Para-Differential Operators in Spaces of Triebel-Lizorkin and Besov Type

T. RUNST

In dieser Arbeit beschäftigen wir uns mit Regularitätsuntersuchungen von Lösungen nichtlinearer partieller Differentialgleichungen in Räumen vom Besov- und Triebel-Lizorkin-Typ. Dabei werden Resultate von J. M. Bony und Y. Meyer auf die hier untersuchten Räume ausgedehnt.

В этой работе мы исследуем регулярность решений нелинейных дифференциальных уравнений в частных производных в пространствах типа Бесова и Трибель-Лизоркина. Мы обобщаем результаты, полученные Аж. М. Бони и И. Мейерем, на исследуемые нами пространства.

In this paper we study the regularity of solutions of non-linear partial differential equations in spaces of Besov and Triebel-Lizorkin type. We extend results obtained by J. M. Bony and Y. Meyer to spaces considered here.

In this paper we study the regularity of solutions of nonlinear partial differential equations. Here we shall extend results of J.M. BONY [1] to Besov and Triebel-Lizorkin spaces, J. M. Bony introduced in [1] the method of para-differential operators in order to prove some theorems about nonlinear partial differential equations. He considered solutions in generalized Sobolev space H_2^s ($s > 0$) and in Hölder spaces C^s , where $s > 0$ is not an integer. In recent years, Y. MEYER extended the regularity results obtained by J. M. Bony to solutions in other classes of functions spaces, cf. $[3-5]$.

In this paper we consider the Besov spaces $B_{p,q}^s$ and Triebel-Lizorkin spaces $F_{p,q}^s$ in the Euclidean n -space \mathbf{R}_n . In Chapter 1 we introduce these spaces, which contain many classical spaces as special cases. In the spaces $F^s_{p,q}$ and $B^s_{p,q}$ we study the regularity of solutions of nonlinear partial differential equations and extend the results of J. M. BONY and Y. MEYER. In order to prove our results we use the method of dyadic decomposition and maximal functions, multiplication properties of Besov and Triebel-Lizorkin spaces and results with respect to the boundedness of pseudodifferential operator of the "exotic" class $L_{1,1}^0$. Applying the theory of para-differential operators introduced by J. M. Bony [1] and Y. MEYER $[3-5]$, we are able to prove our regularity results. All immaterial positive numbers are denoted by c or c'etc.

1. Besov and Triebel-Lizorkin spaces on \mathbf{R}_n

1.1. Definitions

 \mathbf{R}_{n} denotes the *n*-dimensional real Euclidean space. $S = S(\mathbf{R}_{n})$ is the Schwartz space of all complex infinitely differentiable rapidly decreasing functions on \mathbf{R}_n , and $S' = S'(\mathbf{R}_n)$ is the corresponding dual space of tempered distributions. Let \mathcal{F}

and \mathcal{F}^{-1} be the Fourier transform in S' and its inverse, respectively. Φ^c is the set of all systems $\varphi = {\varphi_k(x)}_{k=0}^{\infty} \subset S$ such that:

(i) supp
$$
\varphi_0 \subset \{y \mid |y| \leq 2\},\
$$

$$
\text{supp } \varphi_k \subset \{ y \mid 2^{k-1} \leq |y| \leq 2^{k+1} \}, \ k = 1, 2, \ldots
$$

(ii) For any multi-index α there exists a constant c_{α} such that

$$
|D^{\alpha}\varphi_k(x)| \leq c_{\alpha}2^{-|\alpha|k}, \qquad k=0,1,2, \ldots
$$

 Φ^0 denotes the set of all systems $\varphi \in \Phi^c$ with

(iii)
$$
\sum_{k=0}^{\infty} \varphi_k(x) = 1
$$
 if $x \in \mathbf{R}_n$.

It is easy to show, that Φ^0 is not empty. We use the following usual abbreviations

$$
||f| |L_p|| = \left(\int_{\mathbf{R}_n} |f(x)|^p dx\right)^{1/p} \quad \text{if} \quad 0 < p < \infty,
$$

$$
||f| L_{\infty}|| = \operatorname*{ess\;sup}_{x \in \mathbf{R}_n} |f(x)|
$$
 (Lebesgue measure).

If $\{a_k(x)\}_{k=0}^{\infty}$ is a sequence of functions then

$$
||a_k | l_q|| = \left(\sum_{k=0}^{\infty} |a_k(x)|^q\right)^{1/q} \quad \text{if } \quad 0 < q < \infty,
$$
\n
$$
||a_k | l_\infty|| = \sup_k |a_k(x)|,
$$
\n
$$
||a_k | l_q(L_p)|| = || ||a_k(\cdot) | L_p|| ||l_q|| \quad \text{if } \quad 0 < p, q \leq \infty,
$$
\n
$$
||a_k | L_p(l_q)|| = || ||a_k(x) | l_q|| |L_p|| \quad \text{if } \quad 0 < p < \infty, \quad 0 < q \leq \infty.
$$

After these preliminaries we can define the spaces

$$
B_{p,q}^{s} = B_{p,q}^{s}(\mathbf{R}_{n}) \text{ and } F_{p,q}^{s} = F_{p,q}^{s}(\mathbf{R}_{n}).
$$

function 1.1: Let $-\infty < s < \infty$ and $0 < a <$

Def $\leq \infty$. (i) If $0 < p \leq \infty$ then

(ii) If $0 < p < \infty$ then

$$
F_{p,q}^{s} = F_{p,q}^{s}(\mathbf{R}_{n}) = \{f \in S' \mid ||f|| F_{p,q}^{s}||^{p}
$$

 := $||2^{sk} \mathcal{F}^{-1}[\varphi_{k}(\cdot) \mathcal{F}f(\cdot)](x) | L_{p}(l_{q})|| < \infty \text{ for some } \varphi \in \Phi^{0}\}.$

Here and in the following, we omit R_n in the notations for spaces, if they are defined on \mathbf{R}_n .

1.2. Properties

1.2. Properties
 1.2.1 Basic properties

(i) $S \hookrightarrow B_{p,q}^s \hookrightarrow S', S \hookrightarrow F_{p,q}^s \hookrightarrow S'$, where \hookrightarrow " denotes the continuous imbedding.

Proofs of (i) and all other in 1.2 listed results may be found in [8-11].

(ii) If $\varphi \$ oofs of (i) and all other in 1.2 listed results may be found in $[8-11]$.
(ii) If $\varphi \in \Phi^0$, then $B^s_{p,q}$, equipped with the quasi-norm $||f||B^s_{p,q}||^{\varphi}$ is a quasi-Banach

space (Banach space if $1 \leq p, q \leq \infty$). All the quasi-norms $||f|| B_{p,q}^s||^p$ with $\varphi \in \Phi^0$ are equivalent to each other. The corresponding assertion is valid for the space $F_{p,q}^s$. 1.2.1 Basic properties

(i) $S \hookrightarrow B_{p,q}^s \hookrightarrow S', S \hookrightarrow F_{p,q}^s \hookrightarrow S'$, where "

Proofs of (i) and all other in 1.2 listed results i

(ii) If $\varphi \in \Phi^0$, then $B_{p,q}^s$, equipped with the qu

space (Banach space if $1 \leq p, q \leq \in$ **Properties**
 B. Bosic properties
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 B if $\bigcup_{p,q} P \in \Phi$, then $B_{p,q}^s$, e here " \leftrightarrow " denotes the continuous imbedding.

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All the quasi-norms $||f||B_{p,q}^s||^p$ with $\varphi \in \Phi^0$ -

pponding assertion is valid for th

(iii) If $0 \le p \le \infty$ and $0 \le q \le \infty$, then S is dense in $B_{p,q}^s$ and $F_{p,q}^s$, respectively.
(iv) The following imbedding theorems hold.

(a), If $0 < p \leq \infty$, $0 < q \leq \infty$ then

$$
B_{p,\min(p,q)}^s\hookrightarrow F_{p,q}^s\hookrightarrow B_{p,\max(p,q)}^s.
$$

$$
B_{p,q_0}^s \hookrightarrow B_{p,q_1}^s \quad \text{and} \quad F_{p,q_0}^s \hookrightarrow F_{p,q_1}^s.
$$

1.2.2 Relations to dassical spaces

(i) The following relations to the classical Hölder spaces C^s and Zygmund spaces \mathcal{E}^s (b)
 $\begin{array}{ccc} & & (b) & \ & & \ddots & \ & & (i) & \text{The} \\ & & \text{are valid:} & \ & & \text{If } 0 < \end{array}$ 2.2 Relations to classical spaces

(i) The following relations to the classical Höld

e valid:

If $0 < s \neq$ integer, then $\mathcal{E}^s = C^s = B^s_{\infty, \infty}$ and

if $s > 0$, then $\mathcal{E}^s = B^s_{\infty, \infty}$.

If
$$
0 < s \neq \text{integer}
$$
, then $\mathcal{E}^s = C^s = B^s_{\infty, \infty}$ and

if $s > 0$, then $\mathscr{E}^s = B^s_{\infty, \infty}$.

(ii) If $1 < p < \infty$ and $0 < s +$ integer, then W_p^s denotes the Slobodeckij spaces and $W_{p}^{s} = B_{p,p}^{s}$.

If $1 < p < \infty$ and $m = 0, 1, 2, ...$, then $W_p{}^m$ denotes the classical Sobolev spaces and $W_p{}^m = H_p{}^m$ ($W_p{}^0 = L_p$); i.e., the Sobolev spaces are special Lebesgue spaces. *1.2.3 Imbedding theorems*
 1.2.3 Imbedding theorems $0 < s \neq$ integer, then $\ell^s =$
 > 0 , then $\ell^s = B_{\infty,\infty}^s$.

If $1 < p < \infty$ and $0 < s$.

If $1 < p < \infty$ and $0 < s$.
 $V_p^s = B_{p,p}^s$.
 $\langle p < \infty$ and $m = 0, 1, 2, ...$
 $V_p^m = H_p^m$ $(W_p^0 = L_p)$; i.e.

Furthermore, if $-\infty <$

space $p < p < \infty$ and $0 < s = 0$
 *B*_{p, p},
 $p < \infty$ and $m = 0, 1, 2, ..., t$
 $= H_p^m (W_p^0 = L_p)$; i.e., the thermore, if $-\infty < s < \infty$

thermore, if $-\infty < s < \infty$

ses are special cases of the space of the space of the space of the space of ger, then W_p^s denotes the Slobot

hen W_p^m denotes the classical So

Sobolev spaces are special Lebess

i and $1 < p < \infty$, then $H_p^s =$

paces $B_{p,q}^s$ and $F_{p,q}^s$.

ngs for different metrics.
 ∞ and $-\infty < s_1 \le s$

(iii) Furthermore, if $-\infty < s < \infty$ and $1 < p < \infty$, then $H_p^s = F_{p,2}^s$, i.e. all these spaces are special cases of the spaces $B_{p,q}^s$ and $F_{p,q'}^s$. (iii) Furthermore, if $-\infty < s < \infty$ and $1 < p < \infty$, then $H_p^s = F_{p,2}^s$, i.e. at these spaces are special cases of the spaces $B_{p,q}^s$ and $F_{p,q}^s$.

1.2.3 Imbedding theorems

The this subsection we describe imbeddings fo

In this subsection we describe imbeddings for different metrics.

(i) Let $0 < p_0 \leq p_1 \leq \infty$, $0 < q \leq \infty$ and $-\infty < s_1 \leq s_0 < \infty$. We have

$$
B_{p_0,q}^{s_0} \hookrightarrow B_{p_1,q}^{s_1} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.
$$

Let $0 < p_0 < p_1 < \infty$, $0 < q$, $r \leq \infty$ and $-\infty < s_1 < s_0 < \infty$. We have

$$
F_{p_0,q}^{s_0} \hookrightarrow F_{p_1,r}^{s_1} \quad \text{if} \quad s_0-\frac{n}{p_0}=s_1-\frac{n}{p_1}
$$

If $1 < p < \infty$ and $m = 0, 1, 2, ...$

If $1 < p < \infty$ and $m = 0, 1, 2, ...$

and $W_p^m = H_p^m$ $(W_p^0 = L_p)$; i.e., th

(iii) Furthermore, if $-\infty < s <$

these spaces are special cases of the

1.2.3 Imbedding theorems

The this subsection (ii) $\sum_{i=1}^{n} P_i$
 $B_{p,q}^{s_0} \hookrightarrow B_{p,q}^{s_1}$
 $\sum_{i=1}^{n} P_i^{s_i}$
 $\sum_{i=1}^{n} P_i^{s_i}$
 $\sum_{i=1}^{n} P_i^{s_i} P_i$ 8, if *s>* 0, and the just-mentioned inclusion it follows that $B_{n,q}^{s+n/p} \hookrightarrow \mathcal{E}^s$ if $s>0, 0.$ (i) Let $0 < p_0 \equiv p_1 \equiv \infty$, $0 \leq q \equiv \infty$ and ∞

Let $0 < p_0 < p_1 < \infty$, $0 < q$, $r \leq \infty$ and $-\infty$
 $F_{p_0,q}^{s_0} \hookrightarrow F_{p_1,r}^{s_1}$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.

