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Generalized Resolvents of an Isometric Operator in a Pontrjagin Space

P. SORJONEN

In dieser Arbeit werden kontrahierende und verallgemeinerte Resolventen eines isometrischen Operators im Pontrjagin-Raum untersucht. Der Definitionsbereich des Operators darf sogar entartet sein. Die dann herzuleitenden Formeln sind analog denjenigen aus dem nichtentarteten Fall, die Werte des charakteristischen Parameters sind aber jetzt lineare Relationen anstelle linearer Operatoren.

В этой работе исследуются сжимающие и обобщенные резольвенты изометрического оператора в пространстве Понтрягина. Область определения оператора может быть вырожденной. Выведенные формулы аналогичны тем в невырожденном случае, однако как значения характеристического параметра выступают теперь линейные отношения вместо линейных операторов.

In this paper the contractive and generalized resolvents of an isometric operator in a Pontrjagin space are studied. Especially, the domain of the operator is allowed to be degenerate. The resulting formulae are analogous to the non-degenerate case, but the values of the characterising 'parameter are now linear relations instead of linear operators.

Introduction

As noticed in the introduction of [11] the studies of a canonical differential equation with an indefinite weight function may lead one to consider not only defined symmetric operators but also non-densily defined symmetric operators or even linear. relations in an indefinite inner product space. In this case it can happen that the defect spaces of the operator or linear relation under consideration are not necessarily non-degenerate. This in turn means that its Cayley transform, which in any case is an isometric operator, has a degenerate domain. Thus in order to be able to use the usual method of investigating symmetric operators or relations via their Cayley transforms one must first study isometric operators with degenerate domains.

In this paper we characterize the generalized resolvents of a closed injective isometric operator with equal defect numbers in a Pontriagin space (for the terminology see below). In order to clarify this result let us recall the essence of Satz 4.1 from [6]: If V is a closed injective isometric operator with equal defect numbers and with nondegenerate domain and range in a Pontriagin space, then all generalized resolvents of V are given by the formula

$$
R(z) = (I - zU)^{-1} + \Gamma_{1/z} P(z) \, \Gamma_{\bar{z}}{}^+ ,
$$

where '

$$
P(z) = (I - E(z))(I - X(\bar{z})^+ E(z))^{-1} (I - X(\bar{z}));
$$

here U is a given unitary extension of V in the original space, Γ is a certain operatorvalued function defined in $\rho(U)$ and X is the characteristic function of V. The para-

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meter E corresponding to the generalized resolvent R is defined in the open unit disk C_0 of the complex plane C and its values are contractive operators in the defect space of V . Thus the value set of E is inside the operator unit ball in case of positive defect space and outside the ball in case of negative defect space.

If the defect space of the operator *V* degenerates the proof of the result mentioned above does not hold and furthermore $(\#)$ loses its meaning. In considering the formula $(*)$ for the degenerate case it turned out that the values of the parameter E can be linear relations, i.e. "multi-valued" operators, and that a natural value set for E is not the operator unit ball but the right operator or linear relation half plane: $E(z)$ is an accretive linear relation... disk C₀ of the complex plane C and its values are contractive operators in the d
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The modification of the characterizing parameter E in the formula (#) forced us to modify also the characteristic function X of the operator V . The substitute for X is so-called θ -function θ , which will be studied more closely elsewhere. After the above mentioned modifications it was found out that the basic formula (*) characterising the generalized resolvents of the isometric operator V holds also in the degen-

$$
P(z) = (\theta(z) + E(z))^{-1};
$$

see Theorem 3.6.

Although it would be possible to prove the above mentioned result directly along the same lines as in [6], we use here a different approach. We first characterize-in Chapter'l the cóhtractive and unitary extensions of an isometric, operator. With the' help of that result we prove in Chapter 2 another representation formula for the generalized' resolvents; see Theorem 2.1. Finally, in Chapter 3 we characterize in the 'above mentioned way not only the generalized resolvents of an isometric operator but also contractive and unitary resolvents. $Y(z) = (v(z) + E(z))^{-1}$,

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Chapter 1 the contractive and unitary extensi

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'This work wasinitiated while the author was visiting the Technical University Dresden. The author wishes to thank professor H. LANGER for stimulating discussions.

We represent here briefly-the notation and terminology used in this paper. For more extensive information about the' results concerning Pontrjagin spaces we refer to [2] and [5].

Throughout this paper $\mathfrak H$ denotes a π_x -space or a *Pontriagin space* (with x negative squares), that is, $\mathfrak S$ is a (complex) linear space equipped with a scalar product (indefinite inner product) $\lceil \cdot \rceil$, which has x *negative squares*. The last property means that the space \tilde{p} admits a decomposition in an orthogonal direct sum $\tilde{p} = \tilde{p}_{+}[\dot{+}] \tilde{p}_{-}$, where \mathfrak{g}_+ with $\lceil \cdot \rceil$ is a Hilbert space and \mathfrak{g}_- with $\lceil \cdot \rceil$ is a x-dimensional Hilbert. space. Note that the Pontrjagin space \tilde{p} is also a Hilbert space with respect to the inner product $(f, g) \rightarrow (f | g)$:

$$
(f | g) := [f_{+} | g_{+}] - [f_{-} | g_{-}], \qquad f_{\pm}, g_{\pm} \in \mathfrak{H}_{\pm}.
$$

All topological notions are to be understood with respect 'to the norm topology induced by this positive definite inner product.

A vector *f* in the Pontriagin space \hat{p} is called *positive (negative, neutral),* if $f \mid f$ > 0 . $($0, =0$). An analogous definition holds for a subset of \mathfrak{D} . Orthogonality in $\mathfrak{D}$$ is defined in the usual way: vectors */* and *g* in \hat{p} are *orthogonal* if $[f \mid g] = 0$. The *orthogonal companion* of a subset \mathfrak{M} is denoted by \mathfrak{M}^1 .

In a Hilbert space a subspace \mathfrak{M} is always *orthocomplemented*: $\mathfrak{M} + \mathfrak{M}^1 = \mathfrak{H}$. In a Pontrjagin space this happens if and only, if the subspace is *non-degenerate.* This means that the *isotropic part* $\mathfrak{M}^0 := \mathfrak{M} \cap \mathfrak{M}^{\perp}$ of \mathfrak{M} is zero. In general case one can form the following decomposition for a degenerate subspace \mathfrak{M} :

$$
\mathfrak{D}=\mathfrak{M}_{1}[+]\,\mathfrak{M}_{2}[\dotplus](\mathfrak{M}^{0}\dotplus\mathfrak{M}_{3}).
$$

Here \mathfrak{M}_1 and \mathfrak{M}_2 are subspaces with the properties

 \mathfrak{M}^0 [$+$] $\mathfrak{M}_1 = \mathfrak{M}$, \mathfrak{M}^0 [$+$] $\mathfrak{M}_2 = \mathfrak{M}^\perp$.

The subspace \mathfrak{M}_3 is skewly linked with \mathfrak{M}^0 , i. e. $\mathfrak{M}_3 \cap \mathfrak{M}^{0} = \mathfrak{M}_3^{-1} \cap \mathfrak{M}^0 = \{0\}$; see [2: Theorem IX.2.5].

An operator *V* in the Pontrjagin space $\tilde{\varphi}$ is called *contractive* if $[V/ |V|] \leq [f | f]$ for all f in the domain $\mathfrak{D}(V)$. The contractive operator *V* is *isometric* if the equalitysign holds everywhere. An everywhere defined isometric operator *V* is called *semiunitary,* and *unitary* if in addition the range $\Re(V)$ is the whole space.

If we extend a given operator *V* in the $\pi_{\mathbf{x}}$ -space \mathfrak{H} to an operator *U* acting in a π_{x} -space \mathcal{R} extending \mathfrak{H} , the extension *U* is called *regular*. Furthermore, *U* is said to be a *dilation* of *V* if $V = PU|_{\mathcal{S}}$, where *P* is the orthogonal projector of \mathcal{R} onto \mathcal{S} . If the extension space R equals to the original space the extension is *canonical.*

Let V be a closed injective isometric operator in the Pontrjagin space \mathfrak{H} . If a bounded operator W in $\mathfrak H$ is a contractive extension of V the mapping $z \to (I - zW)^{-1}$. $z^{-1} \in \varrho(W)$, is called a *contractive resolvent* of the operator *V*. Analogously one can define *semiunitary* and *unitary resolverils of V.* Furthermore, if a unitary operator *^U* acting in an extension space \Re is a regular extension of *V* then the mapping $z \rightarrow R(z)$: be a closed injective isometric operatoperator *W* in \tilde{p} is a contractive extension in \tilde{p} is a contractive resolvent of the *miunitary* and *unitary resolvents* of *V*. **I** an extension space \hat{R} is a regu For all the generalized resolvent is two called space \tilde{X} is all the extend in $\tilde{X}(X)$. The contractive operator Y is the whole space.

