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Deformations of Hyperbolic Structures without the Completeness Condition

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Im allgemeinen Fall von Deformationen ohne Vollständigkeitsbedingung zeigen wir, daß die komplexe Dimension des Raumes $\mathrm{Def}_* M^3$ gleich der Anzahl der isolierten Ränder der Mannigfaltigkeit M^3 ist. Für $n \geqq 4$ liegt eine starke Starrheit der Deformationen, ähnlich der Starrheit von vollständigen hyperbolischen Strukturen, vor.

В общем случае деформаций без условия полноты мы показываем, что комплексная размерность пространства $\mathrm{Def}_* M^3$ равна числу изолированных границ многообразия M^3. Для $n \geqq 4$ получается сильная жесткость деформации, подобная жесткости полных гиперболических структур.

In the general case of deformations without the completeness condition we will show that the complex dimensionality of the space $\mathrm{Def}^* M^3$ is equal to the number of isolated ends of the three-manifold M^3. For $n \geqq 4$ there exists a strong rigidity of deformations, similar to the rigidity of complete hyperbolic structures.

1. Set-up of the problem

Let on a manifold M there be introduced a complete hyperbolic structure of finite volume with a holonomy H which maps a fundamental group $\pi_1(M)$ on a discrete subgroup $G = H(\pi_1(M)) \subset \mathrm{Isom}\, H^n$ of the group of isometries of the hyperbolic space H^n, $n \geqq 2$. *Deformation* of the hyperbolic structure on M (cf. [1]) is a homeomorphism f of the manifold M onto some hyperbolic manifold M' of finite volume which is, generally speaking, incomplete but satisfying the "maximality" condition: any geodesic ray l going to an end of the manifold M' raises up to the maximal geodesic ray $\tilde{l}$ in the hyperbolic space H^n (possibility of continuation). Two deformations $f_1: M \to M_1$ and $f_2: M \to M_2$ are equivalent, i.e. they define the same point of the space $\mathrm{Def}_* M$, if the contraction of the homeomorphism $f_2 f_1^{-1}: M_1 \to M_2$ onto enough neighbourhoods $U_1 \subset M_1$ and $U_2 \subset M_2$ is raised in the hyperbolic space H^n up to its isometry.

Exact description of the subspace $\mathrm{Def}\, M \subset \mathrm{Def}_* M$, i.e. of the space of deformations of complete hyperbolic structures on the manifold M, $\mathrm{vol}\, M < \infty$, is given in the Mostow rigidity theorem from which it follows that $\mathrm{Def}\, M$ consists of one point [13, 15]. In the present paper it will be shown that in the general case of deformations without the completeness condition the complex dimensionality of the space $\mathrm{Def}_* M^3$ equals the number of isolated ends of the three-manifold M^3. But in this case the subspace in $\mathrm{Def}_* M^3$ consisting of quasiconformal deformations consists (just as in the complete case) of one point. In the dimension $n \geqq 4$ it will be shown that $\mathrm{Def}_* M$ coincides with $\mathrm{Def}\, M$ and consists of one point. In other words, for $n \geqq 4$ there is strong rigidity of deformations, similar to the rigidity of complete hyperbolic structures — cf. [6].

2. Whitehead link

Consider a 3-manifold M_w which is the complement in the 3-dimensional sphere S^3 to the Whitehead link (Fig. 1) with the corepresentation

$$\langle a, b : (b^{-1}aba^{-1})(b^{-1}a^{-1}ba)(ba^{-1}b^{-1}a)(bab^{-1}a^{-1}) = 1\rangle .$$

The special role of M_w is clear from the following theorem of W. Thurston [18].

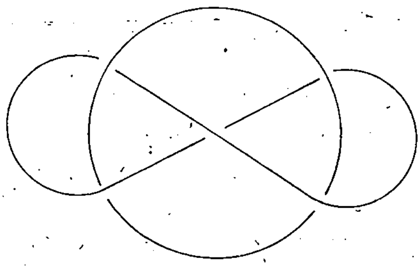

Abb. 1

Theorem 2.1: *The Whitehead link is universal. In other words, every closed, oriented three-manifold M contains a link $L \subset M$ such that $M - L$ is homeomorphic to a finite-sheeted covering space of M_w. Moreover, for every link $L \subset S^3$, there is a link $L' \supset L$ whose complement is a finite-sheeted covering of the Whitehead link complement.*

On the manifold M_w one may introduce a complete hyperbolic structure (non-compact, of finite volume) which defines the mapping of the holonomy $H : \pi_1(M_w) \to G \subset \mathrm{Isom}\, H^3$, where the discrete group G is generated by isometries identifying the sides of an octahedron $P(G) \subset H^3$ with vertices on the absolute ∂H^3 and with right dihedral angles, as it is shown in Fig. 2 — see, for instance, [11: Examples 59 and 66]. Thus assigned is the complete filling of H^3, without overlappings, by octahedra $g(P(G))$, $g \in G$.