(ii) Using $B_{\infty,\infty}^s \leq B^s$, if $s > 0$, and the just-m
 entioned
.
. *F*<sub> $p_0, q \to B_{p_1, q}^s$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.
 • B<sub> $p_0, q \to B_{p_1, q}^s$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.
 i Et $0 < p_0 < p_1 < \infty, 0 < q, r \le \infty$ and $-\infty < s_1 < s_0 < F_{p_0, q}^s \hookrightarrow F_{p_1, r}^s$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n$ $0 < p_0 \ge p_1 \ge \infty$, $0 < q \ge \infty$ and $-\infty < s_1 \ge s_0$
 $B_{p_0,q}^{s_0} \hookrightarrow B_{p_1,q}^{s_1}$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.
 $\le p_0 < p_1 < \infty$, $0 < q, r \le \infty$ and $-\infty < s_1 < s_0$
 $F_{p_0,q}^{s_0} \hookrightarrow F_{p_1,r}^{s_1}$ if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ $B_{p_0,q}^{s_0} \hookrightarrow B_{p_0,q}^{s_0}$

Let $0 < p_0 < p$
 $F_{p_0,q}^{s_0} \hookrightarrow B_{p_0,q}^{s+n}$

(ii) Using $B_{p,\alpha}^{s}$
 $B_{p,q}^{s+n+p} \hookrightarrow B_{p,1}^{s+n+p}$
 \vdots (iii) If $1 \leq p \leq B_{p,1}^{0} \hookrightarrow D_{p,q}^{s+n+p}$
 $(m = 0, 0)$

where
 $C^m = \begin{cases} f & \text{if } n \neq 0 \\$

$$
1 \leq p \leq \infty \text{ then}
$$

\n
$$
B_{p,1}^{0} \hookrightarrow L_{p} \hookrightarrow B_{p,\infty}^{0}, \quad B_{\infty,1}^{0} \hookrightarrow C \hookrightarrow B_{\infty,\infty}^{0}, \quad B_{\infty,1}^{m} \hookrightarrow C^{m} \hookrightarrow B_{\infty,\infty}^{m}
$$

\n
$$
(m = 0, 1, ...),
$$

 $\frac{1}{2}$

$$
B_{p,q}^{s+n/p} \hookrightarrow \mathcal{E}^s \quad \text{if} \quad s > 0, \quad 0 < p, \quad q \leq \infty.
$$

\n1 (iii) If $1 \leq p \leq \infty$ then
\n
$$
B_{p,1}^0 \hookrightarrow L_p \hookrightarrow B_{p,\infty}^0, \quad B_{\infty,1}^0 \hookrightarrow C \hookrightarrow B_{\infty,\infty}^0, \quad B_{\infty,1}^m \hookrightarrow C^m \hookrightarrow B_{\infty,\infty}^m
$$

\n
$$
(m = 0, 1, ...),
$$

\nwhere
\n
$$
C^m = \left\{ f \mid D^s f \in C \text{ for all } |\alpha| \leq m, ||f| |C^m|| = \sum_{|\alpha| \leq m} \sup_{x \in R_n} |D^s f| < \infty \right\}.
$$

'• ^S

 560 T. RUNST Here D° are classical derivatives and $C = C^{\circ}$ is the set of all bounded uniformly continuous functions f on \mathbf{R}_n with $||f||C|| = \sup |f(x)|$. $\begin{split} &\text{RUNST}\ &\text{re}\ &\text{classical derivatives and}\ &\ C=C\ &\text{functions}\ &\text{for}\ &\text{R}_n\ &\text{with}\ &\|f\mid C\|=\sup_{x\in\mathbb{R}^n}\ &\text{(i)}\ \text{it follows}\ &\text{for}\ &\text{R}_{p_1,p_1}=B_{p_1,p_1}^{s_1}\ &\text{for}\ &\text{R}_{p_1,p_1}\ &\text{for}\ &\text{R}_{p_1,p_1}\ &\text{for}\ &\text{R}_{p_1,p_1}\ &\text{for}\ &\text{R}_{p_1,p_1}\ &\text{for}\ &\text{R}_{p_1$

S..

- (iv) Using (i) it follows

$$
F_{p_0,q}^{s_0}\hookrightarrow F_{p_1,p_1}^{s_1}=B_{p_1,p_1}^{s_1}
$$

 $\begin{aligned} \text{F.RUNST}\ \text{are classical deriva} \ \text{as functional of } \text{F2} \ \text{S1} \ \text{S2} \ \text{S3} \ \text{S4} \ \text{F3} \ \text{F3} \ \text{F4} \ \text{F5} \ \text{F6} \ \text{F7} \ \text{F8} \ \text{F8} \ \text{F9} \ \text{F9} \ \text{F9} \ \text{F1} \ \text{F2} \ \text{F3} \ \text{F1} \ \text{F2} \ \text{F3} \ \text{F1} \ \text{F2} \ \text{F3} \ \text{F1} \ \text{F2} \ \text$ $\begin{aligned} \mathcal{L}_{p_0, q} \sim \mathcal{L}_{p_1, p_1} = D_{p_1, p_1} \ \text{if}\ 0 < p_0 < p_1 < \infty, \ -\infty < s_1 < s_0 < \infty, \ 0 < q \leqq \infty, \, s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \end{aligned}$ **For all the CP** are classical derivatives and $C = C^0$ is the set of all bounded renorminous functions *f* on **R**_n with $||f| C|| = \sup_{x \in \mathbb{R}_n} |f(x)|$.

(iv) Using (i) it follows
 $F_{p_n, q}^s \hookrightarrow F_{p_1, p_1}^s = B_{p_1, p_1}^s$

if (v) Let $0 < p_0 < p < p_1 \leq \infty$, $0 < q \leq \infty$, $-\infty < s_1 < s < s_0 < \infty$ and D° are classical derivat

nuous functions f on \mathbf{R}_n
 P<sub>p_p, $q \hookrightarrow F_{p_1, p_1}^{s_1} = B_{p_1, p_2}^{s_1}$
 $\leq p_0 < p_1 < \infty, -\infty <$

Let $0 < p_0 < p < p$

Let $0 < p_0 < p < p$
 $\frac{n}{p_0} = s_1 - \frac{n}{p_1} = s - \frac{n}{p}$ </sub> $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} = s - \frac{n}{p}$. Then (cf. [2])
 $B_{p_0, p}^{s_0} \hookrightarrow F_{p, q}^{s} \hookrightarrow B_{p_1, p}^{s_1}$.

1.2.4 Maximal junctions and maximal inequalities

In the following, we use the technique of maximal function in order to answer the $B_{p,q}^s$ or $F_{p,q}^s$ is a (quasi-normed) algebra under pointwise multiplication. In the case of $B_{p,q}^s$ it is also possible to modify the proof in [1]. Let φ be an infinitely differentiable function in $\mathbf{R}_n \setminus \{0\}$ such that *xximal functions and maximal inequalities*

llowing, we use the technique of maximal function in order to answer the

whether $B_{p,q}^s$ or $F_{p,q}^s$ is a (quasi-normed) algebra under pointwise multi-

In the case of B_{p flowing, we use the techniq
whether $B_{p,q}^s$ or $F_{p,q}^s$ is a (
In the case of $B_{p,q}^s$ it is also
differentiable function in **F**
sup $(|x|^L + |x|^{-L}) \sum_{|\alpha| \leq L} |D^{\alpha} \varphi$
a natural number, which va
...
ition: If $a > 0, f \in$

$$
\sup_{x\in\mathbf{R}_n\setminus\{0\}}(|x|^L+|x|^{-L})\sum_{|x|\leq L}|D^x\varphi(x)|=c_\varphi<\infty.
$$
\n(1)

Here L is a natural number, which value we shall choose later on. Let $\varphi_k(x) = \varphi(2^{-k}x)$, $k = 0, 1, \ldots$

Definition: If $a>0, f\in S'$, then the *maximal function* φ_k^{\ast} *f* is given by

$$
\begin{aligned}\n\text{it:} \quad & \text{if } a > 0, f \in S', \text{ then the } \text{maximal function } \varphi_k^* f \text{ is given} \\
(\varphi_k^* f)(x) &= \sup_{y \in \mathbb{R}_n} \frac{|(\mathcal{F}^{-1} \varphi_k \mathcal{F} f)(x - y)|}{1 + 2^{ka} |y|^a} \quad (x \in \mathbb{R}_n; \ k = 0, 1, \ldots).\n\end{aligned}
$$

We recall the maximal inequalities proved in a more general framework in [9]. Theorem: (i) *If* $0 < p$, $q \le \infty$, $-\infty < s < \infty$, $a > \frac{n}{p}$, $L > L^{p}(s, p, q) = |s|$ **h** $+ n + 4 + \frac{6n}{n}$, then there exists a positive number c such that for all φ with (1) and $+ n + 4 + \frac{6n}{n}$. $\sum_{z \in \mathbb{R}_n \setminus \{0\}} \frac{|z|^2 - 1}{|z|} \sum_{i=1}^{n} \frac{1}{|z|^2} \int_{-\varphi(x)} \frac{1}{|z|^2} = c_{\varphi} < \infty.$

Here *L* is a natural number, which value we shall choose later on. Let $\varphi_k(x) = 0, 1, ...$

Definition: If $a > 0, f \in S'$, then the *max* $n+4+\frac{6n}{p}$, then there exists a positi
 $f \in B_{p,q}^s$
 $||2^{sk} \varphi_k * f||l_q(L_p)|| \leq c c_{\varphi} ||f|| B_{p,q}^s||.$ We recall the maximal inequalities proved in a more general f

Theorem: (i) If $0 < p$, $q \le \infty$, $-\infty < s < \infty$, $a > \frac{n}{p}$, L
 $+n+4+\frac{6n}{p}$, then there exists a positive number c such that for
 $all f \in B_{p,q}^{s}$
 $||2^{sk}p_k^*f$ $\sup_{\eta \in \mathbf{R}_n} \frac{\left| \left(\mathcal{F}^{-1} \varphi_k \mathcal{F} f \right) \right| (x)}{1 + 2^{k a} |y|}$
 nal inequalities pre
 $\leq p, q \leq \infty, -\infty$
 here exists a positionally
 $\|\varphi\| \leq c c_{\varphi} \|f| \|B_{p,q}^s\|$
 g of (1).
 $\leq q \leq \infty, -\infty < 6n$
 $\frac{6n}{p$ We recall the maximal ine

Theorem: (i) If $0 < p$,
 $+ n + 4 + \frac{6n}{p}$, then there ex
 $all f \in B_{p,q}^s$
 $||2^{sk} \varphi_k * f||l_q(L_p)|| \leq c$

Here c_{φ} has the meaning of (1

(ii) If $0 < p < \infty$, $0 < q \le$
 $= |s| + n + 4 + \frac{6n}{\min(p, q)}$
 p wi

$$
||2^{sk}\varphi_k * f || l_q(L_p)|| \leq c c_{\varphi} ||f|| B_{p,q}^s||.
$$

(ii) *I/O < p <* ∞ , 0 < $q \leq \infty$, $-\infty < s' < \infty$, $a > \frac{n}{\min(n, q)}$ and $L > L^r(s, p, q)$ **m**i**n** *(p, q)* $\infty, 0 < q \leq \infty, -\infty < s < \infty, a > 6n$

min (p, q) , then there exists a position
 $\lim_{n \to \infty} F_n$. , *then there exists a positive number c such that for all* φ with (1) and all $f \in F_{p,q}^s$ Theorem: (i) If $0 < p, q \le \infty, -\infty < s < \infty, a$
 $- n + 4 + \frac{6n}{p}$, then there exists a positive number c sue $||1 \notin B_{p,q}^s||$.
 $||2^{sk}p_k * f||l_q(L_p)|| \leq c c_{\varphi} ||f|| B_{p,q}^s||$.

Lere c_{φ} has the meaning of (1).

(ii) If $0 < p < \infty, 0 < q$

1•

$$
||2^{sk}\varphi_k * f | L_p(l_q)|| \leq c c_{\varphi} ||f|| F_{p,q}^s||.
$$

Here c_{φ} *has the meaning of (1).*

2. Multiplication properties, the first example of linearization of nonlinear problems

In our further considerations, we use essentially the fact that the spaces $F_{p,q}^s$ and $B_{p,q}$ are a (quasi-normed) algebra under pointwise multiplication if the numbers ¹ Para-Differential Operators

2. Multiplication properties, the first example of linearization

of nonlinear problems

In our further considerations, we use essentially the fact that the spaces
 $B_{p,q}^s$ are a (quasiof non

In our fu
 $B_{p,q}^s$ are
 s, p, q are
 \ddotsc
 \ddotsc

Proofs of these assertions and references may be found in $[11: p. 145 - 146]$ and in, [2]. In the following, we shall apply essentially the treatment given in [10].

