U. HAMANN and G. WILDENHAIN

Es sei $\Omega \subset \mathbb{R}^n$ ein beschränktes, glattes Gebiet, Γ eine abgeschlossene, glatte, $(n-1)$ -dimensionale Fläche mit Rand im Inneren von Ω und V eine offene Teilmenge des Randes $\partial\Omega$. In Ω werde ein eigentlich elliptischer Differentialoperator L der Ordnung 2m mit glatten Koeffizienten betrachtet. $(B_1, ..., B_m)$ sei ein normales System von Randoperatoren auf $\partial\Omega$, welches der klassischen Wurzelbedingung genügt. $L_V(\Gamma)$ bezeichne den Raum der Einschränkungen der Funktionen des Raumes

$$
L_V(\Omega) = \{u \in C^\infty(\overline{\Omega}) : Lu = 0 \text{ in } \Omega, B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \text{ in } \partial\Omega \setminus V\}
$$

auf Γ . Es wird bewiesen, daß $L_V(\Gamma)$ im Raum $W_p^{2m-1/p}(\Gamma)(p>1)$ dicht liegt.

Пусть $\Omega \subset \mathbb{R}^n$ ограниченная гладкая область, Γ - заминутая гладкая (n - 1)-мерная площадь с краем внутри области Ω , и V - открытое подмножество края д Ω . Рассматривается в Ω собственный эллиптический оператор L порядка $2m$ с гладкими коэффициентами. Пусть $(B_1, ..., B_m)$ - нормальная система краевых операторов на $\partial\Omega$, удовлетворяющая классическому условию на корнях, а $L_v(\Gamma)$ обозначает пространство ограничений на Γ функций пространства

$$
L_V(\Omega) = \{u \in C^\infty(\overline{\Omega}) : Lu = 0 \text{ B } \Omega, B_1u|_{\partial\Omega} = \cdots = B_m u|_{\partial\Omega} = 0 \text{ B } \partial\Omega \setminus V\}.
$$

Доказывается, что $L_V(\Gamma)$ плотно в пространстве $W_p^{2m-1/p}(\Gamma)$ $(p > 1)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, Γ a closed, smooth, $(n-1)$ -dimensional surface with boundary in the interior of Ω and V an open subset of the boundary $\partial\Omega$. In Ω we consider a porperly elliptic differential operator L of order $2m$ with smooth coefficients. Let $(B_1, ..., B_m)$ be a normal system of boundary operators on $\partial\Omega$, which fulfils the classical root condition. $L_V(\Gamma)$ denote the space of the restrictions on Γ of the functions from

$$
L_V(\Omega) = \{u \in C^\infty(\overline{\Omega}) : Lu = 0 \text{ in } \Omega, B_1u|_{\partial\Omega} = \cdots = B_mu|_{\partial\Omega} = 0 \text{ in } \partial\Omega \setminus V\}.
$$

It is proved that $L_V(\Gamma)$ is dense in the space $W_p^{2m-1/p}(\Gamma)$ $(p > 1)$.

1. In a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial \Omega$ a linear elliptic boundary value problem for a differential operator L of order $2m$ is considered. Let $\Gamma \subset \Omega$ be a smooth, $(n - 1)$ -dimensional closed surface with boundary in the interior of Ω . Generalizing earlier results for equations of the second order of H. BECKERT [4] and A. GÖPFERT [8, 9] in [18] the density of some sets of solutions in the Sobolev space $W_2^{2m-1}(\Gamma)$ was proved. Changing for instance the boundary values on an arbitrary small part V of the boundary, one can generate such a dense set.

In the present paper the results of [18] are generalized for the trace spaces. W_p^{2m-} $\overline{P}(F)(p>1)$. Analogous results for uniform approximation are given in [16,

17]. For the case of second order see G. ANGER [3] and G. WANKA [14]. For approximation theorems of another type for higher order elliptic equations we refer to F. E. BROWDER [6, 7] and [12].

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\n2. Let
\n
$$
L = \sum_{|a| \leq 2m} a_a(x) D^a
$$
\n $(m > 0 \text{ an integer}, \alpha = (\alpha_1, ..., \alpha_n), \alpha_i \geq 0 \text{ integers}, |\alpha| = \alpha_1 + \cdots + \alpha_N, D^a = D_1^{\alpha_1} ... D_n^{\alpha_n}, D_i = \frac{\partial}{\partial x_i}, x = (x_1, ..., x_n) \in \mathbb{R}^n)$
\na properly elliptic differential operator with real coefficients in $C^{\infty}(\mathbb{R}^n)$, i.e. th
\nlynomial.
\n
$$
L^0(x, \xi + \tau \eta) = \sum_{|a| = 2m} a_a(x) (\xi + \tau \eta)^a,
$$

\nniich corresponds to the main part
\n $I^0 = \sum_{|a| = 2m} a(a) D^a$

be a properly elliptic differential operator with real coefficients in $C^{\infty}(\mathbb{R}^n)$, i.e. the polynomial.

$$
L^{0}(x, \xi + \tau \eta) = \sum_{|\alpha| = 2m} a_{\alpha}(x) (\xi + \tau \eta)^{\alpha},
$$

which corresponds to the main part

$$
L^0 = \sum_{|\alpha|=2m} a_\alpha(x) D
$$

of the differential operator, for any pair $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ ($\xi \neq 0, \eta \neq 0$) of linearly or the differential operator, for any pair $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ ($\xi \neq 0, \eta \neq 0$) or intearly independent vectors and any $x \in \mathbb{R}^n$ has exactly *m* roots with positiv imaginary part with respect to τ .
For part with respect to τ . $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $D^a = D_1^{a_1} \cdots D_n^{a_n}$, $D_i = \frac{\partial}{\partial x_i}$,

a properly elliptic differential operator with real coeffici

lynomial.
 $L^0(x, \xi + \tau \eta) = \sum_{|\alpha| = 2m} a_\alpha(x) (\xi + \tau \eta)^s$,

aich corresponds to the main part
 $L^0 = \$

$$
L^*u=\sum_{|a|\leq 2m}(-1)^{|a|} D^*(a_a(x)u)
$$

we suppose the "condition for uniqueness in the small". This means, if *u* is a' solution of $L^*u = 0$ in a connected open set Ω , vanishing on a non-vacuous open subset $\Omega' \subset \Omega$, then *u* must be identically zero in Ω . The condition for instance is fulfilled, if the coefficients of *L* are analytic.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial \Omega$ and Γ a smooth, $(n - 1)$ dimensional surface (C^{∞} -manifold) in the interior of Ω , which does not split up the domain Ω . On the boundary $\partial\Omega$ we suppose a normal system of boundary operators $B_1, ..., B_m$ with smooth (infinitely differentiable) coefficients boundary operators $B_i, ..., B_m$ with smooth (infinitely differentiable) coefficients and $m_i = \text{ord } B_i \leq 2m - 1$ ($j = 1, ..., m; m_i + m_j$ for $i + j$). Further we suppose the classical root condition. For the definition of the notions see [12]. The system $(B_i)_{i=1,\dots,m}$ can be completed to a Dirichlet system $(B_1, \dots, B_m, C_1, \dots, C_m)$ of order 2*m* on $\partial\Omega$ by a. (not uniquely determined) normal system $(C_j)_{j=1,\dots,m}$ (ord $C_j = l_j$ $\leq 2m - 1$) (see [11]). This means, that the completed system is a normal system and the set of the orders of the operators is $\{0, 1, ..., 2m - 1\}$. If the operators C_j $(j = 1, ..., m)$ are fixed, then in an unique way one can find 2m boundary operators B_f', C_f' $(j = 1, ..., m)$ with smooth coefficients on $\partial \Omega$, such that the following properties hold: the completed system

rs is $\{0, 1, ..., 2m - 1\}$

que way one can find

fficients on $\partial \Omega$, such that

d $C_j' = l_j' = 2m - 1$

dirichlet system of orc
 $\sum_{j=1}^{m} \int_{\partial \Omega} C_j u B_j' v \, d\sigma - \sum_{j=1}^{m}$ the classical floot condition. It
 $(B_j)_{j=1,\dots,m}$ can be completed to
 $2m$ on $\partial\Omega$ by a (not uniquely
 $\leq 2m - 1$) (see [11]). This mean

the set of the orders of the $(j = 1, ..., m)$ are fixed, then in
 B_j', C_j' ($j = 1, ..., m$) m is a norm
- 1). If the
- 1). If the
d 2*m* bounds the f
- 1 - m_j.
order 2*m* of
 $\sum_{i=1}^{m} \int_{\partial \Omega} B_i u C_i$ Expressed by the state of $\partial \Omega$ and Γ
 $\partial \sigma$

(i) ord $B_i' = m_i' = 2m - 1 - l_i$, ord $C_i' = l_i' = 2m - 1 - m_i$.

