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An Existence Theorem for a Quasilinear Hyperbolic Boundary Value Problem Solved by Semidiscretization in Time

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Unter Verwendung der Rothe-Methode wird die Existenz einer Lösung eines quasilinearen hyperbolischen Randwertproblems bewiesen. Über die Koeffizienten und die rechte Seite wird dabei lediglich Integrierbarkeit vorausgesetzt. Unter diesen schwachen Voraussetzungen kann die Existenz einer Teilfolge der Rothe-Approximationen gezeigt werden, welche (in einem schwachen Sinne) gegen eine Lösung des Problems konvergiert.

С помощью метода Роте доказывается существование решения краевой задачи для квазилинейного гиперболического дифференциального уравнения. О коэффициентах и правой части уравнения при этом предполагается только интегрируемость. При указанном предположении можно доказать существование сходящейся в определённом (слабом) смысле подпоследовательности аппроксимации Роте к решению проблемы.

By the use of Rothe's method there is proved the existence of a solution of a quasilinear hyperbolic boundary value problem. On the coefficients and the right-hand side only integrability is assumed. With such weak assumptions there exists a subsequence of Rothe approximations that converges (in a weak sense) to a solution of the problem.

1. Introduction

The Rothe method developed in [6] has been used by many authors in the investigation of parabolic differential equations (e.g. [1, 3, 5]). In recent years this method has been applied to prove existence of solutions of hyperbolic problems, too [2, 5]. The principle of the Rothe method, also called semidiscretization in time, consists in discretization of the time variable, whereby the hyperbolic problem is approximated by a sequence of elliptic problems. The aim of the present paper is to prove existence of a solution of the initial-boundary value problem for a quasilinear hyperbolic differential equation system with weak assumptions on regularity of the coefficients.

In the cylinder $Q_T = G \times (0, T) \subset \mathbf{R}^2 \times \mathbf{R}^+$, $[0, T] = I$, we consider the system

$$\begin{aligned}
&-\frac{\partial}{\partial x}(A_1u_x + A_2u_y + A_3v_x + A_4v_y) - \frac{\partial}{\partial y}(A_5u_y + A_2u_x + A_6v_y + A_7v_x)\\
&+ B_{11}u_x + B_{12}u_y + B_{13}u_t + B_{14}v_x + B_{15}v_y + B_{16}v_t + \frac{\partial^2 u}{\partial t^2} = f_1,\\
&-\frac{\partial}{\partial x}(A_8v_x + A_9v_y + A_3u_x + A_7u_y) - \frac{\partial}{\partial y}(A_{10}v_y + A_9v_x + A_6u_y + A_4u_x)\\
&+ B_{21}u_x + B_{22}u_y + B_{23}u_t + B_{24}v_x + B_{25}v_y + B_{26}v_t + \frac{\partial^2 v}{\partial t^2} = f_2,
\end{aligned} \tag{1}$$

where $A_\nu = A_\nu(x, y, t)$, $B_{ik} = B_{ik}(x, y, t, u, v)$, $f_k = f_k(x, y, t, u, v)$, $\nu = 1, 2, \ldots, 10$, $i = 1, 2$, $k = 1, 2, \ldots, 6$. Additionally we prescribe initial values for $t = 0$ and a Dirichlet boundary condition on $\Gamma = \partial G \times I$. By means of complex combination $z = x + iy$, $z^* = x - iy$, $w = u + iv$ with the partial complex derivatives

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

we transform the system (1) into one complex differential equation

$$-\frac{\partial}{\partial z^*}\left[a_1 w_z + a_2 w_{z^*} + a_3 (w_z)^* + a_4 (w_{z^*})^*\right]$$
$$-\frac{\partial}{\partial z}\left[a_5 w_{z^*} + a_2^* w_z + a_6 (w_{z^*})^* + a_4 (w_z)^*\right]$$
$$+ b_1 w_z + b_2 w_{z^*} + b_3 (w_z)^* + b_4 (w_{z^*})^* + b_5 w_t + b_6 (w_t)^* + \frac{\partial^2 w}{\partial t^2} = f. \qquad (2)$$

