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On the Majorization Method for Holomorphic Solutions of Linear Partial Differential Equations

L. Berg

Die Anwendung der Majorantenmethode zum Nachweis holomorpher Lösungen von Anfangswertproblemen für zweidimensionale partielle Differentialgleichungen erfordert die asymptotische Abschätzung inverser Matrizen. Der Beitrag berichtet über zwei Beispiele hierzu.

Применение метода мажорант для доказательства существования голоморфных решений задачи Коши для двумерных дифференциальных уравнений в частных производных требует асимптотической оценки обратных матриц. В данной работе приводятся два примера к этой проблеме.

The application of the majorization method to proving the existence of holomorphic solutions of initial value problems for two-dimensional partial differential equations requires the asymptotic estimation of inverse matrices. The article presents two examples concerning this subject.

The majorization method was first used by S. v. Kowalevsky [6] to prove the existence of holomorphic solutions of the Cauchy problem for partial differential equations, and it is still contained in modern textbooks like L. Hörmander [5]. In the two-dimensional case it is possible to generalize the known results using asymptotic estimates for Toeplitzian band matrices [3]. In what follows we first improve one result of [1] concerning the Goursat problem, and second we sketch the transfer of the method to the Cauchy problem with an analytic boundary. The majorization method for ordinary differential equations you find e.g. in W. W. Golubew [4].

Though the considerations can be done for more general cases, we restrict ourselves for simplicity to the special differential equation with constant coefficients

$$\sum_{\nu=0}^{n} a_\nu \frac{\partial^n}{\partial x^\nu \, \partial y^{n-\nu}} z(x, y) = f(x, y), \tag{1}$$

where x, y are complex variables and $f(x, y)$ is holomorphic for $|x| < r$, $|y| < \varrho$. We ask for solutions

$$z(x, y) = \sum_{i,j=0}^{\infty} z_{ij} \frac{x^i}{i!} \frac{y^j}{j!}, \tag{2}$$

which are also holomorphic for $x = y = 0$ and satisfy given initial conditions. Substituting (2) into (1) we obtain the difference equation

$$\sum_{\nu=0}^{n} a_\nu z_{i+\nu, j+n-\nu} = f_{ij}$$

for $i, j = 0, 1, \ldots$, where at the right-hand side we have the coefficients of $f(x, y)$ in an expansion analogous to (2). For $i + j = m$ we introduce the notations

$$z_{i+\nu}^{(m)} = z_{i+\nu, m+n-\nu-i}, \qquad f_i^{(m)} = f_{i, m-i} \tag{3}$$

so that we have to solve the equations

$$\sum_{\nu=0}^{n} a_\nu z_{i+\nu}^{(m)} = f_i^{(m)} \tag{4}$$

for $i = 0, 1, \ldots, m$ and $m = 0, 1, \ldots$.

1. The Goursat problem

For a certain integer k with $0 \leqq k \leqq n$ Goursat's initial conditions read in the homogeneous case

$$\frac{\partial^\nu}{\partial x^\nu} z(0, y) = 0, \qquad \frac{\partial^\mu}{\partial y^\mu} z(x, 0) = 0 \tag{5}$$

for $\nu = 0, 1, \ldots, k-1$ and $\mu = 0, 1, \ldots, n-k-1$. According to (2) and (3) these conditions imply

$$z_0^{(m)} = \cdots = z_{k-1}^{(m)} = 0, \qquad z_{m+k+1}^{(m)} = \cdots = z_{m+n}^{(m)} = 0. \tag{6}$$

Hence the system of linear equations (4), (6) possesses for a fixed m the Toeplitzian band matrix

$$A_m = \begin{pmatrix} a_k & \cdots & a_n & & & \mathbf{0} \\ \vdots & \ddots & & \ddots & & \\ a_0 & & \ddots & & \ddots & \\ & \ddots & & \ddots & & a_n \\ & & \ddots & & \ddots & \vdots \\ \mathbf{0} & & & a_0 & \cdots & a_k \end{pmatrix} \tag{7}$$

of size $(m+1) \times (m+1)$ as coefficient matrix. We always assume that $n \geqq 1$.

