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On Duality and the Maximum Principle for Continuous Linear Programming Problems

F. Tröltzsch

In dieser Arbeit wird das Grinoldsche Maximumprinzip für lineare stetige Optimierungsprobleme erweitert auf Fälle, wo die bekannten Dualitätssätze beim Beweis der Existenz einer Optimallösung des Dualproblems in den gebräuchlichen reflexiven L_p -Räumen versagen. Das erweiterte Maximumprinzip wird dann zur Untersuchung eines. parabolisehen Randsteuerproblems benutzt, bei dem Beschränkungen an die Steuerung und an den Zustand gegeben In dieser Arbeit wird das Grinoldsche Maximumprinzip für 1
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Optimallösung des Dualproblems in den gebräuchlichen reflex
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B этой работе принцип максимума Гринольда для задач пепрерывного линейного программирования расширяется на случай, где известные теоремы двойственности недостаточны для доказательства существования оптимального решения двойственной проблемы в рефлексиеных пространствах L_p . После этого расширенный принцип максимума изпользуется для изучения проблемы оптимального управления, описываемой параболическим уравнением, где управление действует на границе области и заданы ограничения на управление и состояние.

In this paper, the Grinold maximum principle for continuous linear programming problems is extended to the case where the known duality theorems do not ensure the existence of an optimal solution of the dual problem in the usual reflexive L_p -spaces. The extended maximum principle is then applied to the investigation of a parabolic boundary control problem with constraints on the control and the state.

1. Introduction'-

This paper is concerned' with applications of duality theorems for continuous linear programs, thus it contributes to a field *of* optimization theory where many interesting results were found within the last fifteen years. We mention only the basic investigatioms by LEVINSON [3], TYNDALL [9], and GRIN0LD [1], which have been continued by many others. The reader may find a short bibliography in the'author's paper [6]. These investigations were focused on the following pair of linear programs: Fore found within the last fifteen years

LEVINSON [3], TYNDALL [9], and GRI

others. The reader may find a short lestigations were focused on the follom

problem:

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 x, thus it contributes to a field of optimization

ere found within the last fifteen years. We me

LEVINSON [3], TYNDALL [9], and GRINOLD [

others. The reader may find a sh

Primal problem:

$$
\int\limits_0^T a(t)^{\top} x(t) dt = \sup!
$$

subject to

$$
B(t) x(t) \le c(t) + \int_{0}^{T} K(t, s) x(s) ds,
$$

\n
$$
x(t) \ge 0
$$

almost everywhere (a.e.) on [0, T], $x(\cdot) \in L_p(0, T; \mathbb{R}^N)$, $p \in [1, \infty)$ (by ^T we shall denote transposition).

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Dual problem/

$$
\int_{0}^{T} c(t)^{\top} v(t) dt = \inf !
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subject to

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\n*Dual problem*
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$$
\int_{0}^{T} c(t)^{T} v(t) dt = \inf !
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\nsubject to
\n
$$
B(t)^{T} v(t) \geq a(t) + \int_{0}^{T} K(s, t)^{T} v(s) ds,
$$
\n
$$
v(t) \geq \theta
$$
\na.e. on [0, T], $v(\cdot) \in L_{q}(0, T; \mathbb{R}^{M}), 1/q + 1/p = 1.$
\nIn this setting, $a(t)$ and $c(t)$ are suitable vector-valued functions, and $B(t), K(t, s)$ are matrix-valued. We will define them in the next section.
\nMost of the authors supposed $K(t, s)$ to be a continuous Volterra kernel (continuous on $0 \leq s \leq t \leq T$ and vanishing for $s > t$), whereas the more general class (1.1) with (weakly singular) Fredholm kernel was considered by the author [5].
\nThe duality theory refers to the following main questions: Do the primal supremum

 $\mathcal{S}=\frac{1}{2}$

a.e. on [0, T], $v(\cdot) \in L_q(0, T; \mathbb{R}^M)$, $1/q + 1/p = 1$.

In this setting, $a(t)$ and $c(t)$ are suitable vector-valued functions, and $B(t)$, $K(t, s)$ are matrix-valued. We will define them in the next section.

with (weakly singular) Fredholm kernel was considered by the author [5].

The duality theory refers to the following main questions: Do the primal supremum and the dual infimum coincide? (Without assumptions we know only sup \leq inf.) Do there exist optimal solutions to one or both of the two dual programs *E* the authors supposed $K(t, s)$ to bè a continuo
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dual infimum coincide? (W

In 1970 GRINOLD [2] established his *two-level maximum principle* for continuous linear programs. The first level, being of interest for our paper, is as follows: Suppose that $x^0(t)$ and $v^0(t)$ are optimal for (1.1) and (1.2), respectively. Then Frequality theor

and the dual infim

Do there exist optii

In 1970 GRINOLI

linear programs. The

that $x^0(t)$ and $v^0(t)$
 $\max\left[a(t)\right]$

subject to
 $B(t)$ $x \leq c$
 $x \geq \theta$
 $(x \in \mathbb{R}^N)$ is attained

problem (1.2). Cle

$$
\max \left[a(t) + \int\limits_0^T K(s,t)^{\mathsf{T}} \, v^0(s) \, ds\right]^{\mathsf{T}} \, x
$$

subject to

V'

• .

• •

$$
B(t) x \leqq c(t) + \int_{0}^{t} K(t,s) x^0(s) ds,
$$

 $x \geqq \theta$

 $(x \in \mathbb{R}^N)$ is attained a.e. on [0, *T*] by $x = x^0(t)$. A similar version holds for the dual $x \leq 0$
 $(x \in \mathbb{R}^N)$ is attained a.e. on $[0, T]$ by $x = x^0(t)$. A similar version holds for the dual

problem (1.2). Clearly, the validity of the maximum principle is intimately linked In 1970 GRINOLD [2] established his *two-level maximum principle* for continuous
linear programs. The first-level, being of interest for our paper, is as follows: Suppose
that $x^0(t)$ and $v^0(t)$ are optimal for (1.1) and problem (1.2). Clearly, the validity of the maximum principle is intimately linked with the existence of a dual optimal solution $v^0(t)$ for (1.2).

Although the duality theory has progressed very fast in recent years, it is often. the existence of an optimal solution to the dual problem (1.2) which cannot be guar-
anteed. Regard, as a typical example, the simple problem
 $I = \int_{\mathcal{L}} g(t) g(t) dt = \text{sin} t$ •

1.2)
\n
$$
v(t) = u(t) + \int \mathbf{n}(s, t) \cdot v(s) ds,
$$
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a.e. on [0, T], where c, $T>0$, $a(\cdot) \in C[0, T]$, and a continuous real function $k(t, s)$ on $[0, T] \times [0, T]$ are given, and $x(t)$ is taken from $L_2(0, T)$. This problem fits in (1.1) by $p = 2$, $N = 1$, $M = 3$,

that *x* (i) and *v* (i) are optimal for (1.1) and (1.2), respectively. Then
\n
$$
\max \left[a(t) + \int_{0}^{T} K(s, t)^\intercal v^0(s) ds \right]^T x
$$
\nsubject to
\n
$$
B(t) x \leq c(t) + \int_{0}^{T} K(t, s) x^0(s) ds,
$$
\n*x* ≥ θ\n*x*\n*x* ≥ θ\n*x*\n*y*\n*y*\n*x* = *x*⁰(*t*). A similar version holds for problem (1.2). Clearly, the validity of the maximum principle is infinite with the existence of a dual optimal solution *v*⁰(*t*) for (1.2). Although the duality theory has progressed very fast in recent years, it the existence of an optimal solution to the dual problem (1.2) which cannot anteed. Regard, as a typical example, the simple problem\n*x*\n*y*\n

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\nThe dual problem is therefore
\n
$$
\int_{0}^{T} [cv_1(t) + cv_2(t) + v_3(t)] dt = \inf!,
$$
\n
$$
v_1(t) - v_2(t) + v_3(t) \ge a(t) + \int_{t}^{T} k(s, t) (v_1(s) - v_2(s)) ds,
$$
\n
$$
v_i(t) \ge 0, \qquad i = 1, 2, 3,
$$
\na.e. on [0, T], where $v(t) = (v_1(t), v_2(t), v_3(t))^\intercal$ is taken from $L_2(0, T; \mathbb{R}^3)$. Using the theory of [1] it is easy to prove that an optimal solution of (1.3) exists, and that (1.3). (1.4) admit the same optimal values, i.e. max = inf. It is not clear, however,

a.e. on [0, *T*], where $v(t) = (v_1(t), v_2(t), v_3(t))$ ^T is taken from $L_2(0, T; \mathbb{R}^3)$. Using the theory of [1] it is easy to prove that an optimal solution of (1.3) exists, and that (1.3) , (1.4) admit the same optimal values, i.e. max = inf. It is not clear, however, whether an optimal solution exists in (1.4). The known duality theorems by Grinold, Levinson and Tyndall do not apply, as B , K , and c contain components with different signs.