The coefficients $a_i = a_i(z, t) \in \mathbf{C}$, where a_1, a_5 are real-valued, $b_i = b_i(z, t; w) \in \mathbf{C}$, and $f = f(z, t, w) \in \mathbf{C}$, have been derived from the coefficients and right-hand side of (1). Briefly, we also write

$$\mathcal{A}w + \mathcal{B}w + w_{tt} = f. \qquad (2)$$

The boundary and initial conditions for (2) are

$$w(z, t) = 0 \quad \text{for } (z, t) \in \Gamma^{1)}, \qquad (3)$$

$$w(z, 0) = \psi_0(z), \qquad \frac{\partial w}{\partial t}(z, 0) = \psi_1(z) \quad \text{for } z \in G. \qquad (4a, b)$$

Remark 1: The system (1) is strongly connected. The specialization of some coefficients is required to obtain an operator $\mathcal{A}$ which generates a symmetric bilinear form $\mathcal{A}(\cdot, \cdot)$. However, many physically important systems are enclosed in (1), e.g. the equations or equilibrium of dynamic elasticity theory $(\lambda + \mu)$ grad div $\vec{u} + \mu \Delta \vec{u} - \vec{u}_{tt} = \vec{f}$, $\vec{u} = (u_1, u_2)$.

2. Notations

Let $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{Q_T}$ be real inner products in the complex-valued spaces $L_2(G)$ and $L_2(Q_T)$, respectively. Then $\|\cdot\|$ and $\|\cdot\|_{Q_T}$ denote the corresponding norms. By $\mathring{W}_2^1(G)$ we denote the well-known Sobolev space obtained by the closure of the set $C_0^\infty(G)$ in the norm of $W_2^1(G)$. We norm the space $\mathring{W}_2^1(G)$ by

$$\|w\|_1 = \left(\iint_G (|w_z|^2 + |w_{z^*}|^2)\, dx\, dy\right)^{1/2}.$$

Furthermore, let $\mathcal{A}(\cdot, \cdot)$ be a real bilinear form on $\mathring{W}_2^1(G)$ obtained by integration over G of the term $\mathcal{A}w \cdot q$ and application of the Ostrogradskii-Gauss formula. We write $\mathcal{A}(\cdot, \cdot)_{Q_T}$, if an integration over the time interval I is carried out additionally. Now we consider a Banach space V with a norm $\|\cdot\|_V$ and a set of abstract functions $w(t): I \to V$.

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1) The homogeneous boundary condition (3) is formulated without loss of generality, because inhomogeneous boundary values with sufficient regularity may be transformed into (3).

Definition 1: We denote by $L_p(I, V)$, $1 \leqq p \leqq \infty$, the set of all Bochner measurable abstract functions $w: I \to V$ such that

$$\|w\|_{L_p(I,V)} = \left(\int_0^T \|w(t)\|_V{}^p \, dt\right)^{1/p} \quad \text{for} \quad 1 \leqq p < \infty,$$

$$\|w\|_{L_\infty(I,V)} = \sup_{t \in I} \text{ess} \, \|w(t)\|_V \quad \text{for} \quad p = \infty.$$

In $L_\infty(I, V)$ the weak* convergence, denoted by $\overset{*}{\rightharpoonup}$, is used.

Definition 2: $C(I, V)$ denotes the set of all continuous functions $w: I \to V$ with the norm

$$\|w\|_{C(I,V)} = \max_{t \in I} \|w(t)\|_V;$$

$C^{0,1}(I, V)$ is the subset of Lipschitz continuous functions with

$$\|w(t) - w(t')\|_V \leqq L \cdot |t - t'| \quad \text{for} \quad t \in I.$$

Detailed information about these spaces of abstract functions can be found in [7, 8].

Now let a subdivision of Q_T by hyperplanes $t = t_j = j \cdot h$, $h = T/n$, $j = 0, 1, \ldots, n$, be given. Furthermore, let $g_j(z)$ be given functions on $G \times \{t_j\}$ or restrictions $g_j(z) = g(z, t_j)$ of with respect to t continuous functions on Q_T, respectively.