The purpose of this section is to give a natural approach to the theory of "paramultiplication" introduced by J. M. BONY [1]. Let $\overline{\{\varphi_k(x)\}_{k=0}^{\infty}} \in \varPhi^0$. We may assume that $\varphi_k(x) = \varphi_0(2^{-k}x)$ if $k = 1, 2, \ldots$ If */* and *g* are functions in $B_{p,q}^s$ and $F_{p,q}^s$, respectively (the values of s , p and q we shall choose later), then we put $F_{p,q}^s \tF_{p,q}^s \hookrightarrow F_{p,q}^s$ if
 $\begin{cases}\n\text{either } 0 < p \leq \\
\text{or } 0 < p \leq\n\end{cases}$

Proofs of these assertions and references may be

[2]. In the following, we shall apply essentially the

The purpose of this section is to give

$$
b_k = \mathcal{F}^{-1} \varphi_k \mathcal{F} g, \qquad c_k = \mathcal{F}^{-1} \varphi_k \mathcal{F} f \qquad (k = 0, 1, \ldots).
$$

We assume temporary that $\mathcal{F}g$ has a compact support. In that case all the sums below are finite. $\mathbf{R} = 1, x \in \mathbf{R}_n$, we have

e finite.
\n
$$
\sum_{k=0}^{\infty} \varphi_k(x) = 1, x \in \mathbb{R}_n, \text{ we have}
$$
\n
$$
g(x) = \sum_{s'}^{\infty} b_k(x), \qquad f(x) = \sum_{s'}^{\infty} c_k(x).
$$
\n2,... then holds

If $k = 1, 2, \ldots$, then holds

0

$$
g(x) = \sum_{s'} \sum_{k=0}^{\infty} b_k(x), \qquad f(x) = \sum_{s'} \sum_{k=0}^{\infty} c_k(x).
$$

If $k = 1, 2, ...$, then holds

$$
[\mathcal{F}^{-1}\varphi_k \mathcal{F}(g)] (x) = \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_k) (y) (gf) (x - y) dy
$$

$$
= 2^{kn} \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0) (2^ky) (gf) (x - y) dy
$$

$$
= \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0) (y) \sum_{k=0}^{\infty} b_k(x - 2^{-k}y) c_k(x - 2^{-k}y) dy.
$$
(1)
The intersection of the supports of φ_k and $\mathcal{F}(b_j c_l)$,

$$
[\mathcal{F}(b_j c_l)] (x) = \int_{\mathbf{R}_n} (\mathcal{F}b_j) (y) (\mathcal{F}c_l) (x - y) dy,
$$

is empty if the non-negative integers l and j do not belong to one of the following
three cases:
(i) $k - 3 \le l \le k + 3$ and $j = 0, ..., k + 3$,
36 Analysis Bd. 4, Het 0. (1985)

The intersection of the supports of φ_k and $\mathcal{F}(b_i c_i)$,

$$
\left[\mathcal{F}(b_jc_l)\right](x) = \int\limits_{\mathbf{R_n}} \left(\mathcal{F}b_j\right)(y) \left(\mathcal{F}c_l\right)(x-y) \, dy,
$$

is empty if the non-negative integers l and j do not belong to one of the following three cases: -• *•* The intersection of the supports
 $[\mathcal{F}(b_j c_l)] (x) = \int_{\mathbf{R}_n} (\mathcal{F}b_j) (x)$

is empty if the non-negative int

three cases:

(i) $k - 3 \leq l \leq k + 3$ and j

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0 0

 $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- *(i)* $k-3 \leq l \leq k+3$ and $j=0, ..., k+3$,
-

(ii)
$$
l = 0, ..., k + 3
$$
 and $k - 3 \leq j \leq k + 3$,

(iii)
$$
l > k + 3
$$
, $j > k + 3$ and $|l - j| < 3$,

i.e., in the sum in (1) there are of interest only values of *j* and *l* given by $(i) - (iii)$.

(iii) $l > k + 3$, $j > k + 3$ and $|l - j| < 3$,
i.e., in the sum in (1) there are of interest only values of *j* and *l*
We use the following composition of $f \cdot g(x) = \sum_{i=1}^{\infty} c_i(x) \sum_{i=1}^{\infty} b_i(x)$:

$$
f\cdot g=T_f g+T_g f+R(f,g),
$$

$$
\quad \text{where} \quad
$$

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\n(ii)
$$
l = 0, ..., k + 3
$$
 and $k - 3 \leq j \leq k + 3$,
\n(iii) $l > k + 3$, $j > k + 3$ and $|l - j| < 3$,
\ni.e., in the sum in (1) there are of interest only values of j and l given by (i)–(iii).
\nWe use the following composition of $f \cdot g(x) = \sum_{l=0}^{\infty} c_l(x) \sum_{j=0}^{\infty} b_j(x)$:
\n $f \cdot g = T_f g + T_g f + R(f, g),$
\nwhere
\n $T_f g = \sum_{l < j-3} c_l b_j, \qquad T_g f = \sum_{j < l-3} b_j c_l, \qquad R(f, g) = \sum_{|l-j| < 3} b_j c_l.$
\nTherefore, it will be sufficient to consider the following three model cases

z **2** *T. Rowsr*

(ii) $l = 0, ..., k + 3$ and $k - 3 \leq j \leq k + 3$,

iii) $l > k + 3, j > k + 3$ and $|l - j| < 3$,

iii) $l > k + 3, j > k + 3$ and $|l - j| < 3$,

We use the following composition of $f \cdot g(x) = \sum_{l=0}^{\infty} c_l(x) \sum_{j=0}^{\infty} b_j(x)$,
 $T_{1}g = \sum_{l \leq j-3} c_{l}o_{j},$ $T_{g}f = \sum_{j \leq l-3} o_{j}c_{l},$ $R_{(j)}g = \sum_{|l-j| \leq 3} o_{j}c_{l}.$
Therefore, it will be sufficient to consider the following three model cases We use the
 $f \cdot g =$

where
 $T_f g =$

Therefore, it
 $(k = 1, 2, ...):$

Case 1:
 $\sum_{k'}$ Therefore, it will be sufficient to cor
 $k = 1, 2, ...$:

Case 1:
 $\sum_{k'} (x) = \sum_{j=0}^{k} \int_{\mathbf{R}_n} (\mathcal{F}^{-1} \varphi_0) (y) b_j(x - \nabla_{\mathbf{R}_n})$
 $\sum_{k'} (x)$ is equivalent to $[\mathcal{F}^{-1} \varphi_k]$

Case 1:'

where
\n
$$
f \cdot g = T_f g + T_g f + R(f, g),
$$
\nwhere
\n
$$
T_f g = \sum_{i < j-3} c_i b_j, \qquad T_g f = \sum_{j < l-3} b_j c_l, \qquad R(f, g) = \sum_{|i-j| < 3} b_j c_l
$$
\nTherefore, it will be sufficient to consider the following to
\n
$$
(k = 1, 2, ...):
$$
\n
$$
Case 1:
$$
\n
$$
\sum_{i=0}^k (x_i - \sum_{j=0}^k \int_{\mathbf{R}_n} (\mathcal{F}^{-1} \varphi_0) (y) b_j (x - 2^{-k} y) c_k (x - 2^{-k} y) dy,
$$
\n
$$
Case 2:
$$
\n
$$
\sum_{l=0}^k (x_l - \sum_{j=0}^k \int_{\mathbf{R}_n} (\mathcal{F}^{-1} \varphi_0) (y) b_k (x - 2^{-k} y) c_l (x - 2^{-k} y) dy.
$$

Case 2:
\n
$$
\int_{0}^{1} \frac{1}{2} \int_{0}^{2} R_{n} \int_{0}^{2} f(x) \int_{0}^{2} f(x
$$

2.
$$
2x \cdot (x) = \sum_{j=0}^{\infty} R_n
$$

\n
$$
\sum_{j=0}^{\infty} R_n
$$

\n
$$
\sum_{k}^{r} (x) \text{ is equivalent to } [\mathcal{F}^{-1}\varphi_k \mathcal{F}(T_g)] (x).
$$

\nCase 2:
$$
\sum_{l=0}^{k} \int_{R_n} (\mathcal{F}^{-1}\varphi_0) (y) b_k(x - 2^{-k}y) c_l(x - 2^{-k}y) dy,
$$

\n
$$
\sum_{k}^{r} (x) \text{ is equivalent to } [\mathcal{F}^{-1}\varphi_k \mathcal{F}(T_g)] (x).
$$

\nCase 3:
$$
\sum_{k}^{r} (x) = \sum_{l=k}^{\infty} \int_{R_n} (\mathcal{F}^{-1}\varphi_0) (y) b_l(x - 2^{-k}y) c_l(x - 2^{-k}y) dy,
$$

\n
$$
\sum_{k}^{r} (x) \text{ is equivalent to } [\mathcal{F}^{-1}\varphi_k \mathcal{F}(R(f \cdot g))] (x).
$$

\nLet
$$
b_j^*(x) = \sup_{y \in R_n} \frac{|b_j(x - y)|}{1 + |2^j y|^{\alpha_1}}
$$

\nand
$$
c_j^*(x) = \sup_{y \in R_n} \frac{|c_j(x - y)|}{1 + |2^j y|^{\alpha_1}}
$$

$$
\sum_{\mathbf{k}}^{\prime\prime\prime}\cdot(x)\text{ is equivalent to }\left[\mathcal{F}^{-1}\varphi_{\mathbf{k}}\mathcal{F}\big(R(f\cdot g)\big)\right](x).
$$

3:
\n
$$
\sum_{k} x''(x) = \sum_{l=k}^{\infty} \int_{R_n} (\mathcal{F}^{-1}\varphi_0) (y) b_l(x - 2^{-k}y) c_l(x - 2^{-k}y) dy,
$$
\n
$$
\sum_{k} x'''(x) \text{ is equivalent to } [\mathcal{F}^{-1}\varphi_k \mathcal{F}(R(f \cdot g))] (x).
$$
\n
$$
b_j^*(x) = \sup_{y \in R_n} \frac{|b_j(x - y)|}{1 + |2^j y|^{a_1}}
$$
\n
$$
c_j^*(x) = \sup_{y \in R_n} \frac{|c_j(x - y)|}{1 + |2^j y|^{a_2}}
$$
\n
$$
\text{maximal functions. We assume } a_1 > 0, a_2 > \frac{n}{p} \text{ if } f \in B_{p,q}^s \text{ and } a_2 > \frac{n}{\min(p, q)}
$$
\n
$$
c = \int_{R_n} |(\mathcal{F}^{-1}\varphi_0) (y) | (1 + |y|)^{a_1 + a_1} dy,
$$

$$
c_j^*(x) = \sup_{y \in \mathbf{R}_n} \frac{|c_j(x - y)|}{1 + |2^j y|^{a_s}} \tag{4}
$$

 $\begin{split} c_j^*(x) = \sup_{y \in \mathbf{R}_n} \frac{|c_j(x-y)|}{1+|2^j y|^a}, \ \text{be the maximal functions. We assume } a_1 > 0, a_2 > \frac{n}{p} \text{ if } f \in B_{p,q}^s \text{ and } a_2 > \frac{n}{p}. \end{split}$ Let $b_j^*(x) = \text{su}$

and $c_j^*(x) = \text{su}$

be the maximal function $f \in F_{p,q}^s$. If
 $c = \int_{\mathbb{R}_+} |(f - f)|^2 dx$ *b*_j*(x) = sup $\frac{|b_j(x - y)|}{\nu \in \mathbb{R}_n}$ (3)

and
 $c_j^*(x) = \sup_{y \in \mathbb{R}_n} \frac{|c_j(x - y)|}{1 + |2^j y|^{\alpha_1}}$ (4)

be the maximal functions. We assume $a_1 > 0$, $a_2 > \frac{n}{p}$ if $f \in B_{p,q}^s$ and $a_2 > \frac{n}{\min(p,q)}$

if $f \in F_{p,q}^s$. If

$$
c = \int_{\mathbf{R_n}} |(\mathcal{F}^{-1} \varphi_0)(y)| (1 + |y|)^{a_1 + a_1} dy,
$$

then we have by [10]

 $\frac{1}{2}$

$$
\sum_{k}^{\infty} |x^{\ast}(x)| \leq c c_k \cdot \frac{k}{j} b_j \cdot \frac{k}{j} (x), \qquad (5)
$$

and
\n
$$
c_{j}^{*}(x) = \sup_{y \in \mathbb{R}_{n}} \frac{|c_{j}(x - y)|}{1 + |2^{j}y|^{a_{s}}}
$$
\n
\nbe the maximal functions. We assume $a_{1} > 0$, $a_{2} > \frac{n}{p}$ if $f \in B_{p,q}^{s}$ and $a_{2} > \frac{n}{\min(p, q)}$
\nif $f \in F_{p,q}^{s}$. If
\n
$$
c = \int_{\mathbb{R}_{n}} |(\mathcal{F}^{-1}\varphi_{0}) (y)| (1 + |y|)^{a_{1} + a_{1}} dy,
$$
\nthen we have by [10]
\n
$$
|\sum_{k} f'(x)| \leq c c_{k}^{*}(x) \sum_{j=0}^{k} b_{j}^{*}(x),
$$
\n
$$
|\sum_{k} f'(x)| \leq c b_{k}^{*}(x) \sum_{l=0}^{k} c_{l}^{*}(x),
$$
\n
$$
|\sum_{k} f''(x)| \leq c \sum_{l=k}^{\infty} 2^{(l-k)(a_{1} + a_{l})} b_{l}^{*}(x) c_{l}^{*}(x).
$$
\n(6)
\n
$$
|\sum_{k} f''(x)| \leq c \sum_{l=k}^{\infty} 2^{(l-k)(a_{1} + a_{l})} b_{l}^{*}(x) c_{l}^{*}(x).
$$
\n(7)

$$
|\angle k \quad (x)| \leq c_0 \sum_{l=0}^{c_0} 2^{(l-k)(a_1 + a_2)} b_l^*(x) c_l^*(x), \tag{7}
$$
\n
$$
|\sum_{l=k}^{c_0} 2^{(l-k)(a_1 + a_2)} b_l^*(x) c_l^*(x).
$$

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 (10)

By (5) we have

$$
2^{ks} |\sum_{k} 'x)| \leq c 2^{ks} c_k * (x) ||b_j^* | l_1(L_{\infty})||. \tag{8}
$$

If $-\infty < s < \infty$, $0 < p$, $q \leq \infty$, then we get by Theorem 1.2.4

$$
2^{ks} \sum_{k} (x) |l_q(L_p)| \leq c \|f \| B_{p,q}^s \| \cdot \|g \| B_{\infty,1}^0 \|.
$$
 (9)

Remark 1: We assumed above that $\mathcal{F}g$ has a compact support. In [10] it was shown that (8) is true for arbitrary functions g , i.e., we have

$$
|2^{ks}\sum_{k} (x) |l_q(L_p)| \leq c ||f||B_{p,q}^s|| \cdot ||g||B_{\infty,1}^0||.
$$

Since this estimate is symmetric, we obtain also

$$
||2^{ks}\sum_{k}^{'}(x) |l_q(L_p)|| \leq c ||g||B_{p,q}^s|| \cdot ||f||B_{\infty,1}^0||.
$$

Applying now the imbedding

$$
B_{p,q}^s \hookrightarrow B_{p,q}^{n/p} \hookrightarrow B_{\infty,1}^0 \hookrightarrow B_{\infty,\infty}^0 \quad \text{if} \quad s > \frac{n}{p},
$$

we have proved

Theorem 1: Let $0 < p, q \leq \infty$ and $s > \frac{n}{n}$. Then

$$
||T_{f}g||B_{p,q}^{s}|| \leqq c||f||B_{p,q}^{s}|| \cdot ||g||B_{p,q}^{s}||
$$

. and

$$
||T_{\mathfrak{g}}f||B_{p,q}^s|| \leq c||g||B_{p,q}^s|| \cdot ||f||B_{p,q}^s||.
$$

Remark 2: We recall that $\mathcal{E}^s = B^s_{\infty,\infty}$ with $s > 0$. Therefore, the above assertion holds also in the case of Hölder-Zygmund spaces.