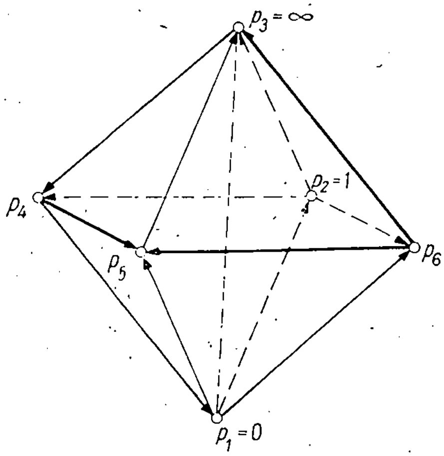

Abb. 2

Further we assume that H^3 is realised as the Poincaré model in the half-space $\mathbf{R}_+^3 = \{x \in \mathbf{R}^3 : x_3 > 0\}$. Here, the group Isom H^3 acts on the extended complex plane $\overline{\mathbf{C}} = \partial H^3$ as the Möbius group $\mathcal{U}_2$ (linear-fractional transformations in $\overline{\mathbf{C}}$).

Theorem 2.2: *The set of hyperbolic structures on the complement to Whitehead link depends on two independent (over* $\mathbf{R}$*) complex parameters* z_1 *and* z_2*. Here, the values* $z_1 = z_2 = i$ *correspond to the only complete structure.*

The proof of the theorem consists in the investigation of identifications of sides of octahedra $P \subset H^3$ with the vertices on the absolute $\partial H^3 = \overline{\mathbf{C}}$. These identifications are similar to those in Fig. 2 and to such ones that the sums of dihedral angles for equivalent edges of the octahedron P equal 2π. To obtain parameters on which such identifications (or, rather, octahedra themselves) depend, we divide an octahedron P into four ideal simplexes having common edge connecting vertices p_1 and p_3. As it is known, such simplexes in H^3 are characterized to within an isometry of H^3 by its three dihedral angles for any vertex (their sum equals π) or, which is the same, by Euclidean triangle by which the simplex intersects an horosphere with some vertex as the centre. One may assume that this triangle lies on a complex plane C so that its vertices are just 0, 1, and the point z, $\operatorname{Im} z > 0$. Therefore, one may assign any ideal simplex in H^3 with the vertices on $\partial H^3 = \overline{\mathbf{C}}$ by the following parameters (see [17: Ch. 4]).

We ascribe to each of the three pairs of non-intersecting edges of the simplex the complex numbers z, v, w with the positive imaginary parts — they are obtained by considering (as above) vertex Euclidean triangles. Then these numbers are connected by obvious relations:

$$v = 1/(1 - z), \qquad w = (z - 1)/z. \tag{2.1}$$

So, we have the simplexes Δ_i, $i = 1, \ldots, 4$ (see Fig. 2):

$$\begin{aligned} \Delta_1 &= (p_1, p_2, p_4, p_3), & \Delta_2 &= (p_1, p_6, p_2, p_3), \\ \Delta_3 &= (p_1, p_5, p_6, p_3), & \Delta_4 &= (p_1, p_4, p_5, p_3), \end{aligned} \tag{2.2}$$

assigned, to within the isometry of H^3, by a set of parameters:

$$\{(z_i, v_i, w_i)\} \subset \{z \in \mathbf{C} : \operatorname{Im} z > 0\}^3, \qquad 1 \leqq i \leqq 4, \tag{2.3}$$

which satisfy eight relations of form (2.1). Besides, let us agree in all the simplexes to ascribe the parameter z_i to the common edge (p_1, p_3) — in Fig. 3 the parameters for Δ_1 are given. The condition of the sums of dihedral angles of each of the four edges of the complex:

$$(p_1, p_3), \rightarrow, \twoheadrightarrow, \ggg,$$

(see Fig. 2) — to be equal to 2π assigns four relations:

$$\begin{aligned} &z_1 z_2 z_3 z_4 = 1, & &w_1 v_4 z_1 z_2 w_2 v_3 = 1, \\ &v_1 w_2 v_3 w_4 w_3 v_4 w_1 v_2 = 1, & &v_2 w_3 z_3 z_4 v_1 w_4 = 1. \end{aligned} \tag{2.4}$$