If we extend a given operator V in the π_x -space \tilde{X} to an operato

$$
R(z):=P(I-zU)^{-1}|_{\mathfrak{H}},\qquad z\in\varrho(U^{-1}),
$$

is a *(regular), generalized resolvent* of the operator *V.* In case *of* a canonical unitary

Along with the Pontrjagin space \hat{p} the product space \hat{p}^2 equipped with the natural structure inherited from \hat{p} is also a Pontrjagin space. A linear manifold of \hat{p}^2 is called a *linear relation*. Identifying opérators with their graphs we see that a linear relation T is an operator if and only if the image $T(0) = \{0\}$. The set $T_\infty := \{0\} \times T(0)$ is the *multivalued part* of *T* and $T_s := T \cap T_{\infty}$ is the *operator part* of *T*. Analogous to the operator case one can define notions like *symmetric* or *contractive linear relation*. Especially, a linear relation T is called accretive if Re $[g|f] \ge 0$ for all $(f, g) \in T$. extension the generalized resolvent is also called *canonical*.

Along with the Pontrjagin space \hat{y} the product space. A linear manifold of \hat{y}^2 is

called a *linear relation*. Identifying opérators with their g If the equality holds for all $(f, g) \in T$, then *T* is said to be *conservative*.

1. Contractive extensions of an isometric operator

The aim of this chapter is to' characterize contractive operator extensions of a given isometric operator in a Pontrjagin space. We start by proving their existence. For this the following result is useful:

Proposition 1.1: *A bounded operator 6n a Pontrjagin space is contractive if and only i/it has a regular unitary dilation.*

Proof: Let *W* be a bounded operator on a π_{κ} -space \mathfrak{S} . If *W* is contractive, it has a regular unitary dilation; see [10: Satz 4.5]. Conversely, let U be a regular dilation of *W*, i.e., *U* is a unitary operator on a π_x -space \hat{X} extending \hat{Y} and $W = PU|_{\hat{Y}}$, where *P* denotes the orthogonal projector of \hat{X} onto \hat{Y} . Then 1. Contractive extensions of an isometric operator

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isometric operator in a Pontrjagin space. We start by proving their e

this the following resul

$$
[Wf | Wf] = [Uf | Uf] - [(I - P) Uf | Uf] \leq [f | f]
$$

for all f in $\mathfrak h$, because the orthogonal companion of $\mathfrak h$ in $\mathfrak R$ is positive definite \blacksquare

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 •
 Proposition 1.2: **Proposition 1.2: A closed injective isometric operator in a Pontrjagin space admits regular unitary extensions.**
 Proof: Let V be a closed injective isometric operator in a π_x -space \mathfrak{H} . By [5: § 9] we can sup *regular unitary extensions.*

Proof: Let *V* be a closed injective isometric operator in a π_{κ} -space \tilde{p} . By [5: § 9] we can suppose that *V* is maximal, i.e., dim $\mathfrak{D}(V)^{\perp} = 0$, for instance. Choose an Proposition 1.2: A closed injective isometric operator in a Pontrjagin space admits
regular unitary extensions.
Troof: Let V be a closed injective isometric operator in a π_x -space \mathfrak{F} . By [5: § 9]
we can suppose **EXECUTE:**
 EXECUTE: A closed injective isometric operator in a Pontriagin space admits equal are unitary extensions.
 Proof: Let *V* be a closed injective isometric operator in a π_x -space \tilde{p} . By [5: § 9]

we quently *V* admits unitary extensions there; see [5: *§ 9]* ^I

By combining the two previous propositions we see that every closed injective by combining the two previous propositions we see that every closed injective
isometric operator in a Pontriagin space has bounded contractive operator exten-
sions. Below we shall characterize them. First, a useful lemma sions. Below we shall characterize them. First; a useful lemma.

Lemma 1.3: Let W be a contractive operator in a Pontrjagin space. If $[Wf_0 | Wf_0]$. $=[f_0 | f_0]$ for f_0 in $\mathfrak{D}(W)$, then $[Wf_0 | W] = [f_0 | f]$ for all f in $\mathfrak{D}(W)$.

The proof is the same as in the definite case; see [12: Lemma 1.1] \mathbf{I}

Let V be a closed injective isometric operator in a Pontriagin space \mathfrak{H} . The domain $\mathfrak{D}(V)$ of *V* induces a decomposition of the whole space:

$$
\tilde{\mathfrak{D}} = \mathfrak{D}_1 \left[\dot{+} \right] \mathfrak{D}_2 \left[\dot{+} \right] \left(\mathfrak{D}_0 + \mathfrak{D}_3 \right); \tag{1.1}
$$

see [2: Theorem IX.2.5]. Here $\mathfrak{D}_0:=D(\mathcal{V})^0$ is the isotropic part of $\mathfrak{D}(V)$, \mathfrak{D}_3 is a neutral subspace skewly linked with \mathfrak{D}_0 , and \mathfrak{D}_1 and \mathfrak{D}_2 are orthocomplemented subspaces such that, **(a)** closed injective isometian

induces a decomposition
 $= \mathfrak{D}_1[+] \mathfrak{D}_2[+] (\mathfrak{D}_0 +$

corem IX.2.5]. Here \mathfrak{D}_0 :

space skewly linked with

uuch that
 $(V) = \mathfrak{D}_1[+] \mathfrak{D}_0$, \mathfrak{D}
 $:= \mathfrak{R}(V)^0 = V(\mathfrak{$ meutral subspace skewly linked with \mathfrak{D}_0 , and \mathfrak{D}_1 and \mathfrak{D}_2 are orthocomplemented
subspaces such that
 $\mathfrak{D}(V) = \mathfrak{D}_1 \left[\frac{1}{2} \right] \mathfrak{D}_0$, $\mathfrak{D}(V)^{\perp} = \mathfrak{D}_2 \left[\frac{1}{2} \right] \mathfrak{D}_0$.
Setting \math

$$
\mathfrak{D}(V) = \mathfrak{D}_1 \left[\dot{+} \right] \mathfrak{D}_0, \qquad \mathfrak{D}(V)^{\perp} = \mathfrak{D}_2 \left[\dot{+} \right] \mathfrak{D}_0.
$$

$$
\mathfrak{F} = \mathfrak{R}_1 \left[+ \right] \mathfrak{R}_2 \left[+ \right] \left(\mathfrak{R}_0 + \mathfrak{R}_3 \right) \tag{1.2}
$$

with the 'components having similar properties to those above. In the following, we keep these decompositions fixed. Setting $\mathfrak{R}_0 := \mathfrak{R}(V)^0 = V(\mathfrak{D}_0)$ and $\mathfrak{R}_1 := V(\mathfrak{D}_1)$ we get an analoge
 $\mathfrak{D} = \mathfrak{R}_1 \left[+ \right] \mathfrak{R}_2 \left[+ \right] (\mathfrak{R}_0 + \mathfrak{R}_3)$

with the components having similar properties to those above. In

keep thes

Theorem *1.4'Lèt V be a closed infective isometric operator in a Pontrjagin space With the decompositions (11) and (1.2).*

If $W \in B(\mathfrak{H})$ is a contractive extension of V, then there exists an operator W' with *the properties* **Example 3** keep these decompositions fixed.
 Figure 1.4: Let V be a closed injective isometric ope

with the decompositions (1:1) and (1.2).

If $W \in B(\mathfrak{H})$ is a contractive extension of V, then there

the properties

 $\mathfrak{D}_\mathfrak{z}$ and its range is in $\mathfrak{R}_\mathbf{1}$ *(ii) the inequality*

$$
2 \operatorname{Re} \left[V f_0 \mid W' f \right] + \left[W' f \mid W' f \right] \leq 2 \operatorname{Re} \left[f_0 \mid f \right] + \left[f \mid f \right] \tag{1.3}
$$

(iii)
$$
W = V + W'
$$
.

Conversely, if an operator W' satisfies (i) *and (ii), 'then the formula in* (iii) *defines a* -,

With the components having similar properties to those
 keep these decompositions fixed.

Theorem 1.4: Let V be a closed injective isometric operation

with the decompositions (1:1) and (1.2).

If $W \in B(\mathfrak{H})$ is a con (iii) $W = V + W'$.
Conversely, if an operator *W'* satisfies (i) and (ii), then the formula in (iii) definate extension $W \in B(\mathfrak{H})$ of V .
For the proof of the first part, define W' as the restriction of W_f to $\mathfrak{D$ By using Lemma 1:3 it is easy to see that this *W'* has the desired properties. For the converse, note that $V + W'$ is closed by [7: Lemma 4.1]. The verification of the other claims is a straightforward calculation \blacksquare

Remarks: 10 If is obvious that basically the same result also holds for not neces- 'sarilyeverywhere defined closed contractive extensions. Thus Theorem *1.4* extends $[12:$ Lemma 1.2] to Pontrjagin spaces and $[5: § 9.2]$ to contactive extensions.