In the following we shall estimate $R(f, g)$ if $\mathcal{F}g$ has a compact support (cf. [10]). If ε and ε' are arbitrary positive numbers with $\varepsilon' < \varepsilon$, then we have

$$
2^{ks} |\sum_{l} \sum_{i} \binom{n}{l} |
$$
\n
$$
\leq c \sum_{l=k}^{\infty} 2^{(l-k)(a_1+a_2+\epsilon-s)} b_l^*(x) 2^{(s-\epsilon(l-k)} c_l^*(x)
$$
\n
$$
\leq c' ||2^{j\max(0,a_1+a_2+\epsilon-s)} b_j^*| l_{\infty}(L_{\infty})||
$$
\n
$$
\times \left(\sum_{l=k}^{\infty} 2^{-\epsilon' (l-k)q} 2^{l s q} c_l^*(x)^q \right)^{1/q} \quad \text{(modification if } q = \infty). \tag{11}
$$

Choosing a_1, a_2 and s is an appropriate way, it is possible that $\frac{a}{n} < \frac{b}{n} + a_1 \le a_1$ $a_1 + a_2 < s$ and, for small positive ε , max $(0, a_1 + a_2 + \varepsilon - s) = 0$. Therefore, we obtain by (11)

$$
||2^{ks}\sum_{k}^{'}(x)||l_{q}(L_{p})|| \leq c||g||B_{\infty,\infty}^{0}||\cdot||f||B_{p,q}^{s}||. \tag{12}
$$

If either $g \in \mathcal{E}^{\varrho}$, $\varrho > 0$ or $g \in B_{p,q}^{s}$ with $s > \frac{n}{p}$, (12) and (10) show

$$
||2^{k(s+\varrho)} \sum_{k}^{N'} (x) |l_q(L_p)|| \leq c ||g|| \mathcal{E}^{\varrho}|| \cdot ||f|| B_{p,q}^{s}||
$$

 $||2^{ks}\sum_{k}^{\ }||x^{s}||(x)||_{q}(L_{p})||\leq c||g||B_{p,q}^{s}||\cdot||f||B_{p,q}^{s}||.$

 $36*$

and.

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Remark 3: The same argument as in [10] yields that (12) is true for arbitrary $g \in B_{p,q}^s$ and $g \in \mathcal{E}^{\varrho}$.

We have proved

Theorem 2: Let
$$
0 < p, q \leq \infty
$$
 and $s > \frac{n}{p}$. Then

$$
||R(f,g) | B_{p,q}^s|| \leqq c ||g | B_{p,q}^s|| \cdot ||f | B_{p,q}^s||
$$

and for $g \in \mathcal{C}^{\varrho}$, $o > 0$.

 $||R(f,g)||B_{p,q}^{s+q}|| \leq c||g||\mathcal{E}^q|| \cdot ||g||B_{p,q}^s||.$

Remark 4: If $p = q = \infty$, it follows by $\mathcal{E}^s = B^s_{\infty, \infty}$ for $s > 0, \varrho > 0$

$$
||R(f,g)||\mathcal{E}^{s+e}|| \leq c||g||\mathcal{E}^{e}|| \cdot ||f||\mathcal{E}^{e}||.
$$

Now we can obtain results related to [1: Théorème 2.5].

Theorem 3: Let $0 < p, q \leq \infty$.

(i) Let
$$
f \in \mathcal{E}^s
$$
 and $g \in \mathcal{E}^e$, $s > 0$, $\rho > 0$. Then

 $f \cdot g = T_f g + T_g f + R(f, g)$.

 $with$

$$
||R(f,g)||\mathcal{E}^{s+e}|| \leq c ||f||\mathcal{E}^{s}|| \cdot ||g||\mathcal{E}^{e}||.
$$

(ii) Let
$$
f \in B_{p,q}^s
$$
 and $g \in B_{p,q}^t$, $s > \frac{n}{p}$, $t > \frac{n}{p}$. Then
\n
$$
f \cdot g = T_f g + T_g f + R(f,g)
$$
\n
$$
f \cdot g = T_f g + T_g f + R(f,g)
$$

wi

$$
||R(f,g)||B^{s+t-n/p}_{p,q}|| \leq c ||f||B^s_{p,q}|| \cdot ||g||B^t_{p,q}||.
$$

Proof: (i) follows from Theorem 1 and 2. (ii) By (2) we have $R(f,g) = f \cdot g - T_f g - R_g f$. Now, (11) with $0 < \epsilon' < \epsilon$ yields

$$
||2^{k(s+t-n/p)} \sum_{k} t'' (x) |l_q(L_p)||
$$

\n
$$
\leq c \sum_{l=k}^{\infty} 2^{(l-k)(a_1+a_1+\epsilon-s-t+n/p)} b_l^{*}(x) 2^{(s-\epsilon(l-k)} c_l^{*}(x)
$$

\n
$$
\leq c' ||2^{j \max(0, a_1+a_1+\epsilon-s-t+n/p)} b_j^{*}(x) |l_{\infty}(L_{\infty}^{*})||
$$

\n
$$
\times \left(\sum_{l=k}^{\infty} 2^{lqs-\epsilon'(l-k)q} c_l^{*}(x)^q \right)^{1/q}.
$$

Because of $a_1 > 0$, $a_2 > \frac{n}{p}$, $s > \frac{n}{p}$, $t > \frac{n}{p}$ and for small positive ε , it is possible that max $(0, a_1 + a_2 + \varepsilon - s - t + n/p) = 0$. Therefore, we obtain $||2^{k(s+t-n/p)} \sum_{k}^{r'''} (x) ||l'_{q}(L_p)|| \leq c ||g|| B_{\infty,\infty}^0 || \cdot ||f|| B_{p,q}^s||.$

Again by the imbedding theorem (10) we get (ii)

Remark 5: Let either $p = q = 2$ or $p = q = \infty$. Then $B_{2,2}^s = H_2^s$ and $B_{\infty, \infty}^s = \mathcal{E}^s$ = C^s for $0 < s$ + integer. Therefore, our Theorem 3 implies the results obtained by $J. M. Bony in [1].$

Remark 6: By Theorem 3 we get for $u \in \mathcal{E}^{\varrho}, \varrho > 0$,

$$
u^{2} = u \cdot u = T_{u}u + T_{u}u + R(u, u) = T_{2u}u + R(u, u)
$$

with $R(u, u) \in \mathcal{E}^{2\rho}$. Analogously, it is possible to show that for $u \in \mathcal{E}^{\rho}$, $\rho > 0$, $G(u)$ $T_{G'(u)}u + r$ with $r \in \mathcal{E}^{2g}$ and G is a polynomial in u with $G(0) = 0$, cf. [1: p. 227]. In Chapter 3 we shall extend this assertion to arbitrary C^{∞} -functions G with $G(0) = 0$ $(C^{\infty} = C^{\infty}(\mathbf{R}_n)$ denotes the set of all infinitely differentiable functions on \mathbf{R}_n).

In the first part of this chapter we have considered spaces of Besov type. From now on we shall be concerned with the spaces $F_{p,q}^s$. We use the methods described
in [2] and in [10]. As above we restrict ourselves to the three model cases. Here we must take in our consideration, that the conditions of Theorem 1.2.4 (ii) are ful-

filled, if we choose $a > \frac{n}{\min(p, q)}$. By (8) we have

$$
||2^{ks}\sum_{k} (x) |L_p(l_q)|| \leq c ||2^{ks}c_k * (x) |L_p(l_q)|| \cdot ||g||B^0_{\infty,1}||.
$$

Theorem 4: Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \frac{n}{p}$. Then

$$
||T_gf||F_{p,q}^s|| \leqq c||g||F_{p,q}^s|| \cdot ||f||F_{p,q}^s||
$$

 and

$$
|||T_{f}g||F_{p,q}^{s}|| \leq c||f||F_{p,q}^{s}|| \cdot ||g||F_{p,q}^{s}||.
$$

Proof: We have $s > \frac{n}{n}$. By using the imbedding 1.2.3 (i) and (iv) it follows that $F_{p,q}^s \hookrightarrow B_{\infty,1}^0$ and by (13)

$$
||T_gf||F_{p,q}^s|| \leqq c||g||F_{p,q}^s|| \cdot ||f||F_{p,q}^s||.
$$

The same arguments with respect to the support of g (cf. Remark 3) yield the first. assertion. Since our estimates are symmetric, we obtain also the second case

In order to show an estimate of $R(f, g)$ we use the methods introduced in [2].

Theorem 5: Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \frac{n}{n}$. Then

$$
||R(f,g) | F_{p,q}^s|| \leqq c ||g | F_{p,q}^s|| \cdot ||f | F_{p,q}^s||
$$

and for $g \in \mathcal{E}^{\varrho}$, $\varrho > 0$,

$$
||R(f,g)||F_{p,q}^{s+q}|| \leq c||g||\mathcal{E}^q|| \cdot ||f||F_{p,q}^{s}||.
$$

Proof: By $[2:3.3]$ we have

$$
||2^{ks}\sum_{k}^{r'}(x)||L_p(l_q)|| \leq c||f||F_{p,q}^{s}|| \cdot ||g||F_{r,\infty}^{n/r}||
$$

if $0 < p, r < \infty, 0 < q \le \infty$ and $s > n \left(\frac{1}{\min(p, 1)} - 1 \right)$. By using the imbedding 1.2.3. (v) we get

$$
F_{p,q}^s \hookrightarrow B_{2p,p}^{s-n/2p} \hookrightarrow B_{2p,p}^{n/2p} \hookrightarrow B_{2p,\infty}^{n/2p}.
$$

Hence, (14) yields

 $||R(f,g)||F_{p,q}^s|| \leq c||g||F_{p,q}^s|| \cdot ||f||F_{p,q}^s||.$

By means of the procedure described in the proof of Theorem 2 we obtain the second result $\mathbb{I}_{\mathcal{X}}$

We are now in a position to carry over the results in Theorem 3 and the results obtained by J. M. Bony [1: Théorème 2.5], respectively, to the spaces $F_{p,q}^s$.

 (13)

 (14)

Theorem 6: Let $0 < p < \infty,$ $0 < q \leqq \infty,$ $s>\frac{n}{x}$, $t>\frac{n}{x}$, $f \in F_{p,q}^{s}$ and $g \in F_{p,q}^{t}.$ $\sum_{i=1}^{n}$ $\sum_{i=1}^{n}$ $f \cdot g = T_f g + T_g f + R(f,g)$ 566 T. RUNST
 Theorem 6: Let $0 < p < \infty$, $0 < q \le \infty$, $s > \frac{n}{p}$, $t > \frac{n}{p}$,
 Then
 f $\cdot g = T_f g + T_g f + R(f, g)$
 with
 $||R(f, g)||F_{p,q}^{s+t-n/p}|| \le c ||f||F_{p,q}^s|| \cdot ||g||F_{p,q}^t||$.
 The proof is analog to that one of Theorem 3

$$
||R(f,g)||F_{p,q}^{s+t-n/p}|| \leq c ||f||F_{p,q}^{s}|| \cdot ||g||F_{p,q}^{t}||.
$$

The proof is analog to that one of Theorem 3 **¹**

3. A second example of linearization

As' mentioned in Remark 2.6, we shall extend Theorem 2.3 and Theorem 2.6 to arbitrary C^{∞} -functions G with $G(0) = 0$. The purpose of this section is to prove an extension of results obtained by Y. MEYER $[3-5]$ and J. M. Bony $[1]$ to $F_{p,q}^s$ and $B_{p,q}^s$.

$$
\qquad \qquad \text{Theorem 1:} \ \textit{Let}
$$

$$
either \quad 0 < p, \, q < \infty \, and \, s > \max\left(\frac{n}{p}, \, n\left(\frac{1}{\min\left(p, q, 1\right)} - 1\right)\right)
$$
\n
$$
or \quad 0 < p < \infty, \, q = \infty \, and \, s > \frac{n}{p}
$$

and $G \in C^{\infty}(\mathbb{R})$ with $G(0) = 0$. Then $G \colon F_{p,q}^s \to F_{p,q}^s$ defined by $G \colon f \to G(f)$ is bounded. $\text{Remark 1: Because of } H_{p^s} = F_{p,2}^s \text{ and } n\left(\frac{1}{\min{(p, 2, 1)}} - 1\right) = 0, \text{ if } 1 < p < \infty.$ Theorem 1 implies the result obtained by *Y.* **MEYER** in [4: Thèoréme 1]. or $0 < p < \infty$, $q = \infty$ and $s > \frac{n}{p}$