The difficulties arising from the treatment of problems like (1.1) are caused by the *fact that x(t) occurs under an* . *integral as well as outside the integral,* as a ,,free term", thus the space for defining the inequality constraints of (1.1) must be as large as that for $x(t)$. Consequently, the well-known Slater-conditions cannot be applied to guarantee the solvability of (1.2), if $x(t)$ is defined in the usual L_p -spaces with $1 \leq p < \infty$. Therefore, one could use a *decomposition procedure separating integrals and free terms of* $x(t)$ in order to overcome the obstacles for proving the existence of a dual optimal solution and to establish a satisfactory maximum principle for optimal solutions of $v_i(t) \geq 0$, $i = 1, 2, 3$,
a.e. on [0, *T*], where $v(t) = (v_1(t), v_2(t), v_3(t))^\intercal$ is taken from $L_2(0, T; \mathbb{R}^3)$. Using the
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(1.3), (1.4) a

We will pursue this idea and its consequences in this paper. Our approach will not lead to entirely new duality results. To a certain extent, our theory is equivalent to the investigation of the problem in the unusual dual space $L_{\infty}(0, T; \mathbb{R}^M)^*$. However, the decomposition trick enables us to avoid the use of this space completely, thus this idea seems to be interesting in its own right.

As a result, we will obtain a useful generalization of the Grinold maximum principle for problems where the known duality theorems fail to ensure the existence of a dual optimal solution. The maximum principle was successfully applied by the author to the numerical solution of a parabolic boundary control problem (see [7]). In Section 4 we shall investigate analogously a more general *boundary control problem* by the extended maximum principle, in order to characterize optimal controls as increasely as a determination of $\alpha f(x)$ in order to overcome the obstacles for presidution and to establish a satisfactory maxim (1.1).
We will pursue this idea and its consequent not lead to entirely new duality results

2. The maximum principle

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At first, we introduce some notations: If X is a Banach space, then we shall denote by $L_n(0, T; X)$, $C[0, T; X]$, or $NBV[0, T; X]$ the spaces of functions on $[0, T]$ with values in X which are p -times integrable, continuous, or of bounded variation and vanishing at $t = 0$ (normalization condition), respectively: \mathbb{R}^N is the Euclidean N-dimensional space (column-vectors) and $\mathbb{R}^{M \times N}$ that of real $M \times N$ -matrices. By $\| \cdot \|_p$ we shall indicate the norm of $L_p(0,T)$. If $f \in X^*$, the dual space to X, then we shall write $\langle f, x \rangle$ for the value of f applied to $x \in X$. All other notations will become clear from the context.

In order to complete the definition of the primal problem (1.1) , we introduce integers $N \geq 1$, $M \geq 1$, real numbers $T> 0$, $p \in [1, \infty)$, and define q by $1/p + 1/q$ $=1.$ We suppose that $a(\cdot) \in L_q(0,T; \mathbb{R}^N)$, $c(\cdot) \in L_\infty(0,T; \mathbb{R}^M)$, and $B(\cdot) \in L_\infty$ \times (0, T; $\mathbf{R}^{M\times N}$) are given. Moreover, we suppose that a measurable function $K(t, s)$:

 $[0, T] \times [0, T] \rightarrow \mathbb{R}^{M \times N}$ is given such that the integral operator

$$
[0, T] \to \mathbf{R}^{M \times N}
$$
 is given

$$
x(\cdot) \mapsto \int_{0}^{T} K(\cdot, s) x(s) ds
$$

is a continuous mapping from $L_p(0, T; \mathbb{R}^N)$ into $C[0, T; \mathbb{R}^M]$, and that its adjoint operator, being a mapping from $NBV[0, T; \mathbb{R}^M]$ into $L_q(0, T; \mathbb{R}^N)$, has the form 528 F. TRÖLTZSOR
 $[0, T] \times [0, T] \rightarrow \mathbf{R}^{M \times N}$ is given such that the integ
 $x(\cdot) \mapsto \int_{0}^{T} K(\cdot, s) x(s) ds$

is a continuous mapping from $L_p(0, T; \mathbf{R}^N)$ into C

operator, being a mapping from $N B V[0, T; \mathbf{R}^M]$ int
 $y(\$

$$
y(\cdot) \mapsto \int\limits_0^T K(s, \cdot)^{\top} dy(s).
$$

$$
\int\limits_0^T z(t)^\top dy(t) := \sum\limits_{i=1}^M \int\limits_0^T z_i(t) dy_i(t).
$$

f z(t) dy(t) =' fz(t) dy1 (t). - Here we used the notation
 $\int_{0}^{T} z(t)^{\top} dy(t) := \sum_{i=1}^{M} \int_{0}^{T} z_i(t) dy_i(t)$.

This property holds, if $K(t, s)$ is continuous on $[0, T] \times [0, T]$ or continuous on $0 \le s$
 $\le t \le T$ and vanishing on $0 \le t < s \le T$ (Volterra kernel). *tion* $\int_{0}^{T} z(t)^{\top} dy(t) := \sum_{i=1}^{M} \int_{0}^{T} z_i(t) dy_i(t)$ *.*

This property holds, if $K(t, s)$ is continuous on $[0, T] \times [0, T]$ or continuous on $0 \le s \le t \le T$ and vanishing on $0 \le t < s \le T$ (Volterra kernel). Further kernels are dis $y(\cdot) \mapsto \int K(s, \cdot)^{\top} dy(s).$

Here we used the notation
 $\int_{0}^{T} z(t)^{\top} dy(t) := \sum_{i=1}^{M} \int_{0}^{T} z_i(t) dy_i(t).$

This property holds, if $K(t, s)$ is continuous on $[0, T] \times [0, T]$ or cont
 $\leq t \leq T$ and vanishing on $0 \leq t < s \leq T$ (Volte $\int_{0}^{1} z(t)^{\top} dy(t) := \sum_{i=1}^{m} \int_{0}^{t} z_{i}(t) dy_{i}(t).$

is property holds, if $K(t, s)$ is continuous on $[0, T] \times [0, T]$ or continuous on $0 \le t \le T$ and vanishing on $0 \le t < s \le T$ (Volterra kernel). Further kernels as

seussed in [will admit another form than (1.2) (see Sect

vestigations will be based on the following t

vestigations will be based on the following t
 $F(t) \in L_{\infty}(0, T; \mathbb{R}^M)$ is given, then any solution
 $B(t) x(t) \leq z(t), \quad x(t) \geq \theta$

(A1) If $z(\cdot) \in L_{\infty}(0, T; \mathbb{R}^M)$ is given, then any solution $x(\cdot) \in L_p(0, T; \mathbb{R}^N)$ of

is bounded and measurable on [0, *T].*

(A2) *(Slater-condition):* There are $\delta > 0$ and $\bar{x}(t) \ge \theta$ from $L_{\infty}(0, T; \mathbb{R}^{N})$ such that the strong inequality
 $B(t) \bar{x}(t) \le c(t) - A_{M} + \int_{0}^{T} K(t, s) \bar{x}(s) ds$ the strong inequality

$$
B(t)\overline{x}(t) \leqq c(t) - A_M + \int\limits_0^T K(t,s)\overline{x}(s) \,ds
$$

holds a.e. on [0, *T*], where Δ_M is the *M*-vector with all entries equal to δ .