Here, the third relation is the result of the three others. Substituting in (2.4) by means of (2.1) the expressions w_i and v_i by z_i we see the dependence of the second and the fourth relations. Therefore, for three complex parameters z_1, z_2, z_3, $\operatorname{Im} z_i > 0$, there remains only one relation connecting them:

$$\frac{(z_1 - 1)(z_2 - 1)}{(z_3 - 1)} = \frac{1 - z_1 z_2 z_3}{z_1 z_2 z_3}. \tag{2.5}$$

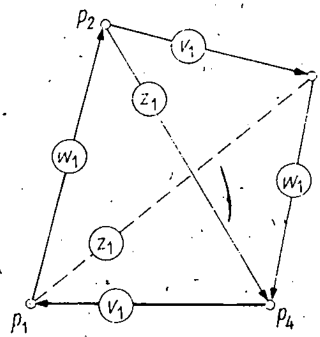

Fig. 3

This proves the first part of the theorem. The proof of the completeness of the manifold, corresponding to the parameters $z_1 = z_2 = i$, is obtained by means of describing the action of stabilizers of non-equivalent vertices of P (for instance, p_3 and p_4) on horospheres with the centres at these vertices (this action is isomorphic to the action $Z \oplus Z$) and by means of the following statement — see [5, 6].

Theorem 2.3: *A hyperbolic structure with the holonomy H on the manifold M^n, $n \geqq 2$, is complete iff the polyhedra $P_1, \ldots, P_k \subset H^n$ with a finite number of sides, whose identification by the generators of the group $G = H(\pi_1(M^n))$, obtained the manifold M^n, do not contain, as their vertices, fixed points of the loxodromic elements of the group G.*

The uniqueness of the obtained complete structure on the complement to Whitehead link follows from the rigidity theorem — see [13, 15].

Remark 2.4: For a more explicit assignment of hyperbolic structures on M_w we describe the obtained octahedra P, excluding the isometric cases by having fixed some three vertices of octahedra. For instance, put

$$p_1 = 0, \qquad p_2 = 1, \qquad p_3 = \infty. \tag{2.6}$$

Then from (2.5) we obtain that the remaining vertices p_4, p_5, p_6 of the octahedron P are just the points of the complex plane $\bar{\mathbf{C}} = \partial H^3$ which satisfy the conditions:

$$\begin{gathered} \frac{(p_4 - 1)(p_4 - p_5)}{(p_6 - 1)(p_6 - p_5)} = \frac{p_4}{p_5 p_6}, \\ \operatorname{Im} p_4 > 0, \quad \operatorname{Im}(p_4/p_5) > 0, \quad \operatorname{Im}(p_6/p_5) > 0. \end{gathered} \tag{2.7}$$

The family of non-complete structures considered in [12] evidently corresponds to one real parameter $t > 0$, namely, among the set of vertices from (2.6) and (2.7) there is pointed out a subset: $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$, $p_4 = it$, $p_5 = -1$, $p_6 = -it$.

Remark 2.5: For the first time such non-complete hyperbolic structures on 3-manifold have been described by W. Thurston on the figure eight-knot complement which may be made up of two ideal simplexes in H^3 — see [17].

3. Dimensionality of the space of deformations of 3-dimensional hyperbolic structures

Let now M be any complete non-compact hyperbolic 3-manifold of finite volume, and $P = P(G) \subset H^3$ be a fundamental polyhedron of the group $G = H(\pi_1(M)) \subset \mathrm{Isom}\, H^3$ for the holonomy H. Deformations $f \in \mathrm{Def}_* M$ assign the deformations of a polyhedron P, which are similar to the deformations of octahedron in Section 2 and give non-complete hyperbolic manifolds M' of finite volume homeomorphic to the complete manifold M.

To answer the question about the dimension of the space $\mathrm{Def}_* M$, consider a structure of ends of homeomorphic manifolds M and M'. In the case of finiteness of volume the manifold M has a finite number of ends $e_1, \ldots, e_m$; their neighbourhoods are homeomorphic to the punctured solid torus $T^2 \times [0, 1)$ — see [2—4], and the ends have Euclidean structure of two-dimensional torus T^2. To non-complete structures of the corresponding ends $e_j' = f(e_j)$, $j = 1, \ldots, m$, of the manifold $M' = f(M)$ there correspond non-complete affine structures on the torus T^2 which are assigned by partitions into quadrangles of the punctured plane $\mathbf{C} - \{0\}$, i.e. by discrete subgroups of the group $R \cdot O(2)$, a set of them is described by one complex parameter — see Theorem 2.3, [5]. This results in the following statement.

