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On Strong Unboundedness of Symmetric Operators

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Es wird gezeigt, daß es zu jeder ungeraden natürlichen Zahl n einen symmetrischen Operator $\mathcal{T}$ im separablen Hilbertraum $\mathcal{H}$ gibt, so daß $\mathcal{T}, \mathcal{T}^3, \ldots, \mathcal{T}^n$ von unten unbeschränkt sind und $\mathcal{T}^k \geqq 0$ für $k > n$ ist.

Доказывается, что для любого нечётного натурального числа n существует симметрический оператор $\mathcal{T}$ в сепарабельном гильбертовом пространстве $\mathcal{H}$, такой, что $\mathcal{T}, \mathcal{T}^3, \ldots, \mathcal{T}^n$ неограничены снизу и $\mathcal{T}^k \geqq 0$ для всех $k > n$.

It will be shown that for each positive odd integer n there is a symmetric operator $\mathcal{T}$ in a separable Hilbert space $\mathcal{H}$ such that $\mathcal{T}, \mathcal{T}^3, \ldots, \mathcal{T}^n$ are unbounded from below and $\mathcal{T}^k \geqq 0$ for $k > n$.

A symmetric operator $\mathcal{T}$ with dense invariant domain $\mathcal{D} \subseteqq D(\mathcal{T})$ is said to be *strongly unbounded from above (below)*, if

$$\sup_{\varphi \in B_k(\mathcal{D})} \langle \mathcal{T}^k \varphi, \varphi \rangle = +\infty \quad \Big(\inf_{\varphi \in B_k(\mathcal{D})} \langle \mathcal{T}^k \varphi, \varphi \rangle = -\infty \Big),$$

for $k = 1, 3, 5, \ldots$, where

$$B_k(\mathcal{D}) = \{\varphi \in \mathcal{D} : |\langle \mathcal{T}^j \varphi, \varphi \rangle| \leqq 1, j = 1, 2, \ldots, k-1\}.$$

It was shown in [1] that two unbounded symmetric operators $\mathcal{T}_1, \mathcal{T}_2$ with dense invariant domains $\mathcal{D}_j = \bigcap_{i=1}^{\infty} D(\overline{\mathcal{T}}_j{}^i)$, $j = 1, 2$, possess dense invariant domains $\mathcal{D}_{j0} \subseteqq \mathcal{D}_j$, $j = 1, 2$, such that $\mathcal{T}_1 \restriction \mathcal{D}_{10}$ and $\mathcal{T}_2 \restriction \mathcal{D}_{20}$ are unitarily equivalent if and only if they are both strongly unbounded from below or both strongly unbounded from above. The latter result has applications to representations of unbounded operator algebras (see [3]). Clearly, a self-adjoint operator $\mathcal{A}$ is strongly unbounded from below (above), if and only if it is unbounded from below (above) in the usual sense. The following theorem shows that the word "strongly" can not be omitted in general. But the theorem seems to be of interest in itself.

Theorem: *Let n be a given positive odd integer. There is a closed symmetric operator $\mathcal{T}$ in a separable Hilbert space $\mathcal{H}$ such that*

(i) $\mathcal{D} = \bigcap_{j=1}^{\infty} D(\mathcal{T}^j)$ *is a core for each* $\mathcal{T}^k$, $k = 1, 2, \ldots$, *i.e.*, $\overline{\mathcal{T}^k \restriction \mathcal{D}} = \mathcal{T}^k$,

(ii) $\inf_{\varphi \in B_k(\mathcal{D})} \langle \mathcal{T}^k \varphi, \varphi \rangle = -\infty$ *if* $k \in \{1, 3, \ldots, n\}$,

(iii) $\langle \mathcal{T}^k \varphi, \varphi \rangle \geqq 0$ *if* $\varphi \in \mathcal{D}$ and $k > n$.

The idea of the proof is to construct T as a restriction of a suitable self-adjoint operator. We start with an auxiliary construction of symmetric operators T_t, $t > 0$.

