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Polysingular Operators and the Topology of Invertible Singular Operators

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Die Homotopie-Gruppen des Raumes der invertierbaren singulären Integraloperatoren mit Cauchy-Kern auf einer geschlossenen Kurve werden berechnet. Es erfolgt die Darstellung einer Index-Formel für einen bisingulären Integraloperator.

Вычисляются гомотопические группы пространства обратимых сингулярных интегральных операторов с ядром Коши на замкнутой кривой. Приводится формула для индекса бисингулярного интегрального оператора.

The homotopy groups of the space of invertible singular integral operators with Cauchy kernel on the closed curve are computed. An index formula for a bisingular integral operator is presented.

1. It is well-known that Fredholm type properties of polysingular operators are connected with the invertibility of certain naturally arising families of lower-dimensional singular operators [1, 2]. It turns out that this link may be effectively used in order to express some basic properties of polysingular integral operators in terms of homotopy groups associated with the corresponding spaces of invertible operators. The computation of these homotopy groups seems to be of an independent interest, and we shall first consider just this topological problem. The main result (Theorem 1) is obtained in a more general setting than is necessary for polysingular integral operator theory. It should be noted that such an abstract setting uses essentially the general theory of singular operators as developed in [3], and our considerations become even more clear in the framework of abstract singular operators. In the last sequel the above-mentioned link is described precisely in the case of a bisingular integral operator with Cauchy kernel (the general case is completely analogous but much more voluminous). In the conclusion an index formula for bisingular operators is presented which provides an example of concrete applications of the preceding topological results, and some generalizations and related results are indicated.

2. We start with the generalities on Fredholm theory and abstract singular operators. Let E be a complex Banach space, and let $L(E)$, $F(E)$ $(F_n(E))$, $C(E)$ denote, respectively, the spaces of all bounded linear operators in E, Fredholm operators (of the index n) and completely continuous (compact) operators endowed with the norm topology. For an arbitrary subalgebra A of $L(E)$ containing the identity operator I let GA denote the group of units of A, that is the set of invertible operators from A. If A doesn't contain I then GA will stand for the set of operators from A which become invertible after adding to them I.

Evidently, $C(E)$ is the closed two-sided ideal of $L(E)$ and the factor-space $L(E)/C(E)$ also has a Banach algebra structure (*Calkin algebra*). Let $q\colon L(E) \to L(E)/C(E)$ be the factor-mapping and let G denote the multiplicative group of invertible elements of the Calkin algebra. The fundamental result on Fredholm operators yields that for $T \in L(E)$ one has $T \in F(E)$ if and only if $q(T) \in G$. Set $G_0 = q(F_0(E))$ — this is a clopen subgroup of G, and if $GL(E)$ is connected, then G_0 is simply the component of the unity of G. Denoting by $p: GL(E) \to GL(E)/GC(E)$ the factor mapping with respect to the Fredholm group and taking into account that the factor group may be identified with G_0, one gets the fundamental commutative diagram:

$$\begin{array}{ccccccc} GL(E) & \to & F_0(E) & \to & F(E) & \to & L(E) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ GL(E)/GC(E) & \xrightarrow{\cong} & G_0 & \to & G & \to & L(E)/C(E). \end{array} \tag{1}$$

Evidently, q is a homotopy equivalence, and if $GL(E)$ is contractible, then p defines the universal $GC(E)$-bundle. The topological structure of the Fredholm group $GC(E)$ and its classifying space has been studied very intensively, and one can obtain a lot of information on subgroups of invertible operators by taking the corresponding restriction of p. This observation will be applied for the class of singular operators defined following [3].

Let R be some closed subalgebra of $L(E)$ and P — a bounded projector in E, $Q = I - P$. We assume that the following conditions are fulfilled:

a) invertible operators are dense in R;
b) if $A \in R$ and at least one of the operators $PAP \mid \operatorname{im} P$ or $QAQ \mid \operatorname{im} Q$ is semi-Fredholm then A is invertible in R;
c) for every $A \in R$ the commutator $[P, A]$ is completely continuous.

