Zeitschrift für Analysis
und ihre Anwendungen
Bd. 4 (6) 1985, S. 543 – 555

Generalized Resolvents of an Isometric Operator in a Pontrjagin Space

P. SORJONEN

In dieser Arbeit werden kontrahierende und verallgemeinerte Resolventen eines isometrischen Operators im Pontrjagin-Raum untersucht. Der Definitionsbereich des Operators darf sogar entartet sein. Die dann herzuleitenden Formeln sind analog denjenigen aus dem nichtentarteten Fall, die Werte des charakteristischen Parameters sind aber jetzt lineare Relationen anstelle linearer Operatoren.

В этой работе исследуются сжимающие и обобщенные резольвенты изометрического оператора в пространстве Понтрягина. Область определения оператора может быть вырожденной. Выведенные формулы аналогичны тем в невырожденном случае, однако как значения характеристического параметра выступают теперь линейные отношения вместо линейных операторов.

In this paper the contractive and generalized resolvents of an isometric operator in a Pontrjagin space are studied. Especially, the domain of the operator is allowed to be degenerate. The resulting formulae are analogous to the non-degenerate case, but the values of the characterising parameter are now linear relations instead of linear operators.

Introduction

As noticed in the introduction of [11] the studies of a canonical differential equation with an indefinite weight function may lead one to consider not only defined symmetric operators but also non-densily defined symmetric operators or even linear relations in an indefinite inner product space. In this case it can happen that the defect spaces of the operator or linear relation under consideration are not necessarily non-degenerate. This in turn means that its Cayley transform, which in any case is an isometric operator, has a degenerate domain. Thus in order to be able to use the usual method of investigating symmetric operators or relations via their Cayley transforms one must first study isometric operators with degenerate domains.

In this paper we characterize the generalized resolvents of a closed injective isometric operator with equal defect numbers in a Pontrjagin space (for the terminology see below). In order to clarify this result let us recall the essence of Satz 4.1 from [6]: If V is a closed injective isometric operator with equal defect numbers and with non-degenerate domain and range in a Pontrjagin space, then all generalized resolvents of V are given by the formula

$$R(z) = (I - zU)^{-1} + \Gamma_{1/z} P(z) \Gamma_{\bar{z}}^{+}, \tag{*}$$

where

$$P(z) = \big(I - E(z)\big) \big(I - X(\bar{z})^{+} E(z)\big)^{-1} \big(I - X(\bar{z})\big); \tag{\#}$$

here U is a given unitary extension of V in the original space, Γ is a certain operator-valued function defined in $\varrho(U)$ and X is the characteristic function of V. The parameter E corresponding to the generalized resolvent R is defined in the open unit disk $\mathbf{C}_0$ of the complex plane $\mathbf{C}$ and its values are contractive operators in the defect space of V. Thus the value set of E is inside the operator unit ball in case of positive defect space and outside the ball in case of negative defect space.

If the defect space of the operator V degenerates the proof of the result mentioned above does not hold and furthermore (#) loses its meaning. In considering the formula (*) for the degenerate case it turned out that the values of the parameter E can be linear relations, i.e. "multi-valued" operators, and that a natural value set for E is not the operator unit ball but the right operator or linear relation half plane: $E(z)$ is an accretive linear relation.

The modification of the characterizing parameter E in the formula (#) forced us to modify also the characteristic function X of the operator V. The substitute for X is so-called θ-function θ, which will be studied more closely elsewhere. After the above mentioned modifications it was found out that the basic formula (*) characterising the generalized resolvents of the isometric operator V holds also in the degenerate case but instead of the function P in (#) one has now

$$P(z) = \big(\theta(z) + E(z)\big)^{-1};$$

see Theorem 3.6.

Although it would be possible to prove the above mentioned result directly along the same lines as in [6], we use here a different approach. We first characterize in Chapter 1 the contractive and unitary extensions of an isometric operator. With the help of that result we prove in Chapter 2 another representation formula for the generalized resolvents; see Theorem 2.1. Finally, in Chapter 3 we characterize in the above mentioned way not only the generalized resolvents of an isometric operator but also contractive and unitary resolvents.

This work was initiated while the author was visiting the Technical University Dresden. The author wishes to thank professor H. LANGER for stimulating discussions.

Notation and terminology

We represent here briefly the notation and terminology used in this paper. For more extensive information about the results concerning Pontrjagin spaces we refer to [2] and [5].

Throughout this paper $\mathfrak{H}$ denotes a $\pi_\varkappa$*-space* or a *Pontrjagin space* (with $\varkappa$ negative squares), that is, $\mathfrak{H}$ is a (complex) linear space equipped with a scalar product (indefinite inner product) $[\cdot \mid \cdot]$, which has $\varkappa$ *negative squares*. The last property means that the space $\mathfrak{H}$ admits a decomposition in an orthogonal direct sum $\mathfrak{H} = \mathfrak{H}_+ [\dot{+}] \mathfrak{H}_-$, where $\mathfrak{H}_+$ with $[\cdot \mid \cdot]$ is a Hilbert space and $\mathfrak{H}_-$ with $-[\cdot \mid \cdot]$ is a $\varkappa$-dimensional Hilbert space. Note that the Pontrjagin space $\mathfrak{H}$ is also a Hilbert space with respect to the inner product $(f, g) \to (f \mid g)$:

$$(f \mid g) := [f_+ \mid g_+] - [f_- \mid g_-], \qquad f_\pm, g_\pm \in \mathfrak{H}_\pm .$$

All topological notions are to be understood with respect to the norm topology induced by this positive definite inner product.

A vector f in the Pontrjagin space $\mathfrak{H}$ is called *positive* (*negative, neutral*), if $[f \mid f] > 0$ (< 0, $= 0$). An analogous definition holds for a subset of $\mathfrak{H}$. Orthogonality in $\mathfrak{H}$ is defined in the usual way: vectors f and g in $\mathfrak{H}$ are *orthogonal* if $[f \mid g] = 0$. The *orthogonal companion* of a subset $\mathfrak{M}$ is denoted by $\mathfrak{M}^\perp$.