(ii) $(B_1',...,B_m',C_1',...,C_m')$ is a Dirichlet system of order $2m$ on $\partial\Omega$ and for $u, v \in C^{\infty}(\overline{\Omega})$ the Green formula

$$
y = 1, ..., m
$$
 with smooth coefficients of 0.25 , such that the book
old:

$$
d B'_j = m'_j = 2m - 1 - l_j
$$
, ord $C'_j = l'_j = 2m - 1 - m_j$.

$$
a_1^{(1)}, ..., B'_m, C_1', ..., C'_m)
$$
 is a Dirichlet system of order $2m$ on ∂S .

$$
\int_{\Omega} (Lu) v dx - \int_{\Omega} uL^* v dx = \sum_{j=1}^m \int_{\partial \Omega} C_j u B'_j v d\sigma - \sum_{j=1}^m \int_{\partial \Omega} B_j u C'_j v d\sigma
$$

boundary value problem

$$
Lu = g
$$
 in Ω , $B_j u|_{\partial \Omega} = \varphi_j$ $(j = 1, ..., m)$

(1)

If the boundary value problem

$$
Lu = g
$$
 in Ω , $B_j u|_{\partial \Omega} = \varphi_j$ $(j = 1, ..., m)$

has an unique solution, under some smoothness conditions on g and φ_j the solution u can be represented by means of a Green function $G = G(x, y)$ in the form

$$
u(x) = \int_{Q} g(y) G(x, y) dy + \sum_{j=1}^{m} \int_{\partial Q} \varphi_{j}(y) C_{j} G(x, y) d\sigma(y)
$$

(see [5, 15]). The operators C_j are applied to y. Under our conditions the function $G = G(x, y)$ for $x \neq y$ has derivatives of arbitrary order with respect to both varbles. Applying of the differential operators to y we have
 $L^*G(x, y) = 0$ for $x, y \in \Omega$ $(x \neq y)$,
 $B_j'G(x, y)|_{y \in \partial \Omega} = 0$ $(j = 1, ..., m)$. bles. Applying of the differential operators to y we have *L*G(x, y)* = 0 *largerify* = 0 *large (y)* $C_i/G(x, y) d\sigma(y)$
 to y. Under our condition
 rbitrary order with respecto y we have
 $x \neq y$,
 n).
 r
 rg that the index is zero.
 he boundary value proble
 $(j = 1, ..., m)$,

blutions $v_1, ..., v_k$ of the ac
 $(i = 1$ Approximation by Solutions of Elliptic Equations 61

thes an unique solution, under some smoothness conditions on g and φ_i the solution
 v can be represented by means of a Green function $G = G(x, y)$ in the form
 \langle

$$
L^*G(x, y) = 0 \quad \text{for} \quad x, y \in \Omega \qquad (x \neq y),
$$

$$
B_j^{\prime}G(x, y)|_{y \in \partial \Omega} = 0 \qquad (j = 1, ..., m). \qquad ,
$$

In the general case we assume for simplicity that the index is zero. If there are *k* linearly independent solutions u_1, \ldots, u_k of the boundary value problem $B_j' G(x, y)|_{y \in \partial \Omega} = 0$ $(j = 1, ..., n)$
 *i*eneral case we assume for simplicit
 independent solutions $u_1, ..., u_k$ of t
 Lu = 0 in Ω , $B_j u|_{\partial \Omega} = 0$ bitrary order with respect to both vari-
 y y we have
 (y, y) (y, y)
 $(y,$

$$
Lu = 0 \text{ in } \Omega, \qquad B_j u|_{\partial \Omega} = 0 \qquad (j = 1, ..., m)
$$

The problem follows
$$
u_1, \ldots, u_k
$$
 of the boundary value problem
\n
$$
Lu = 0
$$
 in Ω , $B_j u|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$,
\nare also k linearly independent solutions v_1, \ldots, v_k of the adjoint problem
\n
$$
L^*v = 0
$$
 in Ω , $B_j' v|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$. (2)
\nthe

We assume that

The area also
$$
k
$$
 linearly independent solutions $v_1, ..., v$
\n $L^*v = 0$ in Ω , $B_j'v|_{\partial\Omega} = 0$ $(j = 1, ..., m)$.
\n
$$
\text{The that}
$$
\n
$$
\int_a v_i(x) v_j(x) dx = \int_a u_i(x) u_j(x) dx = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}
$$

 $(i, j = 1, ..., k)$. There is a generalized Green function $\tilde{G} = \tilde{G}(x, y)$ of the boundary value problem (1) with the following properties (see [5, 10]):

re are also k linearly independent solutions
$$
v_1, ..., v_k
$$
 of the adjoint problem
\n $L^*v = 0$ in Ω , $B_j'v|_{\partial \Omega} = 0$ $(j = 1, ..., m)$. (2)
\nme that
\n
$$
\int_{\Omega} v_i(x) v_j(x) dx = \int_{\Omega} u_i(x) u_j(x) dx = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}
$$
\n..., k). There is a generalized Green function $\tilde{G} = \tilde{G}(x, y)$ of the boundary
\noblem (1) with the following properties (see [5, 10]):
\n $L^*_{(y)}\tilde{G}(x, y) = -\sum_{i=1}^k u_i(x) u_i(y)$ $(x, y \in \Omega, x \neq y)$, (3)
\n $B'_{j,(y)}\tilde{G}(x, y)|_{y \in \partial \Omega} = 0$ $(j = 1, ..., m)$,
\n
$$
\int_{\Omega} \tilde{G}(x, y) v_i(y) dy = 0
$$
 $(i = 1, ..., k)$,
\n $L_{(x)}\tilde{G}(x, y) = -\sum_{i=1}^k v_i(x) v_i(y)$ $(x, y \in \Omega, x \neq y)$,
\n $B_{j,(x)}\tilde{G}(x, y)|_{x \in \partial \Omega} = 0$ $(j = 1, ..., m)$,
\n
$$
\int_{\Omega} \tilde{G}(x, y) u_i(x) dx = 0
$$
 $(i = 1, ..., k)$.

$$
\int \tilde{G}(x, y) v_i(y) dy = 0 \qquad (i = 1, ..., k),
$$

Substituting the values for sum of summing the values. If there are
$$
k
$$
 independent solutions u_1, \ldots, u_k of the boundary value problem.

\nLet $u = 0$ in Ω , $B_j u|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$, the result of x_1, \ldots, x_k of the adjoint problem.

\nLet $x = 0$ in Ω , $B_j v|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$.

\nLet $x = 0$ in Ω , $B_j v|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$.

\nLet $x = 0$ in Ω , $B_j v|_{\partial \Omega} = 0$ $(j = 1, \ldots, m)$.

\nLet $y = 0$ in $x = 0$ and $y = 0$ and $y = 0$.

\nTherefore, the following properties (see [5, 10]):

\nLet $y = 0$ and $y = 0$ and $y = 0$ and $y = 0$.

\nTherefore, $y = 0$ and $y = 1$ and $y = 0$.

\nTherefore, $y = 1$ and $y = 0$ and $y = 0$ and $y = 1$ and $y = 0$.

\nTherefore, $y = 1$ and $y = 0$ and $y = 1$ and $y = 0$.

\nTherefore, $y = 1$ and $y = 0$ and $y = 1$ and $y = 0$.

\nTherefore, <

$$
\int_{2} \tilde{G}(x, y) u_{i}(x) dx = 0 \quad (i = 1, ..., k).
$$

 \tilde{G} has the same smoothness properties as G in the case of uniqueness. The problem (1) has a solution if and only if the conditions

$$
\int\limits_{\Omega} g(y) \, v_i(y) \, dy \, + \, \sum\limits_{j=1}^m \int\limits_{\partial \Omega} \varphi_j(y) \, C_j' v_i(y) \, d\sigma(y) = 0
$$

 (4)
 (5)
 (5)

5

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\n
$$
(i = 1, ..., k) \text{ are fulfilled. Then every solution of (1) can be represented in the form}
$$
\n
$$
u(x) = \int_{\Omega} g(y) \tilde{G}(x, y) dy
$$
\n
$$
+ \sum_{j=1}^{m} \int_{\partial \Omega} \varphi_{j}(y) C_{j} \tilde{G}(x, y) d\sigma(y) + \sum_{i=1}^{k} c_{i} u_{i}(x)
$$
\n(the differential operators C_{j} ' are applied to $y, c_{i} \in \mathbb{R}^{1}, i = 1, ..., k$). Furthermore, if