Definition [3]: An operator of the form

$$C = AP + BQ + T, \tag{2}$$

where $A, B \in R$, $T \in C(E)$, is called *abstract singular operator* (with coefficients in R). If $T = 0$, then C is called a *coupling operator* with coefficients A and B.

The operators of the form (2) form a subalgebra $S(R)$ of $L(E)$, and we shall consider the group $GS(R)$.

Proposition 1 [3]: *An operator C of the form* (2) *is Fredholm if and only if its coefficients A and B are invertible in R.*

It is already evident that the topological structure of $GS(R)$ is determined by the group GR, in particular, working on homotopy groups' level one has to compute $\pi_n(GR)$. This may be accomplished for an arbitrary finitely generated subalgebra with commutative symbols, but for our purposes the following special case is sufficient.

Let $U \in GL(E)$ be an invertible operator with the properties:

1. both the operators U and U^{-1} have the spectral radius equal to 1;
2. there exists a projector $P \in L(E)$ such that

$$UP = PUP, \qquad UP \neq PU, \qquad PU^{-1} = PU^{-1}P;$$

3. $\dim \operatorname{coker} (U \mid \operatorname{im} P) < \infty$.

Let $R(U)$ denote the closure of the subalgebra generated by U and U^{-1}. It was proved in [3] that the algebra $R(U)$ has the properties a)—c) and, moreover, the symbol homomorphism $h: R(U) \to C(S^1)$ is defined, where $S^1 = \{z \in \mathbf{C}: |z| = 1\}$ is the unit circle. If $R(U)$ has trivial radical then h is an isomorphism into. For example, this is just the case if U is a unitary operator in the Hilbert space H.

One can form the algebra $S(U) = S(R(U))$ and define the *symbol* $h(C)$ of an operator (2) to be the pair $(h(A), h(B))$. The symbol is called *non-degenerate* if both the functions $h(A)$ and $h(B)$ are non-vanishing on S^1. Recall that for a non-vanishing function $f: S^1 \to \mathbf{C}$ the index is defined to be the integer

$$\operatorname{ind} f = \frac{1}{2\pi} [\arg f]_{S^1},$$

where usual notation for the increment of the expression in square brackets along S^1 in the positive direction is used. Now the *index* of the symbol is defined to be

$$\operatorname{ind} h(C) = \operatorname{ind} h(A) - \operatorname{ind} h(B),$$

and the main result of the Fredholm theory for abstract singular operators with coefficients in $R(U)$ may be stated as follows.

Proposition 2 [3]: *An operator $C \in S(U)$ is Fredholm if and only if the symbol $h(C)$ is non-degenerate, and in this case* $\operatorname{ind} C = \operatorname{ind} h(C)$. *The coupling operator C of the form* (2) *is invertible if and only if* $\operatorname{ind} h(C) = 0$.

Using this proposition one can easily verify that the restriction of the bundle projection p on $GS(U)$ defines a trivial bundle and its base may be identified with the set $H(U)$ of non-degenerate symbols. Our problem is thus reduced to the computation of $\pi_n(H(U))$. Note first that $H(U)$ is a topological group and it is easy to see that its group of (connected) components $\pi_0(H(U))$ is naturally isomorphic to $\mathbf{Z}$, the isomorphism being defined by assigning to $h(C)$ the integer $\operatorname{ind} h(A)$. Further, dealing with homotopy groups one has to fix a point in $H(U)$. Evidently, all components are homeomorphic, so that the answer depends neither on the choice of the component, nor on the choice of the point within the component. We shall assume that the unity, that is the point $e = (1, 1)$, is fixed, where 1 stands for the function identically equal to 1. Throughout the text we deal only with spaces which are groups and we always assume that the distinguished point is the unity.

In the case of a unitary operator U in the Hilbert space H one has the isomorphism h of $R(U)$ and $C(S^1, \mathbf{C})$, so that the symbols may be arbitrary pairs of continuous functions on S^1. Using the usual radial retraction of $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ on S^1 it is easy to verify that

$$\pi_n(H(U)) = \pi_n(D), \quad \text{where} \quad D = \{(f, g) \in (C(S^1, S^1))^2 : \operatorname{ind} f = \operatorname{ind} g\}.$$

It may be proved by some additional homotopy arguments that the same relation holds for an arbitrary (not necessarily unitary) admissible invertible operator U, and it remains to compute $\pi_n(D)$.

