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Approximation by Solutions of Elliptic Equations

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Es sei $\Omega \subset \mathbf{R}^n$ ein beschränktes, glattes Gebiet, Γ eine abgeschlossene, glatte, $(n-1)$-dimensionale Fläche mit Rand im Inneren von Ω und V eine offene Teilmenge des Randes $\partial\Omega$. In Ω werde ein eigentlich elliptischer Differentialoperator L der Ordnung $2m$ mit glatten Koeffizienten betrachtet. $(B_1, \ldots, B_m)$ sei ein normales System von Randoperatoren auf $\partial\Omega$, welches der klassischen Wurzelbedingung genügt. $L_V(\Gamma)$ bezeichne den Raum der Einschränkungen der Funktionen des Raumes

$$L_V(\Omega) = \{u \in C^\infty(\bar{\Omega}) \colon Lu = 0 \text{ in } \Omega,\ B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \text{ in } \partial\Omega \setminus V\}$$

auf Γ. Es wird bewiesen, daß $L_V(\Gamma)$ im Raum $W_p^{2m-1/p}(\Gamma)$ $(p > 1)$ dicht liegt.

Пусть $\Omega \subset \mathbf{R}^n$ ограниченная гладкая область, Γ — замкнутая гладкая $(n-1)$-мерная площадь с краем внутри области Ω, и V — открытое подмножество края $\partial\Omega$. Рассматривается в Ω собственный эллиптический оператор L порядка $2m$ с гладкими коэффициентами. Пусть $(B_1, \ldots, B_m)$ — нормальная система краевых операторов на $\partial\Omega$, удовлетворяющая классическому условию на корнях, а $L_V(\Gamma)$ обозначает пространство ограничений на Γ функций пространства

$$L_V(\Omega) = \{u \in C^\infty(\bar{\Omega}) \colon Lu = 0 \text{ в } \Omega,\ B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \text{ в } \partial\Omega \setminus V\}.$$

Доказывается, что $L_V(\Gamma)$ плотно в пространстве $W_p^{2m-1/p}(\Gamma)$ $(p > 1)$.

Let $\Omega \subset \mathbf{R}^n$ be a bounded, smooth domain, Γ a closed, smooth, $(n-1)$-dimensional surface with boundary in the interior of Ω and V an open subset of the boundary $\partial\Omega$. In Ω we consider a porperly elliptic differential operator L of order $2m$ with smooth coefficients. Let $(B_1, \ldots, B_m)$ be a normal system of boundary operators on $\partial\Omega$, which fulfils the classical root condition. $L_V(\Gamma)$ denote the space of the restrictions on Γ of the functions from

$$L_V(\Omega) = \{u \in C^\infty(\bar{\Omega}) \colon Lu = 0 \text{ in } \Omega,\ B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \text{ in } \partial\Omega \setminus V\}.$$

It is proved that $L_V(\Gamma)$ is dense in the space $W_p^{2m-1/p}(\Gamma)$ $(p > 1)$.

1. In a bounded domain $\Omega \subset \mathbf{R}^n$ with a smooth boundary $\partial\Omega$ a linear elliptic boundary value problem for a differential operator L of order $2m$ is considered. Let $\Gamma \subset \Omega$ be a smooth, $(n-1)$-dimensional closed surface with boundary in the interior of Ω. Generalizing earlier results for equations of the second order of H. BECKERT [4] and A. GÖPFERT [8, 9] in [18] the density of some sets of solutions in the Sobolev space $W_2^{2m-1}(\Gamma)$ was proved. Changing for instance the boundary values on an arbitrary small part V of the boundary, one can generate such a dense set.

In the present paper the results of [18] are generalized for the trace spaces, $W_p^{2m-\frac{1}{p}}(\Gamma)$ $(p > 1)$. Analogous results for uniform approximation are given in [16, 17]. For the case of second order see G. ANGER [3] and G. WANKA [14]. For approximation theorems of another type for higher order elliptic equations we refer to F. E. BROWDER [6, 7] and [12].

2. Let

$$L = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$$

$(m > 0$ an integer, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geqq 0$ integers,

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,\ D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n},\ D_i = \frac{\partial}{\partial x_i},\ x = (x_1, \ldots, x_n) \in \mathbf{R}^n)$$

be a properly elliptic differential operator with real coefficients in $C^\infty(\mathbf{R}^n)$, i.e. the polynomial

$$L^0(x, \xi + \tau\eta) = \sum_{|\alpha| = 2m} a_\alpha(x) (\xi + \tau\eta)^\alpha,$$

which corresponds to the main part

$$L^0 = \sum_{|\alpha| = 2m} a_\alpha(x) D^\alpha$$

of the differential operator, for any pair $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n$ $(\xi \neq 0, \eta \neq 0)$ of linearly independent vectors and any $x \in \mathbf{R}^n$ has exactly m roots with positiv imaginary part with respect to τ.

