Hypercomplex Factorization of the Helmholtz Equation

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Hauptergebnis der Arbeit ist die Angabe einer globalen hyperkomplexen Faktorisierung des Dirichlet-Problems für die Helmholtz-Gleichung. Dieses Resultat gestattet die Zurückführung des betrachteten Randwertproblems auf zwei Randwertprobleme erster Ordnung für verallgemeinert analytische Funktionen.

Основным результатом работы является построение глобальной гиперкомплексной факторизации задачи Дирихле для уравнения Гельмгольца. Таким образом названная задача может быть сведена к двум краевым задачам первого порядка для обобщенно аналитических функций.

The main result of this paper is the construction of a global hypercomplex factorization of the Dirichlet problem for the Helmholtz equation. This result permits the reduction of the considered boundary value problem into two boundary value problems of first order for generalized analytic functions.

1. Introduction

In the last years factorizing principles for the solution of elliptic boundary value problems have got an increasing importance. Using the concept of generalized analytic functions [1] and an operator calculus [7] global factorizations of such problems for the Laplace equation and for the equations of linear elasticity were found in [4]. The aim of this paper is to state analogous results for the Helmholtz equation.

2. Notations

Let Q be the quaternionic algebra with basic elements $e_1 = (1, 0, 0, 0), \ldots, e_4 = (0, 0, 0, 1)$. Each element $a \in Q$ has an unique representation $a = \sum_{i=1}^{4} a_i \cdot e_i$ with real coefficients a_i. The conjugate element $\bar{a}$ is defined by $\bar{a} = a_1 e_1 - \sum_{i=2}^{4} a_i e_i$ and a norm by $|a| = (a \circ \bar{a})^{1/2}$, where $\circ$ denotes the multiplication of quaternions. Let $G \subset \mathbf{R}^3$ be a bounded domain with smooth boundary Γ. A quaternionic-valued function f will be written as

$$f(x) = \sum_{i=1}^{4} f_i(x)\, e_i \quad \text{with} \quad f_i : G \to \mathbf{R}^1.$$

Define the differential operator $D(\nabla)$ as $D(\nabla) = \sum_{i=2}^{4} e_i\, \dfrac{\partial}{\partial x_i}$, let the differential ope-

rator $D(V)_r$ be given by $D(V)_r = \sum\limits_{i=2}^{4} \dfrac{\partial}{\partial x_i} e_i$ and, finally, denote by $D_\lambda(V)$, $D_\lambda*(V)$, and $D_\lambda(V)_r$ the operators $D_\lambda(V) = \lambda e_1 + D(V)$, $D_\lambda*(V) = \lambda e_1 - D(V)$, and $D_\lambda(V)_r = \lambda e_1 + D(V)_r$ with a real number λ. The normalized outer normal to Γ will be denoted by n. Elements $x \in \mathbb{R}^3$ will be written in the form $x = (x_2, x_3, x_4)$ to get simplicity in index-calculation. With every $x \in \mathbb{R}^3$ we associate a corresponding element $D(x) = \sum\limits_{i=2}^{4} x_i e_i \in Q$.

3. Fundamental solutions

The action of $(\varDelta + \lambda^2) I$ on quaternionic-valued function f will be defined by action of $(\varDelta + \lambda^2)$ in usual sense on each component of f. A formal factorization of this operator is given by

$$(\varDelta + \lambda^2) I = D_\lambda(V) D_\lambda*(V) = D_\lambda*(V) D_\lambda(V).$$

The knowledge of fundamental solutions of the operators $D_\lambda(V)$ and $D_\lambda*(V)$ is necessary for defining suitable integral operators and for constructing complete function systems. For the sake of simplicity, all following investigations will be carried out for the operator $D_\lambda(V)$ only.