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 $\frac{2^0 \text{ For a non-degenerate domain } \mathfrak{D}(V)}{2^0 \text{ for a non-degenerate domain } \mathfrak{D}(V)}$ the sum $V + W'$ is orthogonal; but this 2° For a non-degenerate domain $\mathfrak{D}(V)$ the sum $V + W'$ is orthogonal; but this is not necessarily true for a degenerate domain. For example, take a basis $\{e_1, e_2\}$ ¹² and make C² a Pontrjagin space by defining

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$$
{}^0
$$
 For a non-degenerate domain $\mathfrak{D}(V)$ to necessarily true for a degenerate d.
\n- \n 2 and make C^2 a Pontrjagin space by de\n $[e_j | e_k] := \begin{cases} \n 0, & \text{if } j = k, \\
 1, & \text{if } j = k = 1, \\
 -1, & \text{if } j = k = 2.\n \end{cases}$ \n
\n

Define further *V* and *W'* as the identity operator of $\langle e_1 + e_2 \rangle$ and $\langle e_1 - e_2 \rangle$, resp. Then they satisfy the condition of Theorem 1.4 but they are not orthogonal.

 $3⁰$ If the domain $\mathfrak{D}(V)$ is non-degenerate, then the inequality (1.3) is equivalent to W' being contractive. In the general case, being contractive or even isometric does not imply the inequality (1.3). For example, let the Pontrjagin space and the operator *V* be as in 2⁰, and define $W'(e_1 - e_2) := e_1 + e_2$. Then W' is isometric, and (1.3) is equivalent to the condition $\text{Re}\left[f_0 | f\right] \geq 0$ for all f_0 in \mathfrak{D}_0 and f in $\mathfrak{D}(W')$. But Re $[e_1 + e_2 \mid e_2 - e_1] < 0$; consequently (1.3) is not true and the operator $V + W'$ is not contractive. *V* \mathbb{R}^2 **Contractive With the Contractive** $V^* + W'$ is not contractive.
 V \mathbb{R}^2 **With** \mathbb{R}^2 **is not contract to the contractive of** \mathbb{R}^2 **Contract is not contract to the imaginal of** \mathbb{R}^2 **an** and $\langle e_1 - e_2 \rangle$, orthogonal.

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(*i*) and *f* in \mathfrak{F}

(*i*) and the operator isometric and the operator isometric

(*em* 1.4 *W* is (i) $\ell_{\ell} + \ell_{\ell}$. If $j = k = 1$,
 $\ell_{\ell} + \ell_{\ell}$ and W' as the identity operator of $\langle e_1 + e_2 \rangle$ and
 W' is the domain $\mathfrak{D}(V)$ is non-degenerate, then the inequality (1
 W' being contractive. In the general **(iii)** is non-degenerate, then the inequality (1.3) is W' being contractive. In the general case, being contractive W' being contractive. In the general case, being contractive or ever
es not imply the inequality (1.

Corollary 1.5: *In the correspondence* $W \leftrightarrow W'$ given by Theorem 1.4 *W* is semi*unitary if and only if W' has the properties*

(ii) Re $[Vf_0 | W'] = \text{Re} [f_0 | f]$ *for all* f_0 *in* \mathfrak{D}_0 *and f in* $\mathfrak{D}(W')$.
Furthermore, W is unitary if and only if- *W'* has the properties (i), (ii) and $\mathfrak{R}(W') + \mathfrak{R}_0 = \mathfrak{R}_1^{-1}$.

2. Generalized resolvents of an isometric operator

Let *V* be a closed injective isometric operator in a Pontrjagin space \tilde{p} and let *R* be a (regular) generalized resolvent of *V,* i.e. alized resolvents of an isometric operator
a closed injective isometric operator
r) generalized resolvent of *V*, i.e.
 $R(z) = P(I - zU)^{-1}|_{\mathfrak{D}}$ ($z \in \varrho(U^+)$),
is a regular unitary extension of *V* i

$$
R(z) = P(I - zU)^{-1} |_{\mathfrak{S}} \qquad (z \in \rho(U^+)),
$$

where U is a orthogonal projector of $\hat{\mathbb{R}}$ onto $\hat{\mathbb{S}}$. Then *R* has the following properties; see [6: § 4]:

1. R is meromorphic outside the unit circle;

2. $R(z)$ has a bounded inverse for almost all *z* in the interior C_0 of the unit circle, i.e., for all z in C_0 with the possible exception of a countable set which does not have any cluster points in C_0 ; 2. $R(z)$ has a boun
i.e., for all z in C_0 wit
any cluster points in
3. $R(z) = I - R(1)$
The last property
is in C_0 . 2. Generalized resolvents of an isometric operator

Let V be a closed injective isometric operator in a Pontriagin space \hat{y} and let R be

a (regular) generalized resolvent of V, i.e.
 $R(z) = P(I - zU)^{-1}|_{\hat{y}}$ $(z \in \varrho(U^+$ $R(z) = P(I - zU)^{-1}|_{\mathfrak{D}}$ $(z \in \varrho(U^+))$,
where U is a regular unitary extension of V in an extension space Ω
orthogonal projector of Ω onto $\tilde{\Omega}$. Then R has the following properties
1. R is meromorphic outs circle;
 • • • *****• 2Reformation* α is a more α , α in α is the unit circle;
 α is a bounded inverse for almost all z in the interior C_0 of the unit circle;
 α is a bounded inverse for almost all z in the interior C

3. $R(z) = I - R(1/\overline{z})^+$, $z \in \rho(U^+) \setminus \{0\}$.
The last property implies that we usually need only to consider the case when z

The following result extends $[3:$ Theorem 3] to Pontrjagin spaces.

Theore *^m* 2. 1: *Let V be a closed infective isometric operator in ' a Pontrjagin space* with the decompositions (1.1) and (1.2). A mapping R is a regular generalized resolven *A* $R(z) = I - R(1/\overline{z})^+$, $z \in \rho(U^+) \setminus \{0\}$.

The last property implies that we usually need only to consider the case where $\sin C_0$.

The following result extends [3: Theorem 3] to Pontriagin spaces.

Theorem 2.1: Let V *i.e., for a*
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3. $R(z)$

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35* is in C_0 .

The following result extends [3: Theorem 3] to Pontrjagin space
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with the decompositions (1.1) and (1.2). A mapping R is a regular g

of g result extends [3: Theorem 3] to

1: Let V be a closed injective isomet
 vositions (1.1) and (1.2): A mapping
 y if it has a representation
 $= \{I - z[V + \Phi(z)]\}^{-1}$

in C_0 , where Φ is meromorphic in C_0
 $\frac{1}{3$

$$
R(z) = \{I - z[V + \Phi(z)]\}^{-1}
$$
 (2.1)

for almost all z in C_0 , where Φ *is meromorphic in* C_0 *and holomorphic in zero with values in* $B(\mathfrak{D}_2 \{\dot{+} \} \mathfrak{D}_3; \mathfrak{R}, \dot{+} \lambda)$, and

$$
2 \text{Re} [Vf_0 | \Phi(z) f] + [\Phi(z) f | \Phi(z) f] \leq 2 \text{Re} [f_0 | f] + [f | f]
$$

*35**

 $\begin{split} \mathbf{X} & \in \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times$

Proof: 1⁰ Let *R* be a regular generalized resolvent of *V*. Then the mapping $z \mapsto W(z)$, *2. ^W*is holomorphic in zero; -

$$
W(z) := z^{-1}(I - R(z)^{-1}) \quad \text{for almost all } z \text{ in } \mathbb{C}_0,
$$

has the following properties (see [6: Lemma 4.1] and [8: Lemma 2.3]:

-
- 1. *W* is meromorphic in C_0 ;
2. *W* is holomorphic in zero;
	- 3. $W(z)$ (\in B(\tilde{p})) is contractive for almost all *z* in C_0 ;
- $V = 4$; *W(z)* extends *V* and *W(z)⁺* extends. *V*⁻¹ for almost all *z* in C₀.

By Theorem 1.4 there exists a mapping $\Phi(z) \in B(\mathfrak{D}_2[\dot{+}] \mathfrak{D}_3, \mathfrak{R}_1^{\perp})$ such that $W(z) = V + \Phi(z)$. Using the definition and the properties of *W* it is easy to see that *R* has the representation (2.1) and Φ has the desired properties. $W(z) := z^{-1}(I - R(z)^{-1})$ for almost all z in C_0 ,

blowing properties (see [6: Lemma 4.1] and [8: Lemma 2.3]:
 s motomorphic in zero;
 s holomorphic in zero;
 i ($\in B(\hat{\xi})$) is contractive for almost all z in C_0 ;
 i

2⁰ In order to prove the converse we must construct an extension space $\hat{\mathbf{x}} \supset \tilde{\mathbf{y}}$ and a unitary extension *U* of *V* such that the corresponding generalized, resolvent *R* equals the mapping $z \mapsto S(z)$,

$$
S(z) := \{I - z[V + \Phi(z)]\}^{-1},\tag{2.2}
$$

for all *z* in an open set in C_0 . This can be done in a similar way as in [6: § 4.5]. So it is enough to give an outline of the proof.

