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Bergman-Vekua Operators and "Generalized Axially Symmetric Potential Theory"

E. Lanckau

Die verallgemeinerten komplexen Integraloperatoren von S. Bergman und I. N. Vekua ergeben eine einheitliche Methode zur Konstruktion der Lösungen verschiedenartiger linearer partieller Differentialgleichungen mit drei unabhängigen Veränderlichen. Diese Operatoren verknüpfen holomorphe Funktionen zweier komplexer Veränderlicher und die Lösungen der genannten Gleichungen. Hier spezialisieren wir die komplexen Integraloperatoren für den dreidimensionalen Fall auf eine Anwendung bei einer speziellen Gleichung axialsymmetrischer Probleme (d. h. instationärer axialsymmetrischer Probleme in der Ebene oder stationärer axialsymmetrischer Probleme im Raum). Weiter werden die Beziehungen zwischen der „Verallgemeinerten Axialsymmetrischen Potentialtheorie" von A. Weinstein u. a. und den Bergman-Vekua-Operatoren angegeben und bei der expliziten Lösungsdarstellung benutzt.

Обобщенные комплексные операторы Бергмана и Векуа представляют собой единый аппарат для построения решений разных линейных дифференциальных уравнений в частных производных с тремя независимыми переменными. Эти операторы связывают голоморфные функции с двумя комплексными переменными и решения этих уравнений. Здесь специализирован метод комплексных интегральных операторов трехмерного случая для применения к специальному уравнению осесимметричных проблем (напр. нестационарных осесимметричных проблем на плоскости или стационарных осесимметричных проблем в пространстве). Далее показаны соотношения между „Обобщенной теорией осесимметричных потенциалов" А. Вейнштейна и др. и операторами Бергмана и Векуа и использованы в явной конструкции решений.

The generalized complex integral operators of Bergman-Vekua type give a uniform approach to construct solutions to various linear partial differential equations with three independent variables. The operators associate holomorphic functions of two complex variables and the solutions of the mentioned equations. Here we specify the complex integral operators for the three-dimensional case for application to a special equation for axisymmetric problems (i.e. instationary axisymmetric problems in the plane or stationary axisymmetric problems in the space). Furthermore, the relations between the "Generalized Axially Symmetric Potential Theory", due to A. Weinstein et al., and the Bergman-Vekua operators are given and used in the explicit representation of the solutions.

1. Introduction

The Laplace potential equation in an x,y-space reads for a function u depending only on the radial distance $r = \sqrt{x^2 + y^2}$ and on the variable τ

$$\Delta u = \frac{\partial^2}{\partial r^2} u + \frac{1}{r} \frac{\partial}{\partial r} u + \frac{\partial^2}{\partial \tau^2} u = 0.$$

The theory of generalized axially symmetric potentials is concerned with equations of the type

$$\frac{\partial^2}{\partial r^2} u + \frac{c}{r} \frac{\partial}{\partial r} u + \frac{\partial^2}{\partial \tau^2} u = 0$$

with a constant c. Here we consider the equation

$$\mathscr{L}\{u\} \equiv \frac{\partial^2}{\partial r^2} u + \frac{2\lambda + 1}{r} \frac{\partial}{\partial r} u + S[u] = 0. \tag{1}$$

S is an arbitrary linear operator not depending on r (this means, an operator which depends only on τ), and $\lambda \geqq -1/2$ is a constant. Let $0 \leqq r \leqq R$ (R constant) and $\tau \in T$ with a simply connected domain T (if τ is complex) or an interval T (if τ is real). Thus, we have an equation (1) describing problems with axial symmetry in an x,y,τ-space or instationary processes with radial symmetry in an x,y-plane.

We use a version of the complex integral operators due to I. N. Vekua and S. Bergman to construct (real or complex) solutions to (1). (If τ is real and $S[u]$ is real-valued, the real part of a solution to (1) is a solution to (1), too.)