Theorem 1: *Let* $x, y \in \mathbb{R}^3$, $x \neq y$ *and*

$$F_\lambda(x) = \frac{\lambda \cos \lambda |x|}{|x|} e_1 + \frac{D(x)}{|x|^2} \lambda \sin \lambda |x| + \frac{D(x)}{|x|^3} \cos \lambda |x|. \tag{1}$$

Then

$$D_\lambda(V_x) F_\lambda(x - y) = F_\lambda(x - y) D_\lambda*(V_x)_r = -4\pi\delta(x - y). \tag{2}$$

Proof: A simple computation using $D(V_x) \dfrac{D(x - y)}{|x - y|^3} = -4\pi\delta(x - y)$ (see [2]) ∎

4. Operators

Let $G \subset \mathbb{R}^3$ be a bounded domain with Ljapunov boundary, f a sufficiently smooth function defined in G or on Γ and let F_λ be given by (1). Define

$$(T_\lambda f)(x) = -\frac{1}{4\pi} \int\limits_{G} F_\lambda(x - y) \circ f(y) \, dG_y, \qquad x \in \mathbb{R}^3, \tag{3}$$

$$(\dot{S}_\lambda f)(x) = \frac{1}{4\pi} \int\limits_{\Gamma} F_\lambda(x - y) \circ D(n)_y \circ f(y) \, d\Gamma_y, \qquad x \notin \Gamma, \tag{4}$$

$$(S_\lambda f)(x) = \frac{1}{2\pi} \int\limits_{\Gamma} F_\lambda(x - y) \circ D(n)_y \circ f(y) \, d\Gamma_y, \qquad x \in \Gamma. \tag{5}$$

Remark 1: The integral operators T_λ, $\dot{S}_\lambda$, and S_λ are defined in analogy to the complex T-operator, the Cauchy integral operator and singular Cauchy integral operator, where the integral $\dot{S}_\lambda f$ is to be understood in the Cauchy principal value sense. In the case $\lambda = 0$ we get the operators investigated in [7].

Lemma 1: *Let $G \subset \mathbb{R}^3$ be a bounded domain with Ljapunov boundary and $f \in C^1(G)$* $\cap \bar{C}(\bar{G})$. *Then*

$$f(x) = (\mathring{S}_\lambda f)(x) + (T_\lambda D_\lambda(\nabla) f)(x) \quad \text{for any } x \in G. \tag{6}$$

Proof: We have

$$(T_\lambda D_\lambda(\nabla) f)(x) = -\frac{1}{4\pi} \int\limits_G F_\lambda(x-y) \circ D_\lambda(\nabla_y) f(y) \, dG_y$$

$$= \lim_{\varepsilon \to 0} \left(-\frac{1}{4\pi} \int\limits_{G_\varepsilon} F_\lambda(x-y) \circ D_\lambda(\nabla_y) f(y) \, dG_y \right)$$

with

$$G_\varepsilon = G \setminus \{y : |x-y| \leq \varepsilon\} = G \setminus K_x(\varepsilon), \quad \Gamma_\varepsilon = \partial K_x(\varepsilon),$$

$$-\frac{1}{4\pi} \int\limits_{G_\varepsilon} F_\lambda(x-y) \circ D_\lambda(\nabla_y) f(y) \, dG_y$$

$$= -\frac{1}{4\pi} \int\limits_{G_\varepsilon} \left(F_\lambda(x-y) D_\lambda^*(\nabla_y)_r \right) \circ f(y) \, dG_y$$

$$-\frac{1}{4\pi} \int\limits_\Gamma F_\lambda(x-y) \circ D(n) \circ f(y) \, d\Gamma_y + \frac{1}{4\pi} \int\limits_{\Gamma_\varepsilon} F_\lambda(x-y) \circ D(n) \circ f(y) \, d\Gamma_y.$$

From this equation we obtain the desired result by using the formula (see [7])

$$\lim_{\varepsilon \to 0} \int\limits_{G_\varepsilon} \left[\frac{D(x-y)}{|x-y|^3} D(\nabla_x)_r \right] \circ f(y) \, dG_y = -4\pi f(x) \quad \blacksquare$$

Lemma 2: *Let γ_0 be the trace operator in the sense of [5]. Under the same hypotheses as in Lemma 1 one has*

$$f(x) = (\mathring{S}_\lambda \gamma_0 f)(x) \quad \forall f \in \operatorname{Ker} D_\lambda(\nabla)(G), \quad x \in G. \tag{7}$$

Proof: Application of (6) to $f \in \operatorname{Ker} D_\lambda(\nabla)(G)$ $\blacksquare$

Lemma 3: *Under the hypotheses of Lemma 1*

$$(D_\lambda(\nabla) T_\lambda f)(x) = f(x) \quad \text{for any } x \in G. \tag{8}$$

Proof: In analogy to the proof of Lemma 1 $\blacksquare$

Remark 2: Formula (6) generalizes the well-known Borel-Pompeiu representation of complex function theory, (7) is an analogue of the Cauchy integral formula. Corresponding results for $\lambda = 0$ are proved in [1, 7].

