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Para-Differential Operators in Spaces of Triebel-Lizorkin and Besov Type

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In dieser Arbeit beschäftigen wir uns mit Regularitätsuntersuchungen von Lösungen nichtlinearer partieller Differentialgleichungen in Räumen vom Besov- und Triebel-Lizorkin-Typ. Dabei werden Resultate von J. M. Bony und Y. Meyer auf die hier untersuchten Räume ausgedehnt.

В этой работе мы исследуем регулярность решений нелинейных дифференциальных уравнений в частных производных в пространствах типа Бесова и Трибель-Лизоркина. Мы обобщаем результаты, полученные Аж. М. Бони и И. Мейерем, на исследуемые нами пространства.

In this paper we study the regularity of solutions of non-linear partial differential equations in spaces of Besov and Triebel-Lizorkin type. We extend results obtained by J. M. Bony and Y. Meyer to spaces considered here.

In this paper we study the regularity of solutions of nonlinear partial differential equations. Here we shall extend results of J. M. Bony [1] to Besov and Triebel-Lizorkin spaces. J. M. Bony introduced in [1] the method of para-differential operators in order to prove some theorems about nonlinear partial differential equations. He considered solutions in generalized Sobolev space H_2^s ($s > 0$) and in Hölder spaces C^s, where $s > 0$ is not an integer. In recent years, Y. Meyer extended the regularity results obtained by J. M. Bony to solutions in other classes of functions spaces, cf. [3–5].

In this paper we consider the Besov spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$ in the Euclidean n-space $\mathbf{R}_n$. In Chapter 1 we introduce these spaces, which contain many classical spaces as special cases. In the spaces $F^s_{p,q}$ and $B^s_{p,q}$ we study the regularity of solutions of nonlinear partial differential equations and extend the results of J. M. Bony and Y. Meyer. In order to prove our results we use the method of dyadic decomposition and maximal functions, multiplication properties of Besov and Triebel-Lizorkin spaces and results with respect to the boundedness of pseudo-differential operator of the "exotic" class $L^0_{1,1}$. Applying the theory of para-differential operators introduced by J. M. Bony [1] and Y. Meyer [3–5], we are able to prove our regularity results. All immaterial positive numbers are denoted by c or c' etc.

1. Besov and Triebel-Lizorkin spaces on $\mathbf{R}_n$

1.1. Definitions

$\mathbf{R}_n$ denotes the n-dimensional real Euclidean space. $S = S(\mathbf{R}_n)$ is the Schwartz space of all complex infinitely differentiable rapidly decreasing functions on $\mathbf{R}_n$, and $S' = S'(\mathbf{R}_n)$ is the corresponding dual space of tempered distributions. Let $\mathcal{F}$

and $\mathcal{F}^{-1}$ be the Fourier transform in S' and its inverse, respectively. Φ^c is the set of all systems $\varphi = \{\varphi_k(x)\}_{k=0}^{\infty} \subset S$ such that:

(i) $\operatorname{supp} \varphi_0 \subset \{y \mid |y| \leqq 2\}$,

$\operatorname{supp} \varphi_k \subset \{y \mid 2^{k-1} \leqq |y| \leqq 2^{k+1}\}, \quad k = 1, 2, \ldots$

(ii) For any multi-index α there exists a constant c_α such that

$$|D^\alpha \varphi_k(x)| \leqq c_\alpha 2^{-|\alpha| k}, \qquad k = 0, 1, 2, \ldots$$

Φ^0 denotes the set of all systems $\varphi \in \Phi^c$ with

(iii) $\sum_{k=0}^{\infty} \varphi_k(x) = 1$ if $x \in \mathbf{R}_n$.

It is easy to show, that Φ^0 is not empty. We use the following usual abbreviations.

$$\|f \mid L_p\| = \Big(\int_{\mathbf{R}_n} |f(x)|^p \, dx\Big)^{1/p} \quad \text{if } 0 < p < \infty,$$

$$\|f \mid L_\infty\| = \operatorname*{ess\,sup}_{x \in \mathbf{R}_n} |f(x)| \quad \text{(Lebesgue measure)}.$$

If $\{a_k(x)\}_{k=0}^{\infty}$ is a sequence of functions then

$$\|a_k \mid l_q\| = \Big(\sum_{k=0}^{\infty} |a_k(x)|^q\Big)^{1/q} \quad \text{if } 0 < q < \infty,$$

$$\|a_k \mid l_\infty\| = \sup_k |a_k(x)|,$$

$$\|a_k \mid l_q(L_p)\| = \big\| \|a_k(\cdot) \mid L_p\| \mid l_q \big\| \quad \text{if } 0 < p, q \leqq \infty,$$

$$\|a_k \mid L_p(l_q)\| = \big\| \|a_k(x) \mid l_q\| \mid L_p \big\| \quad \text{if } 0 < p < \infty, \; 0 < q \leqq \infty.$$

After these preliminaries we can define the spaces

$$B^s_{p,q} = B^s_{p,q}(\mathbf{R}_n) \quad \text{and} \quad F^s_{p,q} = F^s_{p,q}(\mathbf{R}_n).$$

Definition 1.1: Let $-\infty < s < \infty$ and $0 < q \leqq \infty$.
(i) If $0 < p \leqq \infty$ then

$$B^s_{p,q} = B^s_{p,q}(\mathbf{R}_n) = \{f \in S' \mid \|f \mid B^s_{p,q}\|^\varphi := \|2^{sk} \mathcal{F}^{-1}[\varphi_k(\cdot)\, \mathcal{F} f(\cdot)]\,(x) \mid l_q(L_p)\| < \infty \quad \text{for some } \varphi \in \Phi^0\}.$$

(ii) If $0 < p < \infty$ then

$$F^s_{p,q} = F^s_{p,q}(\mathbf{R}_n) = \{f \in S' \mid \|f \mid F^s_{p,q}\|^\varphi := \|2^{sk} \mathcal{F}^{-1}[\varphi_k(\cdot)\, \mathcal{F} f(\cdot)]\,(x) \mid L_p(l_q)\| < \infty \text{ for some } \varphi \in \Phi^0\}.$$

Here and in the following, we omit $\mathbf{R}_n$ in the notations for spaces, if they are defined on $\mathbf{R}_n$.

1.2. Properties

1.2.1 Basic properties

(i) $S \hookrightarrow B^s_{p,q} \hookrightarrow S'$, $S \hookrightarrow F^s_{p,q} \hookrightarrow S'$, where "$\hookrightarrow$" denotes the continuous imbedding. Proofs of (i) and all other in 1.2 listed results may be found in [8–11].

(ii) If $\varphi \in \Phi^0$, then $B^s_{p,q}$, equipped with the quasi-norm $\|f \mid B^s_{p,q}\|^\varphi$ is a quasi-Banach space (Banach space if $1 \leqq p, q \leqq \infty$). All the quasi-norms $\|f \mid B^s_{p,q}\|^\varphi$ with $\varphi \in \Phi^0$ are equivalent to each other. The corresponding assertion is valid for the space $F^s_{p,q}$.

(iii) If $0 < p < \infty$ and $0 < q < \infty$, then S is dense in $B^s_{p,q}$ and $F^s_{p,q}$, respectively.

(iv) The following imbedding theorems hold.