• and $G \in C^{\infty}(\mathbf{R})$ with $G(0) = 0$. Then $G: F_{p,q}^s \to F_{p,q}^s$ defined by $G: f \to G(f)$ is bounded

– Remark 1: Because of $H_{p}^s = F_{p,2}^s$ and $n\left(\frac{1}{\min{(p, 2, 1)}} - 1\right) = 0$, if $1 <$ or $0 \leq p \leq \infty$, $q = \infty$ and $s > \frac{n}{p}$

and $G \in C^{\infty}(\mathbb{R})$ with $G(0) = 0$. Then $G: F_{p,q}^s \to F_{p,q}^s$ defined by $G:$

Remark 1: Because of $H_{p}^s = F_{p,2}^s$ and $n \left(\frac{1}{\min(p, 2, 1)} - 1 \right)$

Theorem 1 implies the resu

Remark 1: Because of $H_p^s = F_{p,2}^s$ an
Theorem 1 implies the result obtained by
Proof of Theorem 1: Step 1: We us
to $G(f)$, cf. [3-5]. Let $\varphi \in C_0^{\infty}(\mathbf{R}_n)$, $\varphi(\xi)$
 $\varphi(\xi) = 0$, if $|\xi| > 1$. Now we define as u $\phi \geq 0$ for all $\xi \in \mathbf{R}_{n}$, $\phi(\xi) = 1$, if = 0, if $|\xi| > 1$. Now we define as usually for $f \in F_{p,q}^s$ and $k = 0, 1$, $\begin{aligned}\n\mu_p^0 &= F_{p,2}^* \text{ and } n \left(\frac{1}{\min(p, 2, 1)} - 1 \right) = 0, \text{ if } 1 < p < \infty. \\
\text{obtained by Y. Meyrstr in [4: Theorem 1].} \\
\text{Step 1: We use the decomposition method with respect} \\
C_0^\infty(\mathbf{R}_n), \varphi(\xi) &\geq 0 \text{ for all } \xi \in \mathbf{R}_n, \varphi(\xi) = 1, \text{ if } |\xi| \leq \frac{1}{2}, \\
\text{where } \xi \in \mathbf{R}_n, \varphi(\xi) &= 0,$

$$
S_k(f) = \mathcal{F}^{-1}\varphi\left(\frac{\xi}{2k}\right) \mathcal{F}f_{k+1}, \varphi(\xi) \equiv 0 \text{ for all } \xi \in \mathbb{R}_n, \varphi(\xi) = 1, \text{ in } |\xi| \geq \frac{1}{2},
$$
\n
$$
S_k(f) = \mathcal{F}^{-1}\varphi\left(\frac{\xi}{2k}\right) \mathcal{F}f_{k+1} \text{ and } \Delta_k(f) = S_{k+1}(f) - S_k(f).
$$
\n
$$
S_k(f) = \mathcal{F}^{-1}\varphi\left(\frac{\xi}{2k}\right) \mathcal{F}f_{k+1} \text{ and } \Delta_k(f) = S_{k+1}(f) - S_k(f).
$$
\n
$$
S_k(f) = \mathcal{F}^{-1}\varphi\left(\frac{\xi}{2k}\right) \mathcal{F}f_{k+1} \text{ and } \Delta_k(f) = S_{k+1}(f) - \Delta_k(f) + \cdots \text{ and}
$$
\n
$$
f_{k+1} := S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f).
$$
\n
$$
S_k(f) = S_{k+1}(f) = S_0(f) + \cdots + S_k(f).
$$
\n
$$
S_k(f) = G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f_k) + \cdots.
$$
\n
$$
S_k(f) = S_k(f).
$$
\n
$$
S_k(f) = S_{k+1}(f) - S_k(f).
$$
\n
$$
S_k(f) = S_k(f).
$$
\n
$$
S_k(f) = S_{k+1}(f) - S
$$

Hence, we have
\n
$$
\text{supp } \mathcal{F} \Delta_k(f) = \{ \xi \mid 2^{k-1} \leq |\xi| \leq 2^{k+1} \},
$$
\n
$$
f = S_0(f) + \Delta_0(f) + \dots + \Delta_k(f) + \dots \text{ and}
$$

$$
f_{k+1} := S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f).
$$

Moreover, we use the following representation formula:

$$
G(f) = G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f_k) + \cdots
$$

Notice that $G(0) = 0$ and $f_0 = S_0(f)$. Hence, it is easy to show an estimate of $G(f_0)$.

Moreover, we have
 $G(f_{k+1}) - G(f_k) = m_k \triangle_k(f)$, $m_k := \int_0^1 G'(f_k + t \triangle_k(f)) dt$. (3) supp $\mathcal{F} \Delta_k(f) \subset \{\xi \mid 2^{k-1} \leq |\xi| \leq 2^{k+1}\},$
 $f \frac{1}{S'} S_0(f) + \Delta_0(f) + \cdots + \Delta_k(f) + \cdots$ and
 $f_{k+1} := S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f).$

Moreover, we use the following representation formula:
 $G(f) = G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f$ $f_{k+1} := S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f).$

Moreover, we use the following representation formula:
 $G(f) = G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f_k) + \cdots$

Notice that $G(0) = 0$ and $f_0 = S_0(f)$. Hence, it is easy to show an Moreover, we hav $\begin{aligned}\nS_0(f) + \Delta_0(f) + \cdots + \Delta_k(f) + \cdots \text{ and} \\
f_{k+1} &:= S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f), \\
\text{reover, we use the following representation formula:} \\
G(f) &:= G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f_k) + \cdots. \\
\text{tice that } G(0) &:= 0 \text{ and } f_0 = S_0(f). \text{ Hence, it is easy to show an estimate of } G(f_0). \\
\text{reover, we have} \\
G(f_{k+1}) - G(f_k) &:= m_k \Delta_k(f), \quad m_k := \int_0$

Moreover, we have
\n
$$
G(f_{k+1}) - G(f_k) = m_k \Delta_k(f), \quad m_k := \int_0^1 G'(f_k + t \Delta_k(f)) dt.
$$
\n
$$
= \int_0^1 G'(f_k + t \Delta_k(f)) dt.
$$
\n
$$
L(g) = \sum_{k=1}^\infty m_k \Delta_k(g)
$$
\n(4)

The operator $L: S(\mathbf{R}_n) \to S(\mathbf{R}_n)$ defined by

$$
L(g) = \sum_{k=0}^{\infty} m_k \Delta_k(g)
$$
 is linear.

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era-Differential Operators 567

 S lep 2: We show that the above operator *L* is a pseudo-differential operator of Fara-Differential Operators 567

Step 2: We show that the above operator *L* is a pseudo-differential operator of

the "exotic" class $L_{1,1}^0$ if $f \in F_{p,q}^s$, $s > \frac{n}{p}$. As usual we say that a function $\sigma(x, \xi)$
 $\in C$ Fara-Differential Operators ϵ
 Step 2: We show that the above operator *L* is a pseudo-differential operator

the "exotic" class $L_{1,1}^0$ if $f \in F_{p,q}^s$, $s > \frac{n}{p}$. As usual we say that a function $\sigma(x)$
 ϵC^{\in and β there exists a positive constant $c_{\alpha,\beta}$ such that $\in C^{\infty}(\mathbb{R}_n \times \mathbb{R}_n)$ belongs to $S_{\varrho,\delta}^m$, $m \in \mathbb{R}$, $0 \leq \varrho \leq \delta \leq 1$, if for each multi-index α exotic class $L_{1,1}^x \text{ if } f \in F_{p,q}$, $s > \frac{1}{p}$. As usual we say that a funct
 ${}^{\infty}(\mathbf{R}_n \times \mathbf{R}_n)$ belongs to $S_{p,\delta}^m$, $m \in \mathbf{R}, 0 \leq \varrho \leq \delta \leq 1$, if for each mul
 β there exists a positive constant $c_{a,\$

$$
|D_{\xi}^{\alpha}D_{x}^{\beta}\sigma(x,\xi)|\leq c_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\alpha|+\delta|\beta|}
$$

holds for all x and ξ in \mathbf{R}_n . If $\sigma \in S_{\varrho,\delta}^m$, then the corresponding pseudodifferential operators $\sigma(x, D)$ is said to be in class $L_{\varrho,\delta}^m$. Here, the pseudodifferential operator $\sigma(x, D)$ with symbol σ operators $\sigma(x, D)$ is said to be in class $L_{\rho, \delta}^m$. Here, the pseudodifferential operator $\sigma(x, D)$ with symbol σ is defined, as usual, by and β there exists a positive constant $c_{\alpha,\beta}$ such t
 $|D_{\xi}^* D_x^{\beta} \sigma(x, \xi)| \leq c_{\alpha,\beta} (1 + |\xi|)^{m - \varrho |\alpha| + \delta |\beta|}$

holds for all x and ξ in \mathbb{R}_n . If $\sigma \in S_{\varrho,\delta}^{m}$, then the

operators $\sigma(x, D)$ is said to $|D_i^*D_x^{\beta}\sigma(x,\xi)| \leq c_{x,\beta}(1+|\xi|)^{m-\epsilon|a|+\delta|\beta|}$
holds for all x and ξ in \mathbb{R}_n . If $\sigma \in S_{\ell,\delta}^m$, then the corresponding pseudodifferential
operators $\sigma(x, D)$ is said to be in class $L_{\ell,\delta}^m$. Here, the pseudodi

$$
\sigma(x, D) f(x) = \int_{\mathbf{R}_n} e^{ix\xi} \sigma(x, \xi) \mathcal{J} f(\xi) d\xi, \qquad x \in \mathbf{R}_n, \quad f \in S.
$$

At first, we observe that the symbol σ of *L* defined by (4) is given by

$$
\sigma(x, D) f(x) = \int_{\mathbf{R}_n} e^{ix\xi} \sigma(x, \xi) \mathcal{F}/(\xi) d\xi, \qquad x \in \mathbf{R}_n; \quad f \in S.
$$

At first, we observe that the symbol σ of L defined by (4) is given by

$$
\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x) \mathcal{V}(2^{-k}\xi), \qquad \mathcal{V}(\xi) := \varphi\left(\frac{\xi}{2}\right) - \varphi(\xi).
$$
(5)
Hence, we have to prove that

$$
|D_{\xi} \circ D_x \circ \sigma(x, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|)^{-|\alpha| + |\beta|},
$$
holds for each multi-index α and β . From $F_{p,q}^s \hookrightarrow B_{\infty,1}^0 \hookrightarrow L_{\infty}$ if $s > \frac{n}{p}$, cf. 1.2.3, it follows that $||f_k| L_{\infty}|| \leq c$. Hence we have
 $||D^{\beta}f_k| L_{\infty}|| \leq c2^{k|\beta|}$ and $||D^{\beta}G'(f_k)| L_{\infty}|| \leq c_{\beta}2^{k|\beta|}.$
(3) yields
 $||D_x^{\beta}m_k(x)| L_{\infty}|| \leq c_{\beta}^{\prime}2^{k|\beta|}.$
From the last estimate and the properties of the functions Ψ follows that $\sigma \in S_{1,1}^0$

 $\sum_{i=1}^{N} \frac{1}{2^{i}n^{i}} \left(\frac{1}{2^{i}n^{i}} - \frac{1}{2^{i}n^{i}} \right)$

$$
|D_{\xi}^{\alpha}D_{x}^{\beta}\sigma(x,\xi)|\leq c_{\alpha,\beta}(1+|\xi|)^{-|\alpha|+|\beta|}\qquad (6)
$$

Hence, we have to prove that
 $|D_{\xi} \cdot D_x \ell_{\sigma}(x, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|)^{-|\alpha| + |\beta|}$,

holds for each multi-index α and β . From $F_{p,q}^s \hookrightarrow B_{\infty,1}^0 \hookrightarrow L_{\infty}$ if $s >$

follows that $||f_k||L_{\infty}|| \leq c$. Hence we have
 $||D^{\$ p

$$
||D^{\beta}f_k||L_{\infty}|| \leq c2^{k|\beta|} \quad \text{and} \quad ||D^{\beta}G'(f_k)||L_{\infty}|| \leq c_{\beta}2^{k|\beta|}.
$$

•

V

$$
||D_x^{\beta}m_k(x)||L_{\infty}|| \leq c_{\beta}^{\prime}2^{k|\beta|}.
$$

From the last estimate and the properties of the functions Ψ follows that $\sigma \in \mathcal{S}$ **^V '**

 $\sigma(x,\xi) = \sum_{k=0} m_k(x) \ \Psi(2^{-k}\xi), \qquad \Psi(\xi) := \varphi\left(\frac{\xi}{2}\right) -$

Hence, we have to prove that
 $|D_i^* D_x^{\beta} \sigma(x,\xi)| \leq c_{a,\beta}(1+|\xi|)^{-|a|+|\beta|}$

holds for each multi-index α and β . From $F_{p,q}^{\delta} \hookrightarrow B_{\infty,1}^0 \hookrightarrow$

follows that *Step* 3: We prove the boundedness of pseudodifferential operators of class $L_{1,1}^0$ in Triebel-Lizorkin spaces $F_{p,q}^s$. The following result was obtained in [7: Theorem 1] by the author: $|D_i^a D_x^b \sigma(x, \xi)| \leq c_{a,\beta}(1 + |\xi|)^{-|a| + |\beta|}$

holds for each multi-index α and β . From $F_{p,q}^s \hookrightarrow B_{\infty,1}^0 \hookrightarrow L_{\infty}$ if $s >$

follows that $||f_k | L_{\infty}|| \leq c$. Hence we have
 $||D^{\beta}f_k | L_{\infty}|| \leq c2^{k|\beta|}$ and $||D^{\beta}G$

yields
\n
$$
||D_x^{\beta} n_k(x) || L_{\infty}|| \leq c_{\beta}^{\alpha} 2^{k|\beta|}.
$$
\n
$$
||D_x^{\beta} m_k(x) || L_{\infty}|| \leq c_{\beta}^{\alpha} 2^{k|\beta|}.
$$
\n
$$
||D_x^{\beta} m_k(x) || L_{\infty}|| \leq c_{\beta}^{\alpha} 2^{k|\beta|}.
$$
\n
$$
f(x, D) \in L_{1,1}^0.
$$
\n
$$
f(x, D) \in
$$

If $T \in L_{1,1}^{\overline{0}}$, then $T: F_{p,q}^s \to F_{p,q}^s$. This assertion completes our proof \blacksquare
The counternart of Theorem 1 is

Theorem 2: Let $0 < p, \, q \leq \infty, \, s > \max\left(\frac{n}{p} \right)$ = ∞ .

n completes our p
 $\frac{n}{p}$, $n\left(\frac{1}{\min{(p, 1)}}\right)$
 $\frac{n}{G}$; $f \rightarrow G(f)$ is both Theorem 2: Let $0 < p, q \le \infty$, $s > \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p,1)}-1\right)\right)$ and $G \in C^{\infty}(\mathbb{R})$. If $T \in \overline{L_{1,1}^0}$, then $T: F_{p,q}^s \to F_{p,q}^s$. This assertion completes our proof \blacksquare

The counterpart of Theorem 1 is

Theorem 2: Let $0 < p$, $q \leq \infty$, $s > \max\left(\frac{n}{p}, n\left(\frac{1}{\min{(p, 1)}} - 1\right)\right)$ as
 with $G(0) = 0$. T *with* $G(0) = 0$. Then $G: B_{p,q}^s \to B_{p,q}^s$ defined by $G: f \to G(f)$ is bounded.