In a Hilbert space a subspace $\mathfrak{M}$ is always *orthocomplemented*: $\mathfrak{M} \dotplus \mathfrak{M}^{\perp} = \mathfrak{H}$. In a Pontrjagin space this happens if and only if the subspace is *non-degenerate.* This means that the *isotropic part* $\mathfrak{M}^0 := \mathfrak{M} \cap \mathfrak{M}^{\perp}$ of $\mathfrak{M}$ is zero. In general case one can form the following decomposition for a degenerate subspace $\mathfrak{M}$:

$$\mathfrak{H} = \mathfrak{M}_1[\dotplus]\,\mathfrak{M}_2[\dotplus]\,(\mathfrak{M}^0 \dotplus \mathfrak{M}_3).$$

Here $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are subspaces with the properties

$$\mathfrak{M}^0\,[\dotplus]\,\mathfrak{M}_1 = \mathfrak{M}, \qquad \mathfrak{M}^0\,[\dotplus]\,\mathfrak{M}_2 = \mathfrak{M}^{\perp}.$$

The subspace $\mathfrak{M}_3$ is skewly linked with $\mathfrak{M}^0$, i. e. $\mathfrak{M}_3 \cap \mathfrak{M}^{0\perp} = \mathfrak{M}_3{}^{\perp} \cap \mathfrak{M}^0 = \{0\}$; see [2: Theorem IX.2.5].

An operator V in the Pontrjagin space $\mathfrak{H}$ is called *contractive* if $[Vf \mid Vf] \leqq [f \mid f]$ for all f in the domain $\mathfrak{D}(V)$. The contractive operator V is *isometric* if the equality-sign holds everywhere. An everywhere defined isometric operator V is called *semi-unitary*, and *unitary* if in addition the range $\mathfrak{R}(V)$ is the whole space.

If we extend a given operator V in the π_κ-space $\mathfrak{H}$ to an operator U acting in a π_κ-space $\mathfrak{K}$ extending $\mathfrak{H}$, the extension U is called *regular*. Furthermore, U is said to be a *dilation* of V if $V = PU|_{\mathfrak{H}}$, where P is the orthogonal projector of $\mathfrak{K}$ onto $\mathfrak{H}$. If the extension space $\mathfrak{K}$ equals to the original space $\mathfrak{H}$ the extension is *canonical.*

Let V be a closed injective isometric operator in the Pontrjagin space $\mathfrak{H}$. If a bounded operator W in $\mathfrak{H}$ is a contractive extension of V the mapping $z \to (I - zW)^{-1}$, $z^{-1} \in \varrho(W)$, is called a *contractive resolvent* of the operator V. Analogously one can define *semiunitary* and *unitary resolvents* of V. Furthermore, if a unitary operator U acting in an extension space $\mathfrak{K}$ is a regular extension of V then the mapping $z \to R(z)$:

$$R(z) := P(I - zU)^{-1}|_{\mathfrak{H}}, \qquad z \in \varrho(U^{-1}),$$

is a (*regular*) *generalized resolvent* of the operator V. In case of a canonical unitary extension the generalized resolvent is also called *canonical.*

Along with the Pontrjagin space $\mathfrak{H}$ the product space $\mathfrak{H}^2$ equipped with the natural structure inherited from $\mathfrak{H}$ is also a Pontrjagin space. A linear manifold of $\mathfrak{H}^2$ is called a *linear relation.* Identifying operators with their graphs we see that a linear relation T is an operator if and only if the image $T(0) = \{0\}$. The set $T_\infty := \{0\} \times T(0)$ is the *multivalued part* of T and $T_s := T \cap T_\infty^{\perp}$ is the *operator part* of T. Analogous to the operator case one can define notions like *symmetric* or *contractive linear relation.* Especially, a linear relation T is called *accretive* if $\operatorname{Re}\,[g \mid f] \geqq 0$ for all $(f, g) \in T$. If the equality holds for all $(f, g) \in T$, then T is said to be *conservative.*

1. Contractive extensions of an isometric operator

The aim of this chapter is to characterize contractive operator extensions of a given isometric operator in a Pontrjagin space. We start by proving their existence. For this the following result is useful:

Proposition 1.1: *A bounded operator on a Pontrjagin space is contractive if and only if it has a regular unitary dilation.*

Proof: Let W be a bounded operator on a π_κ-space $\mathfrak{H}$. If W is contractive, it has a regular unitary dilation; see [10: Satz 4.5]. Conversely, let U be a regular dilation of W, i.e., U is a unitary operator on a π_κ-space $\mathfrak{K}$ extending $\mathfrak{H}$ and $W = PU|_{\mathfrak{H}}$, where P denotes the orthogonal projector of $\mathfrak{K}$ onto $\mathfrak{H}$. Then

$$[Wf \mid Wf] = [Uf \mid Uf] - [(I - P)\,Uf \mid Uf] \leqq [f \mid f]$$

for all f in $\mathfrak{H}$, because the orthogonal companion of $\mathfrak{H}$ in $\mathfrak{K}$ is positive definite ∎

Proposition 1.2: *A closed injective isometric operator in a Pontrjagin space admits regular unitary extensions.*

Proof: Let V be a closed injective isometric operator in a $\pi_\varkappa$-space $\mathfrak{H}$. By [5: § 9] we can suppose that V is maximal, i.e., $\dim \mathfrak{D}(V)^\perp = 0$, for instance. Choose an infinite-dimensional Hilbert space $\mathfrak{G}$ with $\dim \mathfrak{G} \geqq \dim \mathfrak{R}(V)^\perp$, and form the $\pi_\varkappa$-space $\mathfrak{K} := \mathfrak{H} \oplus \mathfrak{G}$. In this space the defect numbers of V are equal and consequently V admits unitary extensions there; see [5: § 9] ∎

By combining the two previous propositions we see that every closed injective isometric operator in a Pontrjagin space has bounded contractive operator extensions. Below we shall characterize them. First, a useful lemma.

Lemma 1.3: *Let W be a contractive operator in a Pontrjagin space. If $[Wf_0 \mid Wf_0] = [f_0 \mid f_0]$ for f_0 in $\mathfrak{D}(W)$, then $[Wf_0 \mid Wf] = [f_0 \mid f]$ for all f in $\mathfrak{D}(W)$.*

The proof is the same as in the definite case; see [12: Lemma 1.1] ∎

Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$. The domain $\mathfrak{D}(V)$ of V induces a decomposition of the whole space:

$$\mathfrak{H} = \mathfrak{D}_1 [\dot{+}] \mathfrak{D}_2 [\dot{+}] (\mathfrak{D}_0 \dot{+} \mathfrak{D}_3); \tag{1.1}$$

see [2: Theorem IX.2.5]. Here $\mathfrak{D}_0 := D(V)^0$ is the isotropic part of $\mathfrak{D}(V)$, $\mathfrak{D}_3$ is a neutral subspace skewly linked with $\mathfrak{D}_0$, and $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are orthocomplemented subspaces such that

$$\mathfrak{D}(V) = \mathfrak{D}_1 [\dot{+}] \mathfrak{D}_0, \qquad \mathfrak{D}(V)^\perp = \mathfrak{D}_2 [\dot{+}] \mathfrak{D}_0.$$

Setting $\mathfrak{R}_0 := \mathfrak{R}(V)^0 = V(\mathfrak{D}_0)$ and $\mathfrak{R}_1 := V(\mathfrak{D}_1)$ we get an analogous decomposition

$$\mathfrak{H} = \mathfrak{R}_1 [\dot{+}] \mathfrak{R}_2 [\dot{+}] (\mathfrak{R}_0 \dot{+} \mathfrak{R}_3) \tag{1.2}$$

with the components having similar properties to those above. In the following we keep these decompositions fixed.