Theorem 3.1: *Let M be a complete hyperbolic 3-manifold of finite volume. Then complex dimensionality of the space of deformations* $\mathrm{Def}_* M$ *is equal to a number of isolated ends of the manifold M.*

Another approach to the proof of this statement is to use Theorem 2.1. Namely, consider that link L in S^3 for which the given complete hyperbolic manifold M is obtained from $S^3 - L$ by Dehn's surgery on a part of link L components which are complementary to some link $L_0 \subset L$. Here, since the statement is trivial in the case of the closed manifold M (Mostow rigidity theorem), then $L_0 \neq \emptyset$. The existence of the link L follows from the classification of T. Jørgensen of hyperbolic 3-manifolds of finite volume — see [17: Ch. 5.13].

According to Theorem 2.1 for the link L there exists a link $L' \subset S^3$, $L_0 \subset L \subset L'$, the complement to it finite-sheetedly covers the manifold M_w. Due to this, the fundamental polyhedron P, whose identification of sides obtained $S^3 - L'$, is a union of a finite number of octahedra with right dihedral angles, these octahedra correspond to the complement to the Whitehead link M_w. A space of deformations $\mathrm{Def}_* M_w$ is parametrized by two complex parameters (independent above R) which correspond to two classes of non-equivalent vertices — see Theorem 2.2 and Remark 2.3. To the ends of the manifold M there correspond some classes of equivalence of vertices of the above octahedra, these classes assign the components of the link L_0. To every such class of equivalence of the vertices there corresponds a family of deformations of non-complete structures of the manifold M parametrized by one complex parameter — cf. (2.5)—(2.7).

4. Absence of quasiconformal deformations without the completeness condition

The subspace of quasiconformal deformations in the space $\mathrm{Def}_* M$ will be characterized by the following result.

Theorem 4.1: *A complete hyperbolic 3-manifold M of finite volume and the homeomorphic to it non-complete hyperbolic manifold M', where each geodesic ray going to an end of M' is raised up to the maximal geodesic ray in H^3, are not quasiconformally equivalent.*

Proof: Suppose there exists a quasiconformal mapping f, $f(M) = M'$. Then it is raised up to the mapping with the bounded distortion $\hat{f}: H^3 \to H^3$ — see [16]. As it is known, such mappings are, in some sense, pseudoisometries of the hyperbolic space H^3 — see, for instance, [15; 4: Ch. 7.6], namely, if by $d(x, y)$ we denote the hyperbolic metric, then there exists a constant $c > 0$ such that for all $x, y \in H^3$

$$d(\hat{f}(x), \hat{f}(y)) \leqq cd(x, y). \tag{4.1}$$

By the conditions of the theorem M' has no boundary, is not the interior of a hyperbolic manifold with boundary and is locally convex. Therefore, M' is a convex manifold — see [17: Ch. 8]. Conjugating a group $G = H(\pi_1(M)) \subset \text{Isom}\, H^3$ one may assume that to the end e of the manifold M transferring at the mapping f to the noncomplete end e' of manifold M', there corresponds on $H^3 = \bar{\mathbf{C}}$ the point ∞. Let γ be a geodesic in H^3 with end in ∞ containing the edge of the fundamental polyhedron $P(G)$. Applying the traditional geometric arguments, to within the isometry of H^3, one may assume that the curve $\hat{f}(\gamma)$ lies in the r-neighbourhood of the unique geodesic $\hat{\gamma}$ with the end in ∞, and in such a case $r > 0$ depends only on f.

Indeed, if r is sufficiently large and $U_r(q)$ is r-neighbourhood of some geodesic $q \subset H^3$, then there exists the upper boundary of lengths of bounded components $\gamma \setminus \hat{f}^{-1}(U_r(q))$. Besides, the orthogonal projection from $H^3 \setminus U_r(q)$ on q decreases the distances by at least Ch (r). Therefore, for $x_{-t_0}, x_t \in \gamma$ and the geodesic q_t connecting the points $\hat{f}(x_{-t_0})$ and $\hat{f}(x_t)$ the segment $\hat{f}([x_{-t_0}, x_t])$ of the curve $\hat{f}(\gamma)$ intersects any plane orthogonal to q_t at the bounded distance of q_t. At $t \to +\infty$ the points $\hat{f}(x_t)$ tend to $\infty \in \partial H^3$ (due to the maximality condition of the geodesics) and, consequently, the limit for the geodesic q_t possesses the required property.