Proof: We use the above methods and a result obtained in $[7:$ Theorem 4) concerning the boundedness of operators of class $L_{1,1}^0$ in Besov spaces $B_{p,q}^s$ *V, V*

' **^V**

^V -

Remark 2: Theorem 3 and 4 in [7] was obtained for general pseudodifferential. operators of class $L_{1,1}^0$, i.e. no restrictions on the structure of the symbol σ . Those 568 T. RUNST

Remark 2: Theorem 3 and 4 in [7] was obtained for general

operators of class $L_{1,1}^0$, i.e. no restrictions on the structure of the

results are contained in

Theorem 3: (i) Let $0 < p < \infty$, $0 < q \le \infty$ and
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Remark 2: Theorem 3 and 4 in [7] was obtained for geoperators of class $L_{1,1}^0$, i.e. no restrictions on the structure

results are contained in

Theorem 3: (i) Let $0 < p < \infty$, $0 < q \le \infty$ and
 $\left\{ n \left$

 $If \circ f \in S_{1,1}^m$, $-\infty < m < \infty$, then the corresponding pseudodifferential operator $T = \sigma(x, D)$ is bounded from $F_{p,q}^{s+m}$ into $F_{p,q}^s$

(ii) Let $0 < p, q \le \infty$ and $s > n\left(\frac{1}{\min{(n-1)}}-1\right)$. If $\sigma \in S_{1,1}^m$, $-\infty < m <$ $\begin{array}{r} e \left(\begin{array}{c} q \ = \infty \end{array} \right) \ \hline \begin{array}{c} \hline \begin{array}{c} \hline \end{array} \left(\begin{array}{c} \begin{array}{c} \hline \end{array} \left(\begin{array}{c} \hline \end{array} \left(\begin{array}{c} \hline \end{array} \right) \end{array} \right) \ \hline \begin{array}{c} \hline \end{array} \left(\begin{array}{c} \hline \end{array} \left(\begin{array}{c} \hline \end{array} \right) \end{array} \right) \ \hline \begin{array}{c} \hline \end{array} \left$ \int *tor* $T = \sigma(x, D)$ is bounded from $B_{q,q}^s$ *into* $B_{p,q}^s$. For the $S_{1,1}^m$, $-\infty < m < \infty$, then the corresponding p.

The $\sigma(\mathbf{x}, D)$ is bounded from $F_{p,q}^{s+m}$ into $F_{p,q}^s$.

(ii) Let $0 < p, q \le \infty$ and $s > n \left(\frac{1}{\min (p, 1)} - 1 \right)$. If

then the corresponding pseudodifferential $=$ $\frac{1}{2}$
and
 $\frac{1}{2}$ $T = \sigma(x, D)$ is bounded from $F_{p,q}^{s,n}$ into $F_{p,q}^s$.

(ii) Let $0 < p, q \le \infty$ and $s > n \left(\frac{1}{\min (p, 1)} - 1 \right)$. If $\sigma \in S_{n,1}^m, -\infty < m < \infty$,

then the corresponding pseudodifferential operator $T = \sigma(x, D)$ is bounded from

Remark 3: Theorem 3 is an extension of results discovered by Y. MEYER $[3-5]$. ations. **Theorem 3** is an extension of results discovered b

rems presented in this chapter are fundamental for o

rk 4: Because of $F_{p,2}^s = H_{p^s}^s, 1 < p < \infty$, Theorem 3

tion is false, if $s = 0$, cf. [4].

rk 5: Pseudodifferen

Remark 4: Because of $F_{p,2}^s = H_p^s$, $1 < p < \infty$, Theorem 3 is valid for $s > 0$.
The assertion is false, if $s = 0$, cf. [4]. ark 4: Because of $F_{p,2}^s = H_{p^s}$, $1 <$
ertion is false, if $s = 0$, cf. [4].
ark 5: Pseudodifferential operators
ark 6: Let $r > 0$ and $T \in L_{1,1}^{-r}$.
ark 6: Let $r > 0$ and $T \in L_{1,1}^{-r}$.
 $0 < p < \infty$, $0 < q \le \infty$, *s* sat theorems presented in this chapter are tundamental t
 Theoremark 4: Because of $F_{p,2}^s = H_p^s$, $1 < p < \infty$, Theore

assertion is false, if $s = 0$, cf. [4].
 Emark 5: Pseudodifferential operators of class $S_{p,3}^m$ act

Remark 5: Pseudodifferential-operators of class $S_{\varrho,\delta}^{m}$ acting in Triebel-Lizorkin *•*

Remark 6: Let $r > 0$ and $T \in L_{1,1}^{-r}$.

(i) If $0 < p < \infty$, $0 < q \leq \infty$, *s* satisfies the conditions of Theorem 1, then

(ii) If $0 < p, q \leq \infty$, *s* satisfies the conditions of Theorem 2, then

$$
T: B_{n,q}^t \to B_{n,q}^{t+r} \quad \text{for all} \quad t > s-r,
$$

4. Para-products of J. M. Bony, a third example of linearization

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4.1. Para-products

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The calculus of para-products was introduced by J. M. BONY in [1].

Definition 1: Let $u, v \in S'$. Then the *para-product* $w = T_u v$ is defined by

 \mathbf{r}

$$
w=\sum_{k=3}^{\infty} S_{k-2}(u) \Delta_k(v)
$$

Remark 1: J. M. Bony denoted the para-product by $w = \pi(u, v)$. Comparing .this definition with Chapter 2, we obtain that the operator of para-multiplication is essentially the operator $T_{u}v$ in the theory of multiplication algebras. Hence, we denote the para-product by the same symbol.

0• • '. **^S**

Theorem 1: Let $0 < p < \infty$, $0 < q \le \infty$, $s > s_r$,

Para-Differential Operators	569				
Theorem 1: Let $0 < p < \infty, 0 < q \leq \infty, s > s_f$,	$s_f := \begin{cases} \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p, q, 1)} - 1\right)\right) & \text{if } q < \infty \\ \frac{n}{p} & \text{if } q = \infty \end{cases}$ \n	(1)			
<i>We put</i> $s = s_f + r, r > 0$. Then for each $f \in F_{p,q}^s$ and $G \in C^\infty(\mathbb{R})$ with $G(0) = 0$					
<i>we have</i>	$G(f) = T_{G'(f)}f + g$,	$G(f) = T_{G'(f)}f + g$,	$G(f) = T_{G'(f)}f + g$,	$G(f) = T_{G'(f)}f + g$,	$G(f) = T_{p,q} \in F_{p,q}^s$.
<i>Remark 2: Because</i> of $F_{p,2}^s = H_p^s$, $1 < p < \infty$, $s_f = \frac{n}{p}$, Theorem 1 yields the result obtained by Y. MEXER [3, -5], $G(0) = 0$ is a necessary condition. By 1.2.1(i)					
$S \hookrightarrow F_{p,q} \hookrightarrow S'$. If $G(x) \equiv a, a \neq 0$, then $G(f) \in S$ holds not for general $f \in S$. In					

We put $s = s_F + r$, $r > 0$. Then for each $f \in F_{p,q}^s$ and $G \in C^{\infty}(\mathbb{R})$ with $G(0) = 0$ *we have*

$$
\neg G(f) = T_{G' \cap f} + g,
$$

 $where \, g \in F_{p,q}^{s'}, \, s' = s_F + 2r.$

Theorem,1 yields the result obtained by Y. MEYER [3-5]. $G(0) = 0$ is a necessary condition. By 1.2.1(i) $S \hookrightarrow F_{p,q} \hookrightarrow S'$. If $G(x) = a, a \neq 0$, then $G(f) \in S$ holds not for general $f \in S$. In *S S F*_{*p***₁},** *S*** =** *s_P* **+ 2***r***.
** *Remark 2:* **Because of** $F_{p,2}^s = H_p^s$ **,** $1 < p < \infty$ **,** $s_F = \frac{n}{p}$ **, Theorem, 1 yields the result obtained by Y. MEYER [3,-5].** $G(0) = 0$ **is a necessary condition. By 1.2.1(i) S \hookrightarrow</sub>** $\mathcal{S}_F := \left\{ \begin{aligned} &\max \left(\frac{n}{p}, n \left(\frac{1}{\min{(p, q, 1)}} - 1 \right) \right) &\text{if } q < \infty \\ &\frac{n}{p} &\text{if } q = \infty \,. \end{aligned} \right.$

We put $s = s_f + r, r > 0$. Then for each $f \in F_{p,q}^s$ and $G \in C^\infty(\mathbf{R})$ with $G(0)$ we have
 $G(f) = T_{G'(f)}f + g,$

where g *Step 1:* $s = s_F + r, r > 0$. Then for each $f \in F_{p,q}^s$ and s
 $G(f) = T_{G'(f)}f + g$,
 $s = s_F + 2r$.

Remark 2: Because of $F_{p,q}^s = H_p^s$, $1 < p < \infty$, $s_F =$

sult obtained by Y. MEYER [3,-5]. $G(0) = 0$ is a necess
 $\hookrightarrow F_{p,q} \hookrightarrow S'$.

Proof of Theorem 1: We use the mapping properties of pseudodifferential operators obtained in Chapter 3 and the methods of $[4:$ Théorème. 4].
Sten 1: By Chapter 3 we get the linearization *Philadelphia <i>C*
 A
 A
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$$
I: L_y
$$
 independent or *sec* and *since*

$$
G(f)=L(f)+S_0(f),
$$

where $L \in L_{1,1}^0$ with the symbol

tors obtained in Chapter 3 and p 1: By Chapter 3 we get the
 $G(f) = L(f) + S_0(f)$,
 $E L \in L_{1,1}^0$ with the symbol
 $\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x) \Psi(2^{-k}\xi)$ *k=O* and

where
$$
L \in L_{1,1}^0
$$
 with the symbol
\n
$$
\sigma(x,\xi) = \sum_{k=0}^{\infty} m_k(x) \ \Psi(2^{-k}\xi)
$$
\nand
\n
$$
m_k(x) = \int_0^1 G'(f_k + t \Delta_k(f)) \, dt, \qquad f_k = S_k(f).
$$
\nBy 1.2.3 we have
\n
$$
F_{p,q}^s \hookrightarrow F_{p,q}^{S_p+r} \hookrightarrow \mathcal{E}^r
$$
\nand hence $G'(f) \in \mathcal{E}^r$. Evidently, $T_a \in L_{1,1}^0$ with the symbol

$$
F_{p,q}^s \hookrightarrow F_{p,q}^{S_P+r} \hookrightarrow \mathcal{E}^r
$$

and hence $G'(f) \in \mathcal{E}^r$. Evidently, $T_a \in L^0_{1,1}$ with the symbol

$$
\sum_{k=2}^{\infty} S_{k-2}(a) \Psi(2^{-k}\xi) \qquad (a \in L_{\infty} \text{ fixed}).
$$

Step 2: We prove $L(f) - T_{G(f)}f = \varrho(x, D)f$, where $\varrho \in S_{1,1}^{-r}$. It is sufficient to show that $\|\cdot\|$ $m_k(x) = \int_0^1 G'(f_k + t \Delta_k(f)) dt$, $f_k = S_k(f)$.

we have
 $F_{p,q}^s \hookrightarrow F_{p,q}^{S_r+r} \hookrightarrow \mathcal{E}^r$
 $\in G'(f) \in \mathcal{E}^r$. Evidently, $T_a \in L_{1,1}^0$ with the symbol
 $\sum_{k=2}^\infty S_{k-2}(a) \Psi(2^{-k}\xi)$ $(a \in L_\infty \text{ fixed})$.