\n
$$
\int_{\Omega} g(y) v_{i}(y) dy = 0 \qquad (i = 1, ..., k),
$$
\nthen

\n
$$
u(x) = \int_{\Omega} \tilde{G}(x, y) g(y) dy
$$

(the differential operators C_j are applied to y, $c_i \in \mathbb{R}^1$, $i = 1, ..., k$). Furthermore, if

$$
\int_{Q} g(y) v_i(y) dy = 0 \qquad (i = 1, \ldots, k), \ldots
$$

then

$$
u(x) = \int\limits_{\Omega} \tilde{G}(x, y) g(y) dy
$$

is the only solution of

$$
\int_{\Omega} g(y) v_i(y) dy = 0 \qquad (i = 1, ..., k),
$$

then

$$
u(x) = \int_{\Omega} \tilde{G}(x, y) g(y) dy
$$

is the only solution of

$$
Lu = g, \qquad B_i u|_{\partial \Omega} = 0 \qquad (j = 1, ..., m)
$$

with
$$
\int_{\Omega} u(x) u_i(x) dx = 0 \text{ for } i = 1, ..., k.
$$

For given open sets $V \subset \partial \Omega$ and $G \subset \overline{G} \subset \Omega \setminus$

$$
L_V(\Omega) = \{u \in C^{\infty}(\overline{\Omega}) : Lu = 0 \text{ in } \Omega, B_i u|,
$$

$$
L_G(\Omega) = \{u \in C^{\infty}(\overline{\Omega}) : g := Lu \in C_0^{\infty}(\Omega),
$$

supp $g \subset G, B_i u|_{\partial \Omega} = 0 \quad (j = 1,$
Let $L_V(\Gamma), L_G(\Gamma)$ be the spaces of the restrictions

For given open sets $V \subset \partial\Omega$ and $G \subset \overline{G} \subset \Omega \setminus \overline{F}$ we define

$$
L_{\mathcal{V}}(\Omega) = \{u \in C^{\infty}(\overline{\Omega}) : Lu = 0 \text{ in } \Omega, B_{j}u|_{\partial\Omega \setminus V} = 0 \ (j = 1, ..., m) \} \text{ and}
$$

$$
L_{G}(\Omega) = \{u \in C^{\infty}(\overline{\Omega}) : g := Lu \in C_{0}^{\infty}(\Omega),
$$

$$
\operatorname{supp} g \subset G, B_j u|_{\partial \Omega} = 0 \quad (j = 1, ..., m) \text{ respectively}.
$$

Let $L_v(F)$, $L_c(\Gamma)$ be the spaces of the restrictions onto Γ . Further we define $L_V(\Gamma)$

 $N(G)=\{g\in C_0^\infty(\Omega):Lu=g \text{ for some }u\in L_G(\Omega)\}.$

 $N(G)$ is the set of all functions $g \in C_0^{\infty}(\Omega)$ with support in *G* and $\int g(x) v_i(x) dx = 0$ $(i = 1, ..., k)$. This follows from (5). We shall prove the density of $L_{\mathcal{V}}(\Gamma)$ and $L_{\mathcal{G}}(\Gamma)$ $\sin W_p^{2m-1}$ is the set of all functions $g \in C_0^{\infty}(\Omega)$ with support in G , k). This follows from (5). We shall prove the density $\frac{1}{P}(F)$. First we give the definition of this space. follows from (5). We s
 $k \leq 1$ we give the definition
 $\mathbf{R} \leq p \leq \infty$) denote the
 $\left[\sum_{|a| \leq 2m} \int_{\Omega} |D^a u(x)|^p dx \right]$

bedding theorems (see

3. Let $W_p^{2m}(\Omega)$ ($1 < p < \infty$) denote the classical Sobolev spaces with the norm

$$
||u||_{2m,p} = \left[\sum_{|a| \leq 2m} \int_{\Omega} |D^a u(x)|^p dx \right]^{\frac{1}{p}}.
$$

In the sense of imbedding theorems (see [13]) for $|\alpha| \leq 2m - 1$ on Γ there exist traces of $D^a u$ for the functions $u \in W_p^{2m}(\Omega)$. More precisely

\n- \n
$$
(i = 1, \ldots, k)
$$
. This follows from (5). We shall prove the density of $L_p(I)$ and $L_o(I)$ in $W_p^{-2m} \cdot \frac{1}{p(I)}$. First we give the definition of this space.\n
\n- \n 3. Let $W_p^{2m}(\Omega)$ $(1 < p < \infty)$ denote the classical Sobolev spaces with the norm\n $||u||_{2m,p} = \left[\sum_{|a| \leq 2m} \int |D^a u(x)|^p \, dx\right]^{\frac{1}{p}}$.\n
\n- \n In the sense of imbedding theorems (see [13]) for $|\alpha| \leq 2m - 1$ on Γ there exist traces of $D^a u$ for the functions $u \in W_p^{2m}(\Omega)$. More precisely\n $D^a u|_{\Gamma} \in C(\Gamma)$ for $|\alpha| < 2m - \frac{n}{p}$,\n $D^a u|_{\Gamma} \in L^q(\Gamma)$ for $2m - \frac{n}{p} \leq |\alpha| \leq 2m - 1 \left(1 < q < \frac{p(n-1)}{n - p(2m - |\alpha|)}\right)$.\n $W_p^{-2m} \cdot \frac{1}{p(I)}$ is defined as the space of all functions φ on Γ , which are restrictions of functions from $W_p^{2m}(\Omega)$ on Γ in the trace sense. The expression\n
\n

$$
\|\varphi\|_{2m-\frac{1}{p},p}:=\inf\|u\|_{2m,p},
$$

where the infimum is taken over all $u \in W_p^{2m}(\Omega)$ with $u|_{\Gamma} = \varphi$, is a norm, such that. $W_p^{2m-\frac{1}{p}}(P)$ becomes a Banach space. The space $C^{\infty}(F)$ is dense in $W_p^{2m-\frac{1}{p}}(P)$. For definition of the dual space for $\psi \in C^{\infty}(\Gamma)$ we consider the norm s taken over all
a Banach space
he dual space f
 $:=\sup_{\substack{2m-1\\x\in W_n}}\frac{1}{|y|}$ Approximation by Solut

infimum is taken over all $u \in W_p^{2m}(\Omega)$ with

becomes a Banach space. The space $C^{\infty}(I)$

mition of the dual space for $\psi \in C^{\infty}(I)$ we dual
 $\psi\|_{-2m+\frac{1}{p},p'} := \sup_{\varphi \in W_p} \frac{|(\psi, \varphi)|}{\|\varphi\|_{2m-\$ Approximation by Solutions of Elliptic Equations
 r infimum is taken over all $u \in W_p^{2m}(\Omega)$ with $u|_{\Gamma} = \varphi$, is a norm, su
 r) becomes a Banach space. The space $C^{\infty}(\Gamma)$ is dense in $W_p^{2m-\frac{1}{p}}(\Gamma)$

finition o

$$
||\psi||_{-2m+\frac{1}{p},p'} := \sup_{\varphi \in W_p} \frac{|(\psi, \varphi)|}{\|\varphi\|_{2m-\frac{1}{p},p}}
$$

 $\|\psi\|_{-2m+\frac{1}{p},p'} := \sup_{\varphi \in W_p} \frac{\|\varphi, \psi\|}{\|\varphi\|_{2m-\frac{1}{p},p}}$ (7)
 $((\psi, \varphi) = \int_{\Gamma} \psi(x) \cdot \varphi(x) d\sigma_{\Gamma}(x), d\sigma_{\Gamma} \text{ surface element with respect to } \Gamma, \frac{1}{p} + \frac{1}{p'} = 1).$ $||\psi||_{-2m+\frac{1}{p},p'} := \sup_{\varphi \in W_p} \frac{|(\psi, \varphi)|}{|\varphi||_{2m-\frac{1}{p},p}}$ (7)
 $\int_{\Gamma} \psi(x) \cdot \varphi(x) d\sigma_{\Gamma}(x) d\sigma_{\Gamma}$ surface element with respect to $\Gamma, \frac{1}{p} + \frac{1}{p'} = 1$).

pletion of $C^{\infty}(\Gamma)$ with respect to (7) will be denoted by

The completion of $C^{\infty}(\Gamma)$ with respect to (7) will be denoted by $W_p^{-2m+\frac{1}{p}}(\Gamma)$. From (7) immediately follows

$$
|(\varphi, \varphi)| \leq ||\psi||_{-2m + \frac{1}{p} \cdot p'} \cdot ||\varphi||_{2m - \frac{1}{p} \cdot p}
$$
\n(8)

for $\psi \in C^{\infty}(\Gamma)$, $\varphi \in W_p^{\frac{2m-1}{p},p}$. Let $\psi \in W_p^{\frac{2m-1}{p}}(\Gamma)$, $\varphi \in W_p^{\frac{2m-1}{p}}(\Gamma)$, $\varphi \in W_p^{\frac{2m-1}{p}}(\Gamma)$. Choosing a se-
quence $\psi_n \in C^{\infty}(\Gamma)$ with $\lim_{n \to \infty} ||\psi_n - \psi||$ = 0 we put $e^{-2m} + \frac{1}{p} p^r$
for $\psi \in C^{\infty}(\Gamma)$, $\varphi \in W_p^{\{2m - \frac{1}{p}}(\Gamma)$. If quence $\psi_n \in C^{\infty}(\Gamma)$ with $\lim_{n \to \infty} ||\psi_n||$ $\|\varphi\|_{2m-\frac{1}{p},p}$
 $\det \psi \in W^{-2m+\frac{1}{p}}_{p}(F), \psi \in W^{-2m+\frac{1}{p}}_{-m+\frac{1}{p},p} = 0$ we put $n \to \infty$ $n \to \infty$ etion of $C^{\infty}(\Gamma)$ with resperant immediately follows
 (ρ, φ) $\leq ||\psi||_{-2m+\frac{1}{p}, p'} \cdot ||\varphi||_{2m-\frac{1}{p}}$
 $(\Gamma), \varphi \in W_p^{-2m-\frac{1}{p}}(\Gamma)$. Let $\psi \in C^{\infty}(\Gamma)$ with $\lim_{n \to \infty} ||\psi_n - \psi||_{-\frac{1}{p}}$
 (φ, φ) .
 φ) : = $\lim_{n \to$ bespect to (1) will be denoted by W_p . $p(I_p)$
 $2m-\frac{1}{p}p$
 $\nu \in W_p^{-2m+\frac{1}{p}}(\Gamma), \varphi \in W_p^{-2m-\frac{1}{p}}(\Gamma)$. Choosing
 $\psi||_{-2m+\frac{1}{p},p} = 0$ we put

from (8). (ψ, φ) does not depend on the second in the second in the secon

$$
(\psi,\varphi):=\lim_{n\to\infty}(\psi_n,\varphi).
$$
 (9)

(c)(T) with $\lim_{n\to\infty} ||\psi_n - \psi||_{-2m + \frac{1}{p},p} = 0$ we put
 $\lim_{n\to\infty} (\psi_n, \varphi)$.