(a) If $0 < p \leqq \infty$, $0 < q \leqq \infty$ then

$$B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}.$$

(b) If $0 < p \leqq \infty$ ($p < \infty$ in the case $F^s_{p,q}$) and $0 < q_0 \leqq q_1 \leqq \infty$ then

$$B^s_{p,q_0} \hookrightarrow B^s_{p,q_1} \quad \text{and} \quad F^s_{p,q_0} \hookrightarrow F^s_{p,q_1}.$$

1.2.2 Relations to classical spaces

(i) The following relations to the classical Hölder spaces C^s and Zygmund spaces $\mathscr{C}^s$ are valid:

If $0 < s \neq$ integer, then $\mathscr{C}^s = C^s = B^s_{\infty,\infty}$ and
if $s > 0$, then $\mathscr{C}^s = B^s_{\infty,\infty}$.

(ii) If $1 < p < \infty$ and $0 < s \neq$ integer, then $W_p{}^s$ denotes the Slobodeckij spaces and $W_p{}^s = B^s_{p,p}$.

If $1 < p < \infty$ and $m = 0, 1, 2, \ldots$, then $W_p{}^m$ denotes the classical Sobolev spaces and $W_p{}^m = H_p{}^m$ ($W_p{}^0 = L_p$); i.e., the Sobolev spaces are special Lebesgue spaces.

(iii) Furthermore, if $-\infty < s < \infty$ and $1 < p < \infty$, then $H_p{}^s = F^s_{p,2}$, i.e. all these spaces are special cases of the spaces $B^s_{p,q}$ and $F^s_{p,q}$.

1.2.3 Imbedding theorems

In this subsection we describe imbeddings for different metrics.

(i) Let $0 < p_0 \leqq p_1 \leqq \infty$, $0 < q \leqq \infty$ and $-\infty < s_1 \leqq s_0 < \infty$. We have

$$B^{s_0}_{p_0,q} \hookrightarrow B^{s_1}_{p_1,q} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Let $0 < p_0 < p_1 < \infty$, $0 < q, r \leqq \infty$ and $-\infty < s_1 < s_0 < \infty$. We have

$$F^{s_0}_{p_0,q} \hookrightarrow F^{s_1}_{p_1,r} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(ii) Using $B^s_{\infty,\infty} = \mathscr{C}^s$, if $s > 0$, and the just-mentioned inclusion it follows that

$$B^{s+n/p}_{p,q} \hookrightarrow \mathscr{C}^s \quad \text{if} \quad s > 0, \quad 0 < p, \ q \leqq \infty.$$

(iii) If $1 \leqq p \leqq \infty$ then

$$B^0_{p,1} \hookrightarrow L_p \hookrightarrow B^0_{p,\infty}, \quad B^0_{\infty,1} \hookrightarrow C \hookrightarrow B^0_{\infty,\infty}, \quad B^m_{\infty,1} \hookrightarrow C^m \hookrightarrow B^m_{\infty,\infty}$$

$$(m = 0, 1, \ldots),$$

where

$$C^m = \left\{ f \mid D^\alpha f \in C \text{ for all } |\alpha| \leqq m, \ \|f \mid C^m\| = \sum_{|\alpha| \leqq m} \sup_{x \in R_n} |D^\alpha f| < \infty \right\}.$$

Here D^α are classical derivatives and $C = C^0$ is the set of all bounded uniformly continuous functions f on $\mathbf{R}_n$ with $\|f \mid C\| = \sup_{x\in\mathbf{R}_n} |f(x)|$.

(iv) Using (i) it follows

$$F^{s_0}_{p_0,q} \hookrightarrow F^{s_1}_{p_1,p_1} = B^{s_1}_{p_1,p_1}$$

if $0 < p_0 < p_1 < \infty$, $-\infty < s_1 < s_0 < \infty$, $0 < q \leqq \infty$, $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$.

(v) Let $0 < p_0 < p < p_1 \leqq \infty$, $0 < q \leqq \infty$, $-\infty < s_1 < s < s_0 < \infty$ and $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} = s - \frac{n}{p}$. Then (cf. [2])

$$B^{s_0}_{p_0,p} \hookrightarrow F^{s}_{p,q} \hookrightarrow B^{s_1}_{p_1,p}.$$

1.2.4 *Maximal functions and maximal inequalities*

In the following, we use the technique of maximal function in order to answer the question whether $B^s_{p,q}$ or $F^s_{p,q}$ is a (quasi-normed) algebra under pointwise multiplication. In the case of $B^s_{p,q}$ it is also possible to modify the proof in [1]. Let φ be an infinitely differentiable function in $\mathbf{R}_n \setminus \{0\}$ such that

$$\sup_{x\in\mathbf{R}_n\setminus\{0\}} (|x|^L + |x|^{-L}) \sum_{|\alpha|\leqq L} |D^\alpha\varphi(x)| = c_\varphi < \infty. \tag{1}$$

Here L is a natural number, which value we shall choose later on. Let $\varphi_k(x) = \varphi(2^{-k}x)$, $k = 0, 1, \ldots$

Definition: If $a > 0$, $f \in S'$, then the *maximal function* $\varphi_k^* f$ is given by

$$(\varphi_k^* f)(x) = \sup_{y\in\mathbf{R}_n} \frac{|(\mathcal{F}^{-1}\varphi_k\mathcal{F}f)(x - y)|}{1 + 2^{ka}|y|^a} \quad (x \in \mathbf{R}_n;\ k = 0, 1, \ldots).$$

We recall the maximal inequalities proved in a more general framework in [9].

Theorem: (i) *If* $0 < p, q \leqq \infty$, $-\infty < s < \infty$, $a > \frac{n}{p}$, $L > L^B(s, p, q) = |s| + n + 4 + \frac{6n}{p}$, *then there exists a positive number c such that for all* φ *with* (1) *and all* $f \in B^s_{p,q}$

$$\|2^{sk}\varphi_k^* f \mid l_q(L_p)\| \leqq c c_\varphi \|f \mid B^s_{p,q}\|.$$

Here c_φ *has the meaning of* (1).

(ii) *If* $0 < p < \infty$, $0 < q \leqq \infty$, $-\infty < s < \infty$, $a > \frac{n}{\min(p, q)}$ *and* $L > L^F(s, p, q) = |s| + n + 4 + \frac{6n}{\min(p, q)}$, *then there exists a positive number c such that for all* φ *with* (1) *and all* $f \in F^s_{p,q}$

$$\|2^{sk}\varphi_k^* f \mid L_p(l_q)\| \leqq c c_\varphi \|f \mid F^s_{p,q}\|.$$

Here c_φ *has the meaning of* (1).