Definition 3: We denote

$$\Delta g_j = g_j - g_{j-1}, \qquad \Delta^2 g_j = \Delta(\Delta g_j),$$

$$\bar{g}^n(z, t) = g_j(z) \quad \text{for} \quad t \in (t_{j-1}, t_j],$$

$$\tilde{g}^n(z, t) = \frac{t_j - t}{h} g_{j-1}(z) + \frac{t - t_{j-1}}{h} g_j(z) \quad \text{for} \quad t \in [t_{j-1}, t_j]$$

($\bar{g}^n$, $\tilde{g}^n$ — piecewise constant and piecewise linear interpolations with respect to t, resp.). Moreover, let

$$\tau_h \bar{g}^n = \bar{g}^n(z, t - h), \qquad \tau_h{}^+ \bar{g}^n = \bar{g}^n(z, t + h) \quad \text{for} \quad t \in I,$$

$$\Delta_h \bar{g}^n = \bar{g}^n - \tau_h \bar{g}^n.$$

If there is no other specification, we set $g(z, t) = g(z, 0) = g_0(z)$ for $t < 0$. For the sake of simplicity we use the abbreviations $I_j = (t_{j-1}, t_j]$, $Q_j = G \times I_j$, and write $g_{jz} = \partial g_j/\partial z$, $g_{jz^*} = \partial g_j/\partial z^*$.

3. Assumptions and discretization

Throughout this paper we impose the following assumptions on the problem (2)—(4):

(i) Let $G \subset \mathbf{C}$ be a simply connected, bounded domain; $\partial G \in C^{0,1}$.

(ii) $\psi_0, \psi_1 \in \mathring{W}_2{}^1(G)$.

(iii) Let

$$a_i \in L_1\big(I, L_\infty(G)\big), \quad \int_0^T \frac{1}{\delta} \|a_i(\cdot, t + \delta) - a_i(\cdot, t)\|_{L_\infty(G)} \, dt \leqq M \tag{5}$$

for all δ, $|\delta| \leqq \delta_0$, where $a_i(z, t) = 0$ for $t \notin I$. Further we assume: *Caratheodory condition*: $b_i(z, t, w)$ and $f(z, t, w)$ are measurable on Q_T for all $w \in \mathbf{C}$ and continuous in w

for a.a. $(z, t) \in Q_T$. *Growth limitation:*

$$|b_i(z, t, w)| \leqq B(z, t), \qquad B \in L_1(I, L_\infty(G)), \tag{6}$$

$$|f(z, t, w)| \leqq K \cdot (F(z, t) + |w|), \qquad F \in L_1(I, L_2(G)), \tag{7}$$

for all $w \in \mathbf{C}$, a.a. $(z, t) \in Q_T$, $i = 1, 2, \ldots, 6$.

(iv) For any $p_1, p_2 \in \mathbf{C}$ let the condition

$$\begin{aligned} &\operatorname{Re}\left[(a_1p_1 + a_2p_2 + a_3p_1^* + a_4p_2^*)\, p_1^*\right. \\ &\left.+ (a_5p_2 + a_2^*p_1 + a_6p_2^* + a_4p_1^*)\, p_2^*\right] \\ &\geqq a \cdot (|p_1|^2 + |p_2|^2), \qquad a > 0, \end{aligned}$$

be fulfilled a.e. in Q_T.

Remark 2: There is no assumption on continuity of a_i, b_i, and f. In particular, condition (5) permits jumps with respect to t of the coefficients a_i. (5) does not imply the existence of weak derivatives $\partial a_i/\partial t$.

Remark 3: Assumption (iv) implies strong ellipticity of the operator $\mathcal{A}$. For the example in Remark 1 this condition is fulfilled with $a = \lambda > 0$.

We now divide the time intervall $I = [0, T]$ by equidistant points $t_j = j \cdot h$, $j = 0, \ldots, n$, into $n \in \mathbf{N}$ subintervals. By semidiscretization of (2)—(4) we obtain $n - 1$ elliptic boundary value problems

$$\mathcal{A}_j w_j + \mathcal{B}_j w_j + \frac{1}{h^2}(w_j - 2w_{j-1} + w_{j-2}) = f_j \quad \text{in } G, \tag*{$(2)_j$}$$

$$w_j = 0 \quad \text{on } \partial G, \tag*{$(3)_j$}$$

$j = 2, 3, \ldots, n$, with the start condition

$$w_0 = \psi_0, \qquad w_1 = \psi_0 + h\psi_1. \tag*{$(4)_0$}$$

Here the operator $\mathcal{A}_j$ is obtained from $\mathcal{A}$ by replacing the coefficients a_i by a_{ij}, $\mathcal{B}_j$ is defined as