Theorem 1.4: *Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$ with the decompositions* (1.1) *and* (1.2).

If $W \in \mathsf{B}(\mathfrak{H})$ is a contractive extension of V, then there exists an operator W' with the properties

(i) *W' is closed and contractive, its domain is $\mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3$ and its range is in $\mathfrak{R}_1{}^\perp$;*

(ii) *the inequality*

$$2 \operatorname{Re} [Vf_0 \mid W'f] + [W'f \mid W'f] \leqq 2 \operatorname{Re} [f_0 \mid f] + [f \mid f] \tag{1.3}$$

holds for all f_0 in $\mathfrak{D}_0$ and f in $\mathfrak{D}(W')$;

(iii) $W = V \dot{+} W'$.

Conversely, if an operator W' satisfies (i) *and* (ii), *then the formula in* (iii) *defines a contractive extension $W \in \mathsf{B}(\mathfrak{H})$ of V.*

For the proof of the first part, define W' as the restriction of W to $\mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3$. By using Lemma 1.3 it is easy to see that this W' has the desired properties. For the converse, note that $V \dot{+} W'$ is closed by [7: Lemma 4.1]. The verification of the other claims is a straightforward calculation ∎

Remarks: 1° It is obvious that basically the same result also holds for not necessarily everywhere defined closed contractive extensions. Thus Theorem 1.4 extends [12: Lemma 1.2] to Pontrjagin spaces and [5: § 9.2] to contactive extensions.

2^0 For a non-degenerate domain $\mathfrak{D}(V)$ the sum $V + W'$ is orthogonal; but this is not necessarily true for a degenerate domain. For example, take a basis $\{e_1, e_2\}$ of $\mathbf{C}^2$ and make $\mathbf{C}^2$ a Pontrjagin space by defining

$$[e_j \mid e_k] := \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k = 1, \\ -1, & \text{if } j = k = 2. \end{cases}$$

Define further V and W' as the identity operator of $\langle e_1 + e_2 \rangle$ and $\langle e_1 - e_2 \rangle$, resp. Then they satisfy the condition of Theorem 1.4 but they are not orthogonal.

3^0 If the domain $\mathfrak{D}(V)$ is non-degenerate, then the inequality (1.3) is equivalent to W' being contractive. In the general case, being contractive or even isometric does not imply the inequality (1.3). For example, let the Pontrjagin space and the operator V be as in 2^0, and define $W'(e_1 - e_2) := e_1 + e_2$. Then W' is isometric, and (1.3) is equivalent to the condition $\operatorname{Re}[f_0 \mid f] \geqq 0$ for all f_0 in $\mathfrak{D}_0$ and f in $\mathfrak{D}(W')$. But $\operatorname{Re}[e_1 + e_2 \mid e_2 - e_1] < 0$; consequently (1.3) is not true and the operator $V + W'$ is not contractive.

Corollary 1.5: *In the correspondence $W \leftrightarrow W'$ given by Theorem 1.4 W is semi-unitary if and only if W' has the properties*

(i) *W' is isometric with the domain $\mathfrak{D}_2 [\dotplus] \mathfrak{D}_3$;*

(ii) $\operatorname{Re}[Vf_0 \mid W'f] = \operatorname{Re}[f_0 \mid f]$ *for all f_0 in $\mathfrak{D}_0$ and f in $\mathfrak{D}(W')$.*

Furthermore, W is unitary if and only if W' has the properties (i), (ii) *and* $\mathfrak{R}(W') + \mathfrak{R}_0 = \mathfrak{R}_1{}^{\perp}$.

2. Generalized resolvents of an isometric operator

Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$ and let R be a (regular) generalized resolvent of V, i.e.

$$R(z) = P(I - zU)^{-1}|_{\mathfrak{H}} \quad (z \in \varrho(U^+)),$$

where U is a regular unitary extension of V in an extension space $\mathfrak{K}$ and P is the orthogonal projector of $\mathfrak{K}$ onto $\mathfrak{H}$. Then R has the following properties; see [6: § 4]:

1. R is meromorphic outside the unit circle;

2. $R(z)$ has a bounded inverse for almost all z in the interior $\mathbf{C}_0$ of the unit circle, i.e., for all z in $\mathbf{C}_0$ with the possible exception of a countable set which does not have any cluster points in $\mathbf{C}_0$;

3. $R(z) = I - R(1/\bar{z})^+$, $z \in \varrho(U^+) \setminus \{0\}$.

The last property implies that we usually need only to consider the case when z is in $\mathbf{C}_0$.

The following result extends [3: Theorem 3] to Pontrjagin spaces.

Theorem 2.1: *Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$ with the decompositions* (1.1) *and* (1.2). *A mapping R is a regular generalized resolvent of V if and only if it has a representation*

$$R(z) = \{I - z[V \dotplus \Phi(z)]\}^{-1} \tag{2.1}$$

for almost all z in $\mathbf{C}_0$, where Φ is meromorphic in $\mathbf{C}_0$ and holomorphic in zero with values in $\mathbf{B}(\mathfrak{D}_2 [\dotplus] \mathfrak{D}_3; \mathfrak{R}_1{}^{\perp})$, and

$$2 \operatorname{Re}[Vf_0 \mid \Phi(z) f] + [\Phi(z) f \mid \Phi(z) f] \leqq 2 \operatorname{Re}[f_0 \mid f] + [f \mid f]$$

for all f_0 in $\mathfrak{D}_0$ and f in $\mathfrak{D}_2 [\dotplus] \mathfrak{D}_3$.

Proof: 1⁰ Let R be a regular generalized resolvent of V. Then the mapping $z \mapsto W(z)$,

$$W(z) := z^{-1}\big(I - R(z)^{-1}\big) \quad \text{for almost all } z \text{ in } \mathbf{C}_0,$$

has the following properties (see [6: Lemma 4.1] and [8: Lemma 2.3]:

1. W is meromorphic in $\mathbf{C}_0$;
2. W is holomorphic in zero;
3. $W(z)$ $\big(\in \mathsf{B}(\mathfrak{H})\big)$ is contractive for almost all z in $\mathbf{C}_0$;
4. $W(z)$ extends V and $W(z)^+$ extends V^{-1} for almost all z in $\mathbf{C}_0$.