It should be mentioned that (A2) implies even the existence of $\delta > 0$ and $\bar{x}(t)$, which additionally satisfies $\bar{x}(t) \geq \Delta_N$ (take $\bar{x}(t) := \bar{x}(t) + \epsilon \Delta_N$ with ϵ sufficiently small). Note that in the example (1.3) these assumptions are met! $B(t) \overline{x}(t) \leq c(t) - A_M + \int_0^T K(t, s) \overline{x}(s) ds$

ds a.e. on [0, *T*], where Δ_M is the *M*-vector with all entries equal to δ .

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over that in the example (1.3) these assumptio

For proving the duality theorem we shall apply the following statement, which is adopted from [8], formulated for a linear constraint..

The orem 2.1: Let V and Z be real Banach spaces, $C \subseteq V$ a convex closed set, $f: V \to \mathbf{R}$, a continuously Frechét differentiable functional, and $T: V \to Z$ a linear, con*tinuous operator. Suppose that v⁰ is optimal for the problem • f f* $f(x) = f(x)$, *v* $f(x) = f(x)$, $f(x) = f(x)$,

and that the regularity conditions

$$
TV = Z \tag{2.2}
$$

and

$$
T(\bar{v} - v^0) = \theta \quad \text{for some} \quad \bar{v} \in \text{int } C \tag{2.3}
$$

are fulfilled. Then there is a Lagrange multiplier $y \in Z^*$ such that

$$
\langle f'(v^0), v - v^0 \rangle + \langle y, T(v - v^0) \rangle \geq 0 \quad \text{for all} \quad v \in C. \tag{2.4}
$$

In the theorem, $f'(v^0)$ is the Frechét-derivative of f at v^0 . In what follows, we shall assume that an optimal solution $x^0(t)$ for the primal problem (1.1) exists. This holds true, if the feasible set of (1.1) is bounded in the norm of L_p (0, *T*; \mathbb{R}^N). **Solution** On Duality and the Maximum Pr

In the theorem, $f'(v^0)$ is the Frechet-derivative of f at v^0 . In what for

assume that an optimal solution $x^0(t)$ for the primal problem (1.1) exit

true, if the feasible set

Now we shall prove a Lagrange multiplier rule, from which all other statements can

Theorem 2.2: Let $x^0(t)$ be optimal for (1.1), and suppose that the assumptions (A1), $(A2)$ are satisfied. Then there is a vectorvalued function $y(t)$ from NBV $[0, T; R^M]$ *(henceforth called Lagrange multiplier) such that the pair* $(x^0(t), z^0(t))$ *. f* **f** $f(x) = f(x)$ *f* $f(x$

$$
z^0(t):=\int\limits_0^T K(t,s)\;x^0(s)\;ds\,,
$$

is a solution to the linear programming problem

$$
\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^{\top} dy(s)]^{\top} x(t) dt - \int_{0}^{T} z(t)^{\top} dy(t) = \max !
$$
 (2.5)

subject to

•

e satisfied. Then there is a vectorvalued function
$$
y(t)
$$
 from NBV [0, T; \mathbb{R}^m]
\nthe called Lagrange multiplier) such that the pair $(x^0(t), z^0(t))$,
\n $z^0(t) := \int_{0}^{T} K(t, s) x^0(s) ds$,
\n
$$
\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^T dy(s)]^T x(t) dt - \int_{0}^{T} z(t)^T dy(t) = \max!
$$
\n(2.5)
\n $B(t) x(t) \leq c(t) + z(t)$, $x(t) \geq \theta$ a.e. on [0, T],
\n $x(\cdot) \in L_{\infty}(0, T; \mathbb{R}^N)$, $z(\cdot) \in C[0, T; \mathbb{R}^M]$.
\n $f(\cdot)$ order to apply Theorem 2.1, we write the minimal problem (1.1) in the

equivalent, decomposed form

subject to
\n
$$
B(t) x(t) \leq c(t) + z(t), \quad x(t) \geq \theta \quad a.e. \text{ on } [0, T],
$$
\n
$$
x(\cdot) \in L_{\infty}(0, T; \mathbb{R}^{N}), \quad z(\cdot) \in C[0, T; \mathbb{R}^{M}].
$$
\nProof: In order to apply Theorem 2.1, we write the primal problem (1.1) in the equivalent, decomposed form
\n
$$
\int_{0}^{T} a(t)^{T} x(t) dt = \max 1, \quad \int_{0}^{T} K(t, s) x(s) ds - z(t) = \theta \quad \text{on } [0, T],
$$
\n
$$
B(t) x(t) - z(t) \leq c(t) \quad \text{a.e. on } [0, T],
$$
\n
$$
x(t) \geq 0.
$$
\nAccording to (A1) and the assumption on $K(t, s)$ we can assume $x(\cdot) \in L_{\infty}(0, T; \mathbb{R}^{N}),$
\n $z(\cdot) \in C[0, T; \mathbb{R}^{M}].$ Now we define
\n
$$
V := L_{\infty}(0, T; \mathbb{R}^{N}) \times Z, \quad Z := C[0, T; \mathbb{R}^{M}],
$$
\ndenote the elements of V by $v(t) := (x(t), z(t))$, and introduce
\n
$$
f(v(\cdot)) := \int_{0}^{T} (-a(t))^{T} x(t) dt, \quad (Tv(\cdot)) (t) := \int_{0}^{T} K(t, s) x(s) ds - z(t),
$$

According to (A1) and the assumption on $K(t, s)$ we can assume $x(\cdot) \in L_{\infty}(0; T; \mathbb{R}^{N})$, $z(\cdot) \in C[0, T; \mathbb{R}^M]$. Now we define Pr
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 $z(\cdot) \in$
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 $V:=L_{\infty}(0, T; \mathbb{R}^{N})\times Z, \qquad Z:=C[0, T; \mathbb{R}^{M}).$

denote the elements of V by $v(t) := (x(t), z(t))$ *, and introduce*

$$
B(t) x(t) - z(t) \leq c(t) \quad \text{a.e. on} \quad [0, T],
$$
\n
$$
x(t) \geq 0.
$$
\nAccording to (A 1) and the assumption on $K(t, s)$ we can assume $x(\cdot) \in L_{\infty}(0, 1)$
\n
$$
z(\cdot) \in C[0, T; \mathbb{R}^M].
$$
\nNow we define\n
$$
V := L_{\infty}(0, T; \mathbb{R}^N) \times Z, \quad Z := C[0, T; \mathbb{R}^M],
$$
\ndenote the elements of V by $v(t) := (x(t), z(t))$, and introduce\n
$$
f(v(\cdot)) := \int_0^T (-a(t))^T x(t) dt, \quad (Tv(\cdot)) (t) := \int_0^T K(t, s) x(s) ds - z(t),
$$
\n
$$
C := \{v(\cdot) \in V \mid B(t) x(t) - z(t) \leq c(t), x(t) \geq 0 \quad \text{a.e. on} \quad [0, T] \}.
$$
\nIn this way, the problem (2.7) becomes equivalent to (2.1), and $v^0(t) = (x^0(t), x(t)) \in K(t)$.