From now on, let n be fixed. Consider a positive bounded operator B in a separable Hilbert space H with the properties that $\ker B = \{0\}$ and $BH \neq H$. Choose some vector $x \in H \setminus BH$. We define self-adjoint operators A_t, $t > 0$, in the Hilbert space $\mathbf{C} \oplus H =: \mathbf{H}$ by $D(A_t) := \mathbf{C} \oplus BH$ and $A_t(\lambda, Bf) = (-t^2\lambda, f)$ for $(\lambda, Bf) \in D(A_t)$. The operators T_t are defined by

$$D(T_t) := \{(\lambda, f) \in D(A_t) : A_t(\lambda, f) \perp (t^{n-1}, x)\},$$

$$T_t := A_t \restriction D(T_t).$$

Since T_t is a densely defined closed symmetric operator and has deficiency indices (1,1), [2: Theorem 1.9] yields

Lemma 1: *$D_t := \bigcap_{j=1}^{\infty} D(T_t{}^j)$ is dense in $\mathbf{H}$ and*

$$\overline{T_t{}^k \restriction D_t} = T_t{}^k, \qquad k = 1, 2, \ldots.$$

In the following we will need a suitable description of $D(T_t{}^k)$, which is given by

Lemma 2: *Let k be a positive odd integer. $(\lambda, f) \in \mathbf{H}$ belongs to $D(T_t{}^k)$ if and only if*

$$(\lambda, f) = \lambda(1, B^k h_t{}^k) + (0, B^k g),$$

where $g \perp B^l x$, $l = 0, 1, \ldots, k-1$, and

$$h_t{}^k = \begin{vmatrix} 0 & x & Bx & \ldots & B^{k-1}x \\ -t^{2k+n-1} & b_0 & b_1 & \ldots & b_{k-1} \\ +t^{2k+n-3} & b_1 & b_2 & \ldots & b_k \\ \vdots & \vdots & \vdots & & \vdots \\ -t^{n+1} & b_{k-1} & b_k & \ldots & b_{2k-2} \end{vmatrix} \cdot \begin{vmatrix} b_0 & b_1 & \ldots & b_{k-1} \\ b_1 & b_2 & \ldots & b_k \\ \vdots & \vdots & & \vdots \\ b_{k-1} & b_k & \ldots & b_{2k-2} \end{vmatrix}^{-1},$$

$$b_j := \langle B^j x, x\rangle, \qquad j = 0, 1, \ldots$$

Proof: The definition of $h_t{}^k$ is correct, since

$$\begin{vmatrix} b_0 & \ldots & b_{k-1} \\ \vdots & & \vdots \\ b_{k-1} & \ldots & b_{2k-2} \end{vmatrix}$$

is the Gramian determinant of the vectors $x, Bx, \ldots, B^{k-1}x$, which are linearly independent since $x \notin BH$.

Consider $(\lambda, f) \in D(T_t{}^k)$. Since $D(T_t{}^k) \subseteqq D(A_t{}^k)$, $(\lambda, f) = \lambda(1, B^k h_t{}^k) + (0, B^k g)$ for some $g \in H$. By definition of $D(T_t{}^k)$,

$$A_t{}^j(\lambda, f) \perp (t^{n-1}, x), \qquad j = 1, 2, \ldots, k,$$

i.e.

$$\Big\langle \lambda\big((-t^2)^j, B^{k-j}h_t{}^k\big) + (0, B^{k-j}g), (t^{n-1}, x)\Big\rangle$$
$$= \lambda\big((-1)^j\, t^{2j+n-1} + \langle h_t{}^k, B^{k-j}x\rangle\big) + \langle g, B^{k-j}x\rangle = 0$$

for $j = 1, 2, \ldots, k$. By definition of $h_t{}^k$, $\langle h_t{}^k, B^{k-j}x\rangle = (-1)^{k-j}\, t^{2j+n-1}$, which implies that $\langle g, B^{k-j}x\rangle = 0$, $j = 1, 2, \ldots, k$.

On the other hand, each vector of the described form belongs to $D(T_t{}^k)$ ∎

Since we want to study the operators T_t for $t \to 0$, the following statement will be useful.

Corollary 3: *$h_t^k = t^{n+1}\big(h^k + O(t)\big)$, where h^k is a fixed non-zero vector and $O(t)$ tends to zero if t tends to zero.*

Next we express the positivity of T_t^k in terms of h_t^k.