2. Multiplication properties, the first example of linearization of nonlinear problems

In our further considerations, we use essentially the fact that the spaces $F^s_{p,q}$ and $B^s_{p,q}$ are a (quasi-normed) algebra under pointwise multiplication if the numbers s, p, q are chosen suitably, i.e.,

$$B^s_{p,q} \cdot B^s_{p,q} \hookrightarrow B^s_{p,q} \quad \text{if} \quad \begin{cases} \text{either } 0 < p \leqq \infty, \ 0 < q \leqq \infty, & \text{and } s > \dfrac{n}{p} \\ \text{or} \quad 0 < p \leqq \infty, \ 0 < q \leqq 1, & \text{and } s = \dfrac{n}{p} \end{cases}$$

$$F^s_{p,q} \cdot F^s_{p,q} \hookrightarrow F^s_{p,q} \quad \text{if} \quad \begin{cases} \text{either } 0 < p < \infty, \ 0 < q \leqq \infty, & \text{and } s > \dfrac{n}{p} \\ \text{or} \quad 0 < p \leqq 1, \ 0 < q \leqq \infty, & \text{and } s = \dfrac{n}{p}. \end{cases}$$

Proofs of these assertions and references may be found in [11: p. 145—146] and in [2]. In the following, we shall apply essentially the treatment given in [10].

The purpose of this section is to give a natural approach to the theory of "para-multiplication" introduced by J. M. Bony [1]. Let $\{\varphi_k(x)\}_{k=0}^\infty \in \Phi^0$. We may assume that $\varphi_k(x) = \varphi_0(2^{-k}x)$ if $k = 1, 2, \ldots$ If f and g are functions in $B^s_{p,q}$ and $F^s_{p,q}$, respectively (the values of s, p and q we shall choose later), then we put

$$b_k = \mathcal{F}^{-1}\varphi_k\mathcal{F}g, \qquad c_k = \mathcal{F}^{-1}\varphi_k\mathcal{F}f \qquad (k = 0, 1, \ldots).$$

We assume temporary that $\mathcal{F}g$ has a compact support. In that case all the sums below are finite.

Using $\sum_{k=0}^\infty \varphi_k(x) \equiv 1$, $x \in \mathbf{R}_n$, we have

$$g(x) \underset{S'}{=} \sum_{k=0}^\infty b_k(x), \qquad f(x) \underset{S'}{=} \sum_{k=0}^\infty c_k(x).$$

If $k = 1, 2, \ldots$, then holds

$$\begin{aligned} [\mathcal{F}^{-1}\varphi_k\mathcal{F}(gf)](x) &= \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_k)(y)\,(gf)(x-y)\,dy \\ &= 2^{kn}\int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0)(2^ky)\,(gf)(x-y)\,dy \\ &= \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0)(y) \sum_{l,j=0}^\infty b_j(x-2^{-k}y)\,c_l(x-2^{-k}y)\,dy. \end{aligned} \tag{1}$$

The intersection of the supports of φ_k and $\mathcal{F}(b_jc_l)$,

$$[\mathcal{F}(b_jc_l)](x) = \int_{\mathbf{R}_n} (\mathcal{F}b_j)(y)\,(\mathcal{F}c_l)(x-y)\,dy,$$

is empty if the non-negative integers l and j do not belong to one of the following three cases:

(i) $k - 3 \leqq l \leqq k + 3$ and $j = 0, \ldots, k + 3$,

(ii) $l = 0, \ldots, k+3$ and $k - 3 \leqq j \leqq k + 3$,

(iii) $l > k + 3$, $j > k + 3$ and $|l - j| < 3$,

i.e., in the sum in (1) there are of interest only values of j and l given by (i)—(iii).

We use the following composition of $f \cdot g(x) = \sum_{l=0}^{\infty} c_l(x) \sum_{j=0}^{\infty} b_j(x)$:

$$f \cdot g = T_f g + T_g f + R(f, g), \tag{2}$$

where

$$T_f g = \sum_{l<j-3} c_l b_j, \quad T_g f = \sum_{j<l-3} b_j c_l, \quad R(f, g) = \sum_{|l-j|<3} b_j c_l.$$

Therefore, it will be sufficient to consider the following three model cases ($k = 1, 2, \ldots$):

Case 1:

$$\sum\nolimits_k{}' (x) = \sum_{j=0}^{k} \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0)(y)\, b_j(x - 2^{-k}y)\, c_k(x - 2^{-k}y)\, dy,$$

$\sum_k{}' (x)$ is equivalent to $[\mathcal{F}^{-1}\varphi_k \mathcal{F}(T_g f)](x)$.

Case 2:

$$\sum\nolimits_k{}'' (x) = \sum_{l=0}^{k} \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0)(y)\, b_k(x - 2^{-k}y)\, c_l(x - 2^{-k}y)\, dy,$$

$\sum_k{}'' (x)$ is equivalent to $[\mathcal{F}^{-1}\varphi_k \mathcal{F}(T_f g)](x)$.

Case 3:

$$\sum\nolimits_k{}''' (x) = \sum_{l=k}^{\infty} \int_{\mathbf{R}_n} (\mathcal{F}^{-1}\varphi_0)(y)\, b_l(x - 2^{-k}y)\, c_l(x - 2^{-k}y)\, dy,$$

$\sum_k{}''' (x)$ is equivalent to $\left[\mathcal{F}^{-1}\varphi_k \mathcal{F}\big(R(f \cdot g)\big)\right](x)$.

Let

$$b_j^*(x) = \sup_{y \in \mathbf{R}_n} \frac{|b_j(x - y)|}{1 + |2^j y|^{a_1}} \tag{3}$$

and

$$c_j^*(x) = \sup_{y \in \mathbf{R}_n} \frac{|c_j(x - y)|}{1 + |2^j y|^{a_2}} \tag{4}$$

be the maximal functions. We assume $a_1 > 0$, $a_2 > \frac{n}{p}$ if $f \in B^s_{p,q}$ and $a_2 > \frac{n}{\min(p, q)}$ if $f \in F^s_{p,q}$. If

$$c = \int_{\mathbf{R}_n} |(\mathcal{F}^{-1}\varphi_0)(y)|\, (1 + |y|)^{a_1 + a_2}\, dy,$$

then we have by [10]

$$\left|\sum\nolimits_k{}' (x)\right| \leqq c c_k^*(x) \sum_{j=0}^{k} b_j^*(x), \tag{5}$$

$$\left|\sum\nolimits_k{}'' (x)\right| \leqq c b_k^*(x) \sum_{l=0}^{k} c_l^*(x), \tag{6}$$

$$\left|\sum\nolimits_k{}''' (x)\right| \leqq c \sum_{l=k}^{\infty} 2^{(l-k)(a_1+a_2)} b_l^*(x)\, c_l^*(x). \tag{7}$$

By (5) we have

$$2^{ks}\,|\textstyle\sum_k{}'(x)| \leqq c2^{ks}c_k^*(x)\,\|b_j^* \mid l_1(L_\infty)\|. \tag{8}$$

If $-\infty < s < \infty$, $0 < p, q \leqq \infty$, then we get by Theorem 1.2.4

$$\|2^{ks}\textstyle\sum_k{}'(x) \mid l_q(L_p)\| \leqq c\,\|f \mid B^s_{p,q}\| \cdot \|g \mid B^0_{\infty,1}\|. \tag{9}$$

Remark 1: We assumed above that $\mathcal{F}g$ has a compact support. In [10] it was shown that (8) is true for arbitrary functions g, i.e., we have

$$\|2^{ks}\textstyle\sum_k{}'(x) \mid l_q(L_p)\| \leqq c\,\|f \mid B^s_{p,q}\| \cdot \|g \mid B^0_{\infty,1}\|.$$