Theorem 1: *If all A_m are regular and the elements $d_{ij}^{(m)}$ of the inverses $A_m^{-1} = (d_{ij}^{(m)})$ possess uniformly for $0 \leqq i, j \leqq m$ the asymptotic estimates*

$$d_{ij}^{(m)} = O(m^l \alpha^{i-j}) \tag{8}$$

with a real number $\alpha > 0$ and an integer $l \geqq 0$, then the initial value problem (1), (5) *possesses exactly one solution* (2), *which for an arbitrary p with $0 < p < 1$ is holomorphic for*

$$|x| < pR/\alpha, \qquad |y| < (1-p) R \tag{9}$$

with $R = \min(\alpha r, \varrho)$.

Proof: Since the coefficients z_{ij} can be uniquely determined from (4) with (6), we only have to prove the convergence of (2) in the circles (9). If we diminish the numbers r and ϱ by an arbitrary small amount (but maintain the same notation) we have for the second coefficients in (3) by Cauchy's inequality

$$f_i^{(m)} = O\left(i!(m-i)!\, r^{-i} \varrho^{i-m}\right).$$

Hence, inverting the system (4), (6) we find

$$z_{i+k}^{(m)} = \sum_{j=0}^{m} d_{ij}^{(m)} f_j^{(m)} = O\left(m^l \varrho^{-m} \sum_{j=0}^{m} \alpha^{i-j} j!(m-j)!\, (\varrho/r)^j\right)$$

and in view of $j!(m-j)! \leqq m!$ and

$$\sum_{j=0}^{m} (\varrho/\alpha r)^j = \begin{cases} O\big((\varrho/\alpha r)^m\big) & \text{for } \varrho > \alpha r, \\ m+1 & \text{for } \varrho = \alpha r, \\ O(1) & \text{for } \varrho < \alpha r, \end{cases}$$

consequently $z_{i+k}^{(m)} = O\big(m^l(m+1)!\,\alpha^i R^{-m}\big)$, or, according to (3),

$$z_{ij} = O\big((i+j)^l\,(i+j)!\,\alpha^i R^{-i-j}\big).$$

The binomial formula implies $(i+j)!\,p^i(1-p)^j \leqq i!j!$ for an arbitrary p with $0 < p < 1$. Thus we obtain

$$\frac{z_{ij}}{i!j!} = O\big((i+j)^l\,(pR/\alpha)^{-i}\,((1-p)\,R)^{-j}\big)$$

and we see the convergence of (2) under the conditions (9), if we let tend r and ϱ to their original values ▮

Remarks: In [1] Theorem 1 was proved under the additional assumptions $l = 0$ and $p = 1/2$. Sufficient conditions for (8) you find in [3]. The statements are sharp, as can be seen from the following

Example: The partial differential equation

$$z_{xx}(x, y) - 2\alpha z_{xy}(x, y) + \alpha^2 z_{yy}(x, y) = -h''(x) - \alpha^2 g''(y)$$

with $\alpha > 0$ possesses under the initial conditions (5) with $n = 2$ and $k = 1$ (i.e. $\nu = \mu = 0$) according to A. Schmidt [7] the unique solution

$$z(x, y) = g(\alpha x + y) + xq(\alpha x + y) - g(y) - h(x) + c \tag{10}$$

with $q(x) = \left(h\left(\frac{x}{\alpha}\right) - g(x)\right)\frac{\alpha}{x}$ and $c = g(0) = h(0)$. Because of $f(x, y) = -h''(x) - \alpha^2 g''(y)$ let r and ϱ be the radii of convergence of $h(x)$ and $g(y)$, respectively. Then $q(x)$ has in general the radius of convergence $R = \min(\alpha r, \varrho)$ and (10) is holomorphic in the intersection of $|\alpha x + y| < R$, $|x| < r$ and $|y| < \varrho$. This domain cannot be enlarged in general. Hence it is possible that the solution (10) possesses for $x = pR/\alpha$ and $y = (1-p)R$, i.e. for $\alpha x + y = R$, a singularity. Condition (8) is satisfied with $l = 1$.