Similarly it is shown that there exists a constant C_0 such that for any plane $L \subset H^3$ orthogonal to the geodesic γ the projection of its image $\hat{f}(L)$ upon the geodesic $\hat{\gamma}$ has a diameter not exceeding C_0. To prove this, consider an arbitrary geodesic ray $l \subset L$ with the beginning at the point $x^0 = \gamma \cap L$, and let the geodesic l_1 connect its end with the positive end of the geodesic γ. Denote $C_1 = \text{arc Ch}\, \sqrt{2} = d(x^0, l_1)$, by y^0 denote the point on $\hat{\gamma}$ that is nearest to $\hat{f}(x^0)$, and by $\hat{l}$, $\hat{l}_1$ and l^* denote parallel geodesics in H^3 such that in r-neighbourhoods of the former two ones there lie the curves $\hat{f}(l)$ and $\hat{f}(l_1)$, and l^* intersects $\hat{\gamma}$ at a right angle at some point $y(l)$. Then from (4.1) it follows that

$$d(y^0, \hat{l}_1) \leqq d(y^0, \hat{f}(\bar{x}^0)) + d(\hat{f}(x^0), \hat{f}(l_1)) + d(\hat{f}(l_1), \hat{l}_1) \leq 2r + C_1 c$$

and therefore

$$d(y^0, y(l)) \leqq 2r + C_1(c + 1) = C_2,$$

besides, the constant C_2 does not depend on l and L. Hence we obtain that any point $\hat{f}(x)$ for $x \in L$ at the projection on $\hat{\gamma}$ is contained in the C_0-neighbourhood of the point y^0, where the constant $C_0 = C_2 + r$ does not depend on L.

Due to the non-completeness of the end $e' = f(e)$ of the manifold M', its invariant $d(e')$ (equal to the minimal distance between images of horospheres in H^3 — see [17]) is positive. Denote it by d_0 and let A be a linear mapping with $\det A > 1$ generating the stabilizer $G'_\infty \subset G'$ of the point $\infty \in \partial H^3$. Here the group $G' \subset \text{Isom}\, H^3$ is the image of the group G at the homomorphism induced by the mapping f and transferring into A some element $g \in G_\infty \subset G$. Assume, without restrictions upon generality, that the length of the transference vector of g equals 1, and consider the orbit $\{g^n(x): n \in \mathbf{Z}\}$ of some fixed point $x \in B_\infty \cap \gamma$. For every n we also consider the plane L_n containing the point $g^n(x)$ and intersecting the geodesic γ orthogonally at some point $x^n = L_n \cap \gamma$. Having assumed $x = (0, 0, 1)$, we obtain for the angle w between the radius-vector of the point $g^n(x)$ and the plane $\mathbf{R}^2 = \{x \in \mathbf{R}^3 : x_3 = 0\}$ in $\mathbf{R}^3$ the equality $\operatorname{Sin} w = (n^2 + 1)^{-1/2}$ and therefore the hyperbolic distance $d(x, x^n)$ may be calculated by the formula

$$\operatorname{Ch} d(x, x^n) = \frac{n^2 + 2}{2(n^2 + 1)^{1/2}}. \tag{4.2}$$

Hence it is clear that with the increase of n the distance may be estimated as follows:

$$d(x, x^n) < \ln(|n| + 1). \tag{4.3}$$

Consider the horosphere

$$S_0 = \{y \in H^3 : y_3 = (\hat{f}(x))_3\}. \tag{4.4}$$

By the definition of the invariant $d(e')$ of the end e', for any horosphere $S_n = A^n(S_0)$ there holds the equality

$$d(S_0, S_n) = |n| \cdot d_0. \tag{4.5}$$

At the same time the horosphere S_n contains the point $\hat{f}(g^n(x)) = A^n(\hat{f}(x))$. Hence and from (4.3)—(4.5) we directly obtain that the projection of the image $\hat{f}(L_n)$ of the plane L_n upon the geodesic $\hat{\gamma}$ has a diameter no less than $|n|\, d_0 - \ln(|n| + 1)$. This contradicts the above-obtained uniform estimate of diameters of such projections and, thus, proves the absence of the quasiconformity property for the mapping f ∎

Corollary 4.2: *A subspace of quasiconformal deformations in the space* $\operatorname{Def}_* M$ *of deformations of the complete hyperbolic 3-manifold M of finite volume consists of one point.*

5. Non-rigidity of three-dimensional hyperbolic structures

Let, as above, a non-complete hyperbolic manifold M' be obtained from the complete hyperbolic manifold $M = H^3/G$ of finite volume by the deformation of the fundamental polyhedron $P(G)$ onto the polyhedron P', i.e. a manifold $M' = \bar{P}'/G'$ has the maximality property of raising the geodesics. What are the properties of the groups G' with the approaching of P' to $P(G)$?