For the adjoint operator

$$L^*u = \sum_{|\alpha| \leqq 2m} (-1)^{|\alpha|} D^\alpha\big(a_\alpha(x) u\big)$$

we suppose the "condition for uniqueness in the small". This means, if u is a solution of $L^*u = 0$ in a connected open set Ω, vanishing on a non-vacuous open subset $\Omega' \subset \Omega$, then u must be identically zero in Ω. The condition for instance is fulfilled, if the coefficients of L are analytic.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$ and Γ a smooth, $(n-1)$dimensional surface (C^∞-manifold) in the interior of Ω, which does not split up the domain Ω. On the boundary $\partial\Omega$ we suppose a normal system of boundary operators $B_1, \ldots, B_m$ with smooth (infinitely differentiable) coefficients and $m_j = \operatorname{ord} B_j \leqq 2m - 1$ $(j = 1, \ldots, m;\ m_i \neq m_j$ for $i \neq j)$. Further we suppose the classical root condition. For the definition of the notions see [12]. The system $(B_j)_{j=1,\ldots,m}$ can be completed to a Dirichlet system $(B_1, \ldots, B_m, C_1, \ldots, C_m)$ of order $2m$ on $\partial\Omega$ by a (not uniquely determined) normal system $(C_j)_{j=1,\ldots,m}$ $(\operatorname{ord} C_j = l_j \leqq 2m - 1)$ (see [11]). This means, that the completed system is a normal system and the set of the orders of the operators is $\{0, 1, \ldots, 2m - 1\}$. If the operators C_j $(j = 1, \ldots, m)$ are fixed, then in an unique way one can find $2m$ boundary operators B_j', C_j' $(j = 1, \ldots, m)$ with smooth coefficients on $\partial\Omega$, such that the following properties hold:

(i) $\operatorname{ord} B_j' = m_j' = 2m - 1 - l_j$, $\operatorname{ord} C_j' = l_j' = 2m - 1 - m_j$.

(ii) $(B_1', \ldots, B_m', C_1', \ldots, C_m')$ is a Dirichlet system of order $2m$ on $\partial\Omega$ and for $u, v \in C^\infty(\bar{\Omega})$ the Green formula

$$\int_\Omega (Lu) v \, dx - \int_\Omega uL^*v \, dx = \sum_{j=1}^m \int_{\partial\Omega} C_j u B_j' v \, d\sigma - \sum_{j=1}^m \int_{\partial\Omega} B_j u C_j' v \, d\sigma$$

holds.

If the boundary value problem

$$Lu = g \text{ in } \Omega, \qquad B_j u|_{\partial\Omega} = \varphi_j \qquad (j = 1, \ldots, m) \tag{1}$$

has an unique solution, under some smoothness conditions on g and φ_j the solution u can be represented by means of a Green function $G = G(x, y)$ in the form

$$u(x) = \int_\Omega g(y)\, G(x, y)\, dy + \sum_{j=1}^m \int_{\partial\Omega} \varphi_j(y)\, C_j' G(x, y)\, d\sigma(y)$$

(see [5, 15]). The operators C_j' are applied to y. Under our conditions the function $G = G(x, y)$ for $x \neq y$ has derivatives of arbitrary order with respect to both varibles. Applying of the differential operators to y we have

$$L^* G(x, y) = 0 \quad \text{for} \quad x, y \in \Omega \qquad (x \neq y),$$

$$B_j' G(x, y)|_{y \in \partial\Omega} = 0 \qquad (j = 1, \ldots, m).$$

In the general case we assume for simplicity that the index is zero. If there are k linearly independent solutions $u_1, \ldots, u_k$ of the boundary value problem

$$Lu = 0 \text{ in } \Omega, \qquad B_j u|_{\partial\Omega} = 0 \qquad (j = 1, \ldots, m),$$

then there are also k linearly independent solutions $v_1, \ldots, v_k$ of the adjoint problem

$$L^* v = 0 \text{ in } \Omega, \qquad B_j' v|_{\partial\Omega} = 0 \qquad (j = 1, \ldots, m). \tag{2}$$