EVALUATE: We are provided to the proof.

1. We is holomorphic in zero,

3. $W(z)$ ($\in B(\mathfrak{D})$) is contractive for almost all z in C_6 ,

4. $W(z)$ extends V and $W(z)$ extends V^{-1} for almost all z in C_6 .

By Theor Let $\widetilde{\mathcal{U}}_0 \subset \widetilde{\mathcal{C}}_0$ be an open neighbourhood of zero, symmetric with respect to the real axis and such that $S(z)$ exists as a bounded operator for all *z* in \mathcal{U}_0 . Extend *S* to the set $\mathcal{U}_{\infty} := \{z \mid 1/z \in \mathcal{U}_0\}$ by setting $S(z) := I - S(1/\overline{z})^+$. We use the notation $\mathcal{U} := \mathcal{U}_0 \cup \mathcal{U}_{\infty} \text{ and } W(z) := V + \Phi(z) \text{ so that } S(z) = [I - zW(z)]^{-1}.$
Define a kernel $H: \mathcal{U} \times \mathcal{U} \to B(\mathfrak{H})$ by Theorem 1.4 there exists a mapping $\Phi(z) \in \mathcal{B}(\mathcal{D})$

(z) = $V + \Phi(z)$. Using the definition and the propertit

at *R* has the representation (2.1) and Φ has the desired

2° In order to prove the converse we must cons and a unitary extension *U* of *V* such that the

equals the mapping $z \mapsto S(z)$,
 $S(z) := \{I - z[V + \Phi(z)]\}^{-1}$,

for all z in an open set in C_0 . This can be don

is enough to give an outline of the proof.

Let $\mathcal{U}_0 \subset C_0$ (2.2)

ie in a similar way as in [6: § 4.5]. So it

of zero, symmetric with respect to the

ided operator for all z in \mathcal{U}_0 . Extend S
 $z) := I - S(1/\overline{z})^+$. We use the notation
 $S(z) = [I - zW(z)]^{-1}$.
 $-I$ (z, $\zeta \in \mathcal{U}$

$$
K(z,\zeta):=(1-\overline{z}\zeta)^{-1}\left(S(z)^{+}+S(\zeta)-I\right)\qquad(z,\zeta\in\mathcal{U}).
$$

Then this kernel has the representation'

$$
K(z,\zeta)=K_1(z,\zeta)+S(z)^+\,S(\zeta)\qquad (z,\,\zeta\in\mathcal U)\,,
$$

where K_1 has the following form:

$$
K(z, \zeta) := (1 - \overline{z}\zeta)^{-1} \left(S(z)^{+} + S(\zeta) - I \right) \qquad (z, \zeta \in \mathcal{U}).
$$

is kernel has the representation

$$
K(z, \zeta) = K_{1}(z, \zeta) + S(z)^{+} S(\zeta) \qquad (z, \zeta \in \mathcal{U}),
$$

$$
K_{1}(z, \zeta) = \overline{z}\zeta S(z)^{+} \frac{I - W(z)^{+} W(\zeta)}{1 - \overline{z}\zeta} S(\zeta) \qquad (z, \zeta \in \mathcal{U}_{0});
$$

$$
K_{1}(z, \zeta) = \overline{z}S(z)^{+} \frac{W(z)^{+} - W(1/\overline{\zeta})^{+}}{1 - \overline{z}\zeta} S(1/\overline{\zeta})^{+} \qquad (z \in \mathcal{U}_{0}, \zeta \in \mathcal{U}_{\infty});
$$

$$
K_{1}(z, \zeta) = S(1/\overline{z}) \frac{W(1/\overline{z}) W(1/\overline{\zeta})^{+} - I}{1 - \overline{z}\zeta} S(1/\overline{\zeta})^{+} \qquad (z, \zeta \in \mathcal{U}_{\infty}).
$$

$$
y \text{ these, put } K_{1}(z, \zeta) := K(z, \zeta) - S(z)^{+} S(\zeta), \text{ use the definitions on}
$$

$$
where \zeta \in \mathcal{U}.
$$

$$
y \text{ these, put } K_{1}(z, \zeta) := K(z, \zeta) - S(z)^{+} S(\zeta), \text{ use the definitions on}
$$

To verify these, put $K_1(z, \zeta) := K(z, \zeta) - S(z)^+ S(\zeta)$, use the definitions of K and S, and calculate; cf. [6: pp. 393-394]. The same method as in the proof of Lemma 2.7

in [8] yields the result that K_1 is a positive definite kernel. Now one can easily prove

that the \mathfrak{H} -kernel K has as many neg in [8] yields the result that K_1 is a positive definite kernel. Now one can easily prove
that the $\tilde{\mathfrak{D}}$ -kernel *K* has as many negative squares as the inner product of $\tilde{\mathfrak{D}}$.
By [10: Satz 3.3] there exist

that the $\tilde{\phi}$ -kernel *K* has as many negative squares as the inner product of $\tilde{\phi}$.
 By [10: Satz 3.3] there exists a Pontrjagin space $\tilde{\phi}'$ extending $\tilde{\phi}$ such that

the reproducing kernel of this space, at the \mathfrak{g} -kernel *K* has as many negative squares as the inner product of \mathfrak{g} .
By [10: Satz 3.3] there exists a Pontrjagin space \mathfrak{g}' extending \mathfrak{g} such that *K* is
e reproducing kernel of this spa the reproducing kernel of this space, i.e.,

$$
[f(z) | u] = [f | K(\cdot, z) u] \quad (f \in \mathfrak{H}', u \in \mathfrak{H}, z \in \mathcal{U}).
$$

This space \tilde{p}' is the completion of the space \tilde{p} , which consists of all mappings

$$
[f(z) | u] = [f | K(\cdot, z) u] \quad (f \in \mathfrak{H}', u \in \mathfrak{S}^{\prime} \text{ is the completion of the space } \mathfrak{H} \text{ with a representation}
$$
\n
$$
f = \sum_{k=1}^{n} K(\cdot, z_k) u_k \quad (z_k \in \mathcal{U}, u_k \in \mathfrak{H});
$$

the inner product of S is given by the formula

$$
[f | f] := \sum_{k,j=1}^{n} [K(z_j, z_k) u_k | u_j].
$$

With the aid of the mapping $u \mapsto K(\cdot, 0)u$ we can regard $\mathfrak h$ as a subspace of $\mathfrak h'$. Define

$$
U := \left\{ \left(\sum_{k=1}^{n} K(\cdot, z_k) u_k, \sum_{k=1}^{n} z_k^{-1} [K(\cdot, z_k) - K(\cdot, 0)] u_k \right) \middle| z_k \right\} = 0 \right\};
$$

then U is a linear relation in \mathfrak{g}' , and a straightforward calculation shows that it is isometric. As $\lim K(\cdot, z) u = u$ and $\lim K(\cdot, z) u = 0$, it follows that *U* is densely defined and the range is also dense. Consequently, *U* must be an operator; see [11: defined and the range is also dense. Consequently, U must be an operator; see [11:
Proposition 2.1.2]. By [2: Theorem IX.3.1] we can continue U so that it becomes a
unitary operator on \mathfrak{g}' .
To prove that U extends V unitary operator on S' . ith the aid of the mapping $u \mapsto K(\cdot, 0)u$ we can regard $\tilde{\psi}$ as a su

sfine
 $U := \left\{ \left(\sum_{k=1}^{n} K(\cdot, z_k) u_k, \sum_{k=1}^{n} z_k^{-1} [K(\cdot, z_k) - K(\cdot, 0)] u_k \right) \middle| z_k \right\} = 0$

en *U* is a linear relation in $\tilde{\psi}'$, and a straightforw *V* := $\left\{ \left(\sum_{k=1}^{n} K(\cdot, z_k) u_k, \sum_{k=1}^{n} z_k^{-1} [K(\cdot, z_k) - K(\cdot, 0)] u_k \right) \middle| z_k = 0 \right\};$

s a linear relation in \mathfrak{F}' , and a straightforward calculation shows

As linear relation in \mathfrak{F}' , and a straightforward calcu *P(I - zU)*⁻¹ *u* = *S(z) u* = *n k*(*i, z) u* = *n k*(*i, z) u* = 0, it follows
 P(i, z) u = *u* and lim $K(i, z)$ *u* = 0, it follows
 P(*ii* - *n*)
 P(*ii* - *n*)
 P(*ii* - *xii* - *n*)
 P(I - zvii

$$
\lim_{z\to 0}z^{-1}[K(\cdot,z)-K(\cdot,0)]\,u=Vu.
$$

For this one can use the relations V one must show the relation
 $\lim_{z\to 0} z^{-1}[K(\cdot, z) - K(\cdot, 0)] u = Vu.$