Lemma 5.1: *The above manifold M' is covered by a subdomain $H_0{}^3 \subset H^3$ obtained by the rejection from H^3 of a countable, nowhere dense set of non-intersecting (in $\bar{H}^3 = H^3 \cup \partial H^3$) geodesics.*

To prove the lemma, denote again by $\hat{f}$ a locally univalent raising on H^3 of the homeomorphism $f: M \to M'$. The mapping $\hat{f}$ is the extension of the homeomorphism $P(G) \to P'$ which is compatible with the groups G and G' and, therefore, we have the homomorphism H_f of the group G on the group G':

$$H_f(g) = \hat{f} g \hat{f}^{-1}. \tag{5.1}$$

In other words, one may say that for the homeomorphism $f: M \to M'$ the homomorphism of the monodromy is defined — cf. [9] — by

$$X_f : \pi_1(M) \to G' \subset \mathcal{U}_3 \tag{5.2}$$

which is the superposition of the isomorphism of the holonomy H and homomorphism (5.1). Here, the monodromy group $X_f(\pi_1(M))$ of conformal automorphisms of the three-dimensional sphere $\mathbf{R}^3 \cup \{\infty\}$ acts upon $H^3 = \mathbf{R}_+{}^3$ as a group of isometries of H^3 which leaves some subdomain $H_0{}^3$ invariant.

Let e be one of a finite number of isolated ends of the manifold M. Due to the finiteness of the volume of M, the neighbourhood of this end is covered by a horoball $B_p \subset H^3$ with the centre at some parabolic vertex $p \in \partial H^3$ of the polyhedron $P(G)$ (i.e. by a family of geodesic rays ending at the point p). The stabilizer $G_p \subset G$ of the point p is the free Abelian group of rank 2 and, assuming $p = \infty$, it is generated by Euclidean translations on linearly independent vectors v and w. To these translations for the isomorphism of the holonomy H there correspond generators x and y of the fundamental group of the end e, $\pi_1(e) \cong Z \oplus Z$, which, for homomorphism (5.2), pass into generators a and b of the stabilizer $G'_{p'} \subset G'$ of the vertex $p' = \hat{f}(p)$ of the polyhedron P'. If to this vertex p' there corresponds a non-complete end $e' = \hat{f}(e)$ of the manifold M', then from Theorem 2.3 it follows that the subgroup

$$X_f(\pi_1(e)) = G'_{p'} \tag{5.3}$$

of the monodromy group G' is a cyclic loxodromic group or its finite extension — see [4: Theorem 3.3], and to complete partitioning of the horosphere $S_p = \partial B_p$ by the traces of polyhedra $G(P(G)) = \{g(P(G)) : g \in G\}$ there corresponds the partitioning of the punctured horosphere $S_{p'} - \{z_0\}$ by the traces of polyhedra $G'(P')$, where z_0 is the intersection point of the horosphere $S_{p'}$ with the axis of the loxodromic generator of group (5.3). The axes of loxodromic elements of the discrete group can have no common ends. At the same time, the orbit of fixed points of the loxodromic element of the discrete group is dense in the limit set of the group — cf. [4: Lemma 3.16]. This shows that the set $H^3 - \hat{f}(H^3)$ is the union of non-intersecting (in $\overline{H}^3$) geodesics which are the axes of loxodromic generators of stabilizers of infinitely remote vertices of polyhedra $G'(P')$, i.e. $H_0{}^3 = \hat{f}(H^3)$ has the required property ∎

Remark 5.2: Simultaneously we have shown that the manifold M' is obtained by the factorization of G'-invariant set $\hat{f}(H^3)$ obtained by rejection from H^3 of the axes of subgroups conjugated to (5.3). A more detailed investigation of monodromy groups leads to a non-rigidity theorem of 3-manifolds.

Theorem 5.3: *For a complete non-compact hyperbolic manifold $M = H^3/G$ of finite volume there exists a sequence of various complete hyperbolic manifolds*

$$M_j = H^3/X_j(G) \tag{5.4}$$

which approximate for $j \to \infty$ the manifold M in the sense that homomorphisms $X_j : G \to \operatorname{Isom} H^3$ converge in the topology of pointwise convergence to the inclusion, and the volumes of the manifolds M_j converge to the volume of M.