2. The Cauchy problem

Now, for the partial differential equation of order $n = 2$

$$az_{xx}(x, y) + bz_{xy}(x, y) + cz_{yy}(x, y) = f(x, y) \tag{11}$$

with $ac \neq 0$ and a curve

$$y = dx + \sum_{\nu=2}^{\infty} d_\nu x^\nu \tag{12}$$

which is holomorphic for $x = 0$ we consider the homogeneous Cauchy problem

$$z(x, y) = 0, \qquad pz_x(x, y) + qz_y(x, y) = 0 \tag{13}$$

on the curve (12) with $p^2 + q^2 = 1$ and

$$pd \neq q. \tag{14}$$

The inequality (14) guarantees that in the point $x = y = 0$ the tangent of the curve (12) does not point in the direction of the derivative in (13).

Since (11) is a special case of (1), the corresponding difference equation (4) simplifies to

$$cz_i + bz_{i+1} + az_{i+2} = f_i \tag{15}$$

for $i = 0, 1, \ldots, m$ if we drop the upper index m. On the curve (12) we have

$$z = \sum_{i,j=0}^{\infty} z_{ij} \frac{x^{i+j}}{i!j!} \left(d + \sum_{\nu=1}^{\infty} d_{\nu+1} x^{\nu}\right)^j,$$

$$pz_x + qz_y = \sum_{i,j=0}^{\infty} (pz_{i+1,j} + qz_{i,j+1}) \frac{x^{i+j}}{i!j!} \left(d + \sum_{\nu=1}^{\infty} d_{\nu+1} x^{\nu}\right)^j,$$

so that the conditions (13) go over into $z_{00} = 0$, $dz_{01} + z_{10} = 0$, $qz_{01} + pz_{10} = 0$, i.e. in view of (14) also $z_{01} = z_{10} = 0$, and

$$\left.\begin{aligned} &\sum_{i=0}^{m+2} \binom{m+2}{i} d^{m+2-i} z_i = f_{m+1}, \\ &\sum_{i=0}^{m+2} \left(q \binom{m+1}{i} + pd \binom{m+1}{i-1}\right) d^{m+1-i} z_i = f_{m+2}, \end{aligned}\right\} \tag{16}$$

where f_{m+1} and f_{m+2} are linear combinations of values z_{ij} with $i + j < m + 2$, which, for a fixed m, are already known if we solve the system (15), (16) recursively.

Lemma: *The determinant Δ of the system* (15), (16) *possesses the value*

$$\Delta = (pd - q)(ad^2 - bd + c)^{m+1}. \tag{17}$$

Proof: By means of permissible transformations of the last two lines, where the cases $p \neq 0$, $q = 0$ and $q \neq 0$ are to be treated in different ways, we can write Δ in the form

$$\Delta = (pd - q) \begin{vmatrix} c & b & a & & & 0 \\ & c & b & a & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & c & b & a \\ d^{m+1} & \binom{m+1}{1} d^m & \cdots & \cdots & 1 & 0 \\ 0 & d^{m+1} & \binom{m+1}{1} d^m & \cdots & \cdots & 1 \end{vmatrix} \tag{18}$$

But the remaining determinant is the resultant of the two polynomials

$$a\lambda^2 + b\lambda + c \tag{19}$$

and $(\lambda + d)^{m+1}$ and therefore, according to B. L. van der Waerden [8], equal to $(ad^2 - bd + c)^{m+1}$ ∎

According to (14) and (17) we have the following

Corollary: *The system* (15), (16) *is uniquely solvable, if and only if* $-d$ *is no zero of the polynomial* (19), i.e. if the tangent $y = dx$ of the curve (13) at the point $x = y = 0$ is no characteristical line of the differential equation (11).