Lemma: *The groups $\pi_n(D)$ are given by the relations*

$$\pi_1 = \mathbf{Z} \oplus \mathbf{Z}, \qquad \pi_n = 0, \qquad n \geqq 2. \tag{3}$$

For the proof it is sufficient to deal only with $C(S^1, S^1)$ because every component of an element of D may be homotoped independently. Having this in mind consider first an arbitrary topological group X, and let $G^n(X)$ denote the group of continuous mappings of the sphere S^n to X with the operation induced from the group law of X. We assume that all spaces of continuous mappings under consideration are endowed with the compact-open topology. Let $G_0{}^n(X)$ be the subgroup of all mappings homotopic to constant mappings. Clearly, $\pi_0(G_0{}^n(X)) = \pi_0(X)$. Let us introduce also analogous groups $G^n(X, e)$ and $G_0{}^n(X, e)$ for the mappings of the pair (S^n, s) into (X, e), where s is some fixed point of S^n and e is the unity of X. Note that in the definition of the latter group we could as well use free homotopies instead of homotopies of pairs. This fact follows from the possibility of shifting the image of s under the homotopy to e, which is due to the group structure of X. One has also $G^n(X) = G^n(X, e) \times G_c$, where the group of constant mappings G_c is naturally isomorphic to X. Consequently, $\pi_0(G^n) = \pi_0(G_0{}^r) + \pi_0(X)$. By definition $\pi_0(G^n(X)) = [S^n, X]$, where square brackets denote free homotopy classes, so that

$$[S^n, X] = \pi_0(G^n(X, e)) + \pi_0(X). \tag{4}$$

Using again the group structure in X it is easy to see that $\pi_0(G^n(X, e)) = \pi_n(X, e)$, which combines with (4) and yields

$$\pi_n(X, e) = [S^n, X]/\pi_0(X). \tag{5}$$

Consequently, for an arbitrary group X the problem is reduced to computation of free homotopy groups. For $X = C(S^1, S^1)$ it is fairly simple due to the remarkable property of the circle $S^1 = K(\mathbf{Z}, 1)$. One has, according to the general property of Eilenberg-McLane spaces,

$$[S^n, C(S^1, S^1)] = [S^n \times S^1, S^1] = H^1(S^n, S^1) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{for} \quad n = 1 \\ \mathbf{Z} & \text{for} \quad n \geqq 2. \end{cases} \tag{6}$$

Now, taking into account (5) and reducing (6) modulo $\pi_0(X) = \mathbf{Z}$, we get finally the relations (3) ∎

Going further let us again restrict ourselves to the case of the unitary operator in the Hilbert space, then by the well-known result of infinite dimensional topology

$$\pi_n(GC(H)) = \begin{cases} 0 & \text{for } n \text{ even} \\ \mathbf{Z} & \text{for } n \text{ odd}. \end{cases}$$

Combining this with the preceding remarks on the product structure of $GS(U)$ and with (3) we obtain immediately our main result.

Theorem 1: *Let U be a unitary operator in the complex separable Hilbert space satisfying the conditions 1—3. Then the homotopy groups $\pi_n(GS(U))$ are expressed by the relations*

$$\pi_0 = \mathbf{Z}, \qquad \pi_1 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}; \qquad \pi_{2k} = 0, \qquad \pi_{2k+1} = \mathbf{Z}, \qquad k \geqq 1. \tag{7}$$

Analogous results are valid for an arbitrary admissible invertible operator U acting in some function space of L_p-type, which is important for applications to singular operators in the spaces of integrable functions. It is also worth noting that the case of matrix singular operators with coefficients in $R(U)$ may be treated as well by the similar arguments.