By Theorem 1.4 there exists a mapping $\Phi(z) \in \mathsf{B}(\mathfrak{D}_2\,[\dot{+}]\,\mathfrak{D}_3; \mathfrak{R}_1{}^\perp)$ such that $W(z) = V \dot{+} \Phi(z)$. Using the definition and the properties of W it is easy to see that R has the representation (2.1) and Φ has the desired properties.

2⁰ In order to prove the converse we must construct an extension space $\mathfrak{K} \supset \mathfrak{H}$ and a unitary extension U of V such that the corresponding generalized resolvent R equals the mapping $z \mapsto S(z)$,

$$S(z) := \{I - z[V \dot{+} \Phi(z)]\}^{-1}, \tag{2.2}$$

for all z in an open set in $\mathbf{C}_0$. This can be done in a similar way as in [6: § 4.5]. So it is enough to give an outline of the proof.

Let $\mathcal{U}_0 \subset \mathbf{C}_0$ be an open neighbourhood of zero, symmetric with respect to the real axis and such that $S(z)$ exists as a bounded operator for all z in $\mathcal{U}_0$. Extend S to the set $\mathcal{U}_\infty := \{z \mid 1/z \in \mathcal{U}_0\}$ by setting $S(z) := I - S(1/\bar{z})^+$. We use the notation $\mathcal{U} := \mathcal{U}_0 \cup \mathcal{U}_\infty$ and $W(z) := V \dot{+} \Phi(z)$ so that $S(z) = [I - zW(z)]^{-1}$.

Define a kernel $H : \mathcal{U} \times \mathcal{U} \to \mathsf{B}(\mathfrak{H})$ by

$$K(z, \zeta) := (1 - \bar{z}\zeta)^{-1}\big(S(z)^+ + S(\zeta) - I\big) \qquad (z, \zeta \in \mathcal{U}).$$

Then this kernel has the representation

$$K(z, \zeta) = K_1(z, \zeta) + S(z)^+ S(\zeta) \qquad (z, \zeta \in \mathcal{U}), \tag{2.3}$$

where K_1 has the following form:

$$K_1(z, \zeta) = \bar{z}\zeta S(z)^+ \frac{I - W(z)^+ W(\zeta)}{1 - \bar{z}\zeta} S(\zeta) \qquad (z, \zeta \in \mathcal{U}_0);$$

$$K_1(z, \zeta) = \bar{z} S(z)^+ \frac{W(z)^+ - W(1/\bar{\zeta})^+}{1 - \bar{z}\zeta} S(1/\bar{\zeta})^+ \qquad (z \in \mathcal{U}_0,\ \zeta \in \mathcal{U}_\infty);$$

$$K_1(z, \zeta) = S(1/\bar{z}) \frac{W(1/\bar{z})\, W(1/\bar{\zeta})^+ - I}{1 - \bar{z}\zeta} S(1/\bar{\zeta})^+ \qquad (z, \zeta \in \mathcal{U}_\infty).$$

To verify these, put $K_1(z, \zeta) := K(z, \zeta) - S(z)^+ S(\zeta)$, use the definitions of K and S, and calculate; cf. [6: pp. 393—394]. The same method as in the proof of Lemma 2.7 in [8] yields the result that K_1 is a positive definite kernel. Now one can easily prove that the $\mathfrak{H}$-kernel K has as many negative squares as the inner product of $\mathfrak{H}$.

By [10: Satz 3.3] there exists a Pontrjagin space $\mathfrak{H}'$ extending $\mathfrak{H}$ such that K is the reproducing kernel of this space, i.e.,

$$[f(z) \mid u] = [f \mid K(\cdot, z)\, u] \qquad (f \in \mathfrak{H}',\ u \in \mathfrak{H},\ z \in \mathcal{U}).$$

This space $\mathfrak{H}'$ is the completion of the space $\mathfrak{L}$, which consists of all mappings $f : \mathcal{U} \to \mathfrak{H}$ with a representation

$$f = \sum_{k=1}^{n} K(\cdot, z_k)\, u_k \qquad (z_k \in \mathcal{U},\ u_k \in \mathfrak{H});$$

the inner product of $\mathfrak{L}$ is given by the formula

$$[f \mid f] := \sum_{k,j=1}^{n} [K(z_j, z_k)\, u_k \mid u_j].$$

With the aid of the mapping $u \mapsto K(\cdot, 0)u$ we can regard $\mathfrak{H}$ as a subspace of $\mathfrak{H}'$. Define

$$U := \left\{\left(\sum_{k=1}^{n} K(\cdot, z_k)\, u_k,\ \sum_{k=1}^{n} z_k^{-1}[K(\cdot, z_k) - K(\cdot, 0)]\, u_k\right) \,\middle|\, z_k \neq 0\right\};$$

then U is a linear relation in $\mathfrak{H}'$, and a straightforward calculation shows that it is isometric. As $\lim_{z\to 0} K(\cdot, z)\, u = u$ and $\lim_{z\to\infty} K(\cdot, z)\, u = 0$, it follows that U is densely defined and the range is also dense. Consequently, U must be an operator; see [11: Proposition 2.1.2]. By [2: Theorem IX.3.1] we can continue U so that it becomes a unitary operator on $\mathfrak{H}'$.

To prove that U extends V one must show the relation

$$\lim_{z\to 0} z^{-1}[K(\cdot, z) - K(\cdot, 0)]\, u = Vu.$$

For this one can use the relations $V = S(z)^+ V - \bar{z}S(z)^+$ and (see [6: p. 396])

$$Vu = \lim_{z\to 0} z^{-1}\big(S(z) - I\big)\, u \qquad \big(u \in \mathfrak{D}(V)\big);$$

To finish the proof we must show the relation

$$P(I - zU)^{-1}\, u = S(z)\, u \qquad (z \in \mathcal{U}_0,\ u \in \mathfrak{H}), \tag{2.4}$$

where P is the orthogonal projector of $\mathfrak{H}'$ onto $\mathfrak{H}$. It is easy to see that the mapping $(z, \zeta) \mapsto K(z, 0)$ is the reproducing kernel of $\mathfrak{H}$. Consequently, the projector P has the form

$$PK(\cdot, z)\, u = K(0, z)\, u = S(z)\, u \qquad (z \in \mathcal{U}_0,\ u \in \mathfrak{H});$$

see [10: Satz 2.7]. This together with the definition of U gives (2.4) ∎

From this theorem and Corollary 1.5 we get

Corollary 2.2: *Let the assumptions of Theorem 2.1 be satisfied. A regular generalized resolvent R of V is canonical if and only if the corresponding mapping Φ is independent of z, and $\Phi_c := \Phi(z)$ has the properties*