Lemma 4: *Let $f \in C^\alpha(\Gamma)$ and suppose the boundary Γ of G is a Ljapunov surface. Then*

$$\lim_{\substack{x \to x_0 \in \Gamma \\ x \in G}} (\mathring{S}_\lambda f)(x) = \frac{1}{2} \left[(S_\lambda f)(x_0) + f(x_0) \right] =: (P_\lambda f)(x_0), \tag{9}$$

$$\lim_{\substack{x \to x_0 \in \Gamma \\ x \notin \bar{G}}} (\mathring{S}_\lambda f)(x) = \frac{1}{2} \left[(S_\lambda f)(x_0) - f(x_0) \right] =: -(Q_\lambda f)(x_0) \quad \text{for any } x_0 \in \Gamma. \tag{10}$$

Proof: A simple computation making use of some properties of weakly singular integral operators shows that

$$\lim_{\substack{x \to x_0 \in \Gamma \\ x \in G}} (\dot{S}_\lambda f)\,(x) =: \frac{1}{4\pi} \int_\Gamma \left[\frac{\lambda \cos \lambda\,|x_0 - y|}{|x_0 - y|} + \frac{D(x_0 - y)}{|x_0 - y|^2}\, \lambda \sin \lambda\,|x_0 - y| \right.$$

$$+ \frac{D(x_0 - y)}{|x_0 - y|^3} \sum_{i=1}^\infty (-1)^i \frac{1}{(2i)!} \, |x_0 - y|^{2i}\, \lambda^{2i} \Bigg] \circ D(n) \circ f(y)\, d\Gamma_y$$

$$+ \lim_{\substack{x \to x_0 \\ x \in G}} \frac{1}{4\pi} \int_\Gamma \frac{D(x - y)}{|x - y|^3} \circ D(n) \circ f(y)\, d\Gamma_y. \tag{11}$$

(11) combined with the result for $\lambda = 0$ contained in [7] gives the assertion of the lemma. Analogous arguments yield (10) ∎

Lemma 5: *Let $f \in C^a(\Gamma)$, Γ the Ljapunov boundary of G. Then*

$$S_\lambda^2 f = f, \tag{12}$$

$$\dot{S}_\lambda P_\lambda f = \dot{S}_\lambda f, \tag{13}$$

$$P_\lambda^2 f = P_\lambda f. \tag{14}$$

Proof: It is obvious that $S_\lambda^2 f = f$ iff $P_\lambda^2 f = P_\lambda f$. Formula (14) can be verified by applying (6) to the function $\dot{S}_\lambda f \in \operatorname{Ker} D_\lambda(\Gamma)\,(G)$. Using (14) and applying (6) to $\dot{S}_\lambda f - \dot{S}_\lambda P_\lambda f$ we arrive at (13) ∎

Remark 3: Note that the above results remain true also under weaker smoothness conditions for the function f.

The purpose of what follows is to investigate the continuity properties of the operators $\dot{S}_\lambda$, S_λ and T_λ. From the representation

$$(T_\lambda f)\,(x) = \frac{1}{4\pi} \int_G \left(D_\lambda(\Gamma) \frac{\cos \lambda\,|x - y|}{|x - y|} \right) \circ f(y)\, dG_y$$

we get

$$T_\lambda f \in H^{s+1}(G) \quad \text{for} \quad f \in H^s(G). \tag{16}$$

Hence,

$$T_\lambda D_\lambda(\Gamma) f \in H^s(G) \quad \text{for} \quad f \in H^s(G). \tag{17}$$

From (16) and (17) it follows that $\dot{S}_\lambda \gamma_0 f \in H^s(G)$ for $f \in H^s(G)$. Well-known theorems about traces together with formula (6) imply that $S_\lambda \gamma_0 f \in H^{s-1/2}(\Gamma)$ and $P_\lambda \gamma_0 f \in H^{s-1/2}(\Gamma)$ for $f \in H^s(G)$. It can be shown that S_λ, $P_\lambda \in L\big(H^s(\Gamma)\big)$.