$$\mathcal{B}_j w_j = b_{1j}w_{jz} + b_{2j}w_{jz^*} + b_{3j}(w_{jz})^* + b_{4j}(w_{jz^*})^* + \frac{1}{h}(b_{5j}\Delta w_j + b_{6j}\Delta w_j^*),$$

where

$$a_{ij}(z) = \frac{1}{h}\int_{I_j} a_i(z, t)\, dt, \qquad b_{ij}(z) = \frac{1}{h}\int_{I_j} b_i(z, t, w_{j-1}(z))\, dt,$$

$$f_j(z) = \frac{1}{h}\int_{I_j} f(z, t, w_{j-1}(z))\, dt, \qquad i = 1, 2, \ldots, 6, \quad j = 2, 3, \ldots, n.$$

This kind of discretization allows the renouncement of continuity with respect to t.

Lemma 1: *For $w_{j-1} \in L_2(G)$ and $h > 0$ holds $a_{ij} \in L_\infty(G)$, $b_{ij} \in L_\infty(G)$, $f_j \in L_2(G)$, $i = 1, 2, \ldots, 6$, $j = 2, 3, \ldots, n$.*

Proof: For a Bochner integrable function $u: I \to V$ the integral $\int_{I_j} u(t)\, dt$ exists and belongs to V. Owing to (iii), therefore we have $a_{ij} \in L_\infty(G)$, $b_{ij} \in L_\infty(G)$, and

$f_j \in L_2(G)$. In particular, the relation

$$\left\| \int_{I_j} u(t)\, dt \right\|_V \leqq \int_{I_j} \|u(t)\|_V \, dt$$

yields

$$\|f_j\| \leqq \frac{K}{h} \int_{I_j} \|F(\cdot, t)\| \, dt + K \, \|w_{j-1}\| \tag{8}$$

(cf. [7]) ∎

Moreover it is possible to choose $h_0 > 0$ such that

$$\frac{1}{h^2} - \frac{1}{h} \left(\|b_{5j}\|_{L_\infty(G)} + \|b_{6j}\|_{L_\infty(G)}\right) \geqq 0 \quad \text{for} \quad h \leqq h_0, \qquad j = 2, 3, \ldots, n.$$

This follows immediately from

Lemma 2: *For every $\varepsilon > 0$ there exists $h_0(\varepsilon) > 0$ such that*

$$\int_{I_j} \|B(\cdot, t)\|_{L_\infty(G)} \, dt < \varepsilon \quad for \quad h \leqq h_0, \quad j = 1, 2, \ldots, n.$$

Proof: We consider the function $g = \|B\|_{L_\infty(G)} \in L_1(I)$, $g = 0$ for $t < 0$. For $\varepsilon > 0$ and every point $t \in I$ there exists an interval

$$I_{\delta,t} = (t - \delta, t) \quad \text{with} \quad \int_{t-\delta}^{t} g \, dt < \frac{\varepsilon}{2}.$$

The set of intervals $I_{\delta,t}$ is an open covering of the compact interval I. Hence, by the Borel theorem, I is covered by a finite number of intervals $I_{\delta,t}$ of length $\delta \geqq \delta_{\min}$. For $h_0(\varepsilon) < \delta_{\min}$ and every I_j there are two of these intervals $I_{\delta,t}$ with $\delta \geqq \delta_{\min}$ which cover I_j. Therefore we have $\int_{I_j} g \, dt < \varepsilon$ ∎

Owing to assumption (iv) and Lemma 2, for a given $w_{j-1} \in L_2(G)$, the linear elliptic boundary value problem $(2)_j$, $(3)_j$ has a unique solution $w_j \in \mathring{W}_2{}^1(G)$ [8: Theorem 25.3], which satisfies the integral relation

$$\mathcal{A}_j(w_j, \varphi) + \langle \mathcal{B}_j w_j, \varphi \rangle + \frac{1}{h^2} \langle \Delta^2 w_j, \varphi \rangle = \langle f_j, \varphi \rangle \quad \text{for} \quad \varphi \in \mathring{W}_2{}^1(G). \tag{9}$$

In this relation we have

$$\mathcal{A}_j(w, \varphi) = \operatorname{Re} \iint_G [a_{1j} w_z + a_{2j} w_{z^*} + a_{3j} (w_z)^* + a_{4j} (w_{z^*})^*] \cdot (\varphi_z)^* \, dx \, dy$$
$$+ \operatorname{Re} \iint_G [a_{5j} w_{z^*} + a_{2j}^* w_z + a_{6j} (w_{z^*})^* + a_{4j} (w_z)^*] \cdot (\varphi_{z^*})^* \, dx \, dy$$

for $w, \varphi \in \mathring{W}_2{}^1(G)$.