2. Construction of the solution

First we give an operator which transforms functions $f = f(\tau)$ into solutions of the equation (1), and we prove the existence of this operator. To do this we consider the equation (1) with $\lambda \geqq -1/2$ (everywhere in this section). (If $\lambda < -1/2$, we replace λ by $-\lambda - 1 > -1/2$, this means, $2\lambda + 1$ by $-2\lambda - 1$, and we find equation (5), see below.)

Definition 1 (*Transform*): Let

$$P_\lambda[f(\tau)]\,(r, \tau) = 2^\lambda \Gamma(\lambda + 1) \left\{\left(r \sqrt{S}\right)^{-\lambda} J_\lambda\left(r \sqrt{S}\right)\right\} [f(\tau)]$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\lambda + 1)}{n!\, \Gamma(n + \lambda + 1)} \left(\frac{r}{2}\right)^{2n} S^n[f(\tau)]. \tag{2}$$

Definition 2 (*Associated functions*): $f = f(\tau)$ is an *associated function* (of the equation (1)) if all $S^n[f(\tau)]$, $n = 0, 1, 2, \ldots$, exist and if constants $\alpha > 0$, $C > 0$ exist with

$$|S^n[f(\tau)]| < \alpha C^n (2n)! \quad \text{for} \quad \tau \in T_0 \subseteqq T.$$

The set of associated functions may be denoted by $\mathscr{F}$.

For example, for the ordinary differential operator of second order, $S = a_0 + a_1\partial/\partial\tau + a_2\partial^2/\partial\tau^2$ (with constant a_0, a_1, a_2) all (in a certain $T_0 \subset T$) holomorphic functions are associated functions, see [5]. In this case the domain T_0 may be every domain inside the domain T with $|\tau - \xi| > \delta > 0$ for $\tau \in T$, $\xi \in T_0$, see again [5, 6] (also for other operators S).

Theorem 1 (Existence): *$P[f]$ exists for $\tau \in T_0$ and $r^2 < 1/C$ if $f \in \mathscr{F}$.*

Indeed, in this case we have

$$P[f(\tau)] = \alpha \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + 1)}{n!\, \Gamma(n + \lambda + 1)} (2n)! \left(\frac{Cr^2}{4}\right)^n$$
$$= \alpha \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + 1)}{\Gamma(n + \lambda + 1)} \cdot \frac{\Gamma(n + 1/2)}{\Gamma(1/2)} (Cr^2)^n$$
$$= \alpha F\left(\frac{1}{2}, 1; \lambda + 1; Cr^2\right),$$

and this hypergeometric series converges for $Cr^2 < 1$ ∎

Theorem 2 (Solution): *If $f = f(\tau)$ is an associated function, $u(r, \tau) = P[f(\tau)]$ solves* (1).

Proof: We prove this theorem by insertion of the series (2) into the equation (1). The absolute and uniform convergence of the series enables us to differentiate term by term with respect to r and also to apply the operator S term by term, see [5]. In this way we have, with the abbreviation

$$a_n = \frac{\Gamma(\lambda + 1)}{n!\,\Gamma(n + \lambda + 1)} \left(-\frac{1}{4}\right)^n,$$

the derivatives

$$ru_r = \sum_{n=1}^{\infty} 2na_n r^{2n} S^n[f],$$

$$r^2 u_{rr} = \sum_{n=1}^{\infty} 2n(2n - 1)\, a_n r^{2n} S^n[f],$$

and

$$r^2 S[u] = \sum_{n=1}^{\infty} (-4)\, a_n (n + \lambda)\, n r^{2n} S^n[f];$$

thus

$$r^2 \mathcal{L}\{u\} = \sum_{n=1}^{\infty} a_n r^{2n} S^n[f] \cdot \{2n(2n - 1) + (2\lambda + 1)\, 2n - 4n(n + \lambda)\} = 0 \quad \blacksquare$$

Remark 1: An inversion formula is seen immediately: $f(\tau) = P[f(\tau)]\,(0, \tau)$. This allows us to construct the solution from the knowledge of its values on the axis $r = 0$: $u(r, \tau) = P[u(0, \tau)]$. This reflects the well-known fact that the solutions to (1) are uniquely determined by their values on the axis $r = 0$, if $2\lambda + 1 \geqq 0$ (but not if $2\lambda + 1 < 0$, see (4) and (5) below).