(9)

of the limit follows from (8). (ψ, φ) does not depend on the sequence

r, the inequality

(10)
 $||\psi_n - \psi||_{-2m + \frac{1}{p},p}$. (9)
 $||\psi$ The existence of the limit follows from (8). (ψ, φ) does not depend on the sequence (ψ_n) . Moreover, the inequality $\lim_{n \to \infty} (\psi_n, \varphi).$

e limit follows inequality
 $\|\psi\|_{-2m+\frac{1}{p}, p'} \cdot \|\varphi\|$

$$
|(\psi, \varphi)| \le ||\psi||_{-2m + \frac{1}{p}, p} \cdot ||\varphi||_{2m - \frac{1}{p}, p}
$$
\n(10)

for $\psi \in C^{\infty}(\Gamma)$, $\varphi \in W_p^{\frac{2m-1}{m-p}}(\Gamma)$. Let $\psi \in W_p^{-2m+\frac{1}{p}}(\Gamma)$, $\varphi \in W_p^{\frac{2m-1}{m-p}}(\Gamma)$. Choosing a sequence $\psi_n \in C^{\infty}(\Gamma)$ with $\lim_{n \to \infty} |\psi_n - \psi| \Big|_{-2m+\frac{1}{p},p} = 0$ we put
 $(\psi, \varphi) := \lim_{n \to \infty} (\psi_n, \varphi)$. (are mutually dual; $\left(W_p^{2m-\frac{1}{p}}(T)\right)' = W_p^{-2m+\frac{1}{p}}(T)$. For a given $F \in \left(W_p^{2m-\frac{1}{p}}(T)\right)'$

there exists an unique element $\psi_F \in W_p^{-2m+\frac{1}{p}}(T)$, such that $F(\varphi) = (\psi_F, \varphi)$ for every
 $\varphi \in W_p^{-2m-\frac{1}{p}}$ and $||F|| = ||\psi$ there, exists an unique element $\psi_F \in W_{p'}^{-2m+\frac{1}{p}}(\Gamma)$, such that $F(\varphi) = (\psi_F, \varphi)$ for every $\varphi \in W_p^{\frac{2m-1}{p}}$ and $||F|| = ||\psi_F||_{\frac{2m+\frac{1}{p},p'}{p}}$.

4. Now .we are able to formulate our main result.

Theorem 1: Under the suppositions of Section 2 for $1 < p < \infty$ the space $L_G(\Gamma)$

Proof: We shall prove the theorem indirectly and suppose $\overline{L_G(\Gamma)} + W_p^{2m-\frac{1}{p}}(\Gamma)$. Then there exists an element $h \in W_{p'}^{2m+p}(P)$, $h \neq 0$, with $(h, u) = 0$ for all $u \in L_G(P)$ in the sense of the scalarproduct (9). Therefore we can choose a sequence $h_j \in C^{\infty}(\Gamma)$ with $\|h_j - h\|_{-2m + \frac{1}{p}, p'} \to$ ement $\psi_F \in W^{-2m+\frac{1}{p}}_p(\Gamma)$, such that P
 $\psi_F \Big|_{-\frac{2m+\frac{1}{p},p'}{p}}$.

formulate our main result.
 e suppositions of Section 2 for 1 <

re the theorem indirectly and suppose

interaction of $\psi_F^{-2m+\frac{1}{p}}(\Gamma)$, $h \neq$ *Jnder the suppositions of Section 2 fo*
 $\frac{1}{p}(F)$.

all prove the theorem indirectly and

an element $h \in W_p^{-2m+\frac{1}{p}}(F)$, $h \neq 0$, we scalar
 product (9). Therefore we can
 $+\frac{1}{p}p' \rightarrow 0$, such that
 $\lim_{j\to\infty} ($

$$
(h, u) = \lim_{j \to \infty} (h_j, u) = \lim_{j \to \infty} \int h_j(x) u(x) d\sigma_r(x) = 0
$$
\n(11)

(7)

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for every $u \in L_G(\Gamma)$. The definition of $L_G(\Gamma)$ and (6) give the representation (with $Lu = g \in N(G)$

$$
u(x) = \int\limits_G g(y) \,\tilde{G}(x, y) \,dy + \sum\limits_{i=1}^k c_i u_i(x)
$$

U. HAMANN and G. WILDENHAIN
 u(*x*) *u*(*x*). The definition of $L_G(\Gamma)$ and (6) gi
 u(*x*) = $\int_G g(y) \tilde{G}(x, y) dy + \sum_{i=1}^k c_i u_i(x)$
 u(*x*) *u* = $\int_G g(y) \tilde{G}(x, y) dy + \sum_{i=1}^k c_i u_i(x)$
 u $\in L_G(\Gamma)$. Because of $u_i|_{\Gamma$ *S.* 64 U. HAMANN and G. WILDENHAIN

for every $u \in L_G(\Gamma)$. The definition of $L_G(\Gamma)$ and (6) give the representation
 $Lu = g \in N(G)$)
 $u(x) = \int_{G} g(y) \tilde{G}(x, y) dy + \sum_{i=1}^{k} c_i u_i(x)$

for every $u \in L_G(\Gamma)$. Because of $u_i|_{\Gamma} \in L_G(\Gamma)$ w $(h, u) = \lim_{\substack{\longrightarrow \\ i \to \infty}} \int h_i(x) \left(\int_G g(y) \tilde{G}(x, y) dy \right) d\sigma_r(x)$

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\n
$$
u \in L_G(\Gamma)
$$
. The definition of $L_G(\Gamma)$ and (6) give the representation (with
\n $N(G)$)
\n $u(x) = \int_{G} g(y) \tilde{G}(x, y) dy + \sum_{i=1}^{k} c_i u_i(x)$
\n $u \in L_G(\Gamma)$. Because of $u_i|_{\Gamma} \in L_G(\Gamma)$ we have $\left(h, \sum_{i=1}^{k} c_i u_i\right) = 0$. Hence
\n $(h, u) = \lim_{j \to \infty} \int_{\Gamma} h_j(x) \left(\int_{G} g(y) \tilde{G}(x, y) dy\right) d\sigma_{\Gamma}(x)$
\n $= \lim_{j \to \infty} \int_{G} g(y) \left(\int_{\Gamma} h_j(x) \tilde{G}(x, y) d\sigma_{\Gamma}(x)\right) dy = 0$ (12)
\n $g \in N(G)$.
\nall prove that the sequence
\n $\int h_j(x) D_{\nu}{}^{\rho} \tilde{G}(x, y) d\sigma_{\Gamma}(x)$
\n $\geq m$ with respect to y is uniformly convergent on every compact set
\n Γ . We fix an open neighbourhood U of Γ with $\Gamma \subset U \subset \Omega$ and $\overline{U} \cap K = \theta$.

for every $g \in N(G)$.