For this one can use the relations $V = S(z)^+ V - \overline{z}S(z)^+$ and (see [6: p. 396])

$$
Vu = \lim_{z\to 0} z^{-1}(S(z) - I) u \qquad (u \in \mathfrak{D}(V));
$$

To finish the proof we must show the relation .,

$$
P(I - zU)^{-1} u = S(z) u \qquad (z \in \mathcal{U}_0, \ u \in \mathfrak{H}), \tag{2.4}
$$

where P is the orthogonal projector of \mathfrak{H}' onto \mathfrak{H} . It is easy to see that the mapping $(z, \zeta) \mapsto K(z, 0)$ is the reproducing kernel of ζ . Consequently, the projector P has the form $P(I - zU)^{-1} u = S(z) u \quad (z \in \mathcal{U}_0, u \in \mathfrak{H})$,

where *P* is the orthogonal projector of \mathfrak{H}' onto \mathfrak{H} . It is easy to see that
 $(z, \zeta) \mapsto R(z, 0)$ is the reproducing kernel of \mathfrak{H} . Consequently, the pr

the fo From this one can use the relations $V = S(z)^+ V - \overline{z}S(z)^+$ and (see [6: p. 396])
 $Vu = \lim_{z \to 0} z^{-1}(S(z) - I)u$ $(u \in \mathfrak{D}(V));$

finish the proof we must show the relation
 $P(I - zU)^{-1}u = S(z)u$ $(z \in \mathcal{U}_0, u \in \mathfrak{H}),$
 $\langle \zeta \rangle \mapsto$ *(i) 0 is isometric;*

$$
PK(\cdot, z) u = K(0, z) u = S(z) u \qquad (z \in \mathcal{U}_0, u \in \mathfrak{H});
$$

Corollary 2.2: *Let the assumptions of Theorem 2.1 be satisfied. A regular generalized resolvent R of V is canonical if and only if the corresponding mapping* Φ *is independent of z, and* $\Phi_c := \Phi(z)$ *has the properties*

- (ii) $\Re(\Phi_c) + \Re_0 = \Re_1$ ¹;
- *(iii)* $\text{Re} \left[V_f_0 \mid \Phi_c f \right] = \text{Re} \left[f_0 \mid f \right]$ *for all* f_0 *in* \mathfrak{D}_0 *and f in* $\mathfrak{D}_2 \left[+ \right] \mathfrak{D}_3$.

3. Contractive resolvents of an isometric operator

Let V be a closed injective isometric operator in a Pontrjagin space \mathfrak{H} . The aim of this chapter is to characterize the contractive resolvents of V . As corollaries we get characterizations for semiunitary and. unitary resolvents. Finally, we achieve a representation 'theorem for the generalized resolvents of *V.* Note that Theorem 1.4 implies immediately one such characterization: *A* mapping *R is* a contractive or (semi)unitary resolvent of V if and only if it has a representation $P(\nu_c) + \nu_0 = \nu_1$,
 $P(\nu_f | \Phi_c) = \text{Re} [f_0 | f]$ for all f_0 in
 $P(\nu_f | \Phi_c) = \text{Re} [f_0 | f]$ for all f_0 in
 $P(\nu_f | \Phi_c) = \text{Re} [f_0 | f]$ for all f_0 in
 $P(\nu_f | \Phi_c)$ is a closed injective isometric operator

izations for semiu

$$
R(z) = [I - z(V + W')]^{-1},
$$

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where W' has the properties mentioned in Theorem 1.4 or in Corollary 1.5, resp. We shall use Theorem 1.4 in order to get another characterization which is more in the spirit of [6: Satz 4.1].

Throughout this chapter we suppose that the defect numbers of the operator V are equal, i.e., dim $\mathfrak{D}(V)^{\perp} = \dim \mathfrak{R}(V)^{\perp}$. Furthermore, we keep the decompositions (1.1) and (1.2) fixed. We use the notation $\Re_z := \Re(I - zV)$ and $\Re_z := \Re_{\bar{z}}$ ¹. Let P_0 denote the projector of \mathfrak{D} onto $\mathfrak{D}(V)^{\perp}$ along $\mathfrak{D}_1[\dot{+}] \mathfrak{D}_3$. Then P_0^{\perp} is the projector of \mathfrak{D} onto $\mathfrak{D}_2[\dot{+}] \mathfrak{D}_3$ along $\mathfrak{D}(V)$; thus $\mathfrak{D}(V)^{\perp}$ and $\mathfrak{D}_2[\dot{+}] \mathfrak{D}_3$ form a dual pair with respect to $\left[\cdot\right]$.

Let U be a fixed canonical unitary extension of V and W (\in B(\mathfrak{D})) an arbitrary contractive extension of V; denote by R_0 and R the corresponding resolvents of U and W .

Lemma 3.1: 1º $R(z)$ maps $\Re(V)^{\perp}$ homeomorphically onto $\Re_{1/z}$ $(1/z \in \varrho(W))$. 2^0 $R(z)^+$ maps $\mathfrak{D}(V)$ ¹ homeomorphically onto $\mathfrak{R}_{\bar{z}}(1/z \in \rho(W))$ 3⁰ $\Re(R(z) - R_0(z)) \subset \Re_{1/z} (1/z \in \varrho(W) \cap \varrho(U)).$ $4^{\circ} \Re(R(z) - R_{\circ}(z)) \supset \Re_z (1/z \in \varrho(W) \cap \varrho(U)).$

The proof is a straightforward calculation; cf. $[6: § 4]$ and $[8]$

In order to get the desired characterization of R we consider the difference $1/z \in \varrho(W) \cap \varrho(U)$

$$
R(z) - R_0(z) = -zR(z) U(I - U^+W) R_0(z)
$$

= $R_0(\overline{z}^{-1})^+ P_0(I - zU) U^{-1} \{I \ge zW\}^{-1} U(I - U^+W) P_0^+ R_0(z)$

$$
= R_0(\overline{z}^{-1})^+ P_0 \left\{ (I - U^+W) \left(R_0(z) \le \frac{1}{2} I \right) + \frac{1}{2} (I + U^+W) \right\}^{-1} (I - U^+W) P_0^+ R_0(z);
$$

see Lemma 3.1. Define

$$
S:=(I-U^+W)|_{\mathfrak{D}_\mathbf{z}+\mathfrak{D}_\mathbf{z}};
$$

then $S \in B(\mathfrak{D}_2[\mathfrak{f}]) \mathfrak{D}_3$; $\mathfrak{D}(V)$ ^{\perp}), and a simple calculation shows that

$$
\frac{1}{2} S^{-1}(I + U^+ W)_{\mathfrak{D}(V)^{\perp}} = P_0^+ \left(S^{-1} - \frac{1}{2} I \right)
$$

Note that S^{-1} is not necessarily an operator but a linear relation; for the calculus of linear relations, see [1, 4, 11]. Define temporarily

 $T := \left[(I - U^+ W) \left(R_0(z) - \frac{1}{2} I \right) + \frac{1}{2} (I + U^+ W) \right]_{\mathfrak{D}(V)};$

$$
T^{-1}S = \left\{ S^{-1} \left[SP_0^+ \left(R_0(z) - \frac{1}{2} I \right) \Big|_{\mathfrak{D}(V)} \right] + \frac{1}{2} (I + U^+ W) \Big|_{\mathfrak{D}(V)} \right\}
$$

=
$$
\left\{ P_0^+ \left(R_0(z) - \frac{1}{2} I \right) \Big|_{\mathfrak{D}(V)} \right\} + P_0^+ \left(S^{-1} - \frac{1}{2} I \right) \right\}^{-1}.
$$

Thus

then

$$
R(z) = R_0(z) + R_0(\overline{z}^{-1}) + P_0 \left\{ P_0^{+} / \left(R_0(z) - \frac{1}{2} I \right) \middle|_{\mathfrak{D}(V)^{\perp}} + P_0^{+} \left(S^{-1} - \frac{1}{2} I \right) \right\}^{-1} P_0^{+} R_0(z).
$$

 (3.1)

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(3.4)

We can express the representation (3.1) in a more transparent form. For this, let $\mathfrak G$ with $(\cdot | \cdot)$ be a fixed Hilbert space of dimension equal to dim $\mathfrak D(V)^{\perp}$. Take a. bijective operator *F* from $B(\mathfrak{B}; \mathfrak{D}(V)^{\perp})$. As $\mathfrak{D}(V)^{\perp}$ and $\mathfrak{D}_2[\dot{+}] \mathfrak{D}_3$ are in duality, *F* We can express the representation
let $\mathcal G$ with $(\cdot | \cdot)$ be a fixed Hilbert space
bijective operator Γ from $B(\mathcal G; \mathfrak{D}(V)^{\perp})$
has an adjoint Γ^{\oplus} ($\in B(\mathfrak{D}_2[+]\cdot \mathfrak{D}_3; \mathcal G)$)):
 $(\Gamma^{\oplus} f | u) = [f | I^u] \qquad (f$ (3.1) in a more transparent form. For this,

(3.1) in a more transparent form. For this,
 πt space of dimension equal to dim $\mathfrak{D}(V)^{\perp}$. Take a
 $\mathfrak{D}(V)^{\perp}$. As $\mathfrak{D}(V)^{\perp}$ and $\mathfrak{D}_2 [\pm] \mathfrak{D}_3$ are in

$$
T^{\oplus} f | u) = [f | T u] \qquad (f \in \mathfrak{D}_2 \left[\dot{+} \right] \mathfrak{D}_3, \ u \in \mathfrak{G}). \tag{3.2}
$$

Note that if in (3.2) the vector *f* is running through \mathfrak{H} , i.e., we consider *F* as a mapping into \tilde{p} , then we get another adjoint Γ^+ (\in B(\tilde{p} ; \tilde{g})) for which Γ^+ .