In this way, the problem (2.7) becomes equivalent to (2.1), and $v^0(t) = (x^0(t), z^0(t))$

 $U := \{v(\cdot) \in V \mid B(t) \ x(t) - z(t) \leq c(t), \ x(t) \geq \theta \text{ a.e. on } [0, T] \}.$

In this way, the problem (2.7) becomes equivalent to (2.1), and $v^0(t) = \left(x^0(t), z^0(t)\right)$

solves (2.1). According to (A2), a pair $v(t) := \left(\bar{x}(t) + \varepsilon A_N, \bar{z}(t)\right)$ wit $\times (\bar{x}(s) + \varepsilon A_{N}) ds$ belongs to the interior of *C* and satisfies the regularity condition (2.3). This was the reason for regarding $x(t)$ as a function of $L_{\infty}(0, T; \mathbb{R}^N)$, as in $L_n(0, T; \mathbb{R}^N)$ the interior of *C* would be empty. Moreover, condition (2.2) is fulfilled (the equation $Tv = z$ is solved by $v = (\theta, z)$). Thus Theorem 2.1 yields the existence 34 Analysis Bd. 4, Heft 6 (1985) In this way, the problem (2.7) becomes equivalent to (2.1), and $v^0(t)$
solves (2.1). According to (A2), a pair $v(t) := (\bar{x}(t) + \varepsilon A_N, \bar{z}(t))$ with $\frac{1}{2} \times (\bar{x}(s) + \varepsilon A_N) ds$ belongs to the interior of C and satisfies the re **0 '** a pair $v(t) := (x(t) + \epsilon Z_N, z)$
he interior of C and satisfies
regarding $x(t)$ as a function
would be empty. Moreover,
by $v = (\theta, z)$). Thus Theore

of $y(t)$ from $Z^* = NBV[0, T; \mathbb{R}^M]$ such that

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\nof
$$
y(t)
$$
 from $Z^* = NBV[0, T; R^M]$ such that
\n
$$
\int_{0}^{T} (-a(t))^T (x(t) - x^0(t)) dt
$$
\n
$$
+ \int_{0}^{T} \left[\int_{-b}^{T} K(t, s) (x(s) - x^0(s)) ds - (z(t) - z^0(t)) \right]^T dy(t) \ge 0
$$
\n50. (2.8)
\n50. (2.8)
\n51. (2.9) $\int_{0}^{T} \left[G(t) + \int_{0}^{T} K(s, t)^T dy(s) + \int_{0}^{T} (x(t) - x^0(t)) dt - \int_{0}^{T} [z(t) - z^0(t)]^T dy(t) \le 0 \right]$
\n52. (3.9) $\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^T dy(s)^T]^{T} (x(t) - x^0(t)) dt - \int_{0}^{T} [z(t) - z^0(t)]^T dy(t) \le 0$
\n63. (3.1) $\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^T dy(s)^T]^{T} (x(t) - x^0(t)) dt - \int_{0}^{T} [z(t) - z^0(t)]^T dy(t) \le 0$
\n64. (3.1) $\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^T dy(s)]^{T} dx$
\n65. (3.1) $\int_{0}^{T} [a(t) + \int_{0}^{T} K(s, t)^T dy(s)]^{T} dx$
\n67. (4) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n68. (4) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n69. (5) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n70. (6) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n81. (7) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n82. (8) $\int_{0}^{T} K(s, t)^T dy(s) dx$
\n83. (9) $\int_{0}^{T} K(s, t)^T dy(s)$

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chan for all $(x(\cdot), z(\cdot)) \in C$. Finally, we multiply (2.8) by (-1) , put $y(t) := -y(t)$, and change the order of integration (here we need the second assumption on $K(t, s)$) so that (2.8) takes the form for all $(x(\cdot), z(\cdot)) \in C$. Finally, we multiply (2.8) by (-1) , put $y(t) := -y(t)$, a
change the order of integration (here we need the second assumption on $K(t, s)$)
that (2.8) takes the form
 $\int_{0}^{T} \left[a(t) + \int_{0}^{T} K(s, t)^{T} dy(s)^{T$

$$
\int_{0}^{T} \left[a(t) + \int_{0}^{T} K(s, t)^{\intercal} dy(s)^{\intercal} \right]^{T} (x(t) - x^{0}(t)) dt - \int_{0}^{T} \left[z(t) - z^{0}(t) \right]^{\intercal} dy(t) \leq 0
$$
\n
$$
(\cdot), z(\cdot) \in C, \text{ being equivalent with (2.5), (2.6)}.
$$
\n
$$
\text{Ising this result we obtain several useful conclusions.}
$$
\n
$$
\text{lary 1 (maximum (principle): Under the assumptions of Theorem 2.2)}
$$
\n
$$
\text{max} \left[a(t) + \int_{0}^{T} K(s, t)^{\intercal} dy(s) \right]^{T} x
$$

for all $(x(\cdot), z(\cdot)) \in C$, being equivalent with (2.5) , (2.6)

Discussing this result we obtain several useful conclusions.

Corollary 1 *(maximum' principle): Under the assumptions of Theorem 2.2*

$$
\max\left[a(t)+\int\limits_0^T K(s,t)^\intercal\ dy(s)\right]^T x
$$

$$
c(\cdot), z(\cdot)) \in C, \text{ being equivalent with } (2.5), (2.6)
$$

using this result we obtain several useful con-
llary 1 (*maximum' principle*): Under the assu

$$
\max \left[a(t) + \int_{0}^{T} K(s, t)^{\top} dy(s)\right]^{T} x
$$

2

$$
B(t) x \leq c(t) + \int_{0}^{T} K(t, s) x^{0}(s) ds, \qquad x \geq 0,
$$

is achieved a.e. on [0, T] by $x = x^{0}(t)$.

 $x \in \mathbb{R}^N$, *is achieved a.e. on* [0, *T*] by $x = x^0(t)$.

Proof: This follows easily from Theorem 2.2 after keeping $z(t) = z^0(t) = \int_{0}^{t} K(t, s) \times x^0(s) ds$ fixed \blacksquare

Corollary 2: The entries $y_1(t), \ldots, y_M(t)$ of the *Lagrange multiplier y(t)* of (2.5) are *monotone non-decreasing on* [0, *T].*

Proof: It follows from Theorem 2.2 that $(x^0(t), z^0(t))$ must achieve the (finite) maximum value in (2.5). In particular, $(x^0(t), z^0(t))$ must be ,,better" than all pairs $(x^0(t), z^0(t) + z(t))$ with $z(t) \ge 0$. This can only hold if $z(t) \ge 0$ implies $\int z(t)^T dy(t) \ge 0$, and this yields in turn the corollary \blacksquare $\max \left[a(t) + \int_0^T K(s, t)^T dy(s)\right]^T x$

subject to
 $B(t) x \leq c(t) + \int_0^T K(t, s) x^0(s) ds, \quad x \geq 0,$
 $x \in \mathbb{R}^N$, is achieved a.e. on $[0, T]$ by $x = x^0(t)$.

Proof: This follows easily from Theorem 2.2 after $x^0(s) ds$ fixed \blacksquare

Corol $x \in \mathbb{R}^N$, is achieved a.e. on $[0, T]$ by $x = x^0(t)$.

Proof: This follows easily from Theorem 2.2 after keeping $z(t)$
 $\times x^0(s)$ ds fixed \blacksquare

Corollary 2: The entries $y_1(t), \ldots, y_M(t)$ of the Lagrange multiple

mono $\begin{split} \text{articular,} & \left\{x^0(t),\, \overset{.}{z}^0(\text{.})\right\} \ \text{This can only hold} \ \text{all any } \blacksquare \ \text{I cone } P_M{}^+ \text{ of the} \ \text{I cone } P_M{}^+ \text{ of the} \ \text{I space } P_M{}^+ \text{ of the} \ \text{II space } P_M{}^+ \text{ of the} \ \text{II space } P_M{}^+ \text{ of the} \ \text{II space } P_M{}^+ \text{ of the} \ \text{I space } P_M{}^+ \text{ of the} \ \text{I space } P_M{}^+ \text{ of the} \ \text{I space } P_M{}^+ \$ **holds.** $\{x^0(t), z^0(t) + z(t)\}$ with $z(t) \ge 0$. This can only hold if $z(t) \ge 0$ implies $\int z(t)^\top dy(t) \ge$ and this yields in turn the corollary **i**
and this yields in turn the corollary **i**
and this yields in turn the corollary

Thus $y(t)$ belongs to the dual cone P_M^+ of the cone P_M of non-negative functions of $C[0, T; \mathbf{R}^M]$.