We shall prove that the sequence

$$
\int_{\Gamma} h_j(x) D_y^{\beta} \tilde{G}(x, y) d\sigma_{\Gamma}(x)
$$

for $|\beta| \leq 2m$ with respect to y is uniformly convergent on every compact set $K\!=\! \varOmega\!\smallsetminus\varGamma.$ We fix an open neighbourhood U of \varGamma with $\varGamma\!=\!U\!=\!\varOmega$ and \overline{U} \cap $K=\emptyset.$ From the smoothness of $\tilde{G} = \tilde{G}(x, y)$ for $x + y$ follows $D_y^{\beta} \tilde{G}(x, y) \in W_p^{2m}(U)$ with rion the smoothness of $\alpha = \alpha(x, y)$ for $x + y$ follows Σ_y $\alpha(x, y) \in \mathbb{R}$, γ is smooth.
respect to $x \in U$ and any fixed $y \in K$. We suppose that the boundary ∂U is smooth. Then we can use a general result from the theory of Sobolev spaces. Namely, because Then we can use a general result from the theory of Sobolev spaces. Namely, because
the boundary ∂U is smooth, there exists a continuous extension operator from
 $W_p^{2m}(U)$ into $W_p^{2m}(\Omega)$, i.e. for every $u \in W_p^{2m}(U)$ $W_p^{2m}(U)$ into $W_p^{2m}(\Omega)$, i.e. for every $u \in W_p^{2m}(U)$ there exists an extension $\tilde{u} \in W_p^{2m}(Q)$ with $u(x) = \tilde{u}(x)$ in *U* and
 $\|\tilde{u}\|_{W_p^{2m}(Q)} \leq \omega \cdot \|u\|_{W_p^{2m}(U)}$ $W_p^{2m}(U)$ into $W_p^{2m}(\Omega)$, i.e. for every $u \in W_p^{2m}(U)$ there $\tilde{u} \in W_p^{2m}(\Omega)$ with $u(x) = \tilde{u}(x)$ in U and
 $||\tilde{u}||_{W_p^{2m}(\Omega)} \leq \omega \cdot ||u||_{W_p^{2m}(U)}$

(ω independent of $u \in W_p^{2m}(U)$). Obviously the estimate $||\beta|| \$

 $\|\tilde{u}\|_{W_{\alpha}^{2m}(\mathcal{Q})}\leq \omega\cdot \|u\|_{W_{\alpha}^{2m}(U)}$

2m)

ndary
$$
\partial U
$$
 is smooth, there exists a continuous extens
\ninto $W_p^{2m}(Q)$, i.e. for every $u \in W_p^{2m}(U)$ there ex
\n ${}^n(\Omega)$ with $u(x) = \tilde{u}(x)$ in U and
\n $||\tilde{u}||_{W_p^{2m}(Q)} \leq \omega \cdot ||u||_{W_p^{2m}(U)}$
\nendent of $u \in W_p^{2m}(U)$). Obviously the estimate $(|\beta| \leq 2$:
\n $||D_p^{\rho}\tilde{G}(\cdot, y)||_{W_p^{2m}(U)}$
\n $\leq \sum_{|\alpha| \leq 2m} \sup_{x \in U, y \in K} |D_x^{\rho}D_y^{\rho}\tilde{G}(x, y)| m(U)^{\frac{1}{p}} = C(U, K, m, p, \beta)$
\nhere $m(U)$ denotes the Lebesgue measure of U and the co
\ne is independent of y . It follows
\n $||D_y^{\rho}\tilde{G}(\cdot, y)||_{2m - \frac{1}{p}, p} = \inf_{u|_{\Gamma} = D_v \beta \tilde{G}(\cdot, y)} ||u|_{W_p^{2m}(U)}$
\n $\leq \omega \cdot ||D_y^{\rho}\tilde{G}(\cdot, y)||_{W_p^{2m}(U)} \leq \tilde{C}(U, K, m,$
\ng (10) we get the estimate

holds, where $m(U)$ denotes the Lebesgue measure of U and the constant on the right hand side is independent of y . It follows $\begin{bmatrix}\n\text{holds, when } \text{the hand side}\n\end{bmatrix}$
Applying

$$
\begin{aligned}\n\lim_{n \to \infty} \mathbb{P}_{\mathbf{p}}(x) &\equiv \mathbb{E} \quad \lim_{n \to \infty} \mathbb{P}_{\mathbf{p}}(y) & \text{Obviously the estimate } (|\beta| \leq 2m) \\
\|D_{\mathbf{p}}{}^{\beta}\tilde{G}(\cdot, y)\|_{W_{\mathbf{p}}^{\mathbf{1} \mathbf{m}}(U)} &\leq \sum_{|\mathbf{q}| \leq 2m} \sup_{x \in U, \mathbf{p} \in K} [D_{x}{}^{\alpha}D_{\mathbf{p}}{}^{\beta}\tilde{G}(x, y)] \ m(U)^{\mathbf{p}} &= C(U, K, m, p, \beta) \\
\text{here } m(U) \text{ denotes the Lebesgue measure of } U \text{ and the constant} \\
\mathbf{e} \text{ is independent of } y. \text{ It follows} \\
\|D_{\mathbf{p}}{}^{\beta}\tilde{G}(\cdot, y)\|_{2m} - \frac{1}{p} \cdot p &= \inf_{\mathbf{u}|_{\mathbf{p}} = D_{\mathbf{p}}{}^{\beta}\tilde{G}(\cdot, y)} \|u\|_{W_{\mathbf{p}}^{\mathbf{1} \mathbf{m}}(D)} \\
&\leq \omega \cdot \|D_{\mathbf{p}}{}^{\beta}\tilde{G}(\cdot, y)\|_{W_{\mathbf{p}}^{\mathbf{1} \mathbf{m}}(U)} \leq \tilde{C}(U, K, m, p, \beta).\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{If } \mathbf{p} \in \mathbb{P}(\mathbf{p}) \text{ and } \mathbf{p} \in \mathbb{P}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{If } \mathbf{p} \in \mathbb{P}(\mathbf{p}) \text{ and } \mathbf{p} \in \mathbb{P}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{If } \mathbf{p} \in \mathbb{P}(\mathbf{p}) \text{ and } \mathbf{p} \in \mathbb{P}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{If } \mathbf{p} \in \mathbb{P}(\mathbf{p}) \text{ and } \mathbf{p} \in \mathbb{P}.\n\end{aligned}
$$
\n
$$
\begin{aligned
$$

Applying (10) we get the estimate

$$
||D_{y}{}^{\beta}\tilde{G}(\cdot,y)||_{2m-\frac{1}{p},p} = \inf_{u|_{\Gamma}=D_{y}\beta\tilde{G}(\cdot,y)} ||u||_{W_{p}^{4m}(\Omega)}
$$

\n
$$
\leq \omega \cdot ||D_{y}{}^{\beta}\tilde{G}(\cdot,y)||_{W_{p}^{4m}(U)} \leq \tilde{C}(U, K, m, p, \beta).
$$

\nApplying (10) we get the estimate
\n
$$
\int_{\Gamma} D_{y}{}^{\beta}\tilde{G}(x,y) (h_{i}(x) - h_{j}(x)) d\sigma_{\Gamma}(x) ||
$$

\n
$$
\leq ||D_{y}{}^{\beta}\tilde{G}(\cdot,y)||_{2m-\frac{1}{p},p} ||h_{i} - h_{j}||_{-2m+\frac{1}{p},p'}.
$$

\n
$$
\leq \tilde{C}(U, K, m, p, \beta) \cdot ||h_{i} - h_{j}||_{-2m+\frac{1}{p},p'}.
$$

\nBecause the right side is independent of y, the assertion follows. The sequel
\n
$$
\int_{\Gamma} h_{j}(x) \tilde{G}(x,y) d\sigma_{\Gamma}(x)
$$

\nespecially converges uniformly with respect to $y \in G$.

Because the right side is independent of y , the assertion follows. The sequence

$$
\int_{\Gamma} h_j(x) \; \tilde{G}(x, y) \; d\sigma_{\Gamma}(x)
$$

Therefore in (12) we can change the limit and the integral and get

Approximation by Solutions of Elliptic Equations
\nover in (12) we can change the limit and the integral and get
\n(*h*, *u*) =
$$
\int_{G} g(y) (\lim_{j\to\infty} \int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_I(x)) dy = 0
$$
 (13)
\n $g \in N(G)$, i.e. for every $g \in C_0^{\infty}(\Omega)$ with supp $g \subset G$ and
\n $\int_{\Omega} g(x) v_i(x) dx = 0$ (*i* = 1, ..., *k*) (14)
\nrows from (5)). Combining (13) and (14) we obtain

for every $g \in N(G)$, i.e. for every $g \in C_0^{\infty}(\Omega)$ with supp $g \subset G$ and

$$
\int_{2} g(x) v_i(x) dx = 0 \qquad (i = 1, ..., k)
$$
\n(14)

Approximation by Solutions of Elliptic Equations 65
\nTherefore in (12) we can change the limit and the integral and get
\n
$$
(h, u) = \int_{G} g(y) \lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x) dy = 0
$$
\n(13) to every $g \in N(G)$, i.e. for every $g \in C_{0}^{\infty}(\Omega)$ with supp $g \subset G$ and
\n
$$
\int_{\Omega} g(x) v_{i}(x) dx = 0 \quad (i = 1, ..., k)
$$
\n(14)
\n(14) follows from (5)). Combining (13) and (14) we obtain
\n
$$
\lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x) = \sum_{i=1}^{k} c_{i} v_{i}(y) =: v(y) \text{ in } G.
$$
\nPutting
\n
$$
w(y) := \lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x) - v(y),
$$
\n
$$
\lim_{j \to \infty} L^* w(y) = L^* \lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$
\n
$$
= \lim_{j \to \infty} L^* \int_{\Gamma} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$
\n
$$
= \lim_{j \to \infty} \int_{\Gamma} L^*_{(y)} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$
\n
$$
= \lim_{j \to \infty} \int_{\Gamma} L^*_{(y)} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$
\n
$$
= \lim_{j \to \infty} \int_{\Gamma} L^*_{(y)} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$
\n
$$
= \lim_{j \to \infty} \int_{\Gamma} L^*_{(y)} \tilde{G}(x, y) h_{j}(x) d\sigma_{\Gamma}(x)
$$

$$
w(y) := \lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_{\Gamma}(x) - v(y),
$$

$$
\lim_{j \to \infty} \int_{I} \tilde{G}(x, y) h_j(x) d\sigma_I(x) = \sum_{i=1}^{k} c_i v_i(y) =: v(y) \text{ in } G. \tag{15}
$$
\nPutting