Since this estimate is symmetric, we obtain also

$$\|2^{ks}\textstyle\sum_k{}''(x) \mid l_q(L_p)\| \leqq c\,\|g \mid B^s_{p,q}\| \cdot \|f \mid B^0_{\infty,1}\|.$$

Applying now the imbedding

$$B^s_{p,q} \hookrightarrow B^{n/p}_{p,q} \hookrightarrow B^0_{\infty,1} \hookrightarrow B^0_{\infty,\infty} \quad \text{if} \quad s > \frac{n}{p}, \tag{10}$$

we have proved

Theorem 1: *Let $0 < p, q \leqq \infty$ and $s > \frac{n}{p}$. Then*

$$\|T_f g \mid B^s_{p,q}\| \leqq c\,\|f \mid B^s_{p,q}\| \cdot \|g \mid B^s_{p,q}\|$$

and

$$\|T_g f \mid B^s_{p,q}\| \leqq c\,\|g \mid B^s_{p,q}\| \cdot \|f \mid B^s_{p,q}\|.$$

Remark 2: We recall that $\mathcal{C}^s = B^s_{\infty,\infty}$ with $s > 0$. Therefore, the above assertion holds also in the case of Hölder-Zygmund spaces.

In the following we shall estimate $R(f, g)$ if $\mathcal{F}g$ has a compact support (cf. [10]). If ε and ε' are arbitrary positive numbers with $\varepsilon' < \varepsilon$, then we have

$$\begin{aligned} &2^{ks}\,|\textstyle\sum_k{}'''(x)| \\ &\leqq c \sum_{l=k}^{\infty} 2^{(l-k)(a_1+a_2+\varepsilon-s)}\, b_l^*(x)\, 2^{ls-\varepsilon(l-k)}\, c_l^*(x) \\ &\leqq c'\,\|2^{j\max(0,a_1+a_2+\varepsilon-s)}\, b_j^* \mid l_\infty(L_\infty)\| \\ &\quad \times \left(\sum_{l=k}^{\infty} 2^{-\varepsilon'(l-k)q} 2^{lsq} c_l^*(x)^q\right)^{1/q} \quad \text{(modification if } q = \infty). \end{aligned} \tag{11}$$

Choosing a_1, a_2 and s is an appropriate way, it is possible that $\frac{n}{p} < \frac{n}{p} + a_1 \leqq a_1 + a_2 < s$ and, for small positive ε, $\max(0, a_1 + a_2 + \varepsilon - s) = 0$. Therefore, we obtain by (11)

$$\|2^{ks}\textstyle\sum_k{}'''(x) \mid l_q(L_p)\| \leqq c\,\|g \mid B^0_{\infty,\infty}\| \cdot \|f \mid B^s_{p,q}\|. \tag{12}$$

If either $g \in \mathcal{C}^\varrho$, $\varrho > 0$ or $g \in B^s_{p,q}$ with $s > \frac{n}{p}$, (12) and (10) show

$$\|2^{k(s+\varrho)}\textstyle\sum_k{}'''(x) \mid l_q(L_p)\| \leqq c\,\|g \mid \mathcal{C}^\varrho\| \cdot \|f \mid B^s_{p,q}\|$$

and

$$\|2^{ks}\textstyle\sum_k{}'''(x) \mid l_q(L_p)\| \leqq c\,\|g \mid B^s_{p,q}\| \cdot \|f \mid B^s_{p,q}\|.$$

Remark 3: The same argument as in [10] yields that (12) is true for arbitrary $g \in B^s_{p,q}$ and $g \in \mathscr{C}^\varrho$.

We have proved

Theorem 2: *Let* $0 < p, q \leqq \infty$ *and* $s > \frac{n}{p}$. *Then*

$$\|R(f, g) \mid B^s_{p,q}\| \leqq c \|g \mid B^s_{p,q}\| \cdot \|f \mid B^s_{p,q}\|$$

and for $g \in \mathscr{C}^\varrho$, $\varrho > 0$

$$\|R(f, g) \mid B^{s+\varrho}_{p,q}\| \leqq c \|g \mid \mathscr{C}^\varrho\| \cdot \|g \mid B^s_{p,q}\|.$$

Remark 4: If $p = q = \infty$, it follows by $\mathscr{C}^s = B^s_{\infty,\infty}$ for $s > 0$, $\varrho > 0$

$$\|R(f, g) \mid \mathscr{C}^{s+\varrho}\| \leqq c \|g \mid \mathscr{C}^\varrho\| \cdot \|f \mid \mathscr{C}^s\|.$$

Now we can obtain results related to [1: Théorème 2.5].

Theorem 3: *Let* $0 < p, q \leqq \infty$.

(i) *Let* $f \in \mathscr{C}^s$ *and* $g \in \mathscr{C}^\varrho$, $s > 0$, $\varrho > 0$. *Then*

$$f \cdot g = T_f g + T_g f + R(f, g).$$

with

$$\|R(f, g) \mid \mathscr{C}^{s+\varrho}\| \leqq c \|f \mid \mathscr{C}^s\| \cdot \|g \mid \mathscr{C}^\varrho\|.$$

(ii) *Let* $f \in B^s_{p,q}$ *and* $g \in B^t_{p,q}$, $s > \frac{n}{p}$, $t > \frac{n}{p}$. *Then*

$$f \cdot g = T_f g + T_g f + R(f, g)$$

with

$$\|R(f, g) \mid B^{s+t-n/p}_{p,q}\| \leqq c \|f \mid B^s_{p,q}\| \cdot \|g \mid B^t_{p,q}\|.$$

Proof: (i) follows from Theorem 1 and 2.
(ii) By (2) we have $R(f, g) = f \cdot g - T_f g - R_g f$. Now, (11) with $0 < \varepsilon' < \varepsilon$ yields

$$\begin{aligned} &\|2^{k(s+t-n/p)} {\textstyle\sum_k}''' (x) \mid l_q(L_p)\| \\ &\leqq c \sum_{l=k}^{\infty} 2^{(l-k)(a_1+a_2+\varepsilon-s-t+n/p)} b_l^*(x)\, 2^{ls-\varepsilon(l-k)} c_l^*(x) \\ &\leqq c' \|2^{j \max(0, a_1+a_2+\varepsilon-s-t+n/p)} b_j^*(x) \mid l_\infty(L_\infty)\| \\ &\quad \times \left(\sum_{l=k}^{\infty} 2^{lqs-\varepsilon'(l-k)q} c_l^*(x)^q \right)^{1/q}. \end{aligned}$$

Because of $a_1 > 0$, $a_2 > \frac{n}{p}$, $s > \frac{n}{p}$, $t > \frac{n}{p}$ and for small positive ε, it is possible that $\max(0, a_1 + a_2 + \varepsilon - s - t + n/p) = 0$. Therefore, we obtain

$$\|2^{k(s+t-n/p)} {\textstyle\sum_k}''' (x) \mid l_q(L_p)\| \leqq c \|g \mid B^0_{\infty,\infty}\| \cdot \|f \mid B^s_{p,q}\|.$$

Again by the imbedding theorem (10) we get (ii) ∎

Remark 5: Let either $p = q = 2$ or $p = q = \infty$. Then $B^s_{2,2} = H_2{}^s$ and $B^s_{\infty,\infty} = \mathscr{C}^s = C^s$ for $0 < s \neq$ integer. Therefore, our Theorem 3 implies the results obtained by J. M. Bony in [1].