We assume that

$$\int_\Omega v_i(x)\, v_j(x)\, dx = \int_\Omega u_i(x)\, u_j(x)\, dx = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

$(i, j = 1, \ldots, k)$. There is a generalized Green function $\tilde{G} = \tilde{G}(x, y)$ of the boundary value problem (1) with the following properties (see [5, 10]):

$$L^*_{(y)} \tilde{G}(x, y) = -\sum_{i=1}^k u_i(x)\, u_i(y) \qquad (x, y \in \Omega,\ x \neq y), \tag{3}$$

$$B'_{j,(y)} \tilde{G}(x, y)|_{y \in \partial\Omega} = 0 \qquad (j = 1, \ldots, m),$$

$$\int_\Omega \tilde{G}(x, y)\, v_i(y)\, dy = 0 \qquad (i = 1, \ldots, k), \tag{4}$$

$$L_{(x)} \tilde{G}(x, y) = -\sum_{i=1}^k v_i(x)\, v_i(y) \qquad (x, y \in \Omega,\ x \neq y),$$

$$B_{j,(x)} \tilde{G}(x, y)|_{x \in \partial\Omega} = 0 \qquad (j = 1, \ldots, m),$$

$$\int_\Omega \tilde{G}(x, y)\, u_i(x)\, dx = 0 \qquad (i = 1, \ldots, k).$$

$\tilde{G}$ has the same smoothness properties as G in the case of uniqueness.

The problem (1) has a solution if and only if the conditions

$$\int_\Omega g(y)\, v_i(y)\, dy + \sum_{j=1}^m \int_{\partial\Omega} \varphi_j(y)\, C_j' v_i(y)\, d\sigma(y) = 0 \tag{5}$$

$(i = 1, \ldots, k)$ are fulfilled. Then every solution of (1) can be represented in the form

$$u(x) = \int_\Omega g(y)\, \tilde{G}(x, y)\, dy + \sum_{j=1}^{m} \int_{\partial\Omega} \varphi_j(y)\, C_j' \tilde{G}(x, y)\, d\sigma(y) + \sum_{i=1}^{k} c_i u_i(x) \tag{6}$$

(the differential operators C_j' are applied to y, $c_i \in \mathbf{R}^1$, $i = 1, \ldots, k$). Furthermore, if

$$\int_\Omega g(y)\, v_i(y)\, dy = 0 \quad (i = 1, \ldots, k),$$

then

$$u(x) = \int_\Omega \tilde{G}(x, y)\, g(y)\, dy$$

is the only solution of

$$Lu = g, \quad B_j u|_{\partial\Omega} = 0 \quad (j = 1, \ldots, m)$$

with $\int_\Omega u(x)\, u_i(x)\, dx = 0$ for $i = 1, \ldots, k$.

For given open sets $V \subset \partial\Omega$ and $G \subset \bar{G} \subset \Omega \setminus \Gamma$ we define

$$L_V(\Omega) = \{u \in C^\infty(\bar{\Omega}) : Lu = \theta \text{ in } \Omega,\ B_j u|_{\partial\Omega \setminus V} = 0\ (j = 1, \ldots, m)\} \text{ and}$$

$$L_G(\Omega) = \{u \in C^\infty(\bar{\Omega}) : g := Lu \in C_0^\infty(\Omega),$$
$$\operatorname{supp} g \subset G,\ B_j u|_{\partial\Omega} = 0\ (j = 1, \ldots, m)\} \text{ respectively.}$$

Let $L_V(\Gamma)$, $L_G(\Gamma)$ be the spaces of the restrictions onto Γ. Further we define

$$N(G) = \{g \in C_0^\infty(\Omega) : Lu = g \text{ for some } u \in L_G(\Omega)\}.$$

$N(G)$ is the set of all functions $g \in C_0^\infty(\Omega)$ with support in G and $\int_\Omega g(x)\, v_i(x)\, dx = 0$ $(i = 1, \ldots, k)$. This follows from (5). We shall prove the density of $L_V(\Gamma)$ and $L_G(\Gamma)$ in $W_p^{2m-\frac{1}{p}}(\Gamma)$. First we give the definition of this space.