Define $\Gamma_z := U(U - zI)^{-1} \Gamma$ ($\in B(\mathfrak{G}; \mathfrak{H})$) for *z* in $\varrho(U)$; then, by Lemma 3.1, $\iota_z = R_0(\bar{z}^{-1})^+ P_0 \Gamma$ maps homeomorphically onto $\Re_{1/z}$, and $\Gamma_{\bar{z}}^+ = \Gamma^{\oplus} P_0^+ R_0(z)$ $f_{1/z} = K_0(z^{-1})^T P_0 I$ maps homeomorphically onto $\mathcal{X}_{1/z}$, and $I_z^+ = I^{\prime\prime\prime} P_0^+ R_0(z)$
for z^{-1} in $\varrho(U)$. To find a substitute for the characteristic function of *V* we define *of* $\mathcal{F}(\mathcal{F}) = \mathcal{F}(\mathcal{F})$ $\mathcal{F}(\mathcal{F}) = \mathcal{F}(\mathcal{F})$. As $\mathcal{L}(\mathcal{F}) = \mathcal{F}(\mathcal{F})$ and $\mathcal{F}(\mathcal{F}) = \mathcal{F}(\mathcal{F})$ ($\mathcal{F}(\mathcal{F}) = [f | I^u]$ $(f \in \mathfrak{D}_2[\pm] \mathfrak{D}_3, u \in \mathfrak{G})$.
 $(f^{\oplus}f | u) = [f | I^u]$ $(f \in \mathfrak{D}_2[\$

$$
\theta(z) := \Gamma^{\oplus} P_0^+ \left(R_0(z) - \frac{1}{2} I \right) \Gamma \qquad (\in \mathsf{B}(\mathfrak{G}))
$$

and call it a θ -*function* of \hat{V} . Define further

$$
E = \Gamma^{\oplus} P_0^+ \left(S^{-1} - \frac{1}{2} I \right) \Gamma = \Gamma^+ \left\{ [(I - U^+ W) |_{\mathfrak{D}_* + \mathfrak{D}_*}]^{-1} - \frac{1}{2} I \right\} \Gamma. \tag{3.3}
$$

This *E* is a closed linear relation in \mathfrak{G} . In addition, it is accretive: Re $(v | u) \geq 0$ for all (u, v) in *E*. To see this, take an arbitrary (u, v) from *E* and put $f := h + \frac{1}{2}Tu$, where *h* is'such that (Tu, h) is in $S^{-1} - \frac{1}{2}I$ and $T^+h = v$; then

$$
\begin{aligned} \n\text{Re}(v \mid u) &= \text{Re}\left[h \mid Fu\right] = \text{Re}\left[f - \frac{1}{2} \, Sf \mid Sf\right] \\ \n&= \frac{1}{2} \left\{ [f \mid f] - [Wf \mid Wf] + \frac{1}{2} \, \text{Re}\left([Wf \mid Uf] - [Uf \mid Wf]\right] \right\} \geq 0. \n\end{aligned}
$$

From the definitions of S and *E* it follows easily that $E+\frac{1}{2}\Gamma^+ \Gamma = \Gamma^{\oplus} S^{-1} \Gamma$, which in turn implies that the inverse of the linear relation $E + \frac{1}{2}T^*F$ exists as a bounded operator on $\mathfrak{G}.$ For the sake of brevity, we call an accretive linear relation E From 'the definitions of *S* and *E*, it follows easily that $E + \frac{1}{2} \Gamma^* \Gamma = \Gamma^* \mathbb{B} S^{-1} \Gamma$
which in turn implies that the inverse of the linear relation $E + \frac{1}{2} \Gamma^* \Gamma$ exists as a
bounded operator on **G**. For the $\Gamma^+ \Gamma = I$, i.e., in the case that $\mathfrak{D}(V)$ ¹ is positive definite, E is F-accretive if and only if it is maximal accretive.' The proof of this fact follows the same lines ag the proof of Theorem 3.4 in [4]; instaed of the Cayley transformation one should use the "Möbius transformation" From 'the definitions of S and E it follows easily that $E + \frac{1}{2} \Gamma^* \Gamma = \Gamma^{\oplus} S^{-1}$

which in turn implies that the inverse of the linear relation $E + \frac{1}{2} \Gamma^* \Gamma$ exists as

bounded operator on (9. For the sake of bre which in turn implies that the inverse of the linear
bounded operator on \bullet . For the sake of brevity, we \cdot
with $\left(E + \frac{1}{2} \Gamma^* \Gamma\right)^{-1} \in \mathcal{B}(\bullet)$ a Γ -accretive linear rel
 $\Gamma^* \Gamma = I$, i.e., in the case that \mathfr tation . • - - $\frac{1}{2}$ I^*I' \in **B(** \textcircled{S} **)** α *I* -accretive linear relation in \textcircled{S} . Note that i, i.e., in the case that $\mathfrak{D}(V)^{\perp}$ is positive definite, *E* is *I*-accretive i
aximal accretive. The proof of this $\Gamma^+ \Gamma \stackrel{\rightharpoonup}{=} I$, i.e., in the case that $\mathfrak{D}(F)$ ¹ is positive definite, *E*
if it is maximal accretive. The proof of this fact follows the
of Theorem 3.4 in [4]; instaed of the Cayley transforma
"Möbius transfor

$$
\mathcal{M}(E):=\left\{\!\!\left(v+\frac{1}{2}\;u,\,v-\frac{1}{2}\;u\!\!\right) \Big|\; (u,\,v)\,\in E\!\!\right\},
$$

which gives a bijective torrespondence between accretive linear relations and con-
tractive operators.

The definitions given in this chapter and formula (3.1) now imply the representation

$$
R(z) = R_0(z) + \Gamma_{1/z}(\theta(z) + E)^{-1} \Gamma_{\bar{z}}.
$$

for a contractive resolvent *R* of *V* with a *T*-accretive linear relation *E* in \otimes . Further-more, as is easy to see, this *E* is unique. \cdot , \cdot

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In order to prove that every Γ -accretive linear relation E in $\mathfrak G$ defines a contractive resolvent of *V* by the formula (3.4) we use Theorem 1.4. For this, let \mathscr{E}_0 tractive resolvent of *V* by the formula (3.4) we use Theorem 1.4. For this, let \mathcal{E}_0 be the set of all *I*-accretive linear relations in \mathcal{B} , and denote by \mathcal{W}_0 the set of all those closed operators $W' : \math$ those closed operators $W' : \mathfrak{D}_2 \left[\frac{1}{2} \right] \mathfrak{D}_3 \to \mathfrak{R}_1^{\perp}$ which satisfy the inequality (1.3).
We can define a bijective correspondence between the sets \mathscr{E}_0 and \mathscr{W}_0 . In fact, we have already seen that the mapping. to prove that every *T*-accretive linear relation *E* in
solvent of *V* by the formula (3.4) we use Theorem 1
of all *I*-accretive linear relations in \mathfrak{G} , and denote by
d operators $W' : \mathfrak{D}_2 \left[\frac{1}{r} \right] \mathfrak{D}_3$

$$
\varphi: W' \mapsto \varGamma^+ \left\{ (I - U^+ W')^{-1} - \frac{1}{2} I \right\} \varGamma, \tag{3.5}
$$

cf. (3.3), maps \mathscr{W}_0 into \mathscr{E}_0 . Furthermore, a straightforward but boring calculation shows that the mapping

shows that the mapping
\n
$$
\psi: E \mapsto U|_{\mathfrak{D}_t + \mathfrak{D}_s} - U\Gamma \left(E + \frac{1}{2} \Gamma^* \Gamma \right)^{-1} \Gamma^{\oplus}
$$
\n
$$
\text{maps } \mathcal{E}_0 \text{ into } \mathcal{W}_0 \text{ and is the inverse of } \varphi.
$$
\nSo let $E \in \mathcal{E}_0$ be arbitrary. Then, by Theorem 1.4, the operator $W := V + \psi(E)$

 $(\in B(\mathfrak{H}))$ is a contractive extension of V. The contractive resolvent induced by this Whas the representation (3.4) with an $E' \in \mathcal{E}_0$. But by the construction $E' = \varphi(W|_{\mathfrak{D}_0 + \mathfrak{D}_0})$ $=\varphi(\psi(E)) = E$. Consequently, the correspondence $R \leftrightarrow E$ in (3.4) is bijective. into \mathscr{W}_0 and is the inverse of φ .
 R($\epsilon \mathscr{E}_0$ be arbitrary. Then, by Theorem 1.4, the operator $W := V + \psi(E)$

is a contractive extension of V . The contractive resolvent induced by this
 representation (3.4

Thus -we have the following result.