Remark 2: If $\lambda = 0$, the transform (2) is the Riemann transform defined by the author in [5, 6],

$$P[f(\tau)]\,(r, \tau) = R[f(\tau)]\,(z, z^*, 0, 0, \tau) \quad \text{with} \quad r^2 = zz^*.$$

Definition 3 (*Conjugate functions*): Let the operator $\sqrt{S}$ exist. A function $v = v(r, \tau)$ with

$$r^{2\lambda+1} \sqrt{S}\,[u] = \frac{\partial}{\partial r} v, \quad r^{2\lambda+1} \frac{\partial}{\partial r} u = -\sqrt{S}\,[v] \tag{3}$$

may be called a *conjugate function* of u.

Definition 1_C (*Transform for conjugate functions*): Let

$$\begin{aligned} \tilde{P}_\lambda[f(\tau)]\,(r, \tau) &= \frac{1}{2\lambda + 2}\, r^{2\lambda+2} P_{\lambda+1}\left[\sqrt{S}\, f(\tau)\right] \\ &= 2^\lambda \Gamma(\lambda + 1)\, r^{2\lambda+1} \left\{\left(r\sqrt{S}\right)^{-\lambda} J_{\lambda+1}\left(r\sqrt{S}\right)\right\} [f(\tau)] \\ &= \Gamma(\lambda + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + \lambda + 2)}\, \frac{1}{2^{2n+1}}\, r^{2n+2\lambda+2} S^n \left[\sqrt{S}\, f(\tau)\right]. \end{aligned} \tag{4}$$

We remark that this transform $\tilde{P}_\lambda$ exists if $\sqrt{S} f$ is an associated function; the proof may be given as above.

Theorem 2_C (Solution): *If $\sqrt{S} f$ is an associated function, $v(r, \tau) = \tilde{P}_\lambda[f(\tau)]$ is a conjugate function of $u = u(r, \tau)$, defined as in Definition 1_C and in Theorem 2.*

The proof can be found as for Theorem 2; we do not repeat this ▮

Theorem 3: *The conjugate function $v = v(r, \tau)$ is a solution of the equation*

$$\frac{\partial^2}{\partial r^2} v - \frac{2\lambda + 1}{r} \frac{\partial}{\partial r} v + S[v] = 0. \tag{5}$$

Again we do not repeat the proof. The technique is as for Theorem 2 ▮

3. Representation of the solutions

One of the most significant problems in the use of the above transform in applications is to give a proper representation of the transforms P and $\tilde{P}_\lambda$.

First we consider an elliptic (or hyperbolic) equation (1). To do this let, with (e.g.) differentiable coefficients $a = a(\tau)$, $b = b(\tau)$,

$$S = \hat{S}^2 \quad \text{with} \quad \hat{S} = a + b \frac{\partial}{\partial \tau},$$

that is

$$S = b^2 \frac{\partial^2}{\partial \tau^2} + (2ab + bb') \frac{\partial}{\partial \tau} + a^2 + a'b.$$

For $b = 0$ the equation (1) is without interest, because it is an ordinary differential equation with the parameter τ,

$$\frac{d^2}{dr^2} u + \frac{2\lambda + 1}{r} \frac{d}{dr} u + a^2(\tau) u = 0,$$

and its solution is — as is well-known — in coincidence with (2)

$$u(r, \tau) = \text{const}\, f(\tau)\, r^{-\lambda} J_\lambda\big(a(\tau)\, r\big)$$

with an arbitrary function $f = f(\tau)$.