\n
$$
w(y) := \lim_{j \to \infty} \int_{I} \tilde{G}(x, y) h_j(x) d\sigma_I(x) - v(y),
$$
\n
$$
\lim_{j \to \infty} L^*v = 0, (3) \text{ and the uniform convergence for } |\beta| \leq 2m, \text{ we have for } y \in \Omega \setminus I
$$
\n
$$
L^*w(y) = L^* \left(\lim_{j \to \infty} \int_{I} \tilde{G}(x, y) h_j(x) d\sigma_I(x)\right) - L^*v(y)
$$
\n
$$
= \lim_{j \to \infty} L^* \int_{I} \tilde{G}(x, y) h_j(x) d\sigma_I(x)
$$
\n
$$
= \lim_{j \to \infty} \int_{I} L^*_{U} \tilde{G}(x, y) h_j(x) d\sigma_I(x)
$$
\n
$$
= -\lim_{j \to \infty} \int_{I} \tilde{L}_W^*(y) \int_{I} u_i(x) h_j(x) d\sigma_I(x)
$$
\n
$$
= -\sum_{j=1}^{k} u_i(y) \cdot \left[\lim_{j \to \infty} \int_{I} u_i(x) h_j(x) d\sigma_I(x)\right] = 0,
$$
\nthe last equality follows from (11), because $u_i \in L_G(Q)$ for $i = 1, ..., k$. From (15),

\n
$$
w = 0 \text{ in } G, \text{ and from the "condition for uniqueness in the small", we obtain}
$$
\n
$$
w = 0 \text{ in } G, \text{ and from the "condition for uniqueness in the small", we obtain}
$$
\n
$$
= 0 \text{ in } \Omega \setminus I, \text{ i.e.}
$$
\n
$$
\lim_{j \to \infty} \int_{I} \tilde{G}(x, y) h_j(x) d\sigma_I(x) = v(y) \text{ in } \Omega \setminus I.
$$
\n
$$
\lim_{j \to \infty} \int_{I} u(x) h_j(x) d\sigma_I(x) = 0
$$
\nand for every $u \in C_0^{\infty}(Q)$. Defining

The last equality follows from (11), because $u_i \in L_G(\Omega)$ for $i = 1, ..., k$. From (15), i.e. $w \equiv 0$ in G, and from the "condition for uniqueness in the small", we obtain $w \equiv 0$ in $\Omega \setminus \Gamma$, i.e. The last equality follows
i.e. $w \equiv 0$ in G, and from
 $w \equiv 0$ in $\Omega \setminus \Gamma$, i.e. $\begin{aligned}\n&= -\sum_{i=1} u_i(y) \cdot \left\{ \lim_{j \to \infty} \int u_i(x) \, h_j(x) \, d\sigma_l(x) \right\} = 0. \n\end{aligned}$ The last equality follows from (11), because $u_i \in L_G(Q)$ for $i = 0$.

i.e. $w = 0$ in G , and from the "condition for uniqueness in the $w = 0$ in

$$
\lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_{\Gamma}(x) = v(y) \text{ in } \Omega \setminus \Gamma.
$$
\nNext step we shall show that

\n
$$
\lim_{j \to \infty} \int_{\Gamma} u(x) h_j(x) d\sigma_{\Gamma}(x) = 0
$$
\n(16)

$$
\lim_{j\to\infty}\int u(x)\,h_j(x)\,d\sigma_r(x)=0
$$

holds for every $u \in C_0^{\infty}(\Omega)$. Defining $f := Lu$, the function $u \in C_0^{\infty}(\Omega)$ can be con-*0•* $\lim_{\delta \to \infty} \int f u(x) h_j(x) d\sigma_r(x) = 0$

every $u \in C_0^{\infty}(\Omega)$. Defining $f := L$

is a solution of the boundary value
 $Lu = f$ in Ω , $B_j u|_{\partial \Omega} = 0$

b) we have

$$
Lu = f \text{ in } \Omega, \qquad B_i u|_{\partial \Omega} = 0 \qquad (j = 1, ..., m)
$$

and by (6) we have

$$
Lu = f \text{ in } \Omega, \qquad B_j u|_{\partial \Omega} = 0 \qquad (j
$$

by we have

$$
u(x) = \int_{\Omega} \tilde{G}(x, y) f(y) dy + \sum_{i=1}^{k} c_i u_i(x).
$$

5 Analysis **Bd. 5, Heft 1 (1986)**

Because of $u_i \in L_G(\Omega)$ it is

$$
\lim_{j\to\infty}\int\limits_{\Gamma}\left(\sum\limits_{i=1}^kc_iu_i(x)\right)h_j(x)\,d\sigma_{\Gamma}(x)=0.
$$

Therefore we must show that

$$
\lim_{\theta \to \infty} \int\limits_{\Gamma} \left(\int\limits_{Q} \tilde{G}(x, y) f(y) dy \right) h_j(x) d\sigma_{\Gamma}(x) = 0.
$$

 $T_{\delta} := \{x \in \Omega : d(x, \Gamma) < \delta\}$ denotes an open δ -neighbourhood of Γ . For every $\delta > 0$, there is a $\varphi_{\delta} \in C_0^{\infty}(\mathbb{R}^n)$ with $0 = \varphi_{\delta}(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\varphi_{\delta}(x) = 1$ for all $x \in \Gamma_{\delta/2}$ Therefore we must show that
 $\lim_{j\to\infty} \int \int \int \tilde{G}(x, y) f(y) dy \Bigg) h_j(x) d\sigma_r(x) = 0.$
 $\Gamma_{\delta} := \{x \in \Omega : d(x, \Gamma) < \delta\}$ denotes an open δ -neighbourhood of Γ . For every $\delta > 0$,

there is a $\varphi_{\delta} \in C_0^{\infty}(\mathbb{R}^n)$ with 0 $\lim_{j\to\infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f(y) dy \right) h_j(x) d\sigma_T(x) = 0.$
 $\Gamma_b := \{ x \in \Omega : d(x, \Gamma) < \delta \}$ denotes an open δ -neighbourhood of Γ . For every $\delta >$

there is a $\varphi_{\delta} \in C_0^{\infty}(\mathbb{R}^n)$ with $0 = \varphi_{\delta}(x) \leq 1$ for all x $\text{supp } f_{\delta} \subset \Gamma_{\delta} \text{ and } \bar{f}_{\delta}(x) = 0 \text{ for } x \in \Gamma_{\delta/2}.$ We have $\text{Because} \ \begin{align*} \text{Therefore} \ \cdot \boldsymbol{\varGamma}_{\delta} := \{x \text{ there, is} \ \text{and} \ \text{supp } f_{\delta} \in \mathbb{R} \end{align*}$

R

$$
\lim_{j\to\infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f(y) dy \right) h_j(x) d\sigma_r(x) = 0.
$$
\n
$$
\Gamma_{\delta} := \{x \in \Omega : d(x, \Gamma) < \delta\} \text{ denotes an open } \delta\text{-neighborhood of } \Gamma. \text{ For every } \delta > 0,
$$
\nthere is a $\varphi_{\delta} \in C_0^{\infty}(\mathbb{R}^n)$ with $0 = \varphi_{\delta}(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\varphi_{\delta}(x) = 1$ for all $x \in \Gamma_{\delta/2}$ and $\sup_{\mathcal{P}_{\delta}} \subset \Gamma_{\delta}$. We define $f_{\delta} := f\varphi_{\delta}$ and $\overline{f}_{\delta} := f(1 - \varphi_{\delta})$. It is $f_{\delta} + \overline{f}_{\delta} = f$,
\n
$$
\sup_{j\to\infty} f_{\delta} = \Gamma_{\delta}
$$
 and $\overline{f}_{\delta}(x) = 0$ for $x \in \Gamma_{\delta/2}$. We have\n
$$
\lim_{j\to\infty} \int \left(\int_{\Omega} \tilde{G}(x, y) f(y) dy \right) h_j(x) d\sigma_r(x)
$$
\n
$$
= \lim_{j\to\infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right) h_j(x) d\sigma_r(x)
$$
\n
$$
+ \lim_{j\to\infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right) h_j(x) d\sigma_r(x) = I_{1,\delta} + I_{2,\delta}.
$$
\nUsing once more the uniform convergence of $\int_{\Gamma} h_j(x) \tilde{G}(x, y) d\sigma_r(x)$ and (16), we obtain $(\overline{f}_{\delta} = 0$ in $\Gamma_{\delta/2}$.)\n
$$
I_{1,\delta} = \lim_{j\to\infty} \int_{\Omega} \tilde{f}_{\delta}(y) \left(\int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_r(x) \right) dy
$$