Lemma 6: *Let G be a bounded domain with Ljapunov boundary Γ and $s > 3/2$. Then*

$$\dot{S}_\lambda \in L\big(H^{s-1/2}(\Gamma), H^s(G) \cap \operatorname{Ker} D_\lambda(\Gamma)\,(G)\big).$$

Proof: It will be shown that

$$\big(H^s(G), H^{s-1}(G), H^{s-1/2}(\Gamma), D_\lambda(\Gamma), \gamma_0\big)$$

is a Sard system with completeness condition (for the definition of Sard systems see [3, 6]). We have

$$D_\lambda(\nabla) \in L\big(H^s(G), H^{s-1}(G)\big),$$

$$\operatorname{im} D_\lambda(\nabla) = \overline{\operatorname{im} D_\lambda(\nabla)} = H^{s-1}(G) \quad \text{(see (8) and (16))},$$

$$\gamma_0 \in L\big(H^s(G), H^{s-1/2}(\Gamma)\big) \quad \text{and} \quad \operatorname{im} \gamma_0 = H^{s-1/2}(\Gamma) \text{ [5]},$$

$$H^s(G) \cap \operatorname{Ker} D_\lambda(\nabla)\,(G) \cap \operatorname{Ker} \gamma_0 = \{0\} \quad \text{(see (12))},$$

$$\gamma_0\big(\operatorname{Ker} D_\lambda(\nabla)\,(G) \cap H^s(G)\big) = \overline{\gamma_0\big(\operatorname{Ker} D_\lambda(\nabla)\,(G) \cap H^s(G)\big)}. \tag{18}$$

To prove (18) we take a sequence

$$\{y_n\}_{n=1}^\infty \subset H^{s-1/2}(\Gamma) \cap \gamma_0\big(\operatorname{Ker} D_\lambda(\nabla)\,(G)\big)$$

with $y_n \xrightarrow[n \to \infty]{} y_0$ in $H^{s-1/2}(\Gamma)$. Then one has:

$$S_\lambda y_n = y_n \Rightarrow S_\lambda y_0 = y_0 \Rightarrow P_\lambda y_0 = y_0$$

$$\Rightarrow \mathring{S}_\lambda y_0 \in \operatorname{Ker} D_\lambda(\nabla)\,(G) \cap H^s(G)$$

and

$$y_0 = \gamma_0 \mathring{S}_\lambda y_0 \in \gamma_0\big(\operatorname{Ker} D_\lambda(\nabla)\,(G)\big) \cap H^{s-1/2}(\Gamma).$$

From [3: Lemma 4] and from what has just been proved we obtain the estimate

$$C_1 \|f\|_{H^s(G)} \leqq \big(\|D_\lambda(\nabla)\,f\|^2_{H^{s-1}(G)} + \|\gamma_0 f\|^2_{H^{s-1/2}(\Gamma)}\big)^{1/2} \leqq C_2 \|f\|_{H^s(G)} \tag{19}$$

or any $f \in H^s(G)$ with $0 < C_1, C_2 < \infty$ independent of f. For $f \in \operatorname{Ker} D_\lambda(\nabla)\,(G) \cap H^s(G)$ the inequality (19) implies that

$$C_1 \|f\|_{H^s(G)} \leqq \|\gamma_0 f\|_{H^{s-1/2}(\Gamma)} \leqq C_2 \|f\|_{H^s(G)}. \tag{20}$$

From (20) and the continuity of P_λ in $H^{s-1/2}(\Gamma)$ the statement of the lemma follows ∎

5. Factorization

In the following we shall show the main result of this paper, namely, that the Dirichlet problem for the Helmholtz equation permits a global factorization into two boundary value problems of first order.