Corollary 3 *(complementary slackness principle): Suppose that there are an open interval* $(a, b) \subseteq [0, T]$, $\delta > 0$, and $j \in \{1, ..., M\}$ such that

$$
(t) + z(t) \text{ with } z(t) \geq \theta. \text{ This can only hold if } z(t) \text{ yields in turn the corollary } \blacksquare
$$
\n
$$
y(t) \text{ belongs to the dual cone } P_M^+ \text{ of the cone}
$$
\n
$$
F; \mathbb{R}^M].
$$
\n
$$
(\mathbf{a}, b) \subseteq [0, T], \delta > 0, \text{ and } j \in \{1, ..., M\} \text{ such}
$$
\n
$$
(B(t) x^0(t))_j < \left(c(t) + \int_0^T K(t, s) x^0(s) \, ds\right)_j - \delta
$$

holds a.e. on (a, b) for an optimal solution $x^0(t)$ *of the primal problem. Then y_i(t)* $\equiv y_j(a)$ *holds on (a, b) for the j-th component of the function y(t) in (2.5). holds a.e. on (a, b) for an optimal solution* $x^0(t)$ *of the primal problem. Then* $y_j(t) \equiv y_j(a)$
holds on (a, b) for the j-th component of the function $y(t)$ *in (2.5).*
Proof: Assume that the corollary is not true. Th

Proof: Assume that the corollary is not true. Then, by Corollary 2, there is an

on Duality and the Maximum Principle 531

on [0, *T]* such that $\bar{z}_j(t) = z_j^0(t)$ on $[0, T] \setminus (a, b), z_j^0(t) - \delta \leq \bar{z}_j(t) \leq z_j^0(t)$ on **(a, a** + e) **u**(b) \overline{f} **c**) and $\overline{z}_j(t) = z_j^0(t)$ on $[0, T] \setminus (a, b), z_j^0(t) - \delta \leq \overline{z}_j(t) \leq z_j^0(t)$ on $[a, a + \varepsilon) \cup (b - \varepsilon, b)$ and $\overline{z}_j(t) = z_j^0(t) - \delta$ on $[a + \varepsilon, b - \varepsilon]$. It is easy to show that

$$
\begin{aligned}\n\text{?} \quad \text{such that} \quad & \bar{z}_j(t) = z_j^0(t) \quad \text{or} \\
\text{s) } \cup (b - \varepsilon, b) \text{ and } \bar{z}_j(t) = z_j^0(t) \\
\int_0^T \bar{z}_j(t) \, dy_j(t) < \int_0^T z_j^0(t) \, dy_j(t) \\
\text{e } z_i(t) := z_i^0(t), \, i + j, \, z_j(t) = z \\
\text{c value in (2.5) than } (x^0(t), z^0(t))\n\end{aligned}
$$

Now take $z_i(t) := z_i^0(t)$, $i' \neq j$, $z_i(t) = z_i(t)$. Then $(x^0(t), z(t))$ satisfies (2.6) but achieves a greater value in (2.5) than $(x^0(t), z^0(t))$, contradicting Theorem 2.2 **I**

To illustrate the theory,,we shall now apply the first two corollaries to the example (1.3). Here the assumption (A1) is trivially true, as $B(t) x(t) \le z(t)$, $x(t) \ge 0$, where $z(t) = (z_1(t), z_2(t), z_3(t))^T$, implies $0 \leq x(t) \leq z_3(t)$. The Slater-condition (A2) is satisfied by $\bar{x}(t) = 0$. Thus the Corollaries 1 and 2 ensure the existence of non-decreasing functions $y_1(t)$, $y_2(t)$, $y_3(t)$ from *NBV* [0, *T*] such that an optimal solution $x^0(t)$ of the primal problem (1.3) is almost everywhere on [0, *T]* the solution of $\int \bar{z}_j(t) dy_j(t) < \int_0^t z_j^0(t) dy_j(t)$
 $\int_0^t e z_i(t) := z_i^0(t), i \neq j, z_j(t) = z_j(t)$. Then $(x^0(t), z(t))$ satisfies (2.6) but achieves
 $\int \bar{x}$ at \bar{x} at $\int (2.5) \tan (x^0(t), z^0(t))$, contradicting Theorem 2.2 \blacksquare .
 $\int \int_0^t e^{-x} f(x) dx$ is $\int \overline{z}_j(t) dy_j(t) < \int z_j^0(t)$

Now take $z_i(t) := z_i^0(t), i + j$,

a greater value in (2.5) than (

To illustrate the theory, we

(1.3). Here the assumption (A
 $z(t) = (z_1(t), z_2(t), z_3(t))^T$, implified by $\overline{x}(t) \equiv 0$. Thus the Confuncti strate the theory, we shall now apply the first two corollaries to the

re the assumption (A1) is trivially true, as $B(t) x(t) \le z(t)$, $x(t) \ge$

(*t*), $z_2(t), z_3(t)$]^T, implies $0 \le x(t) \le z_3(t)$. The Slater-condition (A2)

(*t*

$$
\max\left[a(t)+\int\limits_{t}^{T}k(s,t)\,d\big(y_1(s)-y_2(s)\big)\right]x\qquad \qquad (2.9)
$$

$$
-c + \int_{0}^{t} k(t, s) x^{0}(s) ds \leq x \leq c + \int_{0}^{t} k(t, s) x^{0}(s) ds,
$$

$$
0 \leq x \leq 1.
$$

It should be noted that $y_1(t)$ refers to the upper bound $x(t) \leq c + \int k(t, s) x(s) ds$ of the integral constraints, $y_2(t)$ refers to the corresponding lower bound, and $y_3(t)$ to the constraint $x(t) \leq 1$. tunctions $y_1(t)$, $y_2(t)$, $y_3(t)$ from NBV [0, T] such that an optimal sprimal problem (1.3) is almost everywhere on [0, T] the solution
 $\max \left[a(t) + \int_t^T k(s, t) d(y_1(s) - y_2(s)) \right] x$

subject to
 $-c + \int_t^t k(t, s) x^0(s) ds \le x \le c + \int_$

3. The dual problem

Naturally, our approach does not lead to a dual problem in the form (1.2). The function $y(t)$ must be the optimal solution of another type of problems. Under the additional assumption of *continuity 0/ B(t) and c(t)* we can show that the Lagrange multiplier *y(t)* **is** the optimal solution of the *dual problem* the integral constraints, $y_2(t)$ refers to the corresponding
the constraint $x(t) \leq 1$.
3. The dual problem
Naturally, our approach does not lead to a dual probl
function $y(t)$ must be the optimal solution of another ty
 em in the form is the form of problem is shown that
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$$
\int c(t)^{\top} dy(t) = \min!
$$

By, our approach does not lead to a dual problem in the form (1.2). The
$$
y(t)
$$
 must be the optimal solution of another type of problems. Under the all assumption of *continuity of B(t)* and $c(t)$ we can show that the Lagrange er $y(t)$ is the optimal solution of the *dual problem*\n
$$
\int_{0}^{T} c(t)^{\top} dy(t) = \min!
$$
\n
$$
\int_{0}^{T} B(t)^{\top} dy(t) - \int_{0}^{T} \left(a(t) + \int_{0}^{T} K(s, t)^{\top} dy(s) \right) dt \in P_N^+,
$$
\n
$$
y(\cdot) \in P_M^+,
$$
\n
$$
y(\cdot) \in P_M^+
$$
\n
$$
\therefore
$$
\ne denote by P_N^+ the dual cone to the non-negative cone P_N of $C[0, T; \mathbb{R}^N]$;\n
$$
\therefore
$$
 consists of the non-decreasing functions of *NPV*[0, T; \mathbb{R}^N].

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where we denote by P_N^+ the dual cone to the non-negative cone P_N of $C[0, T; \mathbb{R}^N]$; thus P_N^+ consists of the non-decreasing functions of *NBV*[0, *T*; **R**^{*N*}].