Starting with w_0 and w_1 from $(4)_0$ the functions $w_2, w_3, \ldots, w_n$ may be calculated successively. By piecewise linear and piecewise constant interpolation in t, respectively, we obtain the approximates $\tilde{w}^n(z, t)$ and $\overline{w}^n(z, t)$, respectively (Def. 3). Thus the quasilinear hyperbolic problem has been approximated by a sequence of linear elliptic problems.

4. A priori estimates

First we formulate a version of Gronwall's lemma.

Lemma 3: *Let $u \in L_\infty(I)$, $g \in L_1(I)$ be real functions, where $g \geqq 0$ on I and $c \in \mathbf{R}$ is a constant. If the inequality*

$$u(t) \leqq c + \int_0^t g(s)\, u(s)\, ds \tag{10}$$

holds for all $t \in I$, then the estimate

$$u(t) \leqq c \cdot \exp\left\{\int_0^t g(s)\, ds\right\}$$

is valid for all $t \in I$.

Proof: Since g in $L_1(I)$ can be approximated by nonnegative continuous functions it is enough to prove the assertion for $g \in C(I)$. By means of

$$\int_0^t g(s)\left[\int_0^s g(s_1)\, ds_1\right]^k ds = \frac{1}{k+1}\left[\int_0^t g(s)\, ds\right]^{k+1}$$

it follows from (10) by induction that

$$u(t) \leqq c \sum_{k=0}^n \frac{1}{k!} \cdot \left(\int_0^t g(s)\, ds\right)^k + R_{n+1}(t)$$

holds, where

$$R_{n+1}(t) = \int_0^t g(s_1) \int_0^{s_1} g(s_2) \ldots \int_0^{s_n} g(s_{n+1})\, u(s_{n+1})\, ds_{n+1} \ldots ds_1.$$

Denoting $M = \operatorname{sup\,ess} \{|u(t)| : t \in I\}$ we get

$$|R_{n+1}(t)| \leqq \frac{M}{(n+1)!}\left(\int_0^T g(s)\, ds\right)^{n+1} \quad \text{for} \quad t \in I.$$

Consequently, because $R_{n+1}(t) \to 0$ for $n \to \infty$ uniformly on I, the assertion follows by passing to the limit $n \to \infty$ ∎

Lemma 4: *The expression $\mathcal{A}_j(\cdot,\cdot)$, $j = 1, 2, \ldots, n$, define real bilinear forms on $\mathring{W}_2^1(G)$ with the properties*

a) *$a\,\|w\|_1^2 \leqq \mathcal{A}_j(w, w)$ for $w \in \mathring{W}_2^1(G)$ (positivity),*
b) *$\mathcal{A}_j(w, q) = \mathcal{A}_j(q, w)$ for $w, q \in \mathring{W}_2^1(G)$ (symmetry).*

Proof: The bilinearity is obvious. Assertion a) is a consequence of assumption (iv), and assertion b) can be proved by an elementary computation since $\mathcal{A}_j(\cdot,\cdot)$, a_{1j}, a_{5j} are real quantities (i.e. $a_{1j} = a_{1j}^*, \ldots$) ∎

Furthermore we need an identity. As is easily seen, for a symmetric bilinear form $(\cdot, \cdot)$ holds

$$2(\Delta w_j, w_j) = (w_j, w_j) - (w_{j-1}, w_{j-1}) + (\Delta w_j, \Delta w_j). \tag{11}$$

We shall now show the boundedness of all w_j independent of the chosen subdivision.

Lemma 5: *There exist constants M_1, M_2 independent of t_j and n such that the estimates*

$$\|w_j\|_1 \leqq M_1, \quad \left\|\frac{\Delta w_j}{h}\right\| \leqq M_2$$

hold for all $h \leqq h_0$, $j = 1, 2, \ldots, n$.