We prove $L(f) - T_{G'(f)}f = \varrho(x, D$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ sufficient
a sufficient $\sum_{k=2}^{\infty} S_{k-2}(a) \Psi(2^{-k}\xi)$ $(a \in L_{\infty} \text{ fixed}).$
 Step 2: We prove $L(f) = T_{G'(f)}f = \varrho(x, D)f$, where $\varrho \in S_{1,1}^{-r}$. It is sufficient to

show that
 $||D^s m_k(x) - D^s S_{k-2}(a) ||L_{\infty}|| \le c_s 2^{k|\alpha|-kr}$.

Here $a = G'(f), f \in F_{p,q}^s$, m *D) f,* where
 $\frac{d|a|-kr}{2}$ now imbedc

e obtain Theo

r, cf. Remark

)

$$
||D^{\alpha}m_{k}(x) - D^{\alpha}S_{k-2}(a)||L_{\infty}|| \leq c_{a}2^{k|a|-k\tau}.
$$
\n(3)

Here $a = G'(f)$, $f \in F_{p,q}^s$, m_k as above. Using now imbedding (2), then (3) follows by the methods in [4: Prop. 2]. and hence $G'(f) \in \mathcal{E}$. Evidently, $T_a \in L_{1,1}^o$ with the symbol
 $\sum_{k=2}^{\infty} S_{k-2}(a) \Psi(2^{-k}\xi)$ $(a \in L_{\infty} \text{ fixed}).$
 $Step 2: \text{ We prove } L(f) - T_{G'(f)}f = \varrho(x, D)f, \text{ where } \varrho \in S_{1,1}^{-r}.$ It is sufficient

show that
 $\|D^m n_k(x) - D^s S_{k-2}(a$

Step 3: Applying now Theorem 3.3(i), we obtain Theorem 1 \blacksquare

Remark 3: $g(x, D)$ is smoothing of order r, cf. Remark 3.6. Using Theorem 3.3(ii),

Step 2: We prove
$$
L(f) - T_{G'(f)}f = \varrho(x, D)f
$$
, where $\varrho \in S_{1,1}^{-r}$. It is sufficient to how that $||D^s m_k(x) - D^s S_{k-2}(a) | L_{\infty}|| \leq c_2 2^{k|a|-kr}$.\n\nHere $a = G'(f), f \in F_{p,q}^s, m_k$ as above. Using now imbedding (2), then (3) follows by the methods in [4: Prop. 2]. *Step 3*: Applying now Theorem 3.3 (i), we obtain Theorem 1 **Example 3**: $\varrho(x, D)$ is smoothing of order r , cf. Remark 3.6. Using Theorem 3.3 (ii), it is not hard to prove f . Theorem 2: Let $0 < p, q \leq \infty, s > s_B$, $s_B := \max \left(\frac{n}{p}, n \left(\frac{1}{\min(p, 1)} - 1 \right) \right)$.

We put $s = s_B + r$, $r > 0$. *Then for each* $f \in B_{p,q}^s$ and $G \in C^{\infty}(\mathbb{R})$ with $G(0) = 0$ we have $G(f) = T_{G'(f)}f + g$, where $g \in B_{p,q}^{s'}$, $s' = s_B + 2r$. *have G(f)* = *T*. Runst
 He put $s = s_B + r$, $r > 0$. Then for each $f \in B_{p,q}^s$ and $G \in C^{\infty}(\mathbb{R})$ with $G(0)$
 have $G(f) = T_{G'(f)}f + g$ *, where* $g \in B_{p,q}^{s'}$, $s' = s_B + 2r$.
 The following theorem generalizes Theorem 1 an

The following theorem generalizes Theorem 1 and 2. We use the concept of localization and micro-localization.

Definition 2: A function $f(x)$ is *locally* of class $B_{p,q}^s(F_{p,q}^s)$ at the point $x = x_0$, if $\Psi f \in B_{p,q}^{s}(F_{p,q}^{s})$ for any C^{∞} -function $\Psi(x)$ not vanishing at x_{0} and supported in a sufficiently small neighborhood of x_0 .

A function $f(x)$ is *locally* of class $B_{p,q}^s(F_{p,q}^s)$, if $\forall f \in B_{p,q}^s(F_{p,q}^s)$ for any function $\Psi(x) \in C_0^{\infty}$. Here $C_0^{\infty} = C_0^{\infty}(\mathbf{R}_n)$ is the set of all complex-valued infinitely differentiable functions with compact support in \mathbf{R}_{n} .

Definition 3: Let $\Psi(x)$ be the function from the first part of the preceding defi*nition. We say* $f(x)$ *is <i>micro-locally* of class $B_{p,q}^s(F_{p,q}^s)$ *at the point* $(x, \xi) = (x_0, \xi_0)$ *in the cotangent space* if the Fourier transform of *I-'/* is equal to the Fourier transform of a $B_{p,q}^s(F_{p,q}^s)$ function in a canonical neighborhood of ξ_0 (i.e., $\xi_{\vert\vert\xi\vert}$ near $\xi_0\vert_{\vert\xi_0\vert}$). ition 2: A function $f(x)$ is loctor $\sum_{p,q}^{s}(F_{p,q}^{s})$ for any C^{∞} -function Ψ
ly small neighborhood of x_0 .
tion $f(x)$ is locally of class B_{p}^{s} ,
 ∞ . Here $C_0^{\infty} = C_0^{\infty}(\mathbf{R}_n)$ is the
unctions with .
....
.... *The intermal of the scaling of class tier* transforming the same $\frac{1}{\delta X} \left\{ \begin{array}{l} Z \subset \infty, \; 0 \leq \frac{1}{\delta X} \left(x, f_1, \ldots, f_m \right)^T \ \text{if} \; s \in \mathbb{R} \setminus \mathbb{R} \setminus$

Theorem 3: (i) Let $0 < p < \infty$, $0 < q \leq \infty$, $s > s_F$ and $s = s_F + r$, $r > 0$. If $f_j \in F_{p,q}^s$ $(j = 1, ..., m)$ and $G = G(x, X_1, ..., X_m) \in C^{\infty}(\mathbb{R}_n \times \mathbb{R}_m)$, then ot a $B_{p,q}^s(F_{p,q}^s)$ function in a canonical neight
 $\Gamma_f \in F_{p,q}^s$ ($j = 1, ..., m$) and $G = G(x, X_1, ...$
 $G(x, f_1, ..., f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, ..., f_m)} f_j$
 Γ_f
 Γ_f where g belongs local to $F_{p,q}^s$, $s' = s_F + 2r$.

(ii) L

$$
G(x, f_1, \ldots, f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, \ldots, f_m)} f_j + g,
$$

(ii) Let $0 < p, q \le \infty$, $s > s_B$ and $s = s_B + r, r > 0$. If $f_j \in B_{p,q}^s$ ($j = 1, ..., m$)

and $G = G(x, X_1, ..., X_m) \in C^{\infty}(\mathbb{R}_n \times \mathbb{R}_m)$, then
 $G(x, f_1, ..., f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, ..., f_m)} f_j + g$, $and G = G(x, X_1, \ldots, X_m) \in C^\infty(\mathbb{R}_n \times \mathbb{R}_m)$, then *where g belongs local to* $F_{p,q}^s$, $s' = s_F + 2r$.

(ii) Let $0 < p$, $q \leq \infty$, $s > s_B$ and $s =$

and $G = G(x, X_1, ..., X_m) \in C^{\infty}(\mathbf{R}_n \times \mathbf{R}_m)$
 $G(x, f_1, ..., f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, ..., f_m)}$

where g belongs local to $B_{p,q}$

$$
G(x, f_1, ..., f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, ..., f_m)} f_j + g, \quad \vdots
$$

Remark 4: Theorem 3 generalizes results of **J. M.** BONY [1] and Y. MEYER [3-5] and $G = G(x, X_1, ..., X_m) \in C^{\infty}(\mathbf{R}_n \times \mathbf{R}_m)$, then
 $G(x, f_1, ..., f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, ..., f_m)} f_j + g$,

where g belongs local to $B_{p,q}^s$, $s' = s_B + 2r$.

Re mark 4: Theorem 3 generalizes results of J. M. Boxy [1] and Y.

4.2. Para-differential operators

Para-differential operators were recently introduced by J. M. BONY [1]. The theory of para-differential operators may be found in $[1, 3-5]$. The theory is also applicable to the function spages considered here. The following definitions and properties $G(x, f_1, ..., f_m) =$

where g belongs local to B

Remark 4: Theorem

to $B_{p,q}^s$ and $F_{p,q}^s$. In the f

4.2. Para-differential operator

Para-differential operator

of para-differential operator

to the function spaces

may may be found in the above quoted papers. + 2r.

s results of J. M. Bony [1] and

and s_B are defined by (1) and

the same of S. M. Bony

found in [1, 3-5]. The theory

here. The following definition

apers.

Then A_r^m is the set of all syn

y^{m-|a|}

|)^{m-|a|}

Definition 1: Let $m \in \mathbb{R}$, $r > 0$. Then A_r^m is the set of all symbols $\sigma = \sigma(x, \xi)$ may be found in the above quantum $\sum_{i=1}^{n} P_i$.
Such that (i) $\text{Definition 1: Let } m \in \mathbb{R}, r > 0.$

(i) $||D_i^*\sigma(\cdot,\xi)||\mathcal{E}^{\eta}|| \leq c_s(1+|\xi|)$

for each multi-index α and

(ii)
$$
|D_{\xi}^{\alpha}D_{x}^{\beta}\sigma(\cdot,\xi)| \leq c_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|+|\beta|-r}
$$

for each multi-index β with $|\beta| > r$ and each multiindex α .

Definition 1: Let $m \in \mathbb{R}$, $r > 0$. Then A_r^m is the set of all symbols $\sigma = \sigma(x, \xi)$
ch that
(i) $\|D_i^*\sigma(\cdot, \xi) \,|\, \mathcal{E}^r\| \leq c_s(1 + |\xi|)^{m-|a|}$
reach multi-index α and
(ii) $|D_i^*\overline{D_x}^{\beta}\sigma(\cdot, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|$ Remark 1: It holds $S_{1,0}^m \subset A_r^m \subset S_{1,1}^m$. Here $A_r^0 = A_r$. We define the corresponding operator class in the usual way and denote it by Op A_r^m .

(1)

Definition 2: $B_r^m \subset A_r^m$ denotes the set of all symbols $\sigma = \sigma(x,\xi)$ such that Definition 2: $B_r^m \subset$
(i) $\qquad \qquad \|D^{\circ}\sigma(\cdot,\,\xi)\|$ if if $||D^{\alpha}\sigma(\cdot,\xi)||\mathcal{E}^{\tau}|| \leq c_{\alpha}(1 + |\xi|)^{m-|\alpha|}$ Para-Differential Operators 5

Fara-Differential Operators 5

f
 $\lim_{n \to \infty}$
 f_{π} $\lim_{n \to \infty} f(x, \xi) \leftarrow \int_{\mathcal{R}} |x| |x| < \frac{|\xi|}{|\xi|}$ Definition 2: $B_r^m \subset A_r^m$

(i) $||D^a\sigma(\cdot,\xi)||\mathcal{E}^r|| \leq c_a(1)$

and

(ii) for all fixed ξ holds as

In [4] may be found the 'foll (i) $||D^{\alpha}\sigma(\cdot,\xi)||g^{\alpha}|| \leq c_{\alpha}(1+|\xi|)^{m-|\alpha|}$

(ii) for all fixed ξ holds supp $\mathcal{F}_{x\to\eta}\sigma(x,\xi) \subset \left\{\eta \mid |\eta| \leq \frac{|\xi|}{10}\right\}$.

$$
\mathbf{and} \quad
$$

(ii) for all fixed
$$
\xi
$$
 holds supp $\mathcal{F}_{x\to\eta}\sigma(x,\xi) \subset \left\{\eta \mid |\eta| \leq \frac{|\xi|}{10}\right\}.$

In [4] may be found the following facts:

1. If L, denotes the above defined operator with symbol $\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x) \Psi(2^{-k}\xi)$. λ *and* $f \in H_p^s = F_{p,2}^s$ *,* $1 < p < \infty$ *,* $s = \frac{n}{p} + r$ *,* $r > 0$ *, then* $L \in \text{Op } A_r^{\frac{k-1}{2}}$ *.* and

(ii) for all fixed ξ holds supp

In [4] may be found the follow

1. If L denotes the above defin

and $f \in H_p^s = F_{p,2}^s, 1 < p < \infty$,

Using the imbedding theorer

where $0 < p < \infty$, $0 < q \le \infty$,
 $r > 0$).

2. If $a \in \math$ or with syml
 $r, r > 0$, the

2.3, we find
 $k = 0$
 $r, r > 0$
 $T_a: f \rightarrow T_a f$
 $\sum_{k=6}^{\infty} S_{k-6}(a)$ \angle
 $\sum_{k=6}^{\infty} S_{k-6}(a)$ \angle
 $\sum_{k=6}^{\infty} S_{k-6}(a)$

Using the imbedding theorems in 1.2.3, we find $L \in \text{Op } A_r$, if $f \in F_{p,q}^s(B_{p,q}^s)$, where $0 < p < \infty$, $0 < q \leq \infty$, $s = s_f + r$, $r > 0$ $(0 < p, q \leq \infty$, $s = s_B + r$, $r>0$).

2. If $a \in \mathcal{E}^r$, $r > 0$, then the operator $T_a : f \to T_a f$ belongs to Op A_r . It holds Using the imbedding theorems in 1.2.3, we find *I* where $0 < p < \infty$, $0 < q \le \infty$, $s = s_F + r$, $r > 0$ (0
 $r > 0$).