For $b \neq 0$ we may assume $b \equiv 1$ without loss of generality (by a transformation of the independent variable τ). Thus we have

$$\hat{S} = a + \frac{\partial}{\partial \tau}. \tag{6}$$

We set for the associated function $f \in \mathcal{F}$

$$f(\tau) = \alpha(\tau)\, h(\tau) \quad \text{with} \quad \alpha(\tau) = \exp\Big(-\int^{\tau} a(\tau)\, d\tau\Big);$$

from this we get $\hat{S}[f] = \alpha(\tau)\, h'(\tau)$ and further

$$S^n[f] = \alpha(\tau)\, h^{(2n)}(\tau). \tag{7}$$

By the use of the Legendre duplication formula of the Gamma function we have from this

$$u(r,\tau) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\lambda+1)\,\Gamma(n+1/2)}{\Gamma(n+\lambda+1)\,\Gamma(1/2)} \frac{1}{(2n)!} r^{2n} S^n[f(\tau)], \tag{8}$$

this is

$$u(r,\tau) = \alpha(\tau) \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\lambda+1)\,\Gamma(n+1/2)}{\Gamma(n+\lambda+1)\,\Gamma(1/2)} \frac{1}{(2n)!} r^{2n} \frac{d^{2n}}{d\tau^{2n}} h(\tau). \tag{9}$$

Let first $\lambda = -1/2$; we have $P[f(\tau)] = \cos\left(r\sqrt{S}\right)[f(\tau)]$ or

$$u(r,\tau) = \alpha(\tau) \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} r^{2n} \frac{d^{2n}}{d\tau^{2n}} h(\tau).$$

This is a Taylor series,

$$u(r,\tau) = \alpha(\tau) \cdot \frac{1}{2} [h(\tau + ir) + h(\tau - ir)],$$

or, if $\operatorname{Im} h(\tau) = 0$, this means, if $h(\tau)$ is real for real arguments τ, $u(r,\tau) = \alpha(\tau) \times \operatorname{Re} h(\tau + ir)$. As can be immediately seen by insertion, this is a solution of the equation $\left(\partial^2/\partial r^2 + (a + \partial/\partial\tau)^2\right) u = 0$.

Now let $\lambda > -1/2$. Let the constant C_λ be

$$C_\lambda = 2\,\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1/2)\,\Gamma(1/2)}.$$

We represent the coefficients of the series (8) by the use of the Euler integral

$$\int_0^{\pi/2} \cos^{2\lambda}\varphi \sin^{2n}\varphi \, d\varphi = \frac{1}{2} \frac{\Gamma(\lambda+1/2)\,\Gamma(n+1/2)}{\Gamma(n+\lambda+1)};$$

this yields

$$u(r,\tau) = C_\lambda \alpha(\tau) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^{\pi/2} (ir\sin\varphi)^{2n} \cos^{2\lambda}\varphi \, d\varphi \cdot \frac{d^{2n}}{d\tau^{2n}} h(\tau),$$

and again this is a Taylor series,

$$u(r,\tau) = \frac{1}{2} C_\lambda \alpha(\tau) \int_0^{\pi/2} [h(\tau + ir\sin\varphi) - h(\tau - ir\sin\varphi)] \cos^{2\lambda}\varphi \, d\varphi,$$

or, if $\operatorname{Im} h(\tau) = 0$ for a real τ,

$$u(r,\tau) = C_\lambda \alpha(\tau) \operatorname{Re} \int_0^{\pi/2} h(\tau + ir\sin\varphi) \cos^{2\lambda}\varphi \, d\varphi. \tag{10}$$

The transform $\tilde{P}_\lambda$ for the conjugate function $v = v(r, \tau)$ can be treated in the same manner. We do not repeat the calculations, we give only the results:

$$v(r, \tau) = \begin{cases} \alpha(\tau) \operatorname{Im} h(\tau + ir) & \text{for} \quad \lambda = -1/2 \\ C_\lambda \alpha(\tau)\, r^{2\lambda+1} & \\ \times \operatorname{Im} \int\limits_0^{\pi/2} h(\tau + ir \sin\varphi) \sin\varphi \cos^{2\lambda}\varphi \, d\varphi & \text{for} \quad \lambda > -1/2. \end{cases} \tag{11}$$

We remark that for $\lambda = -1/2$ we have

$$\tilde{P}_{-1/2}[f(\tau)] = \sin\left(r\sqrt{S}\right)[f(\tau)].$$

Further we remark: The proof for the functions u (constructed by (10)) and v (constructed by (11)) being conjugate functions (with the same associated function f) may also be given by partial integration in (10), using

$$ir \cos\varphi \cdot h'(\tau + ir \sin\varphi) = \frac{\partial}{\partial\varphi} h(\tau + ir \sin\varphi).$$