Theorem 3.2: *Let V be a dosed injective isometric operator in a Pontrjagin space with the defect numbers equal to* n. *Choose a Hubert space '03 with* dim (3 = n *and a canonical unitary extension U of V. Let* θ *be a* θ *-function of V and define.*

$$
\Gamma_z:=U(U-zI)^{-1}\Gamma\qquad(z\in\varrho(U)),
$$

where Γ (\in $B(\mathfrak{G}; \mathfrak{D}_2[+]\mathfrak{D}_3)$) *is bijective. The formula*

$$
R(z) = (I - zU)^{-1} + \Gamma_{1/z}(\theta(z) + E)^{-1} \Gamma_{\bar{z}}{}^{+}
$$

gives a bijective correspondence between the set of all contractive resolvents B of V and the set \mathscr{E}_0 *of all I-accretive linear relations E in* \mathfrak{B} .

W has the representation (3.4
 $= \varphi(\psi(E)) = E$. Consequent

Thus we have the followin

Theorem 3.2: Let V be a
 with the defect numbers equa

canonical unitary extension $\Gamma_z := U(U - zI)^{-1}$

where $\Gamma (\in \mathcal{B}(\mathfrak{G}; \mathfrak{D}_2 [$ The' proof of the converse part of this theorem needs the following observation: Let $E \in \mathscr{E}_0$ be arbitrary; then the inverse $(\theta(z) + E)^{-1}$ is a bounded operator on \mathfrak{G} for 'all *z* in a neighbourhood of zero. To see this, we first decompose the linear rela-Theo

with the

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where I

where I

gives a

the set of

The

Let E (

for all :

tion E:

where

operato *e* defect numbers equal to n. Choose a Hilbert space \mathcal{B} with dim $\mathcal{B} = \pi$ and a
 al unitary extension U of V . Let θ be a θ -function of V and define
 $\Gamma_z := U(U - zI)^{-1} \Gamma$ $(z \in \varrho(U))$,
 $T(\epsilon B(\mathcal{B}; \mathcal{D}_$ gives a orjective correspondence between the set of all contractive resolvents It of *Y* and
the set \mathcal{E}_0 of all *Γ*-accretive linear relations *E* in \mathfrak{B} .
The proof of the converse part of this theorem needs t

$$
\cdot E = E, \bigoplus E_{\infty},
$$

where $E_{\infty} := \{0\} \times E(0)$ is the multi-valued part of *E* and $E_s := E \bigoplus E_{\infty}$ is the operator part of *E*; see [11: Theorem 2.4]. One can show that E_s is a closed accretive operator in $\mathfrak{G}_1 := \mathfrak{G} \ominus E(0)$; cf. [11: Lemma 2.8]. Furthermore, by using the assumption $E \in \mathcal{E}_0$, the decomposition (3.8) and some calculations one sees that $(E_s + \frac{1}{2} Q P^T P_{\vert \mathfrak{G}_s})^{-1}$ is in B(\mathfrak{G}_1); here Q is the orthogonal projector of \mathfrak{G} onto \mathfrak{G}_1 . Let $E \in \mathcal{E}_0$ be arbitrary; then the inverse $[v(z) + E]^{-1}$ is a bounded operator on the for all z in a neighbourhood of zero. To see this, we first decompose the linear relation E :
 $E = E_s \oplus E_\infty$, (3.8 m)

where E *•* $\left(E_s + \frac{1}{2} Q \varGamma^+ \varGamma|_{\mathfrak{G}_i}\right)^{-1}$ is ir
Then the perturbation the
 $A := \left(E_s + \frac{1}{2} Q\right)$
belongs to B(\mathfrak{G}_1) for suffic
implies the desired result:
 $\{E + \theta(z)\}^{-1} = \{E_s + \theta(z)\}^{-1}$ $\hat{E}_s = \{0\} \times E(0)^p$ is the multi-valued part of E ; see [11: Theorem 2.4]. One can show that E_s in $\mathfrak{G}_1 := \mathfrak{G} \oplus E(0)$; cf. [11: Lemma 2.8]. Furtherm on $E \in \mathscr{E}_0$, the decomposition (3.8) and some calcula Q $\therefore E \ominus E_{\infty}$ is

is a closed accre

nore, by using

tions one sees

ector of $\circled{9}$ onto

operator

and [9: p. 137]

$$
A:=\left(E_s+\frac{1}{2}Q\varGamma^*\varGamma|_{\mathfrak{G}_s}\right)+zQ\varGamma^*R_0(z)\varGamma|_{\mathfrak{G}_1}
$$

belongs to B(\mathfrak{G}_1) for sufficiently small *z*. But $A = E_s + Q\theta(z)|_{\mathfrak{G}_1}$, and [9: p. 137] now implies the desired result:

$$
\{E + \theta(z)\}^{-1} = \{E_s + Q\theta(z)|_{\mathfrak{B}_s}\}^{-1} \ Q \in \mathsf{B}(\mathfrak{G}) \quad \blacksquare
$$

Corollary 3.3: *Let the assumptions of Theorem* 3.2 *be satisfied. A contractive resolvent R of V is semiunitary if and only if the corresponding E in* \mathcal{E}_0 *is conservative.* ssumptions of Theorem 3.2 be satisfied. A

y if and only if the corresponding E in \mathscr{E}_0 is co

lvent if and only if E is conservative and $\Re\left(E\right)$

i of a conservative linear relation one can

and only if $W' := \psi(E) \$ Corollary 3.3: Let the assumptions of Theorem

olvent R of V is semiunitary if and only if the corres

addition, R is a unitary resolvent if and only if E is co
 \mathfrak{G} .

Proof: With the notation of a conservative line

In addition, R is a unitary resolvent if and only if E is conservative and $\Re\left(E-\frac{1}{2}\Gamma^{\ast}\Gamma\right)$ $=$ \mathfrak{B} .

 $Proof:$ With the notation of a conservative linear relation one can prove the following facts': Re a distribution of a conservative linear relation of

facts:

E₀ is conservative if and only if $W' := \psi(E) \in \mathscr{W}_0$, see (3.

Re $[Vf_0 \mid W'] = \text{Re}[f_0 \mid f]$ $(f_0 \in \mathfrak{D}_0, f \in \mathfrak{D}_2 \mid + \infty)$.
 $V' + \mathfrak{R}_0 = R_1^{-1}$ if and **2.** $\mathfrak{R}(W') + \mathfrak{R}_0 = R_1^{\perp}$ if and only if $W' := \psi(E) \in \mathcal{W}_0$, see (3.6), is
 $\mathbb{R}e[V/\circ | W'] = \text{Re}[f_0 | f]$ $(f_0 \in \mathfrak{D}_0, f \in \mathfrak{D}_2 [\div] \mathfrak{D}_3)$.
 $2. \mathfrak{R}(W') + \mathfrak{R}_0 = R_1^{\perp}$ if and only if $\mathfrak{R}(U^*W') + \mathfrak{$ **3.** Proof: With the notation

lowing facts:
 3. *R* in \mathcal{E}_0 is conservative if
 $Re[Vf_0 | W' f] = Re$
 3. $\Re(W') + \Re_0 = R_1^{\perp}$ if a.
 3. $\Re(U^+ W') + \Re_0 = \Re_1^{\perp}$
 4. $I^+ (\Re(U^+ W')) = \Re(E - E)$

1. E in \mathscr{E}_0 is conservative if and only if $W' := \psi(E) \in \mathscr{W}_0$, see (3.6), is isometric and

$$
\mathop{\mathrm{Re}}\left[Vf_0 \mid W'\right] = \mathop{\mathrm{Re}}\left[f_0 \mid f\right] \qquad (f_0 \in \mathfrak{D}_0, \ f \in \mathfrak{D}_2 \left[+ \right] \mathfrak{D}_3).
$$

-
-

4.
$$
\Gamma^+(\mathfrak{R}(U^+W'))=\mathfrak{R}\left(E-\frac{1}{2}\Gamma^+\Gamma\right).
$$

These together with Corollary 1.5, imply the result \blacksquare

As noted above, the parameter *B* in (3.7) is generally a linear relation. We shall now investigate the case when E is an operator. For this, we need the following extension of [13: Proposition 1.3.1]. Less together with Corollary 1.5 imply the result \blacksquare
As noted above, the parameter E in (3.7) is generally a linear relation. We shall
w investigate the case when E is an operator. For this, we need the following

adjoint W^+ have the same invariant vectors.