It is easy to see that the derivative $v(t) := y'(t)$ is an optimal solution of the dual (1.2), if $y(t)$ is additionally absolutely continuous with derivative in $L_q[0, T; \mathbb{R}^M]$. We will not show that $y(t)$ solves indeed (3.1). The proof can be derived, for instance; from Theorem 2.2 and the observation that the subset of all $v(\cdot) = (x(\cdot), z(\cdot)) \in C$ $\int_{0}^{T} B(t)^{\top} dy(t) - \int_{0}^{T} \left(a(t) + \int_{0}^{T} K(s, t)^{\top} dy(s) \right) dt \in P_{N}^{+},$ (3.1
 $y(\cdot) \in P_{M}^{+},$

where we denote by P_{N}^{+} the dual cone to the non-negative cone P_{N} of $C[0, T; \mathbb{R}^{N}]$

thus P_{N}^{+} consists of the no

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with continuous part $x(t)$ is dense in C (the latter follows with some effort from Lusin's theorem). Note that the objective functional of (3.1) is defined only for continuous $c(t)$. If $c(t)$ is not continuous, then all considerations are more difficult. The main task of this paper is, however, to generalize and to apply the Grinold maximum principle rather than to establish a satisfactory duality theory. Therefore, we will not further consider the problem of-duality and refer to a recent paper by PAPAGEORGIOU [4], where similar questions are investigated. with continuous part $x(t)$ is dense in C (the latter follows with
theorem). Note that the objective functional of (3.1) is defi
 $z(t)$. If $c(t)$ is not continuous, then all considerations are more
of this paper is, however mous part $x(t)$ is dense in C (the latter follable that the objective functional of (3
 i is not continuous, then all considerations

to establish a satisfactory duality theorem to establish a satisfactory duality theor

4. Application to a-parabolic boundary control problem

$$
\int (w(T, x) - z(x))^2 dx = \min! \tag{4.1}
$$

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subject to the parabolic initial-boundary value problem

Consider the problem of calculus

\nwhere similar questions are investigated.

\n4. Application to a parabolic boundary control problem

\nIn this section we consider the problem.

\n
$$
\int_{0}^{1} (w(T, x) - z(x))^{2} dx = \min !
$$

\nsubject to the parabolic initial-boundary value problem

\n
$$
w_{t}(t, x) = w_{xx}(t, x) \qquad \text{on } (0, T] \times (0, 1)
$$

\n
$$
w(0, x) = 0 \qquad \text{on } (0, 1),
$$

\n
$$
w_{x}(t, 0) = 0 \qquad \text{on } (0, T],
$$

\n(the subscripts indicate derivatives with respect to t and x) and to the constraints

\n
$$
|u(t) - w(t, 1)| \leq \delta
$$
 a.e. on $[0, T],$

\n(the subscripts indicate derivatives with respect to t and x) and to the constraints

\n
$$
|u(t) - w(t, 1)| \leq \delta
$$
 a.e. on $[0, T],$

\n(4.4)

\nwhere we take the *control* $u(t)$ from $L_{\infty}(0, T)$ and define the corresponding *state* $w(t, x)$ from $C([0, T] \times [0, 1])$ as generalized solution of (4.1) by the expression (4.5) below. In this setting $T > 0$, $c > 0$, and z .) (4.4) we express $\text{loc}(0, T)$ to $c > 0$, and $\text{loc}(0, T)$ be expressed as a heating process, then $u(t)$ is a time-dependent heating law, $w(t, x)$ is the temperature within an infinite plate of thickness one, and the state-constant (4.3) is imposed in order to bound thermal stresses occurring in the plate.

(the subscripts indicate derivatives with respect to t and x) and to the constraints

$$
w_x(t, 1) = \alpha[u(t) - w(t, 1)]
$$
 on $(0, T]$
scripts indicate derivatives with respect to t and x) and to the constraints
 $|u(t) - w(t, 1)| \leq \frac{7}{6}$ a.e. on $[0, T]$,
 $0 \leq u(t) \leq 1$ a.e. on $[0, T]$,
(4.4)

where we take the *control* $u(t)$ from $L_{\infty}(0, T)$ and define the corresponding state $w(t, x)$ from $C([0, T] \times [0, 1])$ as *generalized solution* of (4.1) by the expression (4.5) below. In this setting $T > 0$, $c > 0$, $\alpha > 0$, and $z(\cdot) \in L_2(0, 1)$ are given.

If this problem is viewed as a heating process, then $u(t)$ is a time-dependent heating. $\lim_{x\to a} w(t,x)$ is the temperature within an infinite plate of thickness one, and the stateconstraint. (4.3) is imposed in order to bound thermal stresses occuring in the plate. In what follows, we shall denote by $u^0(t)$ an optimal control for $(4.1) - (4.4)$, and $w^0(t, x)$ is the corresponding state. We define the generalized solution $w(t, x)$ of (4.2) by $\begin{array}{ll} & \text{(the sub}\\ & \text{where } \mathbf{w}\\ & \text{from } C(\text{ this set})\\ & \text{If this set}\\ & \text{law}, w(t)\\ & \text{constra}\\ & w^0(t,x)\\ & \text{where } \end{array}$ From $C([0, T] \times [0, 1])$ as generatized solution of (4.1) by the this setting $T > 0$, $c > 0$, $\alpha > 0$, and $z(\cdot) \in L_2(0, 1)$ are giv if this problem is viewed as a heating process, then $u(t)$ is $\text{law}, w(t, x)$ is the tempera

$$
w(t, x) = \alpha \int_{0}^{t} G(x, 1; t - s) u(s) ds,
$$
\n(4.5)

$$
G(x, \xi; t) = \sum_{n=1}^{\infty} N_n^{-1} \cos (c_n x) \cos (c_n \xi) \exp (-c_n^2 t),
$$

and $0 < c_1 < c_2 < \ldots$ are the non-negative solutions to $x \tan x = \alpha$, $N_n := 1/2$ $+ \sin (2c_n)/4c_n$. If $u(t)$ is continuous, then $w(t, x)$ is a classical solution of (4.2), but we need the extension to bounded and measurable controls $u(t)$. It can be shown that by (4.5) a linear continuous transformation from $L_p(0, T)$ into $C([0, T] \times [0, 1])$ is defined, if $p > 2$ (see [8: Section 5]). Now we take $p > 2$ fixed, regard $u(t)$ formally, as a function of $L_p(0, T)$, and introduce an operator $S: L_p(0, T) \to L_p(0, 1)$ by (Su(.)) ($L_p(0, T)$, and introduce an operator $S: \tilde{B}(Su(\cdot))$ (\tilde{x}) := $w(T, x)$.

(Su(.)) (\tilde{x}) := $w(T, x)$.

(ay, the control problem admits the form $f(u) := \|Su - z\|_2^2 = \min!$

S.

$$
(Su(\cdot))(x):=w(T,x).
$$

In this way, the control problem admits the form

$$
f(u) := \|Su - z\|_2^2 = \min!
$$

(on Dua)

(figure $\int (Su(\cdot)) (x) dx = w(T, x)$)

(give $\int (Su(\cdot)) (x) dx = w(T, x)$)

(give $\int (u) dx = ||S u - z||_2^2 = \min !$

(give $\int (4.3)$, (4.4). Now we obtain from the conditions (apply Theorem 2.1 to f

(4.3) (4.4)) that $v^0(t)$ must be ont ty and the Maximun

rator S: $L_p(0, T) \rightarrow$

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the well-known first

as defined above, Δ

. subject to (4.3) , (4.4) . Now we obtain from the well-known first order necessary optimality conditions (apply Theorem 2.1 to f as defined above, $T = \theta$, and C described by (4.3) , (4.4)) that $u^0(t)$ must be optimal for the linear continuous programming problem on of $L_p(0, T)$, and introduce an operator $S: L_p(0, T) \to L_2(0, 1)$ by $u(\cdot)$ $(\dot{x}) := w(T, x)$.
 t, the control problem admits the form
 $a) := ||Su - z||_2^2 = \min!$