Lemma 4: *Let k be a positive odd integer. $\langle T_t^k\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D_t$ if and only if*

$$1 - \sup_{\substack{0 \neq f \perp B^l x \\ l=0,1,\dots,k-1}} \frac{|\langle B^k h_t^k, f\rangle|^2}{\langle B^k f, f\rangle \langle B^k h_t^k, h_t^k\rangle} \geqq t^{2k} \langle B^k h_t^k, h_t^k\rangle^{-1}. \tag{1}$$

Proof: $\langle T_t^k\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D_t$ iff $\langle T_t^k\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D(T_t^k)$, since D_t is a core for T_t^k by Lemma 1. Obviously $\langle T_t^k(0, B^k g), (0, B^k g)\rangle \geqq 0$. Thus we conclude by Lemma 2 that $\langle T_t^k\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D_t$ iff

$$\big\langle T_t^k\big(1, B^k(h_t^k + g)\big), \big(1, B^k(h_t^k + g)\big)\big\rangle = -t^{2k} + \langle B^k(h_t^k + g), h_t^k + g\rangle \geqq 0$$

for all $g \perp B^l x$, $l = 0, 1, \dots, k-1$. But

$$\begin{aligned} &\min_{\lambda \in \mathbf{C}} \langle B^k(h_t^k + \lambda g), h_t^k + \lambda g\rangle \\ &= \langle B^k h_t^k, h_t^k\rangle \big(1 - |\langle B^k h_t^k, g\rangle|^2 (\langle B^k h_t^k, h_t^k\rangle \langle B^k g, g\rangle)^{-1}\big) \end{aligned}$$

for $g \neq 0$. Using this, (1) follows from the last inequality ∎

Now we consider the left-hand side of inequality (1) for small t.

Lemma 5: *It is*

$$\liminf_{t \to +0} \left(1 - \sup_{\substack{0 \neq f \perp B^l x \\ l=0,1,\dots,k-1}} \frac{|\langle B^k g_t^k, f\rangle|^2}{\langle B^k h_t^k, h_t^k\rangle \langle B^k f, f\rangle}\right) > 0$$

for $k = 3, 5, \dots$.

Proof: Let H_* denote the completion of H with respect to the inner product $\langle \cdot, \cdot\rangle_* := \langle B^k \cdot, \cdot\rangle$. With some abuse of notation we denote the continuous extension of B to H_* again by B. Since $k \geqq 3$,

$$|\langle f, B^{k-1}x\rangle| \leqq \langle B^k f, f\rangle \langle B^{k-2}x, x\rangle$$

for all $f \in H$, i.e., the linear functional $f \mapsto \langle f, B^{k-1}x\rangle$ has a continuous extension to H_*. Thus, there is a vector $x_* \in H_*$ such that $\langle f, B^{k-1}x\rangle = \langle f, x_*\rangle_*$ for all $f \in H$. Obviously, $Bx_* = x$.

Consider

$$a_t := \sup_{\substack{\langle f, x_*\rangle = 0 \\ 0 \neq f \in H_*}} \frac{|\langle h_t^k, f\rangle_*|^2}{\langle f, f\rangle_* \langle h_t^k, h_t^k\rangle_*}.$$

If we denote the angle between h_t^k and x_* by α_t, then a_t is nothing but $\sin^2 \alpha_t$. Hence

$$\begin{aligned} a_t &= 1 - \cos^2 \alpha_t \\ &= 1 - |\langle h_t^k, x_*\rangle|^2 (\langle x_*, x_*\rangle_* \langle h_t^k, h_t^k\rangle_*)^{-1} \\ &= 1 - |\langle B^{k-1}h_t^k, x\rangle|^2 (\langle B^{k-2}x, x\rangle \langle B^k h_t^k, h_t^k\rangle)^{-1}. \end{aligned}$$

Since by definition of h_t^k

$$|\langle B^{k-1}h_t^k, x\rangle| = |\langle h_t^k, B^{k-1}x\rangle| = t^{n+1},$$

we obtain

$$a_t = 1 - t^{2n+2}(\langle B^{k-2}x, x\rangle \langle B^k h_t^k, h_t^k\rangle)^{-1}.$$