(i) *Φ_c is isometric;*
(ii) $\mathfrak{R}(\Phi_c) + \mathfrak{R}_0 = \mathfrak{R}_1{}^{\perp}$;
(iii) $\operatorname{Re}[Vf_0 \mid \Phi_c f] = \operatorname{Re}[f_0 \mid f]$ *for all f_0 in $\mathfrak{D}_0$ and f in $\mathfrak{D}_2\,[\dot{+}]\,\mathfrak{D}_3$.*

3. Contractive resolvents of an isometric operator

Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$. The aim of this chapter is to characterize the contractive resolvents of V. As corollaries we get characterizations for semiunitary and unitary resolvents. Finally, we achieve a representation theorem for the generalized resolvents of V. Note that Theorem 1.4 implies immediately one such characterization: A mapping R is a contractive or (semi)unitary resolvent of V if and only if it has a representation

$$R(z) = [I - z(V \dot{+} W')]^{-1},$$

where W' has the properties mentioned in Theorem 1.4 or in Corollary 1.5, resp. We shall use Theorem 1.4 in order to get another characterization which is more in the spirit of [6: Satz 4.1].

Throughout this chapter we suppose that the defect numbers of the operator V are equal, i.e., $\dim \mathfrak{D}(V)^\perp = \dim \mathfrak{R}(V)^\perp$. Furthermore, we keep the decompositions (1.1) and (1.2) fixed. We use the notation $\mathfrak{R}_z := \mathfrak{R}(I - zV)$ and $\mathfrak{N}_z := \mathfrak{R}_{\bar z}{}^\perp$. Let P_0 denote the projector of $\mathfrak{H}$ onto $\mathfrak{D}(V)^\perp$ along $\mathfrak{D}_1 [\dot+] \mathfrak{D}_3$. Then $P_0{}^+$ is the projector of $\mathfrak{H}$ onto $\mathfrak{D}_2 [\dot+] \mathfrak{D}_3$ along $\mathfrak{D}(V)$; thus $\mathfrak{D}(V)^\perp$ and $\mathfrak{D}_2 [\dot+] \mathfrak{D}_3$ form a dual pair with respect to $[\cdot \mid \cdot]$.

Let U be a fixed canonical unitary extension of V and W $(\in \mathsf{B}(\mathfrak{H}))$ an arbitrary contractive extension of V; denote by R_0 and R the corresponding resolvents of U and W.

Lemma 3.1: 1° *$R(z)$ maps $\mathfrak{R}(V)^\perp$ homeomorphically onto $\mathfrak{N}_{1/z}$ $(1/z \in \varrho(W))$.*
2° *$R(z)^+$ maps $\mathfrak{D}(V)^\perp$ homeomorphically onto $\mathfrak{N}_{\bar z}$ $(1/z \in \varrho(W))$.*
3° *$\mathfrak{R}(R(z) - R_0(z)) \subset \mathfrak{N}_{1/z}$ $(1/z \in \varrho(W) \cap \varrho(U))$.*
4° *$\mathfrak{N}(R(z) - R_0(z)) \supset \mathfrak{R}_z$ $(1/z \in \varrho(W) \cap \varrho(U))$.*

The proof is a straightforward calculation; cf. [6: § 4] and [8] ∎

In order to get the desired characterization of R we consider the difference $(1/z \in \varrho(W) \cap \varrho(U))$

$$\begin{aligned} R(z) - R_0(z) &= -zR(z)\,U(I - U^+W)\,R_0(z) \\ &= R_0(\bar z^{-1})^+ P_0(I - zU)\,U^{-1}\{I - zW\}^{-1}\,U(I - U^+W)\,P_0{}^+R_0(z) \\ &= R_0(\bar z^{-1})^+ P_0 \left\{(I - U^+W)\left(R_0(z) - \frac{1}{2} I\right)\right. \\ &\quad \left. + \frac{1}{2}(I + U^+W)\right\}^{-1} (I - U^+W)\,P_0{}^+R_0(z); \end{aligned}$$

see Lemma 3.1. Define

$$S := (I - U^+W)|_{\mathfrak{D}_2+\mathfrak{D}_3};$$

then $S \in \mathsf{B}(\mathfrak{D}_2 [\dot+] \mathfrak{D}_3; \mathfrak{D}(V)^\perp)$, and a simple calculation shows that

$$\frac{1}{2} S^{-1}(I + U^+W)_{\mathfrak{D}(V)^\perp} = P_0{}^+\left(S^{-1} - \frac{1}{2} I\right).$$

Note that S^{-1} is not necessarily an operator but a linear relation; for the calculus of linear relations, see [1, 4, 11]. Define temporarily

$$T := \left[(I - U^+W)\left(R_0(z) - \frac{1}{2} I\right) + \frac{1}{2}(I + U^+W)\right]\Bigg|_{\mathfrak{D}(V)^\perp};$$

then

$$\begin{aligned} T^{-1}S &= \left\{S^{-1}\left[SP_0{}^+\left(R_0(z) - \frac{1}{2} I\right)\Big|_{\mathfrak{D}(V)^\perp} + \frac{1}{2}(I + U^+W)|_{\mathfrak{D}(V)^\perp}\right]\right\}^{-1} \\ &= \left\{P_0{}^+\left(R_0(z) - \frac{1}{2} I\right)\Big|_{\mathfrak{D}(V)^\perp} + P_0{}^+\left(S^{-1} - \frac{1}{2} I\right)\right\}^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} R(z) = R_0(z) + R_0(\bar z^{-1})^+ P_0 &\left\{P_0{}^+\left(R_0(z) - \frac{1}{2} I\right)\Big|_{\mathfrak{D}(V)^\perp} \right. \\ &\left. + P_0{}^+\left(S^{-1} - \frac{1}{2} I\right)\right\}^{-1} P_0{}^+R_0(z). \end{aligned} \tag{3.1}$$

We can express the representation (3.1) in a more transparent form. For this, let $\mathfrak{G}$ with $(\cdot \mid \cdot)$ be a fixed Hilbert space of dimension equal to dim $\mathfrak{D}(V)^\perp$. Take a bijective operator Γ from $\mathsf{B}(\mathfrak{G}; \mathfrak{D}(V)^\perp)$. As $\mathfrak{D}(V)^\perp$ and $\mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3$ are in duality, Γ has an adjoint $\Gamma^\oplus$ $\left(\in \mathsf{B}(\mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3; \mathfrak{G})\right)$:

$$(\Gamma^\oplus f \mid u) = [f \mid \Gamma u] \qquad (f \in \mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3,\ u \in \mathfrak{G}). \tag{3.2}$$

Note that if in (3.2) the vector f is running through $\mathfrak{H}$, i.e., we consider Γ as a mapping into $\mathfrak{H}$, then we get another adjoint Γ^+ $\left(\in \mathsf{B}(\mathfrak{H}; \mathfrak{G})\right)$ for which $\Gamma^+ = \Gamma^\oplus P_0{}^+$.