(4.3), (4.4). Now we obtain from the well-known first order necessar $f(u) :=$
 $f(u) :=$
 $i \text{ to } (4.3)$
 $i \text{ by } (4.3)$,
 $i \text{ to } (4.3)$
 $i \text{ to } (4$ $(x) := w(T, x).$
 $x \in \text{control problem admits the form}$
 $\|Su - z\|_2^2 = \min!$
 $\|Au - z\|_2^2 = \min!$
 $\|Au - z\|_2^2 = \min!$
 $\|Au - w\|_2^2 = \min!$
 $\|Au - w\|_2^2 = \min!$
 $\|Au - w\|_2^2 = \min!$
 $\|u(t) - w\|_2^2 = \$

$$
\int_{0}^{T} a(t) u(t) dt = \max!,
$$
\n
$$
-c \leq u(t) - \int_{0}^{t} k(t, s) u(s) ds \leq c,
$$
\n(0, T), where\n
$$
a(t) := -\sum_{n=1}^{\infty} N_n^{-1} \cos(c_n) \exp\left(-c_n^2\right)
$$
\n
$$
k(t, s) := \alpha G(1, 1; t - s).
$$
\n
$$
\text{formally the same linear programming}
$$
\n
$$
k(t, s) \text{ are not continuous. We know, how}
$$

 $u(\cdot) \in L_p(0, T)$, where

$$
a(t) := -\sum_{n=1}^{\infty} N_n^{-1} \cos (c_n) \exp \left(-c_n^2(T-t)\right) \int_{0}^{1} (Su^0 - z) (x) \cos (c_n x) dx,
$$

\n
$$
k(t, s) := \alpha G(1, 1; t - s).
$$

This is formally the same linear programming problem as in our example (1.3), but $a(t)$ and $k(t, s)$ are not continuous. We know, however, that $a(t)$ is continuous on [0, *T*), bounded on [0, *T*], and that $k(t, s)$ is continuous for $s < t$ with a weak singularity in $s = t$. Therefore it can be checked that all assumptions imposed on the data of the $s = t$. $s \doteq t$. Therefore it can be checked that all assumptions imposed on the data of the primal problem (1.1) are satisfied by the problem (4.6) for $p > 2$. On account of this, (4.6) can be treated completely analogous to (1 this, (4.6) can be treated completely analogous to (1.3) . Thus, for $u^0(t)$ the maximum principle (2.9) must hold. In the next statement we shall apply this maximum prin-This is formally the same linear programming problem as in our exar $a(t)$ and $k(t, s)$ are not continuous. We know, however, that $a(t)$ is contin
bounded on [0, T], and that $k(t, s)$ is continuous for $s < t$ with a weak
 $s \d$ *h*(4.6) ior $p \ge 2$. On account this, (4.6) ior $p \ge 2$. On account this, (4.6) can be treated completely analogous to (1.3). Thus, for $w \ge 2$ *that* principle (2.9) must hold. In the next statement we shall apply this this, (4.6) can be treated completely analogous to (1.3). Thus, for
inciple (2.9) must hold. In the next statement we shall apply
ciple in order to obtain a far reaching characterization of optin
 \therefore Theorem 4.1: Suppos

circle in order to obtain a far reaching characterization of optimal controls.
\nTheorem 4.1: Suppose that
$$
\int_{0}^{1} (w^0(T, x) - z(x))^2 dx > 0
$$
, that (A1), (A2) are satisfied, and $u^0(t)$ is piecewise continuous. Then there cannot exist any interval $(a, b) \subseteq [0, T]$
\nwhere

/

fied, and $u^0(t)$ *is piecewise continuous. Then there cannot exist any interval* $(a, b) \subseteq [0, T]$

Theorem 4.1. Suppose that
$$
\int_{0}^{1} (w^{2}(1, x) - 2(x))^{-} dx > 0
$$
, that (A 1), (A 2) are satisfied, and $u^{0}(t)$ is piecewise continuous. Then there cannot exist any interval $(a, b) \subseteq [0, T]$ where\n
$$
\max\left(0, -c + \int_{0}^{t} k(t, s) u^{0}(s) ds\right) < u^{0}(t) < \min\left(1, c + \int_{0}^{t} k(t, s) u^{0}(s) ds\right)
$$
\n(4.7)\nholds for all $t \in (a, b)$.\n(Precisely, we must say that $u^{0}(t)$ is supposed to be a.e. equal to a piecewise continuous function).\nProof: Suppose the contrary, i.e., (4.7) is satisfied on (a, b) . We can assume that all items in (4.7) differ at least by $\delta > 0$, so that the inequalities are uniformly strict on (a, b) .\nThe optimal control must satisfy the maximum principle (2.9) with certain functions $y_{1}(t), y_{2}(t), y_{3}(t)$. By Corollary 3 and the note after (2.9) we find\n
$$
y_{1}(t) = y_{1}(a), y_{2}(t) = y_{2}(a)
$$
 on (a, b) ,\n(4.8)

(Precisely, we must say that $u^0(t)$ is supposed to be a.e. equal to a piecewise continuous function).

Proof: Suppose the contrary, i.e., (4.7) is satisfied on (a, b) . We can assume that all items in (4.7) differ at least by $\delta > 0$, so that the inequalities are uniformly strict

The optimal control must satisfy the maximum principle (2.9) with certain functions

$$
y_1(t) = y_1(a), y_2(t) = y_2(a)
$$
 on $(a, b),$

5.

while the maximum principle (2.9) asserts that (4.7) can only hold if

$$
a(t) + \int\limits_t^T k(s,t) \, d(y_1(s) - y_2(s)) = 0 \quad \text{a.e. on} \quad (a, b).
$$

According to (4.8) and the series representations of $a(t)$ and $k(t, s)$, this amounts to

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\nwhile the maximum principle (2.9) asserts that (4.7) can only hold if
\n
$$
a(t) + \int_{t}^{T} k(s, t) d(y_1(s) - y_2(s)) = 0 \quad \text{a.e. on } (a, b).
$$
\nAccording to (4.8) and the series representations of $a(t)$ and $k(t, s)$, this amounts to
\n
$$
\sum_{n=1}^{\infty} \exp (c_n^{2}t) \left[N_n^{-1} \alpha_n \exp (-c_n^{2}T) \langle v_n, v^0 \rangle \right.
$$
\n
$$
- N_n^{-1} \alpha_n^2 \int_{0}^{T} \exp (-c_n^{2}s) dy(s) \right] = 0
$$
\non
$$
\begin{bmatrix} a, b - \varepsilon \end{bmatrix}
$$
 with $\varepsilon > 0$ sufficiently small, where we have introduced $\alpha_n := \cos (c_n)$,
\n
$$
y(s) := y_1(s) - y_2(s), v_n(x) := \cos (c_n x), v^0(x) := w^0(T, x) - z(x).
$$
 Hence, by the
\nlinear independence of the system $(\exp (c_n^{2}t)),$
\n
$$
\langle v_n, v^0 \rangle = \alpha_n \int_{0-t}^{T} \exp (c_n^{2}(T-s)) dy(s), \qquad n = 1, 2, ...
$$
\n(4.9)
\nWe will show that (4.9) implies
\n
$$
\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = +\infty
$$
\n(4.10)

on $[a, b - \varepsilon]$ with $\varepsilon > 0$ sufficiently small, where we have introduced $\alpha_n := \cos(c_n)$, $y(s) := y_1(s) - y_2(s), v_n(x) := \cos(c_n x), v^0(x) := w^0(T, x) - z(x).$ Hence, by the linear independence of the system $\{ \exp (c_n^2 t) \},\$ g to (4.8) and the scries representations of $a(t)$ and $k(t, s)$, this amounts to
 $\sum_{i=1}^{\infty} \exp (c_n^{2}t) \left[N_n^{-1} \alpha_n \exp (-c_n^{2}T) \langle v_n, v^0 \rangle \right.$
 $- N_n^{-1} \alpha_n^{2} \int_{0}^{T} \exp (-c_n^{2} s) dy(s) \right] = 0$
 $- \varepsilon$] with $\varepsilon > 0$ sufficiently s *f* $\left[\begin{array}{c} T \ -N_n^{-1} \alpha_n^2 \int \exp \left(-c_n^2 s\right) dy(s) \right] = 0$
 5 $\left[\begin{array}{c} -\varepsilon \end{array}\right]$ with $\varepsilon > 0$ sufficiently small, where we have int (*s*) $\left[\begin{array}{c} y_2(s), v_n(x) := \cos \left(c_n x\right), v^0(x) := w^0(T, x) \right]$