Let us denote the matrix of the system (15), (16) by A and introduce the notation $A^{-1} = (g_{ij})$ with $i, j = 0, 1, \ldots, m + 2$. Further we denote the zeros of (19) by α, β, and we introduce the notation

$$\alpha_+{}^k = \begin{cases} \alpha^k & \text{for } k \geqq 0, \\ 0 & \text{for } k < 0. \end{cases} \tag{20}$$

Theorem 2: *In the case when* (14) *holds and we have pairwise different values* α, β, $-d$, *the elements* g_{ij} *of* A^{-1} *have the representations*

$$g_{ij} = \frac{1}{a(\alpha - \beta)} \left[\alpha_+{}^{i-j-1} - \beta_+{}^{i-j-1} - \sum_{k=j+1}^{m+1} \binom{m+1}{k} d^{m+1-k} \left(\frac{\alpha^{i+k-j-1}}{(d+\alpha)^{m+1}} - \frac{\beta^{i+k-j-1}}{(d+\beta)^{m+1}}\right)\right] \tag{21}$$

for $j \leqq m$ *as well as*

$$\left.\begin{aligned} g_{i,m+1} &= \frac{1}{(\alpha - \beta)(pd - q)} \left(-\frac{(q + p\beta)\,\alpha^i}{(d+\alpha)^{m+1}} + \frac{(q + p\alpha)\,\beta^i}{(d+\beta)^{m+1}}\right), \\ g_{i,m+2} &= \frac{1}{(\alpha - \beta)(pd - q)} \left(\frac{(d+\beta)\,\alpha^i}{(d+\alpha)^{m+1}} - \frac{(d+\alpha)\,\beta^i}{(d+\beta)^{m+1}}\right). \end{aligned}\right\} \tag{22}$$

Proof: With the shift operator V defined by $Vz_i = z_{i-1}$ equation (15) reads $a(1 - \alpha V)(1 - \beta V)\, z_i = f_{i-2}$ and has in view of

$$\frac{1}{(1 - \alpha V)(1 - \beta V)} = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 - \alpha V} - \frac{\beta}{1 - \beta V}\right)$$

the special solution

$$z_i = \frac{1}{a(\alpha - \beta)} \sum_{\nu=0}^{i-2} (\alpha^{\nu+1} - \beta^{\nu+1})\, f_{i-\nu-2}$$

and, consequently, for $j = i - \nu - 2$ the general solution

$$z_i = \frac{1}{a(\alpha - \beta)} \sum_{j=0}^{i-2} (\alpha^{i-j-1} - \beta^{i-j-1})\, f_j + \alpha^i \sum_{j=0}^{m+2} u_j f_j + \beta^i \sum_{j=0}^{m+2} v_j f_j. \tag{23}$$

The coefficients u_j and v_j are to be determined in such a way that the equations (16) are satisfied, as well, i.e. that

$$\begin{pmatrix} d^{m+2} & \binom{m+2}{1} d^{m+1} & \ldots & \binom{m+2}{m-1} d & 1 \\ qd^{m+1} & q\binom{m+1}{1} d^m + pd^{m+1} & \ldots & q + p\binom{m+1}{m} d & p \end{pmatrix}$$

$$\times \frac{1}{a(\alpha - \beta)} (\alpha_+{}^{i-j-1} - \beta_+{}^{i-j-1})$$

$$+ \begin{pmatrix} (d+\alpha)^{m+2} & (d+\beta)^{m+2} \\ (q + p\alpha)(d+\alpha)^{m+1} & (q + p\beta)(d+\beta)^{m+1} \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

$i, j = 0, 1, \ldots, m + 2$. The last two columns of the matrix $(\alpha_+^{i-j-1} - \beta_+^{i-j-1})$ contain only zero elements. Hence we find for the last two components of $\tilde{u}_j = (d + \alpha)^{m+1} \times u_j$, $\tilde{v}_j = (d + \beta)^{m+1} v_j$

$$\begin{pmatrix} \tilde{u}_{m+1} & \tilde{u}_{m+2} \\ \tilde{v}_{m+1} & \tilde{v}_{m+2} \end{pmatrix} = \begin{pmatrix} d + \alpha & d + \beta \\ q + p\alpha & q + p\beta \end{pmatrix}^{-1} = \frac{-1}{(\alpha - \beta)(pd - q)} \begin{pmatrix} q + p\beta & -d - \beta \\ -q - p\alpha & d + \alpha \end{pmatrix},$$