Remark 6: By Theorem 3 we get for $u \in \mathscr{C}^\varrho$, $\varrho > 0$,

$$u^2 = u \cdot u = T_u u + T_u u + R(u, u) = T_{2u} u + R(u, u)$$

with $R(u, u) \in \mathcal{C}^{2\varrho}$. Analogously, it is possible to show that for $u \in \mathcal{C}^{\varrho}$, $\varrho > 0$, $G(u) = T_{G'(u)}u + r$ with $r \in \mathcal{C}^{2\varrho}$ and G is a polynomial in u with $G(0) = 0$, cf. [1: p. 227]. In Chapter 3 we shall extend this assertion to arbitrary C^∞-functions G with $G(0) = 0$ ($C^\infty = C^\infty(\mathbf{R}_n)$ denotes the set of all infinitely differentiable functions on $\mathbf{R}_n$).

In the first part of this chapter we have considered spaces of Besov type. From now on we shall be concerned with the spaces $F^s_{p,q}$. We use the methods described in [2] and in [10]. As above we restrict ourselves to the three model cases. Here we must take in our consideration, that the conditions of Theorem 1.2.4 (ii) are fulfilled, if we choose $a > \frac{n}{\min(p, q)}$. By (8) we have

$$\|2^{ks} \textstyle\sum_k' (x) \mid L_p(l_q)\| \leqq c\, \|2^{ks} c_k^*(x) \mid L_p(l_q)\| \cdot \|g \mid B^0_{\infty,1}\|. \tag{13}$$

Theorem 4: *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $s > \frac{n}{p}$. *Then*

$$\|T_g f \mid F^s_{p,q}\| \leqq c\, \|g \mid F^s_{p,q}\| \cdot \|f \mid F^s_{p,q}\|$$

and

$$\|T_f g \mid F^s_{p,q}\| \leqq c\, \|f \mid F^s_{p,q}\| \cdot \|g \mid F^s_{p,q}\|.$$

Proof: We have $s > \frac{n}{p}$. By using the imbedding 1.2.3 (i) and (iv) it follows that $F^s_{p,q} \hookrightarrow B^0_{\infty,1}$ and by (13)

$$\|T_g f \mid F^s_{p,q}\| \leqq c\, \|g \mid F^s_{p,q}\| \cdot \|f \mid F^s_{p,q}\|.$$

The same arguments with respect to the support of g (cf. Remark 3) yield the first assertion. Since our estimates are symmetric, we obtain also the second case ∎

In order to show an estimate of $R(f, g)$ we use the methods introduced in [2].

Theorem 5: *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $s > \frac{n}{p}$. *Then*

$$\|R(f, g) \mid F^s_{p,q}\| \leqq c\, \|g \mid F^s_{p,q}\| \cdot \|f \mid F^s_{p,q}\|$$

and for $g \in \mathcal{C}^{\varrho}$, $\varrho > 0$,

$$\|R(f, g) \mid F^{s+\varrho}_{p,q}\| \leqq c\, \|g \mid \mathcal{C}^{\varrho}\| \cdot \|f \mid F^s_{p,q}\|.$$

Proof: By [2: 3.3] we have

$$\|2^{ks} \textstyle\sum_k''' (x) \mid L_p(l_q)\| \leqq c\, \|f \mid F^s_{p,q}\| \cdot \|g \mid B^{n/r}_{r,\infty}\| \tag{14}$$

if $0 < p, r < \infty$, $0 < q \leqq \infty$ and $s > n\left(\frac{1}{\min(p, 1)} - 1\right)$. By using the imbedding 1.2.3 (v) we get

$$F^s_{p,q} \hookrightarrow B^{s-n/2p}_{2p,p} \hookrightarrow B^{n/2p}_{2p,p} \hookrightarrow B^{n/2p}_{2p,\infty}.$$

Hence, (14) yields

$$\|R(f, g) \mid F^s_{p,q}\| \leqq c\, \|g \mid F^s_{p,q}\| \cdot \|f \mid F^s_{p,q}\|.$$

By means of the procedure described in the proof of Theorem 2 we obtain the second result ∎

We are now in a position to carry over the results in Theorem 3 and the results obtained by J. M. Bony [1: Théorème 2.5], respectively, to the spaces $F^s_{p,q}$.

Theorem 6: *Let* $0 < p < \infty$, $0 < q \leqq \infty$, $s > \frac{n}{p}$, $t > \frac{n}{p}$, $f \in F^s_{p,q}$ *and* $g \in F^t_{p,q}$. *Then*

$$f \cdot g = T_f g + T_g f + R(f, g)$$

with

$$\|R(f, g) \mid F^{s+t-n/p}_{p,q}\| \leqq c \, \|f \mid F^s_{p,q}\| \cdot \|g \mid F^t_{p,q}\|.$$

The proof is analog to that one of Theorem 3 ∎

3. A second example of linearization

As mentioned in Remark 2.6, we shall extend Theorem 2.3 and Theorem 2.6 to arbitrary C^∞-functions G with $G(0) = 0$. The purpose of this section is to prove an extension of results obtained by Y. Meyer [3–5] and J. M. Bony [1] to $F^s_{p,q}$ and $B^s_{p,q}$.

Theorem 1: *Let*

either $0 < p, q < \infty$ *and* $s > \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p, q, 1)} - 1\right)\right)$

or $0 < p < \infty$, $q = \infty$ *and* $s > \frac{n}{p}$

and $G \in C^\infty(\mathbf{R})$ *with* $G(0) = 0$. *Then* $G: F^s_{p,q} \to F^s_{p,q}$ *defined by* $G: f \to G(f)$ *is bounded.*

Remark 1: Because of $H^s_p = F^s_{p,2}$ and $n\left(\frac{1}{\min(p, 2, 1)} - 1\right) = 0$, if $1 < p < \infty$, Theorem 1 implies the result obtained by Y. Meyer in [4: Théoréme 1].

Proof of Theorem 1: *Step* 1: We use the decomposition method with respect to $G(f)$, cf. [3–5]. Let $\varphi \in C_0^\infty(\mathbf{R}_n)$, $\varphi(\xi) \geqq 0$ for all $\xi \in \mathbf{R}_n$, $\varphi(\xi) = 1$, if $|\xi| \leqq \frac{1}{2}$, $\varphi(\xi) = 0$, if $|\xi| > 1$. Now we define as usually for $f \in F^s_{p,q}$ and $k = 0, 1, \ldots$

$$S_k(f) = \mathcal{F}^{-1}\varphi\left(\frac{\xi}{2^k}\right)\mathcal{F}f \text{ and } \Delta_k(f) = S_{k+1}(f) - S_k(f). \tag{1}$$

Hence, we have

$$\operatorname{supp} \mathcal{F}\, \Delta_k(f) \subset \{\xi \mid 2^{k-1} \leqq |\xi| \leqq 2^{k+1}\},$$

$$f \underset{S'}{=} S_0(f) + \Delta_0(f) + \cdots + \Delta_k(f) + \cdots \text{ and}$$

$$f_{k+1} := S_{k+1}(f) = S_0(f) + \cdots + \Delta_k(f). \tag{2}$$

Moreover, we use the following representation formula:

$$G(f) = G(f_0) + G(f_1) - G(f_0) + \cdots + G(f_{k+1}) - G(f_k) + \cdots.$$

Notice that $G(0) = 0$ and $f_0 = S_0(f)$. Hence, it is easy to show an estimate of $G(f_0)$. Moreover, we have

$$G(f_{k+1}) - G(f_k) = m_k \, \Delta_k(f), \quad m_k := \int_0^1 G'\big(f_k + t\,\Delta_k(f)\big)\,dt. \tag{3}$$

The operator $L: S(\mathbf{R}_n) \to S(\mathbf{R}_n)$ defined by

$$L(g) = \sum_{k=0}^{\infty} m_k \, \Delta_k(g) \tag{4}$$

is linear.