We are now going to show how these results apply to singular integral operators with Cauchy kernel on closed curves. It follows from [3] that such operators have the form (2) for suitable choice of U and P, so that we have only to reproduce some definitions which are necessary for dealing with the index problem for bisingular operators.

Let K be a closed Ljapunov curve in $\mathbf{C}$. For simplicity assume that K is connected, without self-intersections (thus homeomorphic to the circle) and the bounded domain defined by K contains the origin. The canonical singular integral operator S_K is defined on the functions of Hölder classes by the formula

$$\left(S_K(f)\right)(t) = \frac{1}{\pi i} \int_K \frac{f(u)}{u-t}\, du, \quad t \in K. \tag{8}$$

It is well-known that (8) defines a bounded linear operator in the spaces $L_p(K)$ for $1 < p < \infty$, and $S_K{}^2 = I$. This enables one to introduce the projectors $P = (I + S_K)/2$ and $Q = I - P$. Recall that classical singular integral operators with continuous coefficients are defined to have the form $\hat{a}I + \hat{b}S_K + T$, where $\hat{a}, \hat{b}$ are operators of multiplying by the continuous functions $a, b \in C(K, \mathbf{C})$, $T \in C\left(L_p(K)\right)$. All such operators form an algebra $\Sigma_p(K)$ which evidently has the form $S(M)$, where M is the algebra of multiplication operators arising from continuous functions on K. In the case when K is the unit circle S^1 the algebra M coincides with $R(U)$, where U is the operator of multiplying by the independent variable: $\left(U(f)\right)(t) = tf(t)$. We see that for $K = S^1$ and $p = 2$ our theorem applies directly to get the desired conclusion.

Corollary: *The homotopy groups $\pi_n\left(\Sigma_2(S^1)\right)$ are given by the formulas* (7).

For $p \neq 2$ one of the mentioned generalizations of the theorem is applicable. For general curves K, of course, $M \neq R(U)$, but the whole scheme remains valid and one may easily obtain an analogous result.

Theorem 2: *The homotopy groups $\pi_n\left(\Sigma_p(K)\right)$ of the space of invertible singular integral operators with Cauchy kernel on a simple closed Ljapunov curve K are given by the formulas* (7).

We would also like to point out that similar results hold for curves with several components and for matrix singular operators.

3. We turn now to bisingular operators. Let K_1 and K_2 be simple closed Ljapunov curves in complex planes of independent variables z_1 and z_2. Suppose again that the point $z_j = 0$ belongs to the bounded domain defined by the curve K_j. Consider a bounded linear operator defined in the space $L_p(K_1 \times K_2)$ by the formula

$$\begin{aligned}(Af)(t_1, t_2) = {} & a_0(t_1, t_2)\, f(t_1, t_2) + \frac{1}{\pi i} \int_{K_1} \frac{a_1(t_1, u_1, t_2)}{u_1 - t_1} f(u_1, t_2)\, du_1 \\ & + \frac{1}{\pi i} \int_{K_2} \frac{a_2(t_1, t_2, u_2)}{u_2 - t_2} f(t_1, u_2)\, du_2 \\ & + \frac{1}{(\pi i)^2} \int_{K_1} \int_{K_2} \frac{a_{12}(t_1, u_1, t_2, u_2)}{(u_1 - t_1)(u_2 - t_2)} f(u_1, u_2)\, du_1\, du_2, \end{aligned} \tag{9}$$

where $a_0 \in C(K_1 \times K_2)$ and the functions a_1, a_2, a_{12} are Hölder with respect to the whole set of their variables. The minimal Banach subalgebra of $L\left(L_p(K_1 \times K_2)\right)$ containing all the operators of the form (9) is called the *algebra of bisingular operators* on $K_1 \times K_2$ [1, 2]. This algebra contains all compact operators, which enables one to apply general considerations on symbols and to introduce a quadruple of operator valued functions:

$$\begin{aligned}\left(A_1^{\pm}(t_1)\,f\right)(t_2) &= \left(a_0(t_1, t_2) \pm a_1(t_1, t_1, t_2)\right) f(t_2)\\ &\quad + \frac{1}{\pi i}\int\limits_{K_2} \frac{a_2(t_1, t_2, u_2) \pm a_{12}(t_1, t_1, t_2, u_2)}{u_2 - t_2} f(u_2)\,du_2,\\ \left(A_2^{\pm}(t_2)\,g\right)(t_1) &= \left(a_0(t_1, t_2) \pm a_2(t_1, t_2, t_2)\right) g(t_1)\\ &\quad + \frac{1}{\pi i}\int\limits_{K_1} \frac{a_1(t_1, u_1, t_2) \pm a_{12}(t_1, u_1, t_2, t_2)}{u_1 - t_1} g(u_1)\,du_1.\end{aligned} \tag{10}$$

Here $f \in L_p(K_2)$, $g \in L_p(K_1)$. Assigning such a quadruple to an operator of the form (9) one gets a homomorphism of the algebra of bisingular operators to the product of algebras of norm continuous operator-valued mappings defined on one of the given curves and taking values in singular integral operators in the L_p-space on another curve. The crucial result on bisingular operators claims that the operator (9) is Fredholm if and only if all the operators $A_j^{\pm}$ are invertible for all values of parameters [1, 2]. One may note that every family defined by (10) generates some element of the fundamental group of the space of invertible one-dimensional operators, and we'll use this observation in order to introduce a sort of numerical invariants for A. For this purpose a system of generators in $\pi_1 G(\Sigma_p(S^1)) = \mathbf{Z}^3$ should be chosen. Let us take the homotopy classes of the following one-parameter families of invertible operators: $A_1(t) = tP + Q$, $A_2(t) = P + tQ$, where P and Q are generated by S_{S^1}, and the third is given by the formula

$$\left(A_3(t)\,f\right)(z) = f(z) + \frac{t-1}{2\pi i}\int\limits_{S^1} \frac{f(u)}{u}\,du, \qquad t \in S^1.$$

This allows us to fix an isomorphism $J: \pi_1(G\Sigma_p(S^1)) \to \mathbf{Z}^3$, and using the natural identification of the spaces of singular operators on the curves K_j with the corresponding spaces for S^1 we are now able to assign to every family of the form (10) a triple of integers (J_1, J_2, J_3), which is called the *generalized* index of such a family. Our assumptions imply that for the components (10) of the symbol of bisingular operator (9) the generalized indices are well-defined, so that we have only to combine the corresponding integers in such a way as to obtain ind A. This problem admits a simplification by means of some special representation of bisingular operators developed in [1]. Namely, every operator of the form (9) may be rewritten in the form $A = A^0 A_0 + T$, where A^0 is a bisingular operator with exterior coefficients, A_0 has the symbol of the special type (with values in the Fredholm group) and T is a compact operator. Recall that an operator with exterior coefficients has the form (9) with the functions a_1, a_2, a_{12} depending only on (t_1, t_2). It is sufficient to solve our problem only for operators A^0 and A_0, as follows from additivity and compact invariance properties of the index. Both these special cases are completely analogous, so that we treat here only operators (9) with exterior coefficients.

Note first that the components of the symbol of an operator A with exterior coefficients are continuous families of coupling operators, and from the definition of the generalized index it is clear that they have J_3 equal to zero. Moreover, we have also $J(A_j^+(t_j)) = J(A_j^-(t_j))$. This may be derived from the general theory of bisingular operators [1] and explicit expressions for the introduced numerical invariants in terms of indices of special combinations of coefficients of A. Now we are able to present the final result of the paper.

Proposition 3: *The Fredholm index of a Fredholm bisingular operator* (9) *with exterior coefficients is given by the formula*

$$\operatorname{ind} A = \big(J_1(A_1^+) - J_2(A_1^+)\big)\big(J_1(A_2^+) - J_2(A_2^+)\big).$$

In order to prove this formula it is sufficient to note that both its sides have homotopy invariance property, so that one can verify it only for the simplest operator with monomial coefficients, which follows easily from explicit formulas for solutions of the corresponding homogeneous linear equation [1] ∎

In conclusion we would like to mention that similar results are valid for polysingular operators and for abstract bisingular operators defined in [1].

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