Define $\Gamma_z := U(U - zI)^{-1}\,\Gamma$ $\left(\in \mathsf{B}(\mathfrak{G}; \mathfrak{H})\right)$ for z in $\varrho(U)$; then, by Lemma 3.1, $\Gamma_{1/\bar{z}} = R_0(\bar{z}^{-1})^+\, P_0 \Gamma$ maps homeomorphically onto $\mathfrak{N}_{1/\bar{z}}$, and $\Gamma_{\bar{z}}{}^+ = \Gamma^\oplus P_0{}^+ R_0(z)$ for z^{-1} in $\varrho(U)$. To find a substitute for the characteristic function of V we define

$$\theta(z) := \Gamma^\oplus P_0{}^+ \left(R_0(z) - \frac{1}{2}\, I\right) \Gamma \qquad \left(\in \mathsf{B}(\mathfrak{G})\right)$$

and call it a *θ-function* of V. Define further

$$E := \Gamma^\oplus P_0{}^+ \left(S^{-1} - \frac{1}{2}\, I\right) \Gamma = \Gamma^+ \left\{[(I - U^+ W)|_{\mathfrak{D}_2 + \mathfrak{D}_3}]^{-1} - \frac{1}{2}\, I\right\} \Gamma. \tag{3.3}$$

This E is a closed linear relation in $\mathfrak{G}$. In addition, it is accretive: $\mathrm{Re}\,(v \mid u) \geq 0$ for all (u, v) in E. To see this, take an arbitrary (u, v) from E and put $f := h + \frac{1}{2}\Gamma u$, where h is such that $(\Gamma u, h)$ is in $S^{-1} - \frac{1}{2}\, I$ and $\Gamma^+ h = v$; then

$$\begin{aligned}\mathrm{Re}\,(v \mid u) &= \mathrm{Re}\,[h \mid \Gamma u] = \mathrm{Re}\left[f - \frac{1}{2}\, Sf \mid Sf\right] \\ &= \frac{1}{2}\left\{[f \mid f] - [Wf \mid Wf] + \frac{1}{2}\,\mathrm{Re}\,([Wf \mid Uf] - [Uf \mid Wf])\right\} \geq 0.\end{aligned}$$

From the definitions of S and E it follows easily that $E + \frac{1}{2}\,\Gamma^+\Gamma = \Gamma^\oplus S^{-1}\Gamma$, which in turn implies that the inverse of the linear relation $E + \frac{1}{2}\,\Gamma^+\Gamma$ exists as a bounded operator on $\mathfrak{G}$. For the sake of brevity, we call an accretive linear relation E with $\left(E + \frac{1}{2}\,\Gamma^+\Gamma\right)^{-1} \in \mathsf{B}(\mathfrak{G})$ *a Γ-accretive* linear relation in $\mathfrak{G}$. Note that in the case $\Gamma^+\Gamma = I$, i.e., in the case that $\mathfrak{D}(V)^\perp$ is positive definite, E is Γ-accretive if and only if it is maximal accretive. The proof of this fact follows the same lines as the proof of Theorem 3.4 in [4]; instaed of the Cayley transformation one should use the "Möbius transformation"

$$\mathcal{M}(E) := \left\{\left(v + \frac{1}{2}\, u, v - \frac{1}{2}\, u\right) \,\middle|\, (u, v) \in E\right\},$$

which gives a bijective correspondence between accretive linear relations and contractive operators.

The definitions given in this chapter and formula (3.1) now imply the representation

$$R(z) = R_0(z) + \Gamma_{1/\bar{z}}\{\theta(z) + E\}^{-1}\, \Gamma_{\bar{z}}{}^+ \tag{3.4}$$

for a contractive resolvent R of V with a Γ-accretive linear relation E in $\mathfrak{G}$. Furthermore, as is easy to see, this E is unique.

In order to prove that every Γ-accretive linear relation E in $\mathfrak{G}$ defines a contractive resolvent of V by the formula (3.4) we use Theorem 1.4. For this, let $\mathscr{E}_0$ be the set of all Γ-accretive linear relations in $\mathfrak{G}$, and denote by $\mathscr{W}_0$ the set of all those closed operators $W': \mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3 \to \mathfrak{R}_1^{\perp}$ which satisfy the inequality (1.3). We can define a bijective correspondence between the sets $\mathscr{E}_0$ and $\mathscr{W}_0$. In fact, we have already seen that the mapping

$$\varphi: W' \mapsto \Gamma^+ \left\{(I - U^+W')^{-1} - \frac{1}{2} I\right\} \Gamma, \tag{3.5}$$

cf. (3.3), maps $\mathscr{W}_0$ into $\mathscr{E}_0$. Furthermore, a straightforward but boring calculation shows that the mapping

$$\psi: E \mapsto U|_{\mathfrak{D}_2 \dot{+} \mathfrak{D}_3} - U\Gamma \left(E + \frac{1}{2} \Gamma^+\Gamma\right)^{-1} \Gamma^{\oplus} \tag{3.6}$$

maps $\mathscr{E}_0$ into $\mathscr{W}_0$ and is the inverse of φ.

So let $E \in \mathscr{E}_0$ be arbitrary. Then, by Theorem 1.4, the operator $W := V + \psi(E)$ $(\in \mathsf{B}(\mathfrak{H}))$ is a contractive extension of V. The contractive resolvent induced by this W has the representation (3.4) with an $E' \in \mathscr{E}_0$. But by the construction $E' = \varphi(W|_{\mathfrak{D}_2 \dot{+} \mathfrak{D}_3}) = \varphi(\psi(E)) = E$. Consequently, the correspondence $R \leftrightarrow E$ in (3.4) is bijective.

Thus we have the following result.

Theorem 3.2: *Let V be a closed injective isometric operator in a Pontrjagin space $\mathfrak{H}$ with the defect numbers equal to $\mathfrak{n}$. Choose a Hilbert space $\mathfrak{G}$ with* $\dim \mathfrak{G} = \mathfrak{n}$ *and a canonical unitary extension U of V. Let θ be a θ-function of V and define*

$$\Gamma_z := U(U - zI)^{-1} \Gamma \quad (z \in \varrho(U)),$$

where Γ $(\in \mathsf{B}(\mathfrak{G}; \mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3))$ is bijective. The formula

$$R(z) = (I - zU)^{-1} + \Gamma_{1/z}\{\theta(z) + E\}^{-1} \Gamma_{\bar{z}}^+ \tag{3.7}$$

gives a bijective correspondence between the set of all contractive resolvents R of V and the set $\mathscr{E}_0$ of all Γ-accretive linear relations E in $\mathfrak{G}$.