Step 2: We show that the above operator L is a pseudo-differential operator of the "exotic" class $L^0_{1,1}$ if $f \in F^s_{p,q}$, $s > \frac{n}{p}$. As usual we say that a function $\sigma(x, \xi) \in C^\infty(\mathbf{R}_n \times \mathbf{R}_n)$ belongs to $S^m_{\varrho,\delta}$, $m \in \mathbf{R}$, $0 \leqq \varrho \leqq \delta \leqq 1$, if for each multi-index α and β there exists a positive constant $c_{\alpha,\beta}$ such that

$$|D_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leqq c_{\alpha,\beta}(1 + |\xi|)^{m-\varrho|\alpha|+\delta|\beta|}$$

holds for all x and ξ in $\mathbf{R}_n$. If $\sigma \in S^m_{\varrho,\delta}$, then the corresponding pseudodifferential operators $\sigma(x, D)$ is said to be in class $L^m_{\varrho,\delta}$. Here, the pseudodifferential operator $\sigma(x, D)$ with symbol σ is defined, as usual, by

$$\sigma(x, D) f(x) = \int_{\mathbf{R}_n} e^{ix\xi} \sigma(x, \xi) \mathcal{F}f(\xi)\, d\xi, \qquad x \in \mathbf{R}_n; \quad f \in S.$$

At first, we observe that the symbol σ of L defined by (4) is given by

$$\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x)\, \Psi(2^{-k}\xi), \qquad \Psi(\xi) := \varphi\left(\frac{\xi}{2}\right) - \varphi(\xi). \tag{5}$$

Hence, we have to prove that

$$|D_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leqq c_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|+|\beta|} \tag{6}$$

holds for each multi-index α and β. From $F^s_{p,q} \hookrightarrow B^0_{\infty,1} \hookrightarrow L_\infty$ if $s > \frac{n}{p}$, cf. 1.2.3, it follows that $\|f_k \mid L_\infty\| \leqq c$. Hence we have

$$\|D^\beta f_k \mid L_\infty\| \leqq c2^{k|\beta|} \quad \text{and} \quad \|D^\beta G'(f_k) \mid L_\infty\| \leqq c_\beta 2^{k|\beta|}.$$

(3) yields

$$\|D_x^\beta m_k(x) \mid L_\infty\| \leqq c_\beta' 2^{k|\beta|}.$$

From the last estimate and the properties of the functions Ψ follows that $\sigma \in S^0_{1,1}$ and $\sigma(x, D) \in L^0_{1,1}$.

Step 3: We prove the boundedness of pseudodifferential operators of class $L^0_{1,1}$ in Triebel-Lizorkin spaces $F^s_{p,q}$. The following result was obtained in [7: Theorem 1] by the author:

Let $0 < p < \infty$, $0 < q \leqq \infty$ and

$$\text{either } s > n\left(\frac{1}{\min(p, q, 1)} - 1\right) \quad \text{and} \quad q < \infty$$

$$\text{or} \quad s > \frac{n}{p} \quad \text{and} \quad q = \infty.$$

If $T \in L^0_{1,1}$, then $T: F^s_{p,q} \to F^s_{p,q}$. This assertion completes our proof ∎

The counterpart of Theorem 1 is

Theorem 2: *Let* $0 < p, q \leqq \infty$, $s > \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p, 1)} - 1\right)\right)$ *and* $G \in C^\infty(\mathbf{R})$ *with* $G(0) = 0$. *Then* $G: B^s_{p,q} \to B^s_{p,q}$ *defined by* $G: f \to G(f)$ *is bounded.*

Proof: We use the above methods and a result obtained in [7: Theorem 4] concerning the boundedness of operators of class $L^0_{1,1}$ in Besov spaces $B^s_{p,q}$ ∎

Remark 2: Theorem 3 and 4 in [7] was obtained for general pseudodifferential operators of class $L^0_{1,1}$, i.e. no restrictions on the structure of the symbol σ. Those results are contained in

Theorem 3: (i) *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and*

$$s > \begin{cases} n\left(\dfrac{1}{\min(p, q, 1)} - 1\right) & \text{if } q < \infty \\ \dfrac{n}{p} & \text{if } q = \infty. \end{cases}$$

If $\sigma \in S^m_{1,1}$, $-\infty < m < \infty$, *then the corresponding pseudodifferential operator* $T = \sigma(x, D)$ *is bounded from* $F^{s+m}_{p,q}$ *into* $F^s_{p,q}$

(ii) *Let* $0 < p, q \leqq \infty$ *and* $s > n\left(\dfrac{1}{\min(p, 1)} - 1\right)$. *If* $\sigma \in S^m_{1,1}$, $-\infty < m < \infty$, *then the corresponding pseudodifferential operator* $T = \sigma(x, D)$ *is bounded from* $B^{s+m}_{q,q}$ *into* $B^s_{p,q}$.

For the proof cf. [7] ∎

Remark 3: Theorem 3 is an extension of results discovered by Y. Meyer [3—5]. The theorems presented in this chapter are fundamental for our further considerations.

Remark 4: Because of $F^s_{p,2} = H_p{}^s$, $1 < p < \infty$, Theorem 3 is valid for $s > 0$. The assertion is false, if $s = 0$, cf. [4].

Remark 5: Pseudodifferential operators of class $S^m_{\varrho,\delta}$ acting in Triebel-Lizorkin spaces $F^s_{p,q}$ was considered by L. Päivärinta [6] and other authors.

Remark 6: Let $r > 0$ and $T \in L^{-r}_{1,1}$.

(i) If $0 < p < \infty$, $0 < q \leqq \infty$, s satisfies the conditions of Theorem 1, then

$$T: F^t_{p,q} \to F^{t+r}_{p,q} \quad \text{for all} \quad t > s - r.$$

(ii) If $0 < p, q \leqq \infty$, s satisfies the conditions of Theorem 2, then

$$T: B^t_{p,q} \to B^{t+r}_{p,q} \quad \text{for all} \quad t > s - r,$$

i.e., T is smoothing of order r.

4. Para-products of J. M. Bony, a third example of linearization

4.1. Para-products

The calculus of para-products was introduced by J. M. Bony in [1].

Definition 1: Let $u, v \in S'$. Then the *para-product* $w = T_u v$ is defined by

$$w = \sum_{k=2}^{\infty} S_{k-2}(u)\, \Delta_k(v).$$

Remark 1: J. M. Bony denoted the para-product by $w = \pi(u, v)$. Comparing this definition with Chapter 2, we obtain that the operator of para-multiplication is essentially the operator $T_u v$ in the theory of multiplication algebras. Hence, we denote the para-product by the same symbol.