Remark 5.4: This theorem was first proved by W. THURSTON [14, 17] by means of the topological procedure known as Dehn's surgery of 3-manifolds — see, for instance, [11: Examples 68 and 69]. Besides, Thurston's formulation states that vol $M_j <$ vol M.

Proof of Theorem 5.3: For the complete non-compact manifold M there exists a sequence of non-complete hyperbolic manifolds M_j^* which are assigned by the identification of the sides of polyhedra P_j' obtained by the deformations of the fundamental polyhedron $P(G)$ of the group G — see Theorem 3.1. The identification of the sides of P_j' is realized by the generators of discrete groups of monodromy (5.2) which correspond to a sequence of deformations $f_j\colon M \to M_j^*$,

$$G_j' = X_{f_j}(\pi_1(M)). \tag{5.5}$$

By construction, the polyhedra P_j' tend at $j \to \infty$ to the polyhedron $P(G)$. Therefore, the polygons (Euclidean) equal to $P_j' \cap S_j$, where S_j are horospheres with centres in the corresponding infinitely remote vertices p_j' of P_j', tend to the polygon which is the intersection of $P(G)$ and the horosphere S_p with the centre at the vertex $p \in \partial H^3$ — see the proof of Lemma 5.1. Here, in the case of non-completeness of the ends of manifolds M_j^*, corresponding to the vertex p, in groups (5.3) there appears a new (as compared to G_p) relation of the form

$$X_{f_j}(x^{m_j}y^{n_j}) = a_j^{m_j}b_j^{n_j} = 1, \tag{5.6}$$

where integers m_j and n_j are characterized by the fact that the non-closed chain of the length $m_j + n_j + 1$ composed of the polyhedra

$$P(G), g_1(P(G)), \ldots, g_1^{m_j}(P(G)), g_2g_1^{m_j}(P(G)), \ldots, g_2^{n_j}g_1^{m_j}(P(G)) \tag{5.7}$$

($g_1 = H(x)$ and $g_2 = H(y)$ are parabolic generators of the stabilizer $G_p \subset G$) turns, while mapping f_j to the closed, without self-intersections, chain of polyhedra

$$P_j', a_j(P_j'), \ldots, a_j^{m_j}(P_j'), b_ja_j^{m_j}(P_j'), \ldots, b_j^{n_j}a_j^{m_j}(P_j') = P_j'. \tag{5.8}$$

From our observation of the behaviour of traces of polyhedra on horospheres it follows that for $j \to \infty$ "the faces of (m_j, n_j)-parallelograms" on horospheres S_j, i.e. the m_j-th and n_j-th constituents of chain (5.7), turning into the closed chain (5.8), become arbitrarily long. In other words, the closeness of the polyhedra P_j' to $P(G)$ is characterized by the closeness of the pair (m_j, n_j) on the projective plane to (∞, ∞) of the projective plane RP^2.

Let the generator g of the group G identify the sides Q and Q' of the polyhedron $P(G)$. Then, for j large enough, the polyhedra P_j' have sides Q_j and Q_j' converging to the sides Q and Q' and identified by the generators g_j of groups (5.5). For $j \to \infty$ the mappings g_j converge uniformly on the compacts from H^3 to the mapping g. Hence it follows that for $j \to \infty$ the groups G_j' converge to the group G (the algebraic convergence), i.e. the homomorphisms

$$X_j : G \to G_j' = X_{f_j}(\pi_1(M)) \subset \text{Isom}\, H^3$$

converge to the inclusion in the topology of pointwise convergence. From the convergence of the polyhedra P_j' to $P(G)$ there follows the convergence of their volumes:

$$\lim_{j\to\infty} \text{vol}\, P_j' = \text{vol}\, P(G). \tag{5.9}$$

Consider now complete hyperbolic manifolds M_j whose holonomies assign discrete groups coinciding with G_j':

$$M_j = H^3/G_j'. \tag{5.10}$$

Then, taking into account the convergence of homomorphisms to the inclusion and (5.9), to complete the proof it is necessary to show that the hyperbolic manifolds M_j^* and the manifolds (5.10) have equal volumes for equal j. This follows from Lemma 5.1 since the polyhedra $G_j'(P_j')$ do not only fill in H^3 a countable, nowhere dense set of geodesics — in other words, to turn P_j' into a fundamental set of the group G_j' (having the volume equal to volume of M_j), it is sufficient to join a set of measure 0. This completes the proof of the theorem ∎

6. Rigidity of hyperbolic structures in dimension $n \geqq 4$

The results of this section show that in the dimension $n \geqq 4$ the space $\mathrm{Def}_* M$ is trivial, i.e. there can exist no deformations similar to those described above for $n = 3$ — there takes place strong rigidity, just as for complete structures.

