Delta_N)\, ds$ belongs to the interior of C and satisfies the regularity condition (2.3). This was the reason for regarding $x(t)$ as a function of $L_\infty(0, T; \mathbf{R}^N)$, as in $L_p(0, T; \mathbf{R}^N)$ the interior of C would be empty. Moreover, condition (2.2) is fulfilled (the equation $Tv = z$ is solved by $v = (\theta, z)$). Thus Theorem 2.1 yields the existence of $y(t)$ from $Z^* = NBV[0, T; \mathbf{R}^M]$ such that

$$\int_0^T (-a(t))^\top (x(t) - x^0(t))\, dt$$

$$+ \int_0^T \left[\int_0^T K(t, s) (x(s) - x^0(s))\, ds - (z(t) - z^0(t)) \right]^\top dy(t) \geqq 0 \tag{2.8}$$

for all $(x(\cdot), z(\cdot)) \in C$. Finally, we multiply (2.8) by (-1), put $y(t) := -y(t)$, and change the order of integration (here we need the second assumption on $K(t, s)$) so that (2.8) takes the form

$$\int_0^T \left[a(t) + \int_0^T K(s, t)^\top\, dy(s)^\top \right]^\top (x(t) - x^0(t))\, dt - \int_0^T [z(t) - z^0(t)]^\top\, dy(t) \leqq 0$$

for all $(x(\cdot), z(\cdot)) \in C$, being equivalent with (2.5), (2.6) ∎

Discussing this result we obtain several useful conclusions.

Corollary 1 (*maximum principle*): *Under the assumptions of Theorem* 2.2

$$\max \left[a(t) + \int_0^T K(s, t)^\top\, dy(s) \right]^T x$$

subject to

$$B(t)\, x \leqq c(t) + \int_0^T K(t, s)\, x^0(s)\, ds\,, \qquad x \geqq \mathit{0}\,,$$

$x \in \mathbf{R}^N$, *is achieved a.e. on* $[0, T]$ *by* $x = x^0(t)$.

Proof: This follows easily from Theorem 2.2 after keeping $z(t) = z^0(t) = \int_0^T K(t, s) \times x^0(s)\, ds$ fixed ∎

Corollary 2: *The entries* $y_1(t), \ldots, y_M(t)$ *of the Lagrange multiplier* $y(t)$ *of* (2.5) *are monotone non-decreasing on* $[0, T]$.

Proof: It follows from Theorem 2.2 that $(x^0(t), z^0(t))$ must achieve the (finite) maximum value in (2.5). In particular, $(x^0(t), z^0(t))$ must be „better" than all pairs $(x^0(t), z^0(t) + z(t))$ with $z(t) \geqq \mathit{0}$. This can only hold if $z(t) \geqq \mathit{0}$ implies $\int_0^T z(t)^\top\, dy(t) \geqq 0$, and this yields in turn the corollary ∎

Thus $y(t)$ belongs to the dual cone P_M^+ of the cone P_M of non-negative functions of $C[0, T; \mathbf{R}^M]$.

Corollary 3 (*complementary slackness principle*): *Suppose that there are an open interval* $(a; b) \subseteqq [0, T]$, $\delta > 0$, *and* $j \in \{1, \ldots, M\}$ *such that*

$$(B(t)\, x^0(t))_j < \left(c(t) + \int_0^T K(t, s)\, x^0(s)\, ds \right)_j - \delta$$

holds a.e. on (a, b) *for an optimal solution* $x^0(t)$ *of the primal problem. Then* $y_j(t) \equiv y_j(a)$ *holds on* (a, b) *for the* j*-th component of the function* $y(t)$ *in* (2.5).

Proof: Assume that the corollary is not true. Then, by Corollary 2, there is an $\varepsilon > 0$ such that $y_j(a + \varepsilon) < y_j(b - \varepsilon)$. We can construct a continuous function $\bar{z}_j(t)$

on $[0, T]$ such that $\bar{z}_j(t) = z_j^0(t)$ on $[0, T] \setminus (a, b)$, $z_j^0(t) - \delta \leqq \bar{z}_j(t) \leqq z_j^0(t)$ on $(a, a + \varepsilon) \cup (b - \varepsilon, b)$ and $\bar{z}_j(t) = z_j^0(t) - \delta$ on $[a + \varepsilon, b - \varepsilon]$. It is easy to show that

$$\int_0^T \bar{z}_j(t)\, dy_j(t) < \int_0^T z_j^0(t)\, dy_j(t).$$

Now take $z_i(t) := z_i^0(t)$, $i \neq j$, $z_j(t) = \bar{z}_j(t)$. Then $(x^0(t), z(t))$ satisfies (2.6) but achieves a greater value in (2.5) than $(x^0(t), z^0(t))$, contradicting Theorem 2.2 ∎