and in view of (23) the representations (22) are proved. The first $m + 1$ columns of (24) are satisfied, if we choose by restricting on these columns

$$(\tilde{u}_j) = \frac{-1}{a(\alpha - \beta)} \left(d^{m+1}, \binom{m+1}{1} d^m, \binom{m+1}{2} d^{m-1}, \ldots, 1, 0\right) (\alpha_+^{i-j-1}),$$

$$(\tilde{v}_j) = \frac{1}{a(\alpha - \beta)} \left(d^{m+1}, \binom{m+1}{1} d^m, \binom{m+1}{2} d^{m-1}, \ldots, 1, 0\right) (\beta_+^{i-j-1})$$

since

$$\alpha(\tilde{u}_j) = \frac{-1}{a(\alpha - \beta)} \left(0, d^{m+1}, \binom{m+1}{1} d^m, \ldots, \binom{m+1}{m} d, 1\right) (\alpha_+^{i-j-1})$$

and a similar equation holds with respect to $\tilde{v}_j$. Hence in view of (23) the representations (21) are proved, too ∎

Corollary: *For $i > j$ we can replace equation* (21) *by*

$$g_{ij} = \frac{1}{a(\alpha - \beta)} \sum_{k=0}^{j} \binom{m+1}{k} d^{m+1-k} \left(\frac{\alpha^{i+k-j-1}}{(d + \alpha)^{m+1}} - \frac{\beta^{i+k-j-1}}{(d + \beta)^{m+1}}\right), \tag{25}$$

and for $i \leq j$ the first two terms at the right-hand side of (21) *vanish. The results also make sense for $\alpha \to \beta$.*

Example: In the case $m = 0$ we find in the usual way that

$$\begin{pmatrix} c & b & a \\ d^2 & 2d & 1 \\ qd & q + pd & p \end{pmatrix}^{-1}$$

$$= \frac{1}{(pd - q)(ad^2 - bd + c)} \begin{pmatrix} pd - q & apd + aq - bp & b - 2ad \\ -(pd - q) d & pc - aqd & ad^2 - c \\ (pd - q) d^2 & bqd - cq - cpd & 2cd - bd^2 \end{pmatrix}$$

and we can check the validity of (21), (22) and (25).

Estimates: Assuming that d, α, β are positive, we find from (21), (25) and (22) the asymptotic estimates

$$g_{ij} = \begin{cases} O\left(\binom{m+1}{m-j} \left(\frac{\alpha^i}{(d + \alpha)^j} + \frac{\beta^i}{(d + \beta)^j}\right) d^{m-j}\right) & \text{for } i \leq j, \\ O\left(\binom{m+1}{j} \left(\frac{\alpha^{i-j}}{(d + \alpha)^{m-j}} + \frac{\beta^{i-j}}{(d + \beta)^{m-j}}\right) d^{m-j}\right) & \text{for } i > j \end{cases}$$

and $j \leq m$ as well as

$$g_{ij} = O\left(\frac{\alpha^i}{(d + \alpha)^m} + \frac{\beta^i}{(d + \beta)^m}\right)$$

for $j = m + 1$ and $j = m + 2$. The correctness of these estimates follows immediately from

$$\binom{m+1}{k} \leqq \binom{m+1}{m-j}\binom{m-j}{k-j-1}, \quad \binom{m+1}{k} \leqq \binom{m+1}{j}\binom{j}{k}.$$

Remarks: After having constructed the inverses A^{-1} and estimated their elements, it is possible to calculate the coefficients z_{ij} of the solution (2) of the initial value problem (11), (13) and to construct a majorant for (2). It is also possible to transfer the method as in [1] to more general equations. We leave this to the reader and mention only that it is further possible to use the special solution of (11) constructed in [2], to add the general solution of the homogeneous equation according to A. Schmidt [7], and to determine the arbitrary functions of this general solution from the initial conditions (13).

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VERFASSER:

Prof. Dr. Lothar Berg
Sektion Mathematik der Wilhelm-Pieck-Universität
DDR-2500 Rostock, Universitätsplatz 1