Proof: As *W* in $B(\mathfrak{h})$ is contractive, W^+ is also contractive; see [6: Lemma 3.1]. By symmetry it is enough to prove the inclusion $\Re(W^+ - I) \subset \Re(W - I)$. So let f in $\mathfrak{R}(W^+ - I)$ and *h* in. be arbitrary, and put $g := W^+h - h$ ($\in \mathfrak{R}(W^+ - I)$). Then 3. $\mathfrak{R}(U^+W') + \mathfrak{D}_0 = \mathfrak{D}_1^{-1}$ if and only if $I^-(\mathfrak{R}(U^+W'))$
4. $I^+(\mathfrak{R}(U^+W')) = \mathfrak{R}\left(E - \frac{1}{2}I^+I\right)$.
These together with Corollary 1.5, imply the result \blacksquare
As noted above, the parameter E in (3.7) is **follogive 12.13.** Proposition 3.4: A contraction M
adjoint W^+ have the same invariant vectors
 P roof: As W in B (\tilde{y}) is contractive,
 B y symmetry it is enough to prove the f in $\Re(W^+ - I)$ and h in \til *(i)* W *int W in B*(\mathfrak{h}) is contractive, W^+ is als

symmetry it is enough to prove the inclusio
 $\mathfrak{R}(W^+ - I)$ and h in \mathfrak{H} be arbitrary, and \mathfrak{g}
 $[W^+(zf + h) | W^+(zf + h)] \leq [zf + h | zf]$

all z in C, *Proof:* As *W* in B(5) is contractive, *W* + is also contractive; see [6: Lemn symmetry it is enough to prove the inclusion $\Re(W^+ - I) \subset \Re(W - I)$.
 $\Re(W^+ - I)$ and *h* in 5 be arbitrary, and put $g := W^+h - h$ ($\in \Re(W^+$
 $\in W^+($

$$
[W^+(zj+h) \mid W^+(zj+h)] \leq [zf+h \mid zj+h]
$$

for all *z* in C, which implies $2 \text{ Re } \{z[f \mid g]\} \leq [h \mid h] - [W^+h \mid W^+h]$ for all *z* in C. But this is possible only if $[f | g] = 0$, i.e., $f \in \mathfrak{R}(W^+ - I)^{\perp} = \mathfrak{R}(W - I)$ **I**

We call two extensions W_1 and W_2 of an operator *V* disjoint if they agree only in $\mathfrak{D}(V)$, i.e., $W_1 f = W_2 f$ implies *f* is in $\mathfrak{D}(V)$.

Corollary 3.5: *Let the assumptions of Theorem* 3.2 *be satisfied. Then the following*

-
-
- *(iii) W and U are disjoint extensions of V;*

(iv) W^+ and U^+ are disjoint extensions of V^{-1} .

Proof: Notice that (iii) is equivalent to $\Re(I - U^+ W) = \mathfrak{D}(V)$, and (iv) is equivalent to $\Re(I - W^+ U) = \mathfrak{D}(V)$. Thus (iii) and (iv) are equivalent by Proposition (ii) *E* is densely defined;

(iii) *W* and *U* are disjoint extensions of *V*;

(iv) *W*⁺ and *U*⁺ are disjoint extensions of *V*⁻¹.

Proof: Notice that (iii) is equivalent to $\mathfrak{R}(I - U^*W) = \mathfrak{D}(V)$, and (iv) i (iii) W and U are disjoint extensions of V ;

(iv) W^+ and U^+ are disjoint extensions of V^{-1} .

Proof: Notice that (iii) is equivalent to $\Re(I - U^+W) = \mathfrak{D}(V)$, and (iv) is equivalent to $\Re(I - W^+U) = \mathfrak{D}(V)$. Thu (3.5), (3.6) and Theorem 1.4, we get that Proof: Notice that (
alent to $\mathfrak{R}(I - W^*U)$)
3.4 (applied to U^*W).
(3.5), (3.6) and Theoren
 $E(0) = I^{\oplus}(\mathfrak{R}(I))$
The first formula implie
of (ii) and (iv)

$$
E(0) = \Gamma^{\oplus}(\mathfrak{N}(W'-U)) \quad \text{and} \quad \Gamma(\mathfrak{D}(E)) = \mathfrak{N}(I - U^+W') = \mathfrak{N}(I - U^+W).
$$

The first formula implies the equivalence of (i) and (iii), the second the equivalence

Thus the multi-valuedness of B measures the disjointness of. the extensions *^W* and *U* in such a way that $E(0) = \{0\}$ exactly when *W* and *U* are disjoint and $E(0) = \emptyset$ exactly when $W = U$.

Using the theorems 1.4 and 3.2 we Can now characterize the generalized resolvents of an isometric operator in a way similar to [6: Satz 4.1]. For this, denote by ε the set of mappings *E* from C₀ into \mathscr{E}_0 such that the function $z \mapsto (E(z) + \frac{1}{2}I^T)^T$ 554 P. SORJONEN
Using the theorems 1.4 and 3.2 we can now characterize the general
of an isometric operator in a way similar to [6: Satz 4.1]. For this, d
set of mappings E from C_0 into \mathcal{E}_0 such that the function / **P.** SORJONEN

the theorems 1.4 and 3.2 we can now characterize the generalized resolvents

metric operator in a way similar to [6: Satz 4.1]. For this, denote by 8 the

appings E from C₀ into 8₀ such that the functio

Theorem 3.6: Let the assumptions of Theorem 3.2 *be satisfied. Then the formula*

$$
R(z) = (I - zU)^{-1} + \Gamma_{1/z} \{ \theta(z) + E(z) \}^{-1} \Gamma_{\bar{z}}^+ \quad (a.a. \ z \in \mathbb{C}_0)
$$
 (3.9)

defines a b'ijective correspondence between the set of all regular generalized resolvents R of V and the set of all E in \mathcal{E} .

Furthermore, B is canonical if and only if the corresponding F is independent of z, $E_0 := E(0)$ is conservative and $\Re\left(E_0 - \frac{1}{2} \int I^* I\right) =$

Proof: Let $\mathscr W$ be the set of those mappings Φ from C_0 into $\mathscr W_0$ which are meromorphic in C₀ and holomorphic in zero. Then the mapping $\varphi' : \varphi'(\Phi)(z) := \varphi(\Phi(z))$, see (3.5), maps *W* bijectively onto *E*. Let *R* be a regular generalized resolvent of *V*. Thus there exist a Pontriagin space $\hat{\mathfrak{g}}' \supset \hat{\mathfrak{g}}$ and a unitary operator $U' \supset V$ on $\hat{\mathfrak{g}}'$ Furthermore, R is canonical if and only if the corresponding E is independent o
 $E_0 := E(0)$ is conservative and $\Re\left(E_0 - \frac{1}{2} \Gamma^+ \Gamma\right) = \mathfrak{G}$.

Proof: Let W be the set of those mappings Φ from C_0 into \mathcal{W}_0 such that $E_0 := E(0)$ is conservative and $\Re\left(E_0 - \frac{1}{2} \Gamma^* \Gamma\right) = \mathfrak{G}.$
 \Box Proof: Let \mathcal{W} be the set of those mappings Φ from C_0 into

morphic in C_0 and holomorphic in zero. Then the mapping φ' :

see (3.5),

$$
R(z) = P(I - zU')^{-1}|_{\mathfrak{H}} \qquad \big(z \in \varrho(U'^{+})\big),
$$

where *P* is the orthogonal projector of $\tilde{\phi}'$ onto $\tilde{\phi}$. Define $W(z) := z^{-1}(I - R(z)^{-1});$ then *W* is meromorphic in $\bar{C_0}$, holomorphic in zero with values in B(\tilde{D}) and $W(z)$ is a contractive extension of V. Thus we can apply Theorem 3.2 to $W(\zeta)$, $\zeta \in \mathfrak{D}(W)$:

$$
(I-zW(\zeta))^{-1}=(I-zU)^{-1}+\varGamma_{1/z}(\theta(z)+E(\zeta))^{-1}\varGamma_{\bar z}{}^+.
$$

This formula holds true for all *z* in C₀ such that $1/z \in \varrho(W(\zeta)) \cap \varrho(U)$. But as $W(\zeta)$ is contractive, its spectrum outside the unit circle is finite. Conèquently, for almost all ζ in C₀ we can choose $z = \zeta$. As $R(z) = (I - zW(z))^{-1}$, we get the representation (3:9).

Define $\Phi(z) := W(z)|_{\mathfrak{D},+\mathfrak{D}}$, $z \in \mathfrak{D}(W)$; then, by the theorems 1.4 and 2.1, Φ belongs to *W.* Furthermore, from the proof of Theorem 3.2 we get

$$
E(z) = \varphi\big(W(z)|_{\mathfrak{D}_1+\mathfrak{D}_2}\big) = \varphi\big(\varPhi(z)\big) = \varphi'(\varPhi)(z),
$$

i.e., E belongs to E . The converse part can be proved similarly. For the proof of the 'rest, use Corollary 3.3 I

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