Second we consider a parabolic equation (1), let $S = \hat{S}$. For the same reasons as above we may assume $b \equiv 1$, and now we have with $S^n[f] = \hat{S}^n[f] = \alpha(\tau)\, h^{[n]}(\tau)$ the solution

$$u(r, \tau) = \alpha(\tau) \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + 1)}{n!\,\Gamma(n + \lambda + 1)} \left(-\frac{r^2}{4}\right)^n \frac{d^n}{d\tau^n} h(\tau).$$

Here we represent the derivatives by the Cauchy integral

$$\frac{d^n}{d\tau^n} h(\tau) = \frac{n!}{2\pi i} \oint_K f(\xi) \frac{d\xi}{(\xi - \tau)^{n+1}},$$

with a circle $K = \{\xi : |\xi - \tau| = \delta > 0\}$ in the complex ξ-plane and $K \subset T_0$ for $\tau \in T_0$. From this we immediately obtain the solution by a Cauchy-type integral

$$u(r, \tau) = P[f(\tau)] = \alpha(\tau) \frac{1}{2\pi i} \oint_K H_\lambda\left(\frac{r^2}{4(\tau - \xi)}\right) h(\xi) \frac{d\xi}{\xi - \tau}, \tag{12}$$

and the generating kernel H_λ is a generalized hypergeometric function, converging everywhere,

$$H_\lambda(w) = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + 1)}{\Gamma(n + \lambda + 1)} w^n = {}_1F_1(1; \lambda + 1; w).$$

We have — proven simply — $H_\lambda(w) = 1 + w \int\limits_0^1 (1 - s)^\lambda e^{sw}\, ds$. This gives $H_0(w) = e^w$ and

$$H_{-1/2}(w) = 1 + 2w\, e^w \int\limits_0^1 e^{-ws^2}\, ds,$$

a function well-known in the theory of the equation of heat conduction $(\partial^2/\partial r^2 + \partial/\partial\tau + a)\, u = 0$. The parabolic equation (5) (with the operator $S = \hat{S}$) for the conjugate function may be transformed into a parabolic equation (5) with

$S = \partial/\partial\tau$ (introducing the new function $\alpha(\tau)\,v(r, \tau)$ instead of the function $v(r, \tau)$). Its solution is given by the expressions (4) and (12) with the associated function

$$\sqrt{S}\,[f]\,(\tau) = \frac{\Gamma(3/2)}{2\pi i} \oint_K f(\xi)\, \frac{d\xi}{\sqrt{\xi - \tau}^3}.$$

Finally we consider

$$S[u] = 2c \int_0^\tau u(r, s)\, ds$$

with a constant c, this means, we have the pseudoparabolic equation of third order

$$\frac{\partial^3}{\partial r^2\, \partial\tau} u + \frac{2\lambda + 1}{r} \frac{\partial^2}{\partial r\, \partial\tau} u + 2cu = 0.$$

Here the functions, integrable with respect to τ in $T_0 = T$, are associated functions, see [5]. With

$$S^n[f] = (2c)^n \frac{1}{n!} \frac{\partial}{\partial\tau} \int_0^\tau (\tau - s)^n f(s)\, ds$$

we have the solution, being a convolution integral,

$$u(r, \tau) = P[f(\tau)] = \frac{\partial}{\partial\tau} \int_0^\tau \hat{H}_\lambda\big(cr^2(s - \tau)\big)\, f(s)\, ds; \tag{13}$$

here the generating kernel $\hat{H}_\lambda$ is again a generalized hypergeometric function,

$$\hat{H}_\lambda(w) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2\, \Gamma(n + \lambda + 1)}\, w^n = {}_0F_2(1, \lambda + 1; w)$$