(*v*_n, *v*⁰) = $\alpha_n \int \exp \left(c_n^$

$$
\langle v_n, v^0 \rangle = \alpha_n \int_{b-\epsilon}^T \exp\left(c_n^2(T-s)\right) dy(s), \qquad n = 1, 2, \ldots \tag{4.9}
$$

$$
\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = +\infty \tag{4.10}
$$

 ${\rm (contradicting } v^0(\cdot) \in L_2(0,1)),$ unless $y(t) \equiv y(a)$ on the whole interval [a, T]. Then, however, we get $\langle v_n, v^0 \rangle = 0$ from (4.9) for all $n = 1, 2, ...$ and hence $v^0(x) = w^0(T, x)$ $-z(x) = 0$ a.e. on [0, *T*], as the system {cos $(c_n x)$ } is complete in $L_2(0, 1)$. This is a contradiction to the assumptions of the theorem; thus the statement must be $\begin{array}{c} \text{We have} \\\\ \text{We have} \[\text{con} \] \[\text{conv} \leftarrow \text{z}(\text{cont} \] \[\text{true} \] \[\text{true$ ve will show that $(* \cdot \cdot \cdot)$ implies
 $\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = +\infty$

(contradicting $v^0(\cdot) \in L_2(0, 1)$), unless $y(t) = y(a)$ on the

however, we get $\langle v_n, v^0 \rangle = 0$ from (4.9) for all $n = 1, 2$,
 $-z(x) = 0$ a.e. on [0, T], We will show that (4.9) implies
 $\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = +\infty$

(contradicting $v^0(\cdot) \in L_2(0, 1)$), unless $y(t) \equiv y(a)$ or

however, we get $\langle v_n, v^0 \rangle = 0$ from (4.9) for all $n = 1$
 $-z(x) = 0$ a.e. on [0, *T*], as the sys

Therefore, we suppose finally that $y(t)$ is not identically constant on $[0, T]$, and it remains to verify (4.10) in this case.

$$
t_0 := \{\sup t \mid y_1(t) = y_1(a) \wedge y_2(t) = y_2(a)\}.
$$

We can assume that $y_1(t)$ and $y_2(t)$ are continuous from the right and introduce the jump $h := y(t_0) - y(t_0 - 0)$ of $y(t)$ in t_0 . Two cases can arise:

a) $t_0 = T$. Then $h^2 > 0$, and (4.9) implies

 \mathbf{v}

It remains to verify (4.10) in this case.
\nAs
$$
y_1
$$
 and y_2 are monotone non-decreasing, there exists
\n $t_0 := \{ \sup t \mid y_1(t) = y_1(a) \land y_2(t) = y_2(a) \}$.
\nWe can assume that $y_1(t)$ and $y_2(t)$ are continuous from the right and introduce the
\nump $h := y(t_0) - y(t_0 - 0)$ of $y(t)$ in t_0 . Two cases can arise:
\na) $t_0 = T$. Then $h^2 > 0$, and (4.9) implies
\n
$$
\sum_{n=1}^{\infty} \langle v_n, v^0 \rangle^2 = \sum_{n=1}^{\infty} \alpha_n^2 h^2 = +\infty
$$
\n(since $\alpha_n \sim (-1)^n$ for $n \to \infty$), i.e. (4.10) holds.
\nb) $t_0 < T$. As $u^0(t)$ is piecewise continuous, there is an $\eta > 0$ such that $u^0(t)$ is
\ncontinuous on $(t_0, t_0) \neq 0$ with the point t_0 can be one of discontinuity). Therefore, one

b) $t_0 < T$. As $u^0(t)$ is piecewise continuous, there is an $\eta > 0$ such that $u^0(t)$ is continuous on $(t_0, t_0 + 2\eta)$ (the point t_0 can be one of discontinuity). Therefore, one As y_1 and y_2 are monotone non-decreasing, there exists
 $t_0 := {\sup l | y_1(l) = y_1(a) \land y_2(l) = y_2(a)}$.

We can assume that $y_1(l)$ and $y_2(l)$ are continuous from the right and introduce the

jump $h := y(l_0) - y(l_0 - 0)$ of $y(l)$ i *Propertionally the dimagret and figure of the of the dimagretic inclusion of* $y(t)$ *in* t_0 *. Two cases can arise:***
** $\sum_{n=1}^{\infty} (v_n, v^0)^2 = \sum_{n=1}^{\infty} \alpha_n^2 h^2 = +\infty$ **

(since** $\alpha_n \sim (-1)^n$ **for** $n \to \infty$ **), i.e. (4.10) holds.**

$$
-c + \int_{0}^{t} k(t,s) u^{0}(s) ds \leq u^{0}(t) \leq c + \int_{0}^{t} k(t,s) u^{0}(s) ds,
$$
 (4.11)

On Duality and the Maximum Princip

say the left one, must be strict on $(t_0, t_0 + 2\eta)$, if η is sufficiently small

Corollary 3, $y_2(t)$ remains constant on $(t_0, t_0 + 2\eta]$, and we obtain in turn say the left one, must be strict on $(t_0, t_0 + 2\eta)$, if η is sufficiently small. Now, by

On Duality and the Maximum Principle 535
\nsay the left one, must be strict on
$$
(t_0, t_0 + 2\eta)
$$
, if η is sufficiently small. Now, by
\nCorollary 3, $y_2(t)$ remains constant on $(t_0, t_0 + 2\eta)$, and we obtain in turn
\n
$$
\int_{t_0+t_1}^{t_1} \exp\left(c_n^2(T-s)\right) dy(s) = \exp\left(c_n^2(T-t_0)\right) h + \int_{t_0}^{t_1} \exp\left(c_n^2(T-s)\right) dy(s)
$$
\n
$$
= \int_{t_0+t_1}^{t_1+t_2} \exp\left(c_n^2(T-s)\right) dy_1(s) + \int_{t_0+t_2}^{T} \exp\left(c_n^2(T-s)\right) dy(s).
$$
\nIf $h \neq 0$, then we choose η so small that $y_1(t_0 + \eta) - y_1(a) < |h|/2$ and find
\n
$$
\left|\int_{t_0-t}^{T} \right| \ge \exp\left(c_n^2(T-t_0)\right) \cdot (|h|/2 + o(c_n^2)).
$$
\nIf $h = 0$, then
\n
$$
\int_{t_0-t}^{T} \ge \exp\left(c_n^2(T-t_0 - \eta)\right) \cdot \left(y_1(t_0 + \eta) - y_1(a) + o(c_n^2)\right).
$$
\nIn both cases, (4.10) is easily obtained. If the right inequality of (4.11) is strict, then
\nthe proof is analogous
\nThus, if $u^0(t)$ is not too irregular, then [0, T] can be divided into countably (or
\neven finitely) many open intervals where one of the equations $u^0(t) = 0$, $u^0(t) = 1$,
\n $u^0(t) - u^0(t, 1) = -c$, $u^0(t) - w^0(t, 1) = c$ is fulfilled. These facts can be used to

If $h \neq 0$, then we choose η so small that $y_1(t_0 + \eta) - y_1(a) < |h|/2$ and find,

$$
\int_{b-\epsilon}^{T} \left| \geq \exp\left(c_n^2(T-t_0)\right) \cdot \left(|h|/2 + o(c_n^2)\right).
$$

If $h=0$, then

$$
\int\limits_{-t}^T\geq \exp\big(c_n^2(T-t_0-\eta)\big)\cdot\big(y_1(t_0+\eta)-y_1(a)+o(c_n^2)\big).
$$

If $h =$
If $h =$
 \therefore
In both In both cases, (4.10) is easily obtained. If the right inequality of (4.11) is strict, then

Thus, if $u^0(t)$ is not too irregular, then [0, T] can be divided into countably (or even finitely) many open intervals where one of the equations $u^0(t) = 0$, $u^0(t) = 1$, $u^0(t) - w^0(t, 1) = -c$, $u^0(t) - w^0(t, 1) = c$ is fulfilled. These facts can be used to construct a numerical method for the solution of $(4.1) - (4.4)$ along the lines of [7].

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