The proof of the converse part of this theorem needs the following observation: Let $E \in \mathscr{E}_0$ be arbitrary; then the inverse $(\theta(z) + E)^{-1}$ is a bounded operator on $\mathfrak{G}$ for all z in a neighbourhood of zero. To see this, we first decompose the linear relation E:

$$E = E_s \oplus E_\infty, \tag{3.8}$$

where $E_\infty := \{0\} \times E(0)$ is the multi-valued part of E and $E_s := E \ominus E_\infty$ is the operator part of E; see [11: Theorem 2.4]. One can show that E_s is a closed accretive operator in $\mathfrak{G}_1 := \mathfrak{G} \ominus E(0)$; cf. [11: Lemma 2.8]. Furthermore, by using the assumption $E \in \mathscr{E}_0$, the decomposition (3.8) and some calculations one sees that $\left(E_s + \frac{1}{2} Q\Gamma^+\Gamma|_{\mathfrak{G}_1}\right)^{-1}$ is in $\mathsf{B}(\mathfrak{G}_1)$; here Q is the orthogonal projector of $\mathfrak{G}$ onto $\mathfrak{G}_1$. Then the perturbation theory guarantees that the inverse of the operator

$$A := \left(E_s + \frac{1}{2} Q\Gamma^+\Gamma|_{\mathfrak{G}_1}\right) + zQ\Gamma^+R_0(z)\,\Gamma|_{\mathfrak{G}_1}$$

belongs to $\mathsf{B}(\mathfrak{G}_1)$ for sufficiently small z. But $A = E_s + Q\theta(z)|_{\mathfrak{G}_1}$, and [9: p. 137] now implies the desired result:

$$\{E + \theta(z)\}^{-1} = \{E_s + Q\theta(z)|_{\mathfrak{G}_1}\}^{-1}\, Q \in \mathsf{B}(\mathfrak{G}) \quad \blacksquare$$

Corollary 3.3: *Let the assumptions of Theorem 3.2 be satisfied. A contractive resolvent R of V is semiunitary if and only if the corresponding E in $\mathcal{E}_0$ is conservative. In addition, R is a unitary resolvent if and only if E is conservative and* $\mathfrak{R}\left(E - \frac{1}{2}\Gamma^+\Gamma\right) = \mathfrak{G}$.

Proof: With the notation of a conservative linear relation one can prove the following facts:

1. E in $\mathcal{E}_0$ is conservative if and only if $W' := \psi(E) \in \mathcal{W}_0$, see (3.6), is isometric and
$$\operatorname{Re}[Vf_0 \mid W'f] = \operatorname{Re}[f_0 \mid f] \qquad (f_0 \in \mathfrak{D}_0,\ f \in \mathfrak{D}_2 [\dot{+}] \mathfrak{D}_3).$$
2. $\mathfrak{R}(W') + \mathfrak{R}_0 = R_1{}^{\perp}$ if and only if $\mathfrak{R}(U^+W') + \mathfrak{D}_0 = \mathfrak{D}_1{}^{\perp}$.
3. $\mathfrak{R}(U^+W') + \mathfrak{D}_0 = \mathfrak{D}_1{}^{\perp}$ if and only if $\Gamma^+(\mathfrak{R}(U^+W')) = \mathfrak{G}$.
4. $\Gamma^+(\mathfrak{R}(U^+W')) = \mathfrak{R}\left(E - \frac{1}{2}\Gamma^+\Gamma\right)$.

These together with Corollary 1.5 imply the result ∎

As noted above, the parameter E in (3.7) is generally a linear relation. We shall now investigate the case when E is an operator. For this, we need the following extension of [13: Proposition 1.3.1].

Proposition 3.4: *A contraction W $(\in \mathsf{B}(\mathfrak{H}))$ in a Pontrjagin space $\mathfrak{H}$ and its adjoint W^+ have the same invariant vectors.*

Proof: As W in $\mathsf{B}(\mathfrak{H})$ is contractive, W^+ is also contractive; see [6: Lemma 3.1]. By symmetry it is enough to prove the inclusion $\mathfrak{N}(W^+ - I) \subset \mathfrak{N}(W - I)$. So let f in $\mathfrak{N}(W^+ - I)$ and h in $\mathfrak{H}$ be arbitrary, and put $g := W^+h - h$ $(\in \mathfrak{R}(W^+ - I))$. Then
$$[W^+(zf + h) \mid W^+(zf + h)] \leq [zf + h \mid zf + h]$$
for all z in $\mathbf{C}$, which implies $2\operatorname{Re}\{z[f \mid g]\} \leq [h \mid h] - [W^+h \mid W^+h]$ for all z in $\mathbf{C}$. But this is possible only if $[f \mid g] = 0$, i.e., $f \in \mathfrak{R}(W^+ - I)^{\perp} = \mathfrak{N}(W - I)$ ∎

We call two extensions W_1 and W_2 of an operator V *disjoint* if they agree only in $\mathfrak{D}(V)$, i.e., $W_1 f = W_2 f$ implies f is in $\mathfrak{D}(V)$.

Corollary 3.5: *Let the assumptions of Theorem 3.2 be satisfied. Then the following facts are equivalent:*

(i) *E is an operator;*
(ii) *E is densely defined;*
(iii) *W and U are disjoint extensions of V;*
(iv) *W^+ and U^+ are disjoint extensions of V^{-1}.*

Proof: Notice that (iii) is equivalent to $\mathfrak{N}(I - U^+W) = \mathfrak{D}(V)$, and (iv) is equivalent to $\mathfrak{N}(I - W^+U) = \mathfrak{D}(V)$. Thus (iii) and (iv) are equivalent by Proposition 3.4 (applied to U^+W). Furthermore, from the correspondence $E \leftrightarrow W' \leftrightarrow W$, see (3.5), (3.6) and Theorem 1.4, we get that
$$E(0) = \Gamma^{\oplus}(\mathfrak{R}(W' - U)) \quad \text{and} \quad \Gamma(\mathfrak{D}(E)) = \mathfrak{N}(I - U^+W') = \mathfrak{N}(I - U^+W).$$
The first formula implies the equivalence of (i) and (iii), the second the equivalence of (ii) and (iv) ∎

Thus the multi-valuedness of E measures the disjointness of the extensions W and U in such a way that $E(0) = \{0\}$ exactly when W and U are disjoint and $E(0) = \mathfrak{G}$ exactly when $W = U$.