Again we get the related conjugate function, it is a solution of the equation

$$\frac{\partial^3}{\partial r^2\, \partial\tau} v - \frac{2\lambda + 1}{r} \frac{\partial^2}{\partial r\, \partial\tau} v + 2cv = 0,$$

by using the expressions (4) and (13); here we have (for $2c = 1$)

$$\sqrt{S}\,[f]\,(\tau) = \frac{1}{\Gamma(1/2)} \int_0^\tau f(s)\, \frac{ds}{\sqrt{\tau - s}}.$$

4. Relations to generalized axially symmetric potential theory

We return to the expressions (10) (and (11)). (10) may be written as

$$u(r, \tau) = \frac{1}{2}\, C_\lambda \alpha(\tau) \int_{-\pi/2}^{\pi/2} h(\tau + ir \sin\varphi) \cos^{2\lambda}\varphi\, d\varphi$$

or

$$u(r, \tau) = \text{const}\; \alpha(\tau) \int_0^\pi h(\tau + ir\cos\varphi) \sin^{2\lambda}\varphi\, d\varphi.$$

Now we use the half-circle $C = \{\xi = e^{i\alpha}: 0 \leqq \alpha \leqq \pi\}$. With this variable ξ we have

$$u(r, \tau) = \text{const} \int\limits_C h\left(\tau + \frac{ir}{2}\left(\xi + \frac{1}{\xi}\right)\right)\left(\xi - \frac{1}{\xi}\right)^{2\lambda} \frac{d\xi}{\xi}.$$

This is the expression mostly used in the generalized axially symmetric potential theory, see GILBERT [1], who uses this integral for the investigation of the singularities of the axially symmetric potentials in the space.

Following the ideas of the generalized axially symmetric potential theory, we remark: the real part $\operatorname{Re} h(\tau + ir) \doteq u_{(-1/2)}(r, \tau)$ is a harmonic function, $(\partial^2/\partial r^2 + \partial^2/\partial\tau^2)\, u_{(-1/2)} = 0$, and with this harmonic function we may write (10) as

$$u(r, \tau) = C_\lambda \int\limits_0^{\pi/2} u_{(-1/2)}(r \sin\varphi, \tau) \cos^{2\lambda}\varphi \, d\varphi. \tag{14}$$

(Here we assumed that $\tau \pm i\varrho \in T_0$ if $\tau \in T_0$ for all $|\varrho| \leqq r$.) Generalizing this representation we have

Theorem 4: *If $u_{(-1/2)} = u_{(-1/2)}(r, \tau)$ is a solution of the equation* (1) *with $\lambda = -1/2$, that is, of the equation*

$$\left(\frac{\partial^2}{\partial r^2} + S\right) u_{(-1/2)} = 0, \tag{15}$$

the integral (14) *solves* (1).

Remark: For the ordinary differential operator of second order with respect to τ, $S \doteq \partial^2/\partial\tau^2 + a(\tau)\, \partial/\partial\tau + c(\tau)$, this result is due to P. HENRICI [2]. He proposes to construct the solution of the equation (15) (with the mentioned S) by a Bergman-Vekua operator in its original version for two-dimensional problems.

Proof of Theorem 4: Let $\varrho = r \sin\varphi$. We insert (14) into the equation (1). First we have

$$S[u] = \int\limits_0^{\pi/2} S[u_{(-1/2)}(\varrho, \tau)] \cos^{2\lambda}\varphi \, d\varphi,$$

further

$$\frac{\partial}{\partial r} u = \int\limits_0^{\pi/2} \frac{\partial}{\partial\varrho} u_{(-1/2)}(\varrho, \tau) \sin\varphi \cos^{2\lambda}\varphi \, d\varphi,$$

and

$$\frac{\partial^2}{\partial r^2} u = \int\limits_0^{\pi/2} \frac{\partial^2}{\partial\varrho^2} u_{(-1/2)}(\varrho, \tau) \sin^2\varphi \cos^{2\lambda}\varphi \, d\varphi.$$

In the second expression we use partial integration. In this way we find

$$\frac{2\lambda + 1}{r} \frac{\partial}{\partial r} u = \int\limits_0^{\pi/2} \frac{\partial^2}{\partial\varrho^2} u_{(-1/2)}(\varrho, \tau) \cos^{2\lambda+2}\varphi \, d\varphi.$$

Insertion of all these terms into the equation (1) gives

$$\mathcal{L}\{u\} = \int_0^{\pi/2} \left[\frac{\partial^2}{\partial \varrho^2} u_{(-1/2)}(\tau, \varrho) + S[u_{(-1/2)}(\varrho, \tau)] \right] \cos^{2\lambda} \varphi \, d\varphi = 0 \; \blacksquare$$

The following theorem, found by the solution (11), may be proved in the same way (by insertion) as above.