To illustrate the theory, we shall now apply the first two corollaries to the example (1.3). Here the assumption (A 1) is trivially true, as $B(t)\,x(t) \leqq z(t)$, $x(t) \geqq 0$, where $z(t) = (z_1(t), z_2(t), z_3(t))^\mathsf{T}$, implies $0 \leqq x(t) \leqq z_3(t)$. The Slater-condition (A 2) is satisfied by $\bar{x}(t) \equiv 0$. Thus the Corollaries 1 and 2 ensure the existence of non-decreasing functions $y_1(t)$, $y_2(t)$, $y_3(t)$ from $NBV\,[0, T]$ such that an optimal solution $x^0(t)$ of the primal problem (1.3) is almost everywhere on $[0, T]$ the solution of

$$\max \left[a(t) + \int_t^T k(s, t)\, d\big(y_1(s) - y_2(s)\big) \right] x \tag{2.9}$$

subject to

$$-c + \int_0^t k(t, s)\, x^0(s)\, ds \leqq x \leqq c + \int_0^t k(t, s)\, x^0(s)\, ds,$$

$$0 \leqq x \leqq 1.$$

It should be noted that $y_1(t)$ refers to the upper bound $x(t) \leqq c + \int_0^t k(t, s)\, x(s)\, ds$ of the integral constraints, $y_2(t)$ refers to the corresponding lower bound, and $y_3(t)$ to the constraint $x(t) \leqq 1$.

3. The dual problem

Naturally, our approach does not lead to a dual problem in the form (1.2). The function $y(t)$ must be the optimal solution of another type of problems. Under the additional assumption of *continuity of $B(t)$ and $c(t)$* we can show that the Lagrange multiplier $y(t)$ is the optimal solution of the *dual problem*

$$\int_0^T c(t)^\mathsf{T}\, dy(t) = \min!$$

subject to

$$\int_0^{\cdot} B(t)^\mathsf{T}\, dy(t) - \int_0^{\cdot} \left(a(t) + \int_0^T K(s, t)^\mathsf{T}\, dy(s) \right) dt \in P_N^+, \tag{3.1}$$

$$y(\cdot) \in P_M^+,$$

where we denote by P_N^+ the dual cone to the non-negative cone P_N of $C[0, T; \mathbf{R}^N]$; thus P_N^+ consists of the non-decreasing functions of $NBV[0, T; \mathbf{R}^N]$.

It is easy to see that the derivative $v(t) := y'(t)$ is an optimal solution of the dual (1.2), if $y(t)$ is additionally absolutely continuous with derivative in $L_q[0, T; \mathbf{R}^M]$. We will not show that $y(t)$ solves indeed (3.1). The proof can be derived, for instance, from Theorem 2.2 and the observation that the subset of all $v(\cdot) = (x(\cdot), z(\cdot)) \in C$

with continuous part $x(t)$ is dense in C (the latter follows with some effort from Lusin's theorem). Note that the objective functional of (3.1) is defined only for continuous $c(t)$. If $c(t)$ is not continuous, then all considerations are more difficult. The main task of this paper is, however, to generalize and to apply the Grinold maximum principle rather than to establish a satisfactory duality theory. Therefore, we will not further consider the problem of duality and refer to a recent paper by Papageorgiou [4], where similar questions are investigated.

4. Application to a parabolic boundary control problem

In this section we consider the problem

$$\int_0^1 \left(w(T, x) - z(x)\right)^2 dx = \min! \tag{4.1}$$

subject to the parabolic initial-boundary value problem

$$\begin{aligned} w_t(t, x) &= w_{xx}(t, x) && \text{on } (0, T] \times (0, 1) \\ w(0, x) &= 0 && \text{on } (0, 1), \\ w_x(t, 0) &= 0 && \text{on } (0, T], \\ w_x(t, 1) &= \alpha[u(t) - w(t, 1)] && \text{on } (0, T] \end{aligned} \tag{4.2}$$

(the subscripts indicate derivatives with respect to t and x) and to the constraints

$$|u(t) - w(t, 1)| \leqq c \quad \text{a.e. on} \quad [0, T], \tag{4.3}$$

$$0 \leqq u(t) \leqq 1 \quad \text{a.e. on} \quad [0, T], \tag{4.4}$$

where we take the *control* $u(t)$ from $L_\infty(0, T)$ and define the corresponding *state* $w(t, x)$ from $C([0, T] \times [0, 1])$ as *generalized solution* of (4.1) by the expression (4.5) below. In this setting $T > 0$, $c > 0$, $\alpha > 0$, and $z(\cdot) \in L_2(0, 1)$ are given.