Using once more the uniform convergence of $\int_{\Gamma} h_j(x) \tilde{G}(x, y) d\sigma_r(x)$ and (16), we obtain $(\bar{f}_\delta \equiv 0 \text{ in } \Gamma_{\delta/2})$

$$
\lim_{j \to \infty} \int_{\Gamma} \int_{\Omega} \tilde{G}(x, y) \tilde{f}_s(y) dy \Big) h_j(x) d\sigma_r(x)
$$

\n
$$
= \lim_{j \to \infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f_s(y) dy \right) h_j(x) d\sigma_r(x)
$$

\n
$$
+ \lim_{j \to \infty} \int_{\Gamma} \left(\int_{\Omega} \tilde{G}(x, y) f_s(y) dy \right) h_j(x) d\sigma_r(x)
$$

\n
$$
= \lim_{j \to \infty} \int_{\Omega} \tilde{f}_s(y) \left(\int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_r(x) \right) dy
$$

\n
$$
= \int_{\Omega} \tilde{f}_s(y) \left(\lim_{j \to \infty} \int_{\Gamma} \tilde{G}(x, y) h_j(x) d\sigma_r(x) \right) dy
$$

\n
$$
= \int_{\Omega} \tilde{f}_s(y) v(y) dy
$$

\nBecause v is a solution of the homogeneous boundary
\ndition (5) gives
\n
$$
0 = \int_{\Omega} v(y) f(y) dy = \int_{\Omega} v(y) \tilde{f}_s(y) dy + \int_{\Gamma} v(y) f_s(x) dx
$$

\nBecause of $\lim_{\Omega} m(\Gamma_0) = 0$ ($m(\Gamma_0)$ denotes the Lebesgu

Because *v* is a solution of the homogeneous boundary. value problem (2), the con-

$$
0 = \int\limits_{\Omega} v(y) f(y) dy = \int\limits_{\Omega} v(y) \tilde{f}_\delta(y) dy + \int\limits_{\Gamma_\delta} v(y) f_\delta(y) dy.
$$

Because of $\lim m(\Gamma_{\delta}) = 0$ ($m(\Gamma_{\delta})$) denotes the Lebesgue measure of Γ_{δ}) there is a Because v is a solution of the homogeneous boundary v
dition (5) gives
 $0 = \int_{\Omega} v(y) f(y) dy = \int_{\Omega} v(y) \int_{\delta}(y) dy + \int_{\Gamma_{\delta}} v(y) f_{\delta}(y)$
Because of $\lim_{\delta \to 0} m(\Gamma_{\delta}) = 0$ (m(Γ_{δ}) denotes the Lebesgue
 $\delta_1(\varepsilon) > 0$ for a give Because v is a
dition (5) gives
 $0 = \int_a$
Because of lin
 $\delta_1(\varepsilon) > 0$ for a
 $|I_{\mathbf{L},\delta}|$
Now we con

of
$$
\lim_{\delta \to 0} m(\Gamma_{\delta}) = 0
$$
 ($m(\Gamma_{\delta})$ denotes the Lebesgue in
\nfor a given $\varepsilon > 0$, such that
\n $|I_{\mathbf{L}^{\delta}}| = \left| \int_{\Omega} v(y) \, \tilde{f}_{\delta}(y) \, dy \right| < \varepsilon$ for $0 < \delta \leq \delta_1(\varepsilon)$.
\nwe consider $I_{z,\delta}$:
\n $\int h_j(x) \int \tilde{G}(x, y) f_{\delta}(y) \, dy) \, d\sigma(\varepsilon)$

Now we consider $I_{2,\delta}$: -

we consider
$$
\begin{aligned}\n&\int_{\Gamma} b_j(x) \left(\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right) d\sigma_r(x) \\
&= \frac{\sum ||h_j||_{-2m + \frac{1}{p}, p'}}{\int_{\Omega} \int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy} \Big|_{r} \Big|_{2m - \frac{1}{p}, p} \\
&\leq C \cdot \Big| \int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \Big|_{2m, p}\n\end{aligned}
$$

LI

(the convergent sequence h_j is bounded!). Now we want to show that

Approximation by Sc
\n
$$
\text{Vergent sequence } h_j \text{ is bounded!}. \text{ Now with}
$$
\n
$$
\lim_{\delta \to 0} \left\| \int_a^{\delta} \tilde{G}(x, y) f_s(y) dy \right\|_{2m, p} = 0.
$$
\n
$$
= \int_a^{\delta} f_s(y) v_i(y) dy \text{ we have } \lim_{\delta \to 0} C_i^{\delta} = 0 \text{ (i}
$$
\n
$$
F_s(y) := f_s(y) - \sum_{i=1}^k C_i^{\delta} v_i(y)
$$
\n
$$
\int F_s(y) v_i(y) dy = 0 \text{ (i = 1, ..., k). Therefore}
$$

Approximation by Solutions of Elliptic Equation

(the convergent sequence h_j is bounded!). Now we want to show that
 $\lim_{\delta \to 0} \left\| \int_a^b \tilde{G}(x, y) f_{\delta}(y) dy \right\|_{2m, p} = 0.$

For $C_i^{\delta} := \int_a^b f_{\delta}(y) v_i(y) dy$ we have \lim_{δ (the convergent sequence h_j is bounded!). Now we want to show
 $\lim_{\delta \to 0} \left\| \int_a \tilde{G}(x, y) f_{\delta}(y) dy \right\|_{\mathbb{Z}m, p} = 0.$

For $C_i^{\delta} := \int_a^{\delta} f_{\delta}(y) v_i(y) dy$ we have $\lim_{\delta \to 0} C_i^{\delta} = 0$ $(i = 1, ..., k)$. We $F_{\delta}(y) := f_{\delta}(y) - \sum_{$

$$
F_{\delta}(y) := f_{\delta}(y) - \sum_{i=1}^{k} C_i^{\delta} v_i(y)
$$

and get $\int F_{\delta}(y) v_i(y) dy = 0$ $(i = 1, ..., k)$. Therefore

$$
W_{\delta}(x) := \int\limits_{Q} \tilde{G}(x, y) F_{\delta}(y) dy
$$

is a solution of the boundary value problem

$$
Lu = F_{\delta} \text{ in } \Omega, \qquad B_j u|_{\partial \Omega} = 0 \qquad (j = 1, ..., m).
$$

 $F_{\delta}(y) := f_{\delta}(y) - \sum_{i=1}^{k} C_{i}{}^{b}v_{i}(y)$
 $\int F_{\delta}(y) v_{i}(y) dy = 0 \quad (i = 1, ..., k).$ Therefore
 $W_{\delta}(x) := \int_{a} \tilde{G}(x, y) F_{\delta}(y) dy$

ion of the boundary value problem
 $Lu = F_{\delta}$ in Ω , $B_{j}u|_{\partial\Omega} = 0$ $(j = 1, ..., m).$
 Ω Schauder From the Schauder estimate (see [2]) we can conclude that there exists a constant $W_{\delta}(x) := \int_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy$

is a solution of the boundary value problem
 $Lu = F_{\delta}$ in Ω , $B_j u|_{\partial \Omega} = 0$ ($j =$

From the Schauder estimate (see [2]) we can cone
 $C < \infty$ with
 $\mathcal{L} = \frac{1}{2} \int_{-\infty}^{+\infty} \mathcal{L} \$ $W_{\delta}(x) := \int_{a} \tilde{G}(x, y) F_{\delta}(y) dy$

is a solution of the boundary value problem
 $Lu = F_{\delta}$ in Ω , $B_{j}u|_{\partial \Omega} = 0$ ($j = 1$,

From the Schauder estimate (see [2]) we can conclude
 $C < \infty$ with
 $\int_{\gamma} ||W_{\delta}||_{2m, p} \leq C \cdot$ and get $\int F_{\delta}(y) v_i(y) dy = 0$ $(i = 1, ..., k)$. Therefore
 $W_{\delta}(x) := \int_{a} \tilde{G}(x, y) F_{\delta}(y) dy$

is a solution of the boundary value problem
 $Lu = F_{\delta}$ in Ω , $B_i u|_{\partial \Omega} = 0$ $(j = 1, ..., m)$.