Using the theorems 1.4 and 3.2 we can now characterize the generalized resolvents of an isometric operator in a way similar to [6: Satz 4.1]. For this, denote by $\mathcal{E}$ the set of mappings E from $\mathbf{C}_0$ into $\mathcal{E}_0$ such that the function $z \mapsto \left(E(z) + \frac{1}{2} \Gamma^+\Gamma\right)^{-1}$ is meromorphic in $\mathbf{C}_0$ and holomorphic in zero.

Theorem 3.6: *Let the assumptions of Theorem 3.2 be satisfied. Then the formula*

$$R(z) = (I - zU)^{-1} + \Gamma_{1/z}\{\theta(z) + E(z)\}^{-1} \Gamma_{\bar{z}}^{+} \quad (a.a.\ z \in \mathbf{C}_0) \tag{3.9}$$

defines a bijective correspondence between the set of all regular generalized resolvents R of V and the set of all E in $\mathcal{E}$.

Furthermore, R is canonical if and only if the corresponding E is independent of z, $E_0 := E(0)$ is conservative and $\Re\left(E_0 - \frac{1}{2}\Gamma^+\Gamma\right) = \mathfrak{G}$.

Proof: Let $\mathcal{W}$ be the set of those mappings Φ from $\mathbf{C}_0$ into $\mathcal{W}_0$ which are meromorphic in $\mathbf{C}_0$ and holomorphic in zero. Then the mapping $\varphi' : \varphi'(\Phi)(z) := \varphi(\Phi(z))$, see (3.5), maps $\mathcal{W}$ bijectively onto $\mathcal{E}$. Let R be a regular generalized resolvent of V. Thus there exist a Pontrjagin space $\mathfrak{H}' \supset \mathfrak{H}$ and a unitary operator $U' \supset V$ on $\mathfrak{H}'$ such that

$$R(z) = P(I - zU')^{-1}|_{\mathfrak{H}} \quad \left(z \in \varrho(U'^{+})\right),$$

where P is the orthogonal projector of $\mathfrak{H}'$ onto $\mathfrak{H}$. Define $W(z) := z^{-1}\left(I - R(z)^{-1}\right)$; then W is meromorphic in $\mathbf{C}_0$, holomorphic in zero with values in $\mathsf{B}(\mathfrak{H})$ and $W(z)$ is a contractive extension of V. Thus we can apply Theorem 3.2 to $W(\zeta)$, $\zeta \in \mathfrak{D}(W)$:

$$\left(I - zW(\zeta)\right)^{-1} = (I - zU)^{-1} + \Gamma_{1/z}\{\theta(z) + E(\zeta)\}^{-1} \Gamma_{\bar{z}}^{+}.$$

This formula holds true for all z in $\mathbf{C}_0$ such that $1/z \in \varrho\left(W(\zeta)\right) \cap \varrho(U)$. But as $W(\zeta)$ is contractive, its spectrum outside the unit circle is finite. Consequently, for almost all ζ in $\mathbf{C}_0$ we can choose $z = \zeta$. As $R(z) = \left(I - zW(z)\right)^{-1}$, we get the representation (3.9).

Define $\Phi(z) := W(z)|_{\mathfrak{D}_2 + \mathfrak{D}_3}$, $z \in \mathfrak{D}(W)$; then, by the theorems 1.4 and 2.1, Φ belongs to $\mathcal{W}$. Furthermore, from the proof of Theorem 3.2 we get

$$E(z) = \varphi\left(W(z)|_{\mathfrak{D}_2 + \mathfrak{D}_3}\right) = \varphi\left(\Phi(z)\right) = \varphi'(\Phi)(z),$$

i.e., E belongs to $\mathcal{E}$. The converse part can be proved similarly. For the proof of the rest, use Corollary 3.3 ∎

REFERENCES:

[1] ARENS, R.: Operational calculus of linear relations. Pacific J. Math. **11** (1961), 9–23.

[2] BOGNAR, J.: Indefinite inner product spaces (Ergebnisse der Mathematik und ihrer Grenzgebiete 78). Berlin–Heidelberg–New York: Springer-Verlag 1974.

[3] CHUMAKIN, M. E.: Generalized resolvents of isometric operators. Siberian Math. J. **8** (1967), 666–678.

[4] DIJKSMA, A., and H. S. V. DE SNOO: Self-adjoint extensions of symmetric subspaces. Pacific J. Math. **54** (1974), 71–100.

[5] IOHVIDOV, I. S., and M. G. KREĬN: Spectral theory of operators in spaces with an indefinite metric I. Amer. Math. Soc. Transl. (2) **13** (1960), 105–175.

[6] KREĬN, M. G., and H. LANGER: Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume $\Pi_\varkappa$. In: Hilbert space

operators and operator algebras. Colloquia mathematica Societatis János Bolyai 5, Tihany 1970 (Ed.: B. Sz.-Nagy). Amsterdam—London: North-Holland Publ. Comp. 1972, 353—399.

[7] Kreĭn, M. G., and Ju. L. Šmul'jan: $\mathfrak{I}$-polar representation of plusoperators. Amer. Math. Soc. Transl. (2) 85 (1969), 115—143.

[8] Langer, H., and P. Sorjonen: Verallgemeinerte Resolventen hermitescher und isometrischer Operatoren im Pontrjaginraum. Ann. Acad. Sci. Fenn. Ser. A I **561** (1974), 1—45.

[9] Langer, H., and B. Textorius: On generalized resolvents and Q-funczions of symmetric linear relations (subspaces) in Hilbert space. Pacific J. Math. **72** (1977), 135—165.

[10] Sorjonen, P.: Pontrjaginräume mit einem reproduzierenden Kern. Ann. Acad. Sci. Fenn. Ser. A I **594** (1974), 1—30.

[11] Sorjonen, P.: On linear relations in an indefinite inner product space. Ann. Acad. Sci. Fenn. Ser. A I **4** (1978/1979), 169—192.

[12] Štraus, A. V.: On the extensions and characteristic function of a symmetric operator. Math. USSR-Izv. **2** (1968), 181—203.

[13] Sz.-Nagy, B., and C. Foias: Haomonic analysis of operators on Hilbert spaces. Budapest: Akadémiai Kiadó 1970, and Amsterdam—London: North-Holland Publ. Comp. 1970.

Manuskripteingang: 29. 08. 1983; in revidierter Fassung 03. 07. 1984

VERFASSER:

Dr. Pekka Sorjonen
Department of Mathematics of the University of Jyväskylä
SF-40100 Jyväskylä 10, Seminaarinkatu 15