If this problem is viewed as a heating process, then $u(t)$ is a time-dependent heating law, $w(t, x)$ is the temperature within an infinite plate of thickness one, and the state-constraint (4.3) is imposed in order to bound thermal stresses occuring in the plate. In what follows, we shall denote by $u^0(t)$ an optimal control for (4.1)—(4.4), and $w^0(t, x)$ is the corresponding state. We define the generalized solution $w(t, x)$ of (4.2) by

$$w(t, x) = \alpha \int_0^t G(x, 1; t - s)\, u(s)\, ds, \tag{4.5}$$

where G is the Green function

$$G(x, \xi; t) = \sum_{n=1}^{\infty} N_n^{-1} \cos(c_n x) \cos(c_n \xi) \exp(-c_n^2 t),$$

and $0 < c_1 < c_2 < \dots$ are the non-negative solutions to $x \tan x = \alpha$, $N_n := 1/2 + \sin(2c_n)/4c_n$. If $u(t)$ is continuous, then $w(t, x)$ is a classical solution of (4.2), but we need the extension to bounded and measurable controls $u(t)$. It can be shown that by (4.5) a linear continuous transformation from $L_p(0, T)$ into $C([0, T] \times [0, 1])$ is defined, if $p > 2$ (see [8: Section 5]). Now we take $p > 2$ fixed, regard $u(t)$ formally as a function of $L_p(0, T)$, and introduce an operator $S: L_p(0, T) \to L_2(0, 1)$ by

$$\big(Su(\cdot)\big)(x) := w(T, x).$$

In this way, the control problem admits the form

$$f(u) := \|Su - z\|_2^2 = \min!$$

subject to (4.3), (4.4). Now we obtain from the well-known first order necessary optimality conditions (apply Theorem 2.1 to f as defined above, $T = \theta$, and C described by (4.3), (4.4)) that $u^0(t)$ must be optimal for the linear continuous programming problem

$$\int_0^T a(t)\, u(t)\, dt = \max!,$$

$$-c \leqq u(t) - \int_0^t k(t, s)\, u(s)\, ds \leqq c, \qquad 0 \leqq u(t) \leqq 1, \tag{4.6}$$

$u(\cdot) \in L_p(0, T)$, where

$$a(t) := -\sum_{n=1}^{\infty} N_n^{-1} \cos(c_n) \exp\big(-c_n^2(T - t)\big) \int_0^1 (Su^0 - z)(x) \cos(c_n x)\, dx,$$

$$k(t, s) := \alpha G(1, 1; t - s).$$

This is formally the same linear programming problem as in our example (1.3), but $a(t)$ and $k(t, s)$ are not continuous. We know, however, that $a(t)$ is continuous on $[0, T)$, bounded on $[0, T]$, and that $k(t, s)$ is continuous for $s < t$ with a weak singularity in $s = t$. Therefore it can be checked that all assumptions imposed on the data of the primal problem (1.1) are satisfied by the problem (4.6) for $p > 2$. On account of this, (4.6) can be treated completely analogous to (1.3). Thus, for $u^0(t)$ the maximum principle (2.9) must hold. In the next statement we shall apply this maximum principle in order to obtain a far reaching characterization of optimal controls.

Theorem 4.1: *Suppose that* $\int_0^1 \big(w^0(T, x) - z(x)\big)^2 dx > 0$, *that* (A 1), (A 2) *are satisfied, and* $u^0(t)$ *is piecewise continuous. Then there cannot exist any interval* $(a, b) \subseteqq [0, T]$ *where*

$$\max\left(0, -c + \int_0^t k(t, s)\, u^0(s)\, ds\right) < u^0(t) < \min\left(1, c + \int_0^t k(t, s)\, u^0(s)\, ds\right) \tag{4.7}$$

holds for all $t \in (a, b)$.
(Precisely, we must say that $u^0(t)$ is supposed to be a.e. equal to a piecewise continuous function).

Proof: Suppose the contrary, i.e., (4.7) is satisfied on (a, b). We can assume that all items in (4.7) differ at least by $\delta > 0$, so that the inequalities are uniformly strict on (a, b).
The optimal control must satisfy the maximum principle (2.9) with certain functions $y_1(t)$, $y_2(t)$, $y_3(t)$. By Corollary 3 and the note after (2.9) we find

$$y_1(t) = y_1(a), \quad y_2(t) = y_2(a) \quad \text{on } (a, b), \tag{4.8}$$

while the maximum principle (2.9) asserts that (4.7) can only hold if

$$a(t) + \int_t^T k(s, t)\, d\big(y_1(s) - y_2(s)\big) = 0 \quad \text{a.e. on} \quad (a, b).$$