Theorem 6.1: *Let n-dimensional, $n \geqq 4$, hyperbolic manifolds M and M' have finite volumes and be homeomorphic, the manifold M be complete, and any geodesic ray going to an end of the manifold M' is raised up to the maximal geodesic ray in the hyperbolic space H^n. Then the manifold M' is complete and isometric to the manifold M.*

Proof: For closed manifolds the statement trivially follows from the rigidity theorem [13, 15]. In the non-compact case any end e of the manifold M has, due to $\mathrm{vol}\, M < \infty$, the Euclidean structure of the $(n-1)$-dimensional torus T^{n-1}, and its neighbourhoods are homeomorphic to the solid cusp torus $T^{n-1} \times [0, 1)$ — see [4: Ch. 8.1]. For the homeomorphism $f : M \to M'$ the end e maps to the end $e' = f(e)$ of the manifold M' which, in the case of completeness of the manifold M', should also have the Euclidean structure of $(n-1)$-dimensional torus. To the non-complete end e' there should correspond the affine structure (non-complete) of the torus T^{n-1} (due to homeomorphy of e and e') induced by the action of a cycle loxodromic group on the set of geodesic rays going out of some infinitely remote point $p' \in \partial H^n$ determined by a finite-length geodesic going in the manifold M' to the end e'. The point p' is determined up to G'-equivalence. The discrete group G' of isometries of H^n is the image of $\pi_1(M)$ at the homomorphism $X_f : \pi_1(M) \to \mathrm{Isom}\, H^n$. The existence of the above geodesic in M' follows from the non-completeness of the end e' [10: Ch. 1, Theorem 10.3] and from the convexity of the manifold M' (of a locally convex hyperbolic manifold without boundary, satisfying the maximality condition of the geodesics of the Theorem — see [17: Ch. 8].

The action of the cyclic loxodromic group $G'_{p'} \subset G'$ on a set of geodesic rays is equivalent to the action of an affine discrete group generated by a linear transformation A, $\det A > 1$, in the Euclidean $(n-1)$-space without the origin. This follows from the fact that the hyperbolic metric of H^n induces on a horosphere $S_{p'} \subset H^n$ with the centre at the point $p' \in \partial H^n$ the Euclidean metric, and the origin corresponds to the point $z_0 \in S_{p'}$ which is the intersection with $S_{p'}$ of the axis of the loxodromic generator of the group $G'_{p'}$. At the same time, it is known [8] that the factorization $N = \{\mathbf{R}^{n-1} - \{0\}\}/\{A^k : k \in \mathbf{Z}\}$ for $\det A > 1$ gives a Hopf manifold diffeomorphic to $S^1 \times S^{n-2}$. Hence it is clear that for $n \geqq 4$ this manifold N cannot be homeomorphic to $(n-1)$-dimensional torus T^{n-1}, to which the end e' should be homeomorphic. This proves the completeness of the manifold M' and, the rigidity theorem for complete hyperbolic manifolds being taken into account, the isometricity of M and M' ∎

Corollary 6.2: *For a complete hyperbolic manifold M^n, $n \geqq 4$, of finite volume the space $\mathrm{Def}_* M^n$ is trivial.*

Remark 6.3: With Theorem 6.1 (for small deformations) the result of G. Garland and M. S. Raghunathan [7] is closely connected. This result is opposite to Theorem 5.3 on non-rigidity of hyperbolic 3-manifolds as follows from the next theorem.

Theorem 6.4: *For any complete hyperbolic manifold $M = H^n/G$, $n \geqq 4$, $G \subset$ Isom H^n, of finite volume there exists no sequence of homomorphisms $X_j : G \to$ Isom H^n converging to the inclusion.*

Remark 6.5: The result of the present paper and, especially, Sect. 6 are closely connected with the author's results [6] on the filling of the hyperbolic space by polyhedra, which generalize the well-known Poincaré-Aleksandrov's theorems. In their context it may be said that (unlike the case $n \leqq 3$) for $n \geqq 4$ there exist no deformations of complete partitionings of H^n (of finite-volume) preserving the conditions of cycles of codimensionality two.

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