Theorem 1: *Let* $0 < p < \infty$, $0 < q \leqq \infty$, $s > s_F$,

$$s_F := \begin{cases} \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p, q, 1)} - 1\right)\right) & \text{if } q < \infty \\ \frac{n}{p} & \text{if } q = \infty. \end{cases} \tag{1}$$

We put $s = s_F + r$, $r > 0$. *Then for each* $f \in F^s_{p,q}$ *and* $G \in C^\infty(\mathbf{R})$ *with* $G(0) = 0$ *we have*

$$G(f) = T_{G'(f)}f + g,$$

where $g \in F^{s'}_{p,q}$, $s' = s_F + 2r$.

Remark 2: Because of $F^s_{p,2} = H^s_p$, $1 < p < \infty$, $s_F = \frac{n}{p}$, Theorem 1 yields the result obtained by Y. Meyer [3—5]. $G(0) = 0$ is a necessary condition. By 1.2.1 (i) $S \hookrightarrow F_{p,q} \hookrightarrow S'$. If $G(x) \equiv a$, $a \neq 0$, then $G(f) \in S$ holds not for general $f \in S$. In this case g belongs locally to $F^{s'}_{p,q}$.

Proof of Theorem 1: We use the mapping properties of pseudodifferential operators obtained in Chapter 3 and the methods of [4: Théorème 4].

Step 1: By Chapter 3 we get the linearization

$$G(f) = L(f) + S_0(f),$$

where $L \in L^0_{1,1}$ with the symbol

$$\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x)\, \Psi(2^{-k}\xi)$$

and

$$m_k(x) = \int_0^1 G'(f_k + t\,\Delta_k(f))\, dt, \qquad f_k = S_k(f).$$

By 1.2.3 we have

$$F^s_{p,q} \hookrightarrow F^{s_F+r}_{p,q} \hookrightarrow \mathscr{C}^r \tag{2}$$

and hence $G'(f) \in \mathscr{C}^r$. Evidently, $T_a \in L^0_{1,1}$ with the symbol

$$\sum_{k=2}^{\infty} S_{k-2}(a)\, \Psi(2^{-k}\xi) \qquad (a \in L_\infty \text{ fixed}).$$

Step 2: We prove $L(f) - T_{G'(f)}f = \varrho(x, D) f$, where $\varrho \in S^{-r}_{1,1}$. It is sufficient to show that

$$\|D^\alpha m_k(x) - D^\alpha S_{k-2}(a) \mid L_\infty\| \leqq c_\alpha 2^{k|\alpha| - kr}. \tag{3}$$

Here $a = G'(f)$, $f \in F^s_{p,q}$, m_k as above. Using now imbedding (2), then (3) follows by the methods in [4: Prop. 2].

Step 3: Applying now Theorem 3.3 (i), we obtain Theorem 1 ∎

Remark 3: $\varrho(x, D)$ is smoothing of order r, cf. Remark 3.6. Using Theorem 3.3 (ii), it is not hard to prove

Theorem 2: *Let* $0 < p, q \leqq \infty$, $s > s_B$,

$$s_B := \max\left(\frac{n}{p}, n\left(\frac{1}{\min(p, 1)} - 1\right)\right). \tag{4}$$

We put $s = s_B + r$, $r > 0$. Then for each $f \in B^s_{p,q}$ and $G \in C^\infty(\mathbf{R})$ with $G(0) = 0$ we have $G(f) = T_{G'(f)}f + g$, where $g \in B^{s'}_{p,q}$, $s' = s_B + 2r$.

The following theorem generalizes Theorem 1 and 2. We use the concept of localization and micro-localization.

Definition 2: A function $f(x)$ is *locally* of class $B^s_{p,q}(F^s_{p,q})$ *at the point* $x = x_0$, if $\Psi f \in B^s_{p,q}(F^s_{p,q})$ for any C^∞-function $\Psi(x)$ not vanishing at x_0 and supported in a sufficiently small neighborhood of x_0.

A function $f(x)$ is *locally* of class $B^s_{p,q}(F^s_{p,q})$, if $\Psi f \in B^s_{p,q}(F^s_{p,q})$ for any function $\Psi(x) \in C_0^\infty$. Here $C_0^\infty = C_0^\infty(\mathbf{R}_n)$ is the set of all complex-valued infinitely differentiable functions with compact support in $\mathbf{R}_n$.

Definition 3: Let $\Psi(x)$ be the function from the first part of the preceding definition. We say $f(x)$ is *micro-locally* of class $B^s_{p,q}(F^s_{p,q})$ *at the point* $(x, \xi) = (x_0, \xi_0)$ *in the cotangent space* if the Fourier transform of Ψf is equal to the Fourier transform of a $B^s_{p,q}(F^s_{p,q})$ function in a canonical neighborhood of ξ_0 (i.e., $\xi|_{|\xi|}$ near $\xi_0|_{|\xi_0|}$).

Theorem 3: (i) *Let $0 < p < \infty$, $0 < q \leqq \infty$, $s > s_F$ and $s = s_F + r$, $r > 0$. If $f_j \in F^s_{p,q}$ $(j = 1, \ldots, m)$ and $G = G(x, X_1, \ldots, X_m) \in C^\infty(\mathbf{R}_n \times \mathbf{R}_m)$, then*

$$G(x, f_1, \ldots, f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, \ldots, f_m)} f_j + g,$$

where g belongs local to $F^{s'}_{p,q}$, $s' = s_F + 2r$.

(ii) *Let $0 < p$, $q \leqq \infty$, $s > s_B$ and $s = s_B + r$, $r > 0$. If $f_j \in B^s_{p,q}$ $(j = 1, \ldots, m)$ and $G = G(x, X_1, \ldots, X_m) \in C^\infty(\mathbf{R}_n \times \mathbf{R}_m)$, then*

$$G(x, f_1, \ldots, f_m) = \sum_{j=1}^m T_{\frac{\partial G}{\partial X_j}(x, f_1, \ldots, f_m)} f_j + g,$$

where g belongs local to $B^{s'}_{p,q}$, $s' = s_B + 2r$.

Remark 4: Theorem 3 generalizes results of J. M. Bony [1] and Y. Meyer [3—5] to $B^s_{p,q}$ and $F^s_{p,q}$. In the following, s_F and s_B are defined by (1) and (4), respectively.

4.2. Para-differential operators

Para-differential operators were recently introduced by J. M. Bony [1]. The theory of para-differential operators may be found in [1, 3—5]. The theory is also applicable to the function spaces considered here. The following definitions and properties may be found in the above quoted papers.

Definition 1: Let $m \in \mathbf{R}$, $r > 0$. Then A_r^m is the set of all symbols $\sigma = \sigma(x, \xi)$ such that

(i) $\quad \|D_\xi^\alpha \sigma(\cdot, \xi) \mid \mathscr{C}^r\| \leqq c_\alpha (1 + |\xi|)^{m-|\alpha|}$

for each multi-index α and

(ii) $\quad |D_\xi^\alpha D_x^\beta \sigma(\cdot, \xi)| \leqq c_{\alpha,\beta} (1 + |\xi|)^{m-|\alpha|+|\beta|-r}$

for each multi-index β with $|\beta| > r$ and each multiindex α.