Hence

$$1 - \sup_{\substack{0 \neq f \perp B^l x \\ f \in H, l=0,1,\ldots,k-1}} \frac{|\langle B^k h_t^k, f\rangle|^2}{\langle B^k h_t^k, h_t^k\rangle \langle B^k f, f\rangle}$$

$$\geqq 1 - \sup_{\substack{0 \neq f \perp B^{k-1}x \\ f \in H}} \frac{\langle B^k h_t^k, f\rangle|^2}{\langle B^k h_t^k, h_t^k\rangle \langle B^k f, f\rangle}$$

$$\geqq 1 - a_t = t^{2n+2}(\langle B^{k-2}x, x\rangle \langle B^k h_t^k, h_t^k\rangle)^{-1}.$$

Thus, by Corollary 3,

$$\liminf_{t \to +0} \left(1 - \sup_{\substack{0 \neq f \perp B^l x \\ l=0,1,\ldots,k-1}} \frac{|\langle B^k h_t^k, f\rangle|^2}{\langle B^k h_t^k, h_t^k\rangle \langle B^k f, f\rangle}\right)$$

$$\geqq \lim_{t \to +0} t^{2n+2}(\langle B^{k-2}x, x\rangle \langle B^k h_t^k, h_t^k\rangle)^{-1} = (\langle B^{k-2}x, x\rangle \langle B^k h^k, h^k\rangle)^{-1},$$

which proves our assertion ∎

Proof of the Theorem: Since

$$\lim_{t \to +0} t^{2n+4}\langle B^{n+2}h_t^{n+2}, h_t^{n+2}\rangle^{-1} = \lim_{t \to +0} t^2\langle B^{n+2}h^{n+2}, h^{n+2}\rangle^{-1} = 0$$

and by Lemma 4 and 5, there is an $\varepsilon > 0$ such that $\langle T_t^{n+2}\varphi, \varphi\rangle \geqq 0$ for all $t \in (0, \varepsilon)$ and all $\varphi \in D_t$.

Since

$$\lim_{t \to +0} t^{2n}\langle B^n h_t^n, h_t^n\rangle^{-1} = \lim_{t \to +0} t^{-2}\langle B^n h^n, h^n\rangle^{-1} = \infty,$$

there is some $t_0 \in (0, \varepsilon)$ such that $t_0^{2n}\langle B^n h_{t_0}^n, h_{t_0}^n\rangle > 1$. Writing T and D instead of T_{t_0} and D_{t_0}, this means by Lemma 4 that there is some $\varphi_0 \in D$ such that $\langle T^n\varphi_0, \varphi_0\rangle < 0$. Obviously, $\langle T^{n-2j}T^j\varphi_0, T^j\varphi_0\rangle < 0$ for $j = 0, 1, \ldots, (n-1)/2$ and, since $\langle T^{n+2}\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D$, $\langle T^k\varphi, \varphi\rangle \geqq 0$ for all $\varphi \in D$ and $k > n$.

Consider the closed symmetric operator $\mathcal{T} = \sum_{j=1}^{\infty} \oplus jT$ in the separable Hilbert space $\mathcal{H} = \sum_{j=1}^{\infty} \oplus \mathbf{H}$. $\mathcal{T}$ satisfies the conditions in the theorem:

(i) is satiesfied, since $\mathcal{D} = \bigcap_{j=1}^{\infty} D(\mathcal{T}^j) = \sum_{j=1}^{\infty} \oplus D$ and since D is a core for each T^k, $k = 1, 2, \ldots$.

(ii) is true, since

$$\inf_{\varphi \in B_k(\mathcal{D})} \langle \mathcal{T}^k\varphi, \varphi\rangle \leqq j \left(\sum_{l=0}^{k-1} |\langle T^l\varphi_0, \varphi_0\rangle|\right)^{-1} \langle T^k\varphi_0, \varphi_0\rangle$$

for $j = 1, 2, \ldots$, and since $\langle T^k\varphi_0, \varphi_0\rangle < 0$ for $k = 1, 3, \ldots, n$.

(iii) follows immediately from the fact that $\langle T^k\varphi, \varphi\rangle \geqq 0$ for $k > n$, which completes the proof ∎

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