From the Schauder estimate (see [2]) we ca

$$
||W_{\delta}||_{2m,p} \leq C \cdot ||LW_{\delta}||_{0,p} = C \cdot ||F_{\delta}||_{0,p}
$$

$$
Lu = F_{\delta} \text{ in } \Omega, \qquad B_{j}u|_{\partial\Omega} = 0 \qquad (j = 1, ..., m).
$$

from the Schauder estimate (see [2]) we can conclude that

$$
\langle \infty \text{ with}
$$

$$
||W_{\delta}||_{2m,p} \leq C \cdot ||LW_{\delta}||_{0,p} = C \cdot ||F_{\delta}||_{0,p}
$$

r every $\delta > 0$ (*C* does not depend on δ). Because of

$$
||F_{\delta}||_{0,p} \leq ||f_{\delta}||_{0,p} + \sum_{i=1}^{k} |C_{i}^{*}|| \cdot ||v_{i}||_{0,p},
$$

$$
\lim_{\delta \to 0} ||f_{\delta}||_{0,p} = 0 \text{ and } \lim_{\delta \to 0} C_{i}^{\delta} = 0 \qquad (i = 1, ..., k)
$$

e have
$$
\lim_{\delta \to 0} ||W_{\delta}||_{2m,p} = 0. \text{ From } \int_{\Omega} \tilde{G}(x, y) v_{i}(y) dy = 0 \text{ (see (1))}
$$

$$
\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy = \int_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy = W_{\delta}(x).
$$

therefore we get
$$
\lim_{\delta \to 0} \left\| \int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right\|_{2m,n} = 0. \text{ For the}
$$

$$
\int_{\delta} \tilde{G}(x, y) f_{\delta}(y) dy = 0 \text{ for the}
$$

i on of the boundary value problem
 Lu = F_{δ} in Ω , $B_{j}u|_{\partial\Omega} = 0$ $(j = 1, ..., m)$.

 Ω Schauder estimate (see [2]) we can conclude that there exit
 $||W_{\delta}||_{2m,p} \leq C \cdot ||LW_{\delta}||_{0,p} = C \cdot ||F_{\delta}||_{0,p}$
 $\delta > 0$ (*C* we have $\lim_{b \to 0} \|W_b\|_{2m,p} = 0$. From $\int_{0}^{b} \tilde{G}(x, y) v_i(y) dy = 0$ (see (4)) we conclude $\lim_{\delta \to 0} ||f_{\delta}||_{0,p} = 0 \text{ and}$

we have $\lim_{\delta \to 0} ||W_{\delta}||_{2m,p} = 0$. From $\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy = \int_{\Omega}$

Therefore we get $\lim_{\delta \to 0} \left| \int_{\Omega} \tilde{G}(x, y) \right|$
 $\int_{\Omega} h_j(x) \cdot \left(\int_{\Omega} \tilde{G}(x, y) \right)$ $\lim_{\delta \to 0} \|W_{\delta}\|_{2m,p} = 0.$ From $\int_{\Omega} \tilde{G}(x, y) v_{i}(y) dy = 0$ (see (4)) we concl
 $\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy = \int_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy = W_{\delta}(x).$
 $\lim_{\delta \to 0} \| \int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \|_{2m,p} = 0.$ For that reason we
 $\left| \int_{\Gamma$

$$
\int\limits_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy = \int\limits_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy = W_{\delta}(x).
$$

We have $\lim_{\delta \to 0} ||W_{\delta}||_{2m,p} = 0$. From $\int_{\Omega} \tilde{G}(x, y) v_i(y) dy = 0$ (see (4)) we conclude
 $\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy = \int_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy = W_{\delta}(x)$.

Therefore we get $\lim_{\delta \to 0} ||\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy||_{2m,p} = 0$. Fo

$$
\left|\int\limits_{\Gamma} h_j(x) \cdot \left(\int\limits_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy\right) d\sigma_T(x)\right| < \epsilon
$$

for $0 < \delta \leq \delta_2(\epsilon)$ and every j. This means,

*5**

$$
\lim_{j\to\infty}\int\limits_{\Gamma} u(x)\,h_j(x)\,d\sigma_{\Gamma}(x)=0.
$$

Now we consider any function $\psi \in C^{\infty}(\Gamma)$. From the smoothness of Γ follows that an extension to a function $u \in C_0^{\infty}(\Omega)$ can be found. From our preceding considerations we obtain $\int_{\Omega} G(x, y) f_{\delta}(y) dy = \int_{\Omega} \tilde{G}(x, y) F_{\delta}(y) dy = W_{\delta}(x).$

Therefore we get $\lim_{\delta \to 0} \left\| \int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right\|_{\ell=m, p} = 0.$ For that reason we can find a
 $\left| \int_{\Gamma} h_i(x) \cdot \left(\int_{\Omega} \tilde{G}(x, y) f_{\delta}(y) dy \right) d\sigma_T(x) \right| < \varepsilon$ $\begin{aligned}\n\left| \int_{\Gamma} h_j(x) \cdot \left(\int_{\Omega} G(x, y) f_b(y) dy \right) d\sigma_r(x) \right| < \varepsilon \\
\delta &\leq \delta_2(\varepsilon) \text{ and every } j. \text{ This means.} \\
\lim_{j \to \infty} \int_{\Gamma} u(x) h_j(x) d\sigma_r(x) &= 0. \\
\delta &\to \infty \int_{\Gamma} \\
\text{we consider any function } \psi \in C^\infty(\Gamma). \text{ From the smoothness} \\
\text{asion to a function } u \in C_0^\infty(\Omega) \text{ can be found. From our we obtain} \\
\lim_{j \to \infty} (h_j$

$$
\lim_{j\to\infty} (h_j,\psi)=0 \text{ for every } \psi\in C^{\infty}(\Gamma).
$$

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\nFinally we get
\n
$$
||h||_{-2m+\frac{1}{p},p'} = \sup_{\varphi \in W_p} \frac{|(h, \varphi)|}{\|\varphi\|_{2m-\frac{1}{p},p}} = \sup_{\varphi \in C^{\infty}(\Gamma)} \frac{|(h, \psi)|}{\|\psi\|_{2m-\frac{1}{p},p}}
$$
\n
$$
= \sup_{\varphi \in C^{\infty}(\Gamma)} \frac{\left|\lim_{t \to \infty} (h_j, \psi)\right|}{\|\psi\|_{2m-\frac{1}{p},p}} = 0.
$$
\ni.e. $h = 0$ and $\overline{L_G(\Gamma)} = W_p^{-2m-\frac{1}{p}}(\Gamma)$
\n5. Let $V \subset \partial \Omega$ be an open subset of the boundary $\partial \Omega$. W
\nsmooth domain $\Omega_1 \supset \Omega$ with $\partial \Omega \setminus V \subset \partial \Omega_1$. In addition to one suppose:
\n(i) The coefficients of B_j can be extended to $\partial \Omega_1 \setminus \partial \Omega$ in such a system $(B_j)_{j=1,...,m}$ of boundary operators on $\partial \Omega_1$ also is normal condition.
\nIf the coefficients of B_j are constant (for instance in the problem) condition (i) is fulfilled.

i.e. $h = 0$ and $\overline{L_G(\overline{\Gamma})} = W_p$ *p p* (P) **l**

5. Let $V \subset \partial\Omega$ be an open subset of the boundary $\partial\Omega$. We construct a larger smooth domain $\Omega_1 \supset \Omega$ with $\partial \Omega \setminus V \subset \partial \Omega_1$. In addition to our earlier assumptions we suppose:

(i) The coefficients of B_j can be extended to $\partial \Omega_j \setminus \partial \Omega$ in such a way that the new system $(B_i)_{i=1,\dots,m}$ of boundary operators on $\partial\Omega_1$ also is normal and satisfies the roots

If the coefficients of *B,* are constant (for instance in the case of the Dirichiet problem) condition (i) is fulfilled.

Theorem 2: We suppose the conditions of Section 2 and (i). Then $L_v(\Gamma)$ is dense *in* $W_p^{2m-\frac{1}{p}}(F)$ $(1 < p < \infty)$. coefficients of B_j are constant (for insta
condition (i) is fulfilled.
 $P = 2$: *We suppose the conditions of Section*
 $P(F)$ ($1 < p < \infty$).

We choose an open subset $G = 0$.

Proof: We choose an open subset $G \subset \mathcal{Q}_1 \setminus \mathcal{Q}$ and consider the space $L_G(\mathcal{Q}_1)$. By Theorem 1 we have $\overline{L_G(Q_1)}|_r = W_p^{2m-\frac{1}{p}}(r)$. Since $L_G(Q_1)|_{\overline{G}} \subset L_{\tilde{V}}(Q)$, we obtain $\overline{L_V(\overline{I})} = W_p^{2m-\frac{1}{p}}(r)$ **I**

If we further suppose that the boundary value problem
 $Lu = 0$ in Ω , $B_iu|_{\partial \Omega} =$ $\overline{L_{V}(\Gamma)} = W_p^{2m-\frac{1}{p}}(\Gamma)$ **i**

If we further suppose that the boundary value problem

(j = 1, ..., m) $\hat{u} = 0$ in Ω , $B_i u|_{\partial \Omega} = 0$ (j = 1, ..., m)

only has the trivial solution, in the same manner as in [18], replacing $W_2^{2m-1}(\Gamma)$ by $W_p^{2m-\frac{1}{p}}(P)$ we can prove the theorems, corresponding Theorem, 3 and 4 in [18].

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