Remark 1: It holds $S^m_{1,0} \subset A_r^m \subset S^m_{1,1}$. Here $A_r^0 = A_r$. We define the corresponding operator class in the usual way and denote it by $\operatorname{Op} A_r^m$.

Definition 2: $B_r^m \subset A_r^m$ denotes the set of all symbols $\sigma = \sigma(x, \xi)$ such that

(i) $\|D^\alpha\sigma(\cdot, \xi) \mid \mathscr{C}^r\| \leqq c_\alpha(1 + |\xi|)^{m-|\alpha|}$

and

(ii) for all fixed ξ holds $\operatorname{supp} \mathscr{F}_{x\to\eta}\sigma(x, \xi) \subset \left\{\eta \mid |\eta| \leqq \frac{|\xi|}{10}\right\}$.

In [4] may be found the following facts:

1. If L denotes the above defined operator with symbol $\sigma(x, \xi) = \sum_{k=0}^{\infty} m_k(x)\, \Psi(2^{-k}\xi)$ and $f \in H_p^s = F_{p,2}^s$, $1 < p < \infty$, $s = \frac{n}{p} + r$, $r > 0$, then $L \in \operatorname{Op} A_r$.

Using the imbedding theorems in 1.2.3, we find $L \in \operatorname{Op} A_r$, if $f \in F_{p,q}^s(B_{p,q}^s)$, where $0 < p < \infty$, $0 < q \leqq \infty$, $s = s_F + r$, $r > 0$ $(0 < p, q \leqq \infty$, $s = s_B + r$, $r > 0)$.

2. If $a \in \mathscr{C}^r$, $r > 0$, then the operator $T_a: f \to T_a f$ belongs to $\operatorname{Op} A_r$. It holds $T_a \in \operatorname{Op} B_r$ if the para-product is defined by $\sum_{k=6}^{\infty} S_{k-6}(a)\ \triangle_k(f)$. We have $\operatorname{Op} A_r \equiv \operatorname{Op} B_r$ (mod r — smoothing).

In Chapter 5 we shall describe micro-local regularity of solutions of nonlinear partial differential equations. There we use the following

Lemma: *Let* $(x_0, \xi_0) \in \mathbf{R}_n \times \mathbf{R}_n \setminus \{0\}$ *and* $\sigma \in B_r$, $\liminf_{\lambda\to+\infty} |\sigma(x_0, \lambda\xi_0)| > 0$. *Then there exist* $\tau \in A_r$, $\varphi \in C_0^\infty$ *and* $\mu \in C^\infty$ *such that*

(a) $\varphi(x_0) = 1$, $\mu(\lambda\xi) = \mu(\xi)$ *if* $|\xi| \geqq R_0$ *and* $\lambda \geqq 1$, $\mu(\lambda\xi_0) \neq 0$, *if* $\lambda \geqq \lambda_0$ *and*

(b) $\tau(x, D) \circ \sigma(x, D) = \varphi(x)\, \mu(D) + \varrho(x, D)$, *where* $\varrho \in S_{1,1}^{-r}$.

Remark 2: A proof may be found in [3: Prop. 4]. $\varphi(x)\, \mu(D)$ is said to be an operator of micro-localization, cf. [5]. We refer to Definition 2 and 3.

5. Applications

Let $N \geqq 1$, $n \geqq 1$, $G \in C^\infty(\mathbf{R}_n \times \mathbf{R}_N)$ a function of variables $X_0 = (x_1, \ldots, x_n)$ and $X_1, \ldots, X_N$ and $f: \mathbf{R}_n \to \mathbf{R}$ a function of class C^m $(m \in \mathrm{N})$ satisfying

$$G(x, f(x), \ldots, D^\alpha f(x), \ldots) = 0, \quad |\alpha| \leqq m. \tag{1}$$

We define

$$p_m(x, \xi) = \sum_{|\alpha|=m} \frac{\partial G}{\partial X_\alpha}(x, f(x), \ldots, D^\beta f(x), \ldots)\, (i\xi)^\alpha.$$

Definition 1: A point $(x_0, \xi_0) \in \mathbf{R}_n \times \mathbf{R}_n \setminus \{0\}$ is said to be *noncharacteristic* with respect to the solution f of (1), if $p_m(x_0, \xi_0) \neq 0$.

Theorem: (i) *Let* $f \in F_{p,q}^s$ *be a solution of* (1), $s = m + s_F + r$, s_F *defined by* 4.1/(1), $0 < p < \infty$, $0 < q \leqq \infty$. *Then f is micro-locally* $F_{p,q}^{s+r}$ *at all noncharacteristic points* (x_0, ξ_0) *with respect to* f.

(ii) *Let* $f \in B_{p,q}^s$ *be a solution of* (1), $s = m + s_B + r$, *where* s_B *defined by* 4.1/(4), $0 < p, q \leqq \infty$. *Then f is micro-locally* $B_{p,q}^{s+r}$ *at all noncharacteristic points* (x_0, ξ_0) *with respect to* f.

Proof: We use the method of Y. MEYER in [3—5]. By Theorem 4.1.3 (i) we obtain

$$G(x, f(x), \ldots, D^\beta f(x), \ldots) = \sum_1^N T_{\frac{\partial G}{\partial X_\alpha}(x, f(x), \ldots, D^\beta f(x), \ldots)} D^\alpha f(x) + g(x),$$

where $g \in F^{s_0+2r}_{p,q}$. We put

$$L_\alpha(u) = T_{\frac{\partial G}{\partial X_\alpha}(x, \ldots, D^\beta f, \ldots)} u.$$

By Theorem 4.1.1/Step 2, f is the solution of

$$\sum_1^N L_\alpha(D^\alpha f) = -g = \varrho(x, D) f; \qquad \varrho \in S^{-r}_{1,1}.$$

Denoting by L the operator

$$L := \sum_1^N L_\alpha \circ D^\alpha \circ (I - \triangle)^{-m/2},$$

then we have by means of the assertions in 4.2 $L \in \mathrm{Op}\, B_r$. Let $\sigma = \sigma(x, \xi)$ be the symbol of L. Then (cf. [5])

$$\lim_{|\xi| \to \infty} \left(\sigma(x, \xi) - \frac{p_m(x, \xi)}{|\xi|^m} \right) = 0. \tag{2}$$

Hence, if $p_m(x_0, \xi_0) \neq 0$, by (2) there exists $r_0 > 0$ such that

$$|\sigma(x_0, r\xi_0)| \geqq \delta > 0 \quad \text{for all} \quad r \geqq r_0. \tag{3}$$

Using (3), we obtain that L satisfies the assumptions of Lemma 4.2 at all non-characteristic points (x_0, ξ_0). Putting now $h = (I - \triangle)^{m/2} f$, then we obtain $L(h) = -g$. According to Lemma 4.2, it follows $\varphi(x)\, \mu(D)\, h \in F^{s_p+2r}_{p,q}$, i.e., h is micro-locally of class $F^{s_p+2r}_{p,q}$ and hence, f is micro-locally of class $F^{s+r}_{p,q}$.

The proof of (ii) is similarly ∎

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