3. Let $W_p^{2m}(\Omega)$ $(1 < p < \infty)$ denote the classical Sobolev spaces with the norm

$$\|u\|_{2m,p} = \left[\sum_{|\alpha| \leq 2m} \int_\Omega |D^\alpha u(x)|^p\, dx\right]^{\frac{1}{p}}.$$

In the sense of imbedding theorems (see [13]) for $|\alpha| \leqq 2m - 1$ on Γ there exist traces of $D^\alpha u$ for the functions $u \in W_p^{2m}(\Omega)$. More precisely

$$D^\alpha u|_\Gamma \in C(\Gamma) \quad \text{for} \quad |\alpha| < 2m - \frac{n}{p},$$

$$D^\alpha u|_\Gamma \in L^q(\Gamma) \quad \text{for} \quad 2m - \frac{n}{p} \leqq |\alpha| \leqq 2m - 1 \left(1 < q < \frac{p(n-1)}{n - p(2m - |\alpha|)}\right).$$

$W_p^{2m-\frac{1}{p}}(\Gamma)$ is defined as the space of all functions φ on Γ, which are restrictions of functions from $W_p^{2m}(\Omega)$ on Γ in the trace sense. The expression

$$\|\varphi\|_{2m-\frac{1}{p},p} := \inf \|u\|_{2m,p},$$

where the infimum is taken over all $u \in W_p^{2m}(\Omega)$ with $u|_\Gamma = \varphi$, is a norm, such that $W_p^{2m-\frac{1}{p}}(\Gamma)$ becomes a Banach space. The space $C^\infty(\Gamma)$ is dense in $W_p^{2m-\frac{1}{p}}(\Gamma)$.

For definition of the dual space for $\psi \in C^\infty(\Gamma)$ we consider the norm

$$\|\psi\|_{-2m+\frac{1}{p},p'} := \sup_{\varphi \in W_p^{2m-\frac{1}{p}}(\Gamma)} \frac{|(\psi, \varphi)|}{\|\varphi\|_{2m-\frac{1}{p},p}} \tag{7}$$

$\left((\psi, \varphi) = \int_\Gamma \psi(x) \cdot \varphi(x)\, d\sigma_\Gamma(x),\ d\sigma_\Gamma\right.$ surface element with respect to Γ, $\left.\frac{1}{p} + \frac{1}{p'} = 1\right)$.

The completion of $C^\infty(\Gamma)$ with respect to (7) will be denoted by $W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$.

From (7) immediately follows

$$|(\psi, \varphi)| \leqq \|\psi\|_{-2m+\frac{1}{p},p'} \cdot \|\varphi\|_{2m-\frac{1}{p},p} \tag{8}$$

for $\psi \in C^\infty(\Gamma)$, $\varphi \in W_p^{2m-\frac{1}{p}}(\Gamma)$. Let $\psi \in W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$, $\varphi \in W_p^{2m-\frac{1}{p}}(\Gamma)$. Choosing a sequence $\psi_n \in C^\infty(\Gamma)$ with $\lim_{n\to\infty} \|\psi_n - \psi\|_{-2m+\frac{1}{p},p'} = 0$ we put

$$(\psi, \varphi) := \lim_{n\to\infty} (\psi_n, \varphi). \tag{9}$$

The existence of the limit follows from (8). (ψ, φ) does not depend on the sequence (ψ_n). Moreover, the inequality

$$|(\psi, \varphi)| \leqq \|\psi\|_{-2m+\frac{1}{p},p'} \cdot \|\varphi\|_{2m-\frac{1}{p},p} \tag{10}$$

holds for $\psi \in W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$, $\varphi \in W_p^{2m-\frac{1}{p}}(\Gamma)$. The spaces $W_p^{2m-\frac{1}{p}}(\Gamma)$ and $W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$ are mutually dual, $\left(W_p^{2m-\frac{1}{p}}(\Gamma)\right)' = W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$. For a given $F \in \left(W_p^{2m-\frac{1}{p}}(\Gamma)\right)'$ there exists an unique element $\psi_F \in W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$, such that $F(\varphi) = (\psi_F, \varphi)$ for every $\varphi \in W_p^{2m-\frac{1}{p}}$ and $\|F\| = \|\psi_F\|_{-2m+\frac{1}{p},p'}$.