Lemma 7: *Let* $G \subset \mathbf{R}^3$ *be a bounded domain,* $\Gamma = \partial G$ *a Ljapunov surface, and* $s > 3/2$. *Then the boundary value problem*

$$\begin{aligned} D_\lambda(\nabla)\, u &= 0 \quad \text{in } G, \\ \gamma_0 u &= g \quad \text{on } \Gamma \end{aligned} \tag{21}$$

has a solution $u \in H^s(G)$ *iff* $g \in H^{s-1/2}(\Gamma) \cap \operatorname{im} P_\lambda$.

Proof: 1. Let $u \in H^s(G)$ be a solution of (21). From (7) and (9) then results that $\gamma_0 u = P_\lambda \gamma_0 u$.
2. Let $g \in H^{s-1/2}(\Gamma) \cap \operatorname{im} P_\lambda$. Then (6), (7), (9), and Lemma 6 imply that $u = \mathring{S}_\lambda g \in \operatorname{Ker} D_\lambda(\nabla)\,(G) \cap H^s(G)$ is a solution of the problem (21) ∎

Remark 4: The solution of (21), if it exists, is unique.

Theorem 2: *Suppose $\lambda^2 > 0$ is not an eigenvalue of the Δ-operator, G is a bounded domain with Ljapunov boundary $\Gamma = \partial G$, $s > 3/2$. Then the boundary value problem*

$$\Delta u + \lambda^2 u = 0 \quad in \ G,$$
$$\gamma_0 u = g \quad on \ \Gamma \tag{22}$$

admits a factorization into two boundary value problems of first order

$$D_\lambda(V) = 0 \quad in \ G, \qquad D_\lambda^*(V)\, w = 0 \quad in \ G,$$
$$\gamma_0 v = P_\lambda g \quad on \ \Gamma, \qquad \gamma_0 T_\lambda w = Q_\lambda g \quad on \ \Gamma \tag{23}, (24)$$

and the solution u of (22) is given by $u = v + T_\lambda w$.

Proof: The existence of a solution of the problem (23) has been shown in Lemma 7. From the solvability of the problem

$$\Delta z + \lambda^2 z = 0 \quad in \ G,$$
$$\gamma_0 z = Q_\lambda g \quad on \ \Gamma$$

and from (6) we deduce that

$$z = \mathring{S}_\lambda \gamma_0 z + T_\lambda D_\lambda(V)\, z = \mathring{S}_\lambda P_\lambda Q_\lambda \gamma_0 z + T_\lambda [D_\lambda(V)\, z] = T_\lambda w.$$

Because $z \in \mathrm{Ker}\,(\Delta + \lambda^2)\,(G)$ we get $w = D_\lambda(V)\, z \in \mathrm{Ker}\, D_\lambda^*(V)\,(G)$ ∎

Remark 5: Corresponding results for the case $\lambda = 0$ are contained in [7]. In [2], GOLDSCHMIDT considered factorization problems for the equation $(\Delta - \lambda^2)\, u = 0$, $\lambda \in \mathbf{R}$.

6. Complete function systems

For the application of a numerical method based upon the result of Theorem 2, it is necessary to construct complete function systems in $\mathrm{Ker}\, D_\lambda(V)\,(G) \cap H^s(G)$. For the sake of brevity the result will be given without proof.

Theorem 3: *Let G and G_A be bounded domains with Ljapunov boundaries $\partial G = \Gamma$ and $\partial G_A = \Gamma_A$ such that $\bar{G} \subset G_A$. Further let $\{x_i\}_{i=1}^\infty$ be a dense subset of Γ_A and the functions $\varphi_i(x) = F_\lambda(x - x_i)$ be defined as in Section 3. Then the system $\{\varphi_i\}_{i=1}^\infty \subset L_2(G) \cap \mathrm{Ker}\, D_\lambda(V)\,(G)$ is complete in $L_2(G) \cap \mathrm{Ker}\, D_\lambda(V)\,(G)$.*

Remark 6: Starting from this theorem, one can show the completeness of the system $\{\varphi_i\}_{i=1}^\infty$ in $H^s(G) \cap \mathrm{Ker}\, D_\lambda(V)\,(G)$ and then, using theorems about traces, its completeness in $H^s(\Gamma) \cap \mathrm{im}\, P_\lambda$.

Remark 7: Using the system $\{\varphi_i\}_{i=1}^\infty$, function systems with better numerical properties can be constructed with the method demonstrated in [4].

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