Theorem 4_C: *If $u_{(-1/2)} = u_{(-1/2)}(r, \tau)$ is a solution of the equation* (5) (*or* (1)) *with $\lambda = -1/2$, that is, of the equation* (15),

$$v(r, \tau) = C_\lambda r^{2\lambda+1} \int_0^{\pi/2} u_{(-1/2)}(r \sin \varphi, \tau) \sin \varphi \cos^{2\lambda} \varphi \, d\varphi$$

solves (5).

5. Examples

We give only some hints concerning the application of the above results by the construction of special solutions.

a) *Parabolic equations:* A solution of the equation (15) with $S = \partial/\partial\tau$ is

$$u_{(-1/2)} = e^{c^2\tau} \cos (cr)$$

with a constant c. The integral (14) gives

$$u(r, \tau) = C_\lambda e^{c^2\tau} \int_0^{\pi/2} \cos (cr \sin \varphi) \cos^{2\lambda} \varphi \, d\varphi .$$

By the use of the well-known representation of the Bessel function

$$J_\lambda(cr) = 2 \frac{\left(\frac{c}{2} r\right)^\lambda}{\Gamma(\lambda + 1/2)\, \Gamma(1/2)} \int_0^{\pi/2} \cos (cr \sin \varphi) \cos^{2\lambda} \varphi \, d\varphi$$

we have a solution of (1)

$$u(r, \tau) = 2^\lambda \Gamma(\lambda + 1)\, (cr)^{-\lambda} J_\lambda(cr)\, e^{c^2\tau} .$$

However, this is a solution of (1) that can also be found by separation of variables.

b) *Elliptic equations:* We get sets of particular solutions to the equation (1) with $S = \partial^2/\partial\tau^2$ in the same manner, this means, by insertion of special associated functions into (10) or (11). We mention without proof that we find, with the help of known integral relations containing special functions, e.g.

$$u(r, \tau) = P[e^{c\tau}] = 2^\lambda \Gamma(\lambda + 1)\, (cr)^{-\lambda} J_\lambda(cr)\, e^{c\tau}$$

(see the former result), or

$$u(r, \tau) = P[\tau^k] = 2^\lambda \Gamma(\lambda + 1)\, r^{-\lambda} R^{k+\lambda} P_{\lambda+k}^{-\lambda} \left(\frac{\tau}{R}\right)$$

with constants c, $k \geqq 0$, $R^2 = r^2 + \tau^2$ and with the associated Legendre functions $P_{\lambda+k}^{-\lambda}$. We get from these relations e.g. with

$$P[e^\tau] = \sum_{k=0}^{\infty} \frac{1}{k!} P[\tau^k]$$

the series

$$J_\lambda(r) = e^{-\tau} \sum_{k=0}^{\infty} \frac{1}{k!} (r^2 + \tau^2)^{1/2(k+\lambda)} P_{\lambda+k}^{-\lambda} \left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right).$$

Using these and other solutions we may treat physical problems with axial symmetry, e.g., with $\lambda = 0$, the axisymmetric flow of an incompressible fluid (see [3]) or, with $\lambda = 1$, the torsion of a body of revolution (see [4]).

c) *Pseudoparabolic equations:* For an equation (1) with a more complicated integral operator S, describing the instationary axisymmetric flow of an incompressible viscous (non-Newtonian) fluid see [7].

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VERFASSER:

Prof. Dr. EBERHARD LANCKAU
Sektion Mathematik der Technischen Hochschule
DDR-9022 Karl-Marx-Stadt, Reichenhainer Str. 41