4. Now we are able to formulate our main result.

Theorem 1: *Under the suppositions of Section 2 for $1 < p < \infty$ the space $L_G(\Gamma)$ is dense in $W_p^{2m-\frac{1}{p}}(\Gamma)$.*

Proof: We shall prove the theorem indirectly and suppose $\overline{L_G(\Gamma)} \neq W_p^{2m-\frac{1}{p}}(\Gamma)$. Then there exists an element $h \in W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$, $h \neq 0$, with $(h, u) = 0$ for all $u \in L_G(\Gamma)$ in the sense of the scalarproduct (9). Therefore we can choose a sequence $h_j \in C^\infty(\Gamma)$ with $\|h_j - h\|_{-2m+\frac{1}{p},p'} \to 0$, such that

$$(h, u) = \lim_{j\to\infty} (h_j, u) = \lim_{j\to\infty} \int_\Gamma h_j(x)\, u(x)\, d\sigma_\Gamma(x) = 0 \tag{11}$$

for every $u \in L_G(\Gamma)$. The definition of $L_G(\Gamma)$ and (6) give the representation (with $Lu = g \in N(G)$)

$$u(x) = \int_G g(y)\, \tilde{G}(x, y)\, dy + \sum_{i=1}^{k} c_i u_i(x)$$

for every $u \in L_G(\Gamma)$. Because of $u_i|_\Gamma \in L_G(\Gamma)$ we have $\left(h, \sum_{i=1}^{k} c_i u_i\right) = 0$. Hence

$$\begin{aligned}(h, u) &= \lim_{j\to\infty} \int_\Gamma h_j(x) \left(\int_G g(y)\, \tilde{G}(x, y)\, dy\right) d\sigma_\Gamma(x)\\ &= \lim_{j\to\infty} \int_G g(y) \left(\int_\Gamma h_j(x)\, \tilde{G}(x, y)\, d\sigma_\Gamma(x)\right) dy = 0 \end{aligned} \tag{12}$$

for every $g \in N(G)$.

We shall prove that the sequence

$$\int_\Gamma h_j(x)\, D_y^\beta \tilde{G}(x, y)\, d\sigma_\Gamma(x)$$

for $|\beta| \leqq 2m$ with respect to y is uniformly convergent on every compact set $K \subset \Omega \setminus \Gamma$. We fix an open neighbourhood U of Γ with $\Gamma \subset U \subset \Omega$ and $\overline{U} \cap K = \emptyset$. From the smoothness of $\tilde{G} = \tilde{G}(x, y)$ for $x \neq y$ follows $D_y^\beta \tilde{G}(x, y) \in W_p^{2m}(U)$ with respect to $x \in U$ and any fixed $y \in K$. We suppose that the boundary ∂U is smooth. Then we can use a general result from the theory of Sobolev spaces. Namely, because the boundary ∂U is smooth, there exists a continuous extension operator from $W_p^{2m}(U)$ into $W_p^{2m}(\Omega)$, i.e. for every $u \in W_p^{2m}(U)$ there exists an extension $\tilde{u} \in W_p^{2m}(\Omega)$ with $u(x) = \tilde{u}(x)$ in U and

$$\|\tilde{u}\|_{W_p^{2m}(\Omega)} \leqq \omega \cdot \|u\|_{W_p^{2m}(U)}$$

(ω independent of $u \in W_p^{2m}(U)$). Obviously the estimate ($|\beta| \leqq 2m$)

$$\begin{aligned}&\|D_y^\beta \tilde{G}(\cdot, y)\|_{W_p^{2m}(U)}\\ &\leqq \sum_{|\alpha|\leqq 2m} \sup_{x\in U, y\in K} |D_x^\alpha D_y^\beta \tilde{G}(x, y)|\, m(U)^{\frac{1}{p}} = C(U, K, m, p, \beta)\end{aligned}$$

holds, where $m(U)$ denotes the Lebesgue measure of U and the constant on the right hand side is independent of y. It follows

$$\begin{aligned}\|D_y^\beta \tilde{G}(\cdot, y)\|_{2m-\frac{1}{p},p} &= \inf_{u|_\Gamma = D_y^\beta \tilde{G}(\cdot, y)} \|u\|_{W_p^{2m}(\Omega)}\\ &\leqq \omega \cdot \|D_y^\beta \tilde{G}(\cdot, y)\|_{W_p^{2m}(U)} \leqq \tilde{C}(U, K, m, p, \beta).\end{aligned}$$

Applying (10) we get the estimate

$$\begin{aligned}&\left|\int_\Gamma D_y^\beta \tilde{G}(x, y) \left(h_i(x) - h_j(x)\right) d\sigma_\Gamma(x)\right|\\ &\leqq \|D_y^\beta \tilde{G}(\cdot, y)\|_{2m-\frac{1}{p},p} \|h_i - h_j\|_{-2m+\frac{1}{p},p'}\\ &\leqq \tilde{C}(U, K, m, p, \beta) \cdot \|h_i - h_j\|_{-2m+\frac{1}{p},p'}.\end{aligned}$$

Because the right side is independent of y, the assertion follows. The sequence

$$\int_\Gamma h_j(x)\, \tilde{G}(x, y)\, d\sigma_\Gamma(x)$$

especially converges uniformly with respect to $y \in G$.