2. If $a \in \mathcal{E}^r$, $r > 0$, then the operator $T_a: f \to T_a f$ b
 $T_a \in \text{Op } B_r$ if the para-product is defined by $T_a \in \text{Op } B_r$ if the para-product is defined by $\sum_{k=6}^{\infty} S_{k-6}(a) \triangle_k(f)$. We have $\text{Op } A_r \equiv \text{Op } B_r$
(mod r - smoothing).
In Chapter 5 we shall describe micro-local regularity of solutions of nonlinear
partial differ (mod r – smoothing).
In Chapter 5 we shall describe micro-local regularity of solutions of nonlinear

partial differential equations. There we use the following

Lemma: Let $(x_0, \xi_0) \in \mathbb{R}_n \times \mathbb{R}_n \setminus \{0\}$ and $\sigma \in B_r$, $\lim_{\lambda \to +\infty}$ inf $|\sigma(x_0, \lambda \xi_0)| > 0$. Then there exist $\tau \in A_r$, $\sigma \in C_0^{\infty}$ and $\mu \in C^{\infty}$ such that (mod r - smoothing).

In Chapter 5 we shall describe micro-lc

partial differential equations. There we use

Lemma: Let $(x_0, \xi_0) \in \mathbb{R}_n \times \mathbb{R}_n \setminus \{0\}$ an

there exist $\tau \in A_r$, $\varphi \in C_0^{\infty}$ and $\mu \in C^{\infty}$ su

(a)
$$
\varphi(x_0) = 1
$$
, $\mu(\lambda \xi) = \mu(\xi)$ if $|\xi| \ge R_0$ and $\lambda \ge 1$, $\mu(\lambda \xi_0) = 0$, if $\lambda \ge \lambda_0$ and

(b) $\tau(x, D) \circ \sigma(x, D) = \varphi(x) \mu(D) + \varrho(x, D)$, where $\varrho \in S_{1,1}^{-1}$.

Remark 2: A proof may be found in [3: Prop. 4]. $\varphi(x)$ $\mu(D)$ is said to be an operator of micro-localization, cf. [5]. We refer to Definition 2 and 3. $\begin{aligned}\n\mathcal{L}_k &\in A_r, \, q \in C_0 \quad \text{and} \quad \mu \in C^{\infty} \text{ such that} \\
\mathcal{L}_0 &\ni = 1, \, \mu(\lambda\xi) = \mu(\xi) \quad \text{if } |\xi| \geq R_0 \text{ and } \lambda \geq 1, \, \mu(\lambda\xi_0) = 0, \, \text{if } \lambda \geq \lambda_0 \text{ and} \\
\mathcal{L}_1 &\ni = \mu(\lambda\xi) = \mu(\lambda\xi) \quad \text{if } \lambda \geq 1, \, \mu(\lambda\xi_0) = 0 \quad \text{if } \lambda \geq 1, \,$

5. Applications

Let $N \ge 1$, $n \ge 1$, $G \in C^{\infty}(\mathbb{R}_n \times \mathbb{R}_N)$ a function of variables $X_0 = (x_1, ..., x_n)$ and
 X_1, \dots, X_N and *f*: $\mathbb{R}_n \to \mathbb{R}$ a function of class C^m $(m \in \mathbb{N})$ satisfying
 $G(x, f(x), ..., D^s f(x), ...) = 0,$ X_1, \ldots, X_N and $f: \mathbf{R}_n \to \mathbf{R}$ a function of class C^m ($m \in \mathbf{N}$) satisfying

$$
G(x, f(x), \ldots, D^{\alpha}f(x), \ldots) = 0, \qquad |\alpha| \leq m.
$$

We define

Equations

\n
$$
\geq 1, \ n \geq 1, \ G \in C^{\infty}(\mathbf{R}_{n} \times \mathbf{R}_{N}) \text{ a function of } \text{var}.
$$
\n
$$
\overline{X}_{N} \text{ and } f: \mathbf{R}_{n} \to \mathbf{R} \text{ a function of class } C^{m} \ (m \in N)
$$
\n
$$
G(x, f(x), \ldots, D^{\circ}f(x), \ldots) = 0, \qquad |\alpha| \leq m.
$$
\nand

\n
$$
p_{m}(x, \xi) = \sum_{|\alpha| = m} \frac{\partial G'}{\partial X_{\alpha}} (x, f(x), \ldots, D^{\beta}f(x), \ldots) (i\xi)^{\circ}.
$$
\ninition 1: A point $(x_{0}, \xi_{0}) \in \mathbf{R}_{n} \times \mathbf{R}_{n} \setminus \{0\}$ is set

Definition 1: A point $(x_0, \xi_0) \in \mathbb{R}_n \times \mathbb{R}_n \setminus \{0\}$ is said to be *noncharacteristic* with respect to the solution *f* of (1), if $p_m(x_0, \xi_0) \neq 0$.

5. Applications

Let $N \ge 1$, $n \ge 1$, $G \in C^{\infty}(\mathbf{R}_n \times \mathbf{R}_N)$ a function of variables X
 X_1, \dots, X_N and $f: \mathbf{R}_n \to \mathbf{R}$ a function of class C^m ($m \in \mathbf{N}$) satisfy
 $G(x, f(x), \dots, D^s f(x), \dots) = 0$, $|\alpha| \le m$.

We Theorem: (i) Let $f \in F_{p,q}^s$ be a solution of (1), $s = m + s_F + r$, s_F defined by 4.1/(1), $0 < p < \infty$, $0 < q \leq \infty$. Then *f* is micro-locally $F_{p,q}^{s+r}$ at all/noncharacteristic points (x_0, ξ_0) with respect to *f*. Let $N \geq 1$, $n \geq 1$, $G \in C^{\infty}(\mathbf{R}_n \times \mathbf{R}_N)$ a
 X_1, \ldots, X_N and $f: \mathbf{R}_n \to \mathbf{R}$ a function of
 $G(x, f(x), \ldots, D^{\circ}f(x), \ldots) = 0$,

We define
 $p_m(x, \xi) = \sum_{|\mathbf{a}| = m} \frac{\partial G'}{\partial X_{\mathbf{a}}}(x, f(x), \ldots)$

Definition 1: A point *n* (*s*) satisfying
 n (*N*) satisfying
 .
 (iξ)^{*s*}.
 l is said to be *n*
 0.
 s = *m* + *s_r* + *r*, *n*
 scally $F_{p,q}^{s+r}$ *at all*_{/*n*}

+ *r*, *where s_R* deft
 mcharacteristic points Theorem: (i) Let $f \in F_{p,q}^s$ be a solution of (1), $s = m + s_F + r$, s_F defined by 4.1/(1), $0 < p < \infty$, $0 < q \le \infty$. Then *f* is micro-locally $F_{p,q}^{s+r}$ at all noncharacteristic points (x_0, ξ_0) with respect to *f*.
(ii) L *refigures* $G(x, f(x), \ldots)$
 Formalies $p_m(x, \xi) = \sum_{|\alpha|=1}$
 refigures $p_m(x, \xi) = \sum_{|\alpha|=1}$
 refigures to the solution 1: A
 refigures refigures (i) Let f $\in B_{p,q}^s$ *be a*
 noints (x_0, ξ_0) *with respect to*

(ii) Let $f \in B_{p,q}^s$ be a solution of (1), $s = m + s_B + r$, where s_B defined by 4.1/(4),

•

S

 $\frac{1}{\sqrt{2}}\sum_{i=1}^{n} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2.$

Proof: We use the method of Y. MEYER in [3–5]. By Theorem 4.1.3(i) we obtain

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\nProof: We use the method of Y. Meyer in [3-5]. By Theorem 4.1.3
\n
$$
G(x, f(x), ..., D^{\beta}f(x), ...) = \sum_{1}^{N} T_{\frac{\partial G}{\partial X_{\alpha}}(x, f(x), ..., D^{\beta}f(x), ...)} D^{\alpha}f(x) + g(x),
$$
\nwhere $g \in F_{p,q}^{s_{s+2}}$. We put
\n
$$
L_{\alpha}(u) = T_{\frac{\partial G}{\partial X_{\alpha}}(x, ..., D^{\beta}f, ...)} u.
$$

\nBy Theorem 4.1.1/Step 2, f is the solution of
\n
$$
\sum_{1}^{N} L_{\alpha}(D^{\alpha}f) = -g = \varrho(x, D) f, \qquad \varrho \in S_{1,1}^{-1}.
$$

where $g \in F^{s_0+2r}_{p,q}$. We put

$$
L_{\alpha}(u)=T_{\frac{\partial G}{\partial X_{\alpha}}(x,\ldots,D\beta f,\ldots)}u.
$$

$$
L_a(u) = T_{\frac{\partial G}{\partial X_a}(x,\ldots,D\beta f,\ldots)} u.
$$

rem 4.1.1/Step 2, *f* is the solution of

$$
\sum_{i=1}^{N} L_a(D^a f) = -g = \varrho(x, D) f, \qquad \varrho \in S_{1,1}^{-1}.
$$

Denoting by *L* the operator

$$
\sum_{1} L_a(D^2 I) = -g = \varrho(x, D) I;
$$

g by L the operator

$$
L := \sum_{1}^{N} L_a \circ D^a \circ (I - \triangle)^{-m/2},
$$

e solution of
 p) *f* $g \in S_{1,1}^{-r}$.
 i $g \in S_{1,1}^{-r}$.
 c
 d) $^{-m/2}$,

assertions in 4.2 *L* \in Op *B_r*. Let $\sigma = \sigma(x, \xi)$ be then we have by means of the assertions in $4.2 L \in Op B_r$. Let $\sigma = \sigma(x, \xi)$ be the symbol of *L*. Then (cf. [5])
 $\lim_{x \to 0} \left(\sigma(x, \xi) - \frac{p_m(x, \xi)}{p_m(x)} \right)$

$$
L_{a}(u) = T_{\frac{\partial G}{\partial X_{\alpha}}(x,...,D\beta f,...)}u.
$$

\n
$$
\text{em 4.1.1/Step 2, } f \text{ is the solution of}
$$

\n
$$
\sum_{1}^{N} L_{a}(D^{\alpha}f) = -g = \varrho(x, D) f, \qquad \varrho \in S_{1,1}^{-}.
$$

\nby *L* the operator
\n
$$
L := \sum_{1}^{N} L_{a} \circ D^{\alpha} \circ (I - \triangle)^{-m/2},
$$

\nhave by means of the assertions in 4.2 $L \in \text{Op } B_r$. Let $\sigma = \sigma(x, \xi)$ be the
\n
$$
L \text{. Then (cf. [5])}
$$

\n
$$
\lim_{\xi \to \infty} \left(\sigma(x, \xi) - \frac{p_{m}(x, \xi)}{|\xi|^{m}} \right) = 0.
$$

\n
$$
p_{m}(x_{0}, \xi_{0}) = 0, \text{ by (2) there exists } r_{0} > 0 \text{ such that}
$$

\n
$$
|\sigma(x_{1}, x_{2})| \geq \delta > 0 \text{ for all } x \geq r.
$$
 (3)

Hence, if $p_m(x_0, \xi_0) \neq 0$, by (2) there exists $r_0 > 0$ such that

$$
|\sigma(x_0, r\xi_0)| \geq \delta > 0 \quad \text{for all} \quad r \geq r_0. \tag{3}
$$

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 $\sum_{i=1}^{N} L_{a}(D^{a}f) = -g = \varrho(x, D) f, \qquad \varrho \in S_{1,1}^{-r}.$

by *L* the operator
 $L := \sum_{i=1}^{N} L_{a} \circ D^{a} \circ (I - \triangle)^{-m/2},$

have by means of the assertions in 4.2 $L \in \text{Op } B_{r}.$ Let $\sigma = \sigma(x, \xi)$ be the
 L. Then (cf. [5])
 $\lim_{n \to$ Using (3), we obtain that *L* satisfies the assumptions of Lemma 4.2 at all noncharacteristic points (x_0, ξ_0) . Putting now $h = (I - \triangle)^{m/2} f$, then we obtain $L(h) = -g$. According to Lemma 4.2, it follows $\varphi(x) \mu(D) h \in F_{p,q}^{s_p+2r}$, i.e., h is micro-*LET* $\lim_{|z| \to \infty} \left(\sigma(x, \xi) - \frac{p_m(x, \xi)}{|\xi|^m} \right) = 0.$ (2)
 LET $\left(\sigma(x, \xi) - \frac{p_m(x, \xi)}{|\xi|^m} \right) = 0.$ (2)
 LET $\left(\sigma(x_0, \xi_0) + 0, \text{ by (2) there exists } r_0 > 0 \text{ such that}$
 $|\sigma(x_0, r\xi_0)| \ge \delta > 0 \text{ for all } r \ge r_0.$ (3)
 LET $\left(\sigma(x_0, r\xi_0) \$ $\text{locally of class } F_{p,q}^{s+2r} \text{ and hence, } f \text{ is micro-locally of class } F_{p,q}^{s+r}$ on we have by means of the assertions in

mbol of *L*. Then (cf. [5])
 $\lim_{|t|\to\infty} \left(\sigma(x,\xi) - \frac{p_m(x,\xi)}{|\xi|^m} \right) = 0.$

ence, if $p_m(x_0, \xi_0) \neq 0$, by (2) there exists r_0
 $|\sigma(x_0, r\xi_0)| \ge \delta > 0$ for all $r \ge r_0.$

sing (class $F_{p,q}^{s_p+2r}$ and hence, *f* is micro-locally of class of of (ii) is similarly. g (3), we obtain that L satisfies the assumptions of lacteristic points (x_0, ξ_0) . Putting now $h = (I - \triangle)$
 $= -g$. According to Lemma 4.2, it follows $\varphi(x) \mu(D) h \in$

ly of class $F_{p,q}^{s+2r}$ and hence, f is micro-locall

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