According to (4.8) and the series representations of $a(t)$ and $k(t, s)$, this amounts to

$$\sum_{n=1}^{\infty} \exp(c_n^2 t) \Bigg[N_n^{-1} \alpha_n \exp(-c_n^2 T) \langle v_n, v^0 \rangle - N_n^{-1} \alpha_n^2 \int_{b-\varepsilon}^{T} \exp(-c_n^2 s)\, dy(s) \Bigg] = 0$$

on $[a, b - \varepsilon]$ with $\varepsilon > 0$ sufficiently small, where we have introduced $\alpha_n := \cos(c_n)$, $y(s) := y_1(s) - y_2(s)$, $v_n(x) := \cos(c_n x)$, $v^0(x) := w^0(T, x) - z(x)$. Hence, by the linear independence of the system $\{\exp(c_n^2 t)\}$,

$$\langle v_n, v^0 \rangle = \alpha_n \int_{b-\varepsilon}^{T} \exp\big(c_n^2 (T - s)\big)\, dy(s), \qquad n = 1, 2, \ldots \tag{4.9}$$

We will show that (4.9) implies

$$\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = +\infty \tag{4.10}$$

(contradicting $v^0(\cdot) \in L_2(0, 1)$), unless $y(t) \equiv y(a)$ on the whole interval $[a, T]$. Then, however, we get $\langle v_n, v^0 \rangle = 0$ from (4.9) for all $n = 1, 2, \ldots$ and hence $v^0(x) = w^0(T, x) - z(x) = 0$ a.e. on $[0, T]$, as the system $\{\cos(c_n x)\}$ is complete in $L_2(0, 1)$. This is a contradiction to the assumptions of the theorem; thus the statement must be true.

Therefore, we suppose finally that $y(t)$ is not identically constant on $[0, T]$, and it remains to verify (4.10) in this case.

As y_1 and y_2 are monotone non-decreasing, there exists

$$t_0 := \{\sup t \mid y_1(t) = y_1(a) \wedge y_2(t) = y_2(a)\}.$$

We can assume that $y_1(t)$ and $y_2(t)$ are continuous from the right and introduce the jump $h := y(t_0) - y(t_0 - 0)$ of $y(t)$ in t_0. Two cases can arise:

a) $t_0 = T$. Then $h^2 > 0$, and (4.9) implies

$$\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = \sum_{n=1}^{\infty} \alpha_n^2 h^2 = +\infty$$

(since $\alpha_n \sim (-1)^n$ for $n \to \infty$), i.e. (4.10) holds.

b) $t_0 < T$. As $u^0(t)$ is piecewise continuous, there is an $\eta > 0$ such that $u^0(t)$ is continuous on $(t_0, t_0 + 2\eta]$ (the point t_0 can be one of discontinuity). Therefore, one of the two inequalities

$$-c + \int_0^t k(t, s)\, u^0(s)\, ds \leqq u^0(t) \leqq c + \int_0^t k(t, s)\, u^0(s)\, ds, \tag{4.11}$$

say the left one, must be strict on $(t_0, t_0 + 2\eta)$, if η is sufficiently small. Now, by Corollary 3, $y_2(t)$ remains constant on $(t_0, t_0 + 2\eta]$, and we obtain in turn

$$\int_{b-\varepsilon}^{T} \exp\left(c_n^2(T-s)\right) dy(s) = \exp\left(c_n^2(T-t_0)\right) h + \int_{t_0}^{t_0+\eta} \exp\left(c_n^2(T-s)\right) dy_1(s)$$

$$+ \int_{t_0+\eta}^{t_0+2\eta} \exp\left(c_n^2(T-s)\right) dy_1(s) + \int_{t_0+2\eta}^{T} \exp\left(c_n^2(T-s)\right) dy(s).$$

If $h \neq 0$, then we choose η so small that $y_1(t_0 + \eta) - y_1(a) < |h|/2$ and find

$$\left| \int_{b-\varepsilon}^{T} \right| \geqq \exp\left(c_n^2(T-t_0)\right) \cdot \left(|h|/2 + o(c_n^2)\right).$$

If $h = 0$, then

$$\int_{b-\varepsilon}^{T} \geqq \exp\left(c_n^2(T-t_0-\eta)\right) \cdot \left(y_1(t_0+\eta) - y_1(a) + o(c_n^2)\right).$$

In both cases, (4.10) is easily obtained. If the right inequality of (4.11) is strict, then the proof is analogous ∎

Thus, if $u^0(t)$ is not too irregular, then $[0, T]$ can be divided into countably (or even finitely) many open intervals where one of the equations $u^0(t) = 0$, $u^0(t) = 1$, $u^0(t) - w^0(t, 1) = -c$, $u^0(t) - w^0(t, 1) = c$ is fulfilled. These facts can be used to construct a numerical method for the solution of (4.1)—(4.4) along the lines of [7].

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