Therefore in (12) we can change the limit and the integral and get

$$(h, u) = \int_G g(y) \left(\lim_{j\to\infty} \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x)\right) dy = 0 \tag{13}$$

for every $g \in N(G)$, i.e. for every $g \in C_0^\infty(\Omega)$ with $\operatorname{supp} g \subset G$ and

$$\int_\Omega g(x)\, v_i(x)\, dx = 0 \qquad (i = 1, \ldots, k) \tag{14}$$

((14) follows from (5)). Combining (13) and (14) we obtain

$$\lim_{j\to\infty} \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x) = \sum_{i=1}^{k} c_i v_i(y) =: v(y) \quad \text{in } G. \tag{15}$$

Putting

$$w(y) := \lim_{j\to\infty} \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x) - v(y),$$

using $L^*v = 0$, (3) and the uniform convergence for $|\beta| \leqq 2m$, we have for $y \in \Omega \setminus \Gamma$

$$\begin{aligned} L^*w(y) &= L^* \left(\lim_{j\to\infty} \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x)\right) - L^*v(y) \\ &= \lim_{j\to\infty} L^* \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x) \\ &= \lim_{j\to\infty} \int_\Gamma L^*_{(y)}\tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x) \\ &= -\lim_{j\to\infty} \sum_{i=1}^{k} u_i(y) \int_\Gamma u_i(x)\, h_j(x)\, d\sigma_\Gamma(x) \\ &= -\sum_{i=1}^{k} u_i(y) \cdot \left\{\lim_{j\to\infty} \int_\Gamma u_i(x)\, h_j(x)\, d\sigma_\Gamma(x)\right\} = 0. \end{aligned}$$

The last equality follows from (11), because $u_i \in L_G(\Omega)$ for $i = 1, \ldots, k$. From (15), i.e. $w \equiv 0$ in G, and from the "condition for uniqueness in the small", we obtain $w \equiv 0$ in $\Omega \setminus \Gamma$, i.e.

$$\lim_{j\to\infty} \int_\Gamma \tilde{G}(x, y)\, h_j(x)\, d\sigma_\Gamma(x) = v(y) \quad \text{in } \Omega \setminus \Gamma. \tag{16}$$

In the next step we shall show that

$$\lim_{j\to\infty} \int_\Gamma u(x)\, h_j(x)\, d\sigma_\Gamma(x) = 0$$

holds for every $u \in C_0^\infty(\Omega)$. Defining $f := Lu$, the function $u \in C_0^\infty(\Omega)$ can be considered as a solution of the boundary value problem

$$Lu = f \text{ in } \Omega, \qquad B_j u|_{\partial\Omega} = 0 \qquad (j = 1, \ldots, m)$$

and by (6) we have

$$u(x) = \int_\Omega \tilde{G}(x, y)\, f(y)\, dy + \sum_{i=1}^{k} c_i u_i(x).$$

Because of $u_i \in L_G(\Omega)$ it is

$$\lim_{j\to\infty} \int_\Gamma \left(\sum_{i=1}^k c_i u_i(x)\right) h_j(x)\, d\sigma_\Gamma(x) = 0.$$

Therefore we must show that

$$\lim_{j\to\infty} \int_\Gamma \left(\int_\Omega \tilde G(x, y)\, f(y)\, dy\right) h_j(x)\, d\sigma_\Gamma(x) = 0.$$

$\Gamma_\delta := \{x \in \Omega: d(x, \Gamma) < \delta\}$ denotes an open δ-neighbourhood of Γ. For every $\delta > 0$, there is a $\varphi_\delta \in C_0^\infty(\mathbf{R}^n)$ with $0 = \varphi_\delta(x) \leqq 1$ for all $x \in \mathbf{R}^n$, $\varphi_\delta(x) = 1$ for all $x \in \Gamma_{\delta/2}$ and $\operatorname{supp} \varphi_\delta \subset \Gamma_\delta$. We define $f_\delta := f\varphi_\delta$ and $\bar f_\delta := f(1 - \varphi_\delta)$. It is $f_\delta + \bar f_\delta = f$, $\operatorname{supp} f_\delta \subset \Gamma_\delta$ and $\bar f_\delta(x) = 0$ for $x \in \Gamma_{\delta/2}$. We have

$$\begin{aligned}
&\lim_{j\to\infty} \int_\Gamma \left(\int_\Omega \tilde G(x, y)\, f(y)\, dy\right) h_j(x)\, d\sigma_\Gamma(x)\\
&= \lim_{j\to\infty} \int_\Gamma \left(\int_\Omega \tilde G(x, y)\, \bar f_\delta(y)\, dy\right) h_j(x)\, d\sigma_\Gamma(x)\\
&\quad + \lim_{j\to\infty} \int_\Gamma \left(\int_\Omega \tilde G(x, y)\, f_\delta(y)\, dy\right) h_j(x)\, d\sigma_\Gamma(x) = I_{1,\delta} + I_{2,\delta}.
\end{aligned}$$

Using once more the uniform convergence of $\int_\Gamma h_j(x)\, \tilde G(x, y)\, d\sigma_\Gamma(x)$ and (16), we obtain ($\bar f_\delta \equiv 0$ in $\Gamma_{\delta/2}$!)

$$\begin{aligned}
I_{1,\delta} &= \lim_{j\to\infty} \int_\Omega \bar f_\delta(y) \left(\int_\Gamma \tilde G(x, y)\, h_j(x)\, d\sigma_\Gamma(x)\right) dy\\
&= \int_\Omega \bar f_\delta(y) \left(\lim_{j\to\infty} \int_\Gamma \tilde G(x, y)\, h_j(x)\, d\sigma_\Gamma(x)\right) dy\\
&= \int_\Omega \bar f_\delta(y)\, v(y)\, dy.
\end{aligned}$$

Because v is a solution of the homogeneous boundary value problem (2), the condition (5) gives

$$0 = \int_\Omega v(y)\, f(y)\, dy = \int_\Omega v(y)\, \bar f_\delta(y)\, dy + \int_{\Gamma_\delta} v(y)\, f_\delta(y)\, dy.$$

Because of $\lim_{\delta\to 0} m(\Gamma_\delta) = 0$ ($m(\Gamma_\delta)$ denotes the Lebesgue measure of Γ_δ) there is a $\delta_1(\varepsilon) > 0$ for a given $\varepsilon > 0$, such that

$$|I_{1,\delta}| = \left|\int_\Omega v(y)\, \bar f_\delta(y)\, dy\right| < \varepsilon \quad \text{for } 0 < \delta \leqq \delta_1(\varepsilon).$$

Now we consider $I_{2,\delta}$:

$$\begin{aligned}
&\left|\int_\Gamma h_j(x) \left(\int_\Omega \tilde G(x, y)\, f_\delta(y)\, dy\right) d\sigma_\Gamma(x)\right|\\
&\leqq \|h_j\|_{-2m+\frac{1}{p},p'} \cdot \left\|\int_\Omega \tilde G(x, y)\, f_\delta(y)\, dy|_\Gamma\right\|_{2m-\frac{1}{p},p}\\
&\leqq C \cdot \left\|\int_\Omega \tilde G(x, y)\, f_\delta(y)\, dy\right\|_{2m,p}
\end{aligned}$$

(the convergent sequence h_j is bounded!). Now we want to show that

$$\lim_{\delta\to 0} \left\| \int_\Omega \tilde{G}(x, y)\, f_\delta(y)\, dy \right\|_{2m,p} = 0.$$

For $C_i^\delta := \int_\Omega f_\delta(y)\, v_i(y)\, dy$ we have $\lim_{\delta\to 0} C_i^\delta = 0$ $(i = 1, \ldots, k)$. We define

$$F_\delta(y) := f_\delta(y) - \sum_{i=1}^{k} C_i^\delta v_i(y)$$

and get $\int_\Omega F_\delta(y)\, v_i(y)\, dy = 0$ $(i = 1, \ldots, k)$. Therefore

$$W_\delta(x) := \int_\Omega \tilde{G}(x, y)\, F_\delta(y)\, dy$$

is a solution of the boundary value problem

$$Lu = F_\delta \text{ in } \Omega, \qquad B_j u|_{\partial\Omega} = 0 \qquad (j = 1, \ldots, m).$$

From the Schauder estimate (see [2]) we can conclude that there exists a constant $C < \infty$ with

$$\|W_\delta\|_{2m,p} \leqq C \cdot \|LW_\delta\|_{0,p} = C \cdot \|F_\delta\|_{0,p}$$

for every $\delta > 0$ (C does not depend on δ). Because of

$$\|F_\delta\|_{0,p} \leqq \|f_\delta\|_{0,p} + \sum_{i=1}^{k} |C_i^\delta| \cdot \|v_i\|_{0,p},$$

$$\lim_{\delta\to 0} \|f_\delta\|_{0,p} = 0 \quad \text{and} \quad \lim_{\delta\to 0} C_i^\delta = 0 \qquad (i = 1, \ldots, k)$$

we have $\lim_{\delta\to 0} \|W_\delta\|_{2m,p} = 0$. From $\int_\Omega \tilde{G}(x, y)\, v_i(y)\, dy = 0$ (see (4)) we conclude

$$\int_\Omega \tilde{G}(x, y)\, f_\delta(y)\, dy = \int_\Omega \tilde{G}(x, y)\, F_\delta(y)\, dy = W_\delta(x).$$

Therefore we get $\lim_{\delta\to 0} \left\| \int_\Omega \tilde{G}(x, y)\, f_\delta(y)\, dy \right\|_{2m,p} = 0$. For that reason we can find a $\delta_2(\varepsilon) > 0$ that

$$\left| \int_\Gamma h_j(x) \cdot \left(\int_\Omega \tilde{G}\,(x, y)\, f_\delta(y)\, dy \right) d\sigma_\Gamma(x) \right| < \varepsilon$$

for $0 < \delta \leqq \delta_2(\varepsilon)$ and every j. This means

$$\lim_{j\to\infty} \int_\Gamma u(x)\, h_j(x)\, d\sigma_\Gamma(x) = 0.$$

Now we consider any function $\psi \in C^\infty(\Gamma)$. From the smoothness of Γ follows that an extension to a function $u \in C_0^\infty(\Omega)$ can be found. From our preceding considerations we obtain

$$\lim_{j\to\infty} (h_j, \psi) = 0 \text{ for every } \psi \in C^\infty(\Gamma).$$

Finally we get

$$\|h\|_{-2m+\frac{1}{p},p'} = \sup_{\varphi \in W_p^{2m-\frac{1}{p}}(\Gamma)} \frac{|(h,\varphi)|}{\|\varphi\|_{2m-\frac{1}{p},p}} = \sup_{\psi\in C^\infty(\Gamma)} \frac{|(h,\psi)|}{\|\psi\|_{2m-\frac{1}{p},p}}$$

$$= \sup_{\psi\in C^\infty(\Gamma)} \frac{\left|\lim_{j\to\infty} (h_j,\psi)\right|}{\|\psi\|_{2m-\frac{1}{p},p}} = 0,$$

i.e. $h = 0$ and $\overline{L_G(\Gamma)} = W_p^{2m-\frac{1}{p}}(\Gamma)$ ∎

5. Let $V \subset \partial\Omega$ be an open subset of the boundary $\partial\Omega$. We construct a larger smooth domain $\Omega_1 \supset \Omega$ with $\partial\Omega \setminus V \subset \partial\Omega_1$. In addition to our earlier assumptions we suppose:

(i) The coefficients of B_j can be extended to $\partial\Omega_1 \setminus \partial\Omega$ in such a way that the new system $(B_j)_{j=1,\dots,m}$ of boundary operators on $\partial\Omega_1$ also is normal and satisfies the roots condition.

If the coefficients of B_j are constant (for instance in the case of the Dirichlet problem) condition (i) is fulfilled.

Theorem 2: *We suppose the conditions of Section* 2 *and* (i). *Then* $L_V(\Gamma)$ *is dense in* $W_p^{2m-\frac{1}{p}}(\Gamma)$ $(1 < p < \infty)$.

Proof: We choose an open subset $G \subset \Omega_1 \setminus \Omega$ and consider the space $L_G(\Omega_1)$. By Theorem 1 we have $\overline{L_G(\Omega_1)|_\Gamma} = W_p^{2m-\frac{1}{p}}(\Gamma)$. Since $L_G(\Omega_1)|_{\bar\Omega} \subset L_V(\Omega)$, we obtain $\overline{L_V(\Gamma)} = W_p^{2m-\frac{1}{p}}(\Gamma)$ ∎

If we further suppose that the boundary value problem

$$Lu = 0 \text{ in } \Omega, \qquad B_j u|_{\partial\Omega} = 0 \qquad (j = 1, \dots, m)$$

only has the trivial solution, in the same manner as in [18], replacing $W_2^{2m-1}(\Gamma)$ by $W_p^{2m-\frac{1}{p}}(\Gamma)$ we can prove the theorems, corresponding Theorem 3 and 4 in [18].

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