Proof: We substitute the test function in (9) by $\varphi = \Delta w_j$ and obtain with $q_j := (1/h)\,\Delta w_j$

$$\mathcal{A}_j(w_j, \Delta w_j) + \langle \Delta q_j, q_j \rangle = h \cdot (\langle f_j, q_j \rangle - \langle \mathcal{B}_j w_j, q_j \rangle).$$

Due to (11) we can estimate

$$\mathcal{A}_j(w_j, w_j) - \mathcal{A}_j(w_{j-1}, w_{j-1}) + \|q_j\|^2 - \|q_{j-1}\|^2$$
$$\leqq 2h \cdot (\langle f_j, q_j \rangle - \langle \mathcal{B}_j w_j, q_j \rangle),$$

and

$$\mathcal{A}_j(w_j, w_j) - \mathcal{A}_{j-1}(w_{j-1}, w_{j-1}) + \|q_j\|^2 - \|q_{j-1}\|^2$$
$$\leqq [\mathcal{A}_j(w_{j-1}, w_{j-1}) - \mathcal{A}_{j-1}(w_{j-1}, w_{j-1})] + 2h(\|f_j\| \cdot \|q_j\| + \|\mathcal{B}_j w_j\| \cdot \|q_j\|).$$

Summation over $j = 2, 3, \ldots, p$ and application of (8) yields

$$\mathcal{A}_p(w_p, w_p) + \|q_p\|^2$$
$$\leqq \mathcal{A}_1(w_1, w_1) + \|q_1\|^2 + \sum_{j=2}^{p} [\mathcal{A}_j(w_{j-1}, w_{j-1}) - \mathcal{A}_{j-1}(w_{j-1}, w_{j-1})]$$
$$+ \sum_{j=2}^{p} \left(2K \int_{I_j} \|F\|\, dt \cdot \|q_j\| + 2Kh\, \|w_{j-1}\| \cdot \|q_j\| + 2h\, \|\mathcal{B}_j w_j\| \cdot \|q_j\|\right). \tag{12}$$

Since we have $B(\cdot, t) \cdot w_j \in L_2(G)$, $B : w_j \in L_1(I_j, L_2(G))$, it is possible to estimate

$$\|\mathcal{B}_j w_j\| \leqq \frac{1}{h} \int_{I_j} \|b_1 w_{jz} + \cdots + b_6 q_j^*\|\, dt$$
$$\leqq \frac{1}{h} \int_{I_j} 2\, \|B(\cdot, t)\|_{L_\infty(G)}\, dt \cdot \left(\sqrt{2}\, \|w_j\|_1 + \|q_j\|\right),$$

$$2h\, \|\mathcal{B}_j w_j\| \cdot \|q_j\| \leqq \int_{I_j} \|B(\cdot, t)\|_{L_\infty(G)}\, dt \cdot (4\, \|w_j\|_1^2 + 5\, \|q_j\|^2).$$

The first items in (12) may be estimated by the use of the start condition $(4)_0$:

$$\begin{aligned}
&\mathcal{A}_1(w_1, w_1) + \|q_1\|^2 \\
&= \mathcal{A}_1(\psi_0, \psi_0) + 2h \cdot \mathcal{A}_1(\psi_0, \psi_1) + h^2\mathcal{A}_1(\psi_1, \psi_1) + \|\psi_1\|^2 \\
&\leqq \frac{1}{h} \iint\limits_G \left[\int\limits_{I_1} \left((a_1\psi_{0z} + \cdots) (\psi_{0z})^* + (a_5\psi_{0z^*} + \cdots) (\psi_{0z^*})^*\right) dt \right] dx\, dy \\
&\quad + 4 \cdot \max_{i=1,2,\ldots,6} \int\limits_{I_1} \|a_i(\cdot, t)\|_{L_\infty(G)}\, dt \cdot (\|\psi_0\|_1^2 + \|\psi_1\|_1^2) \\
&\quad + 4h \cdot \max_{i=1,2,\ldots,6} \int\limits_{I_1} \|a_i(\cdot, t)\|_{L_\infty(G)}\, dt \cdot \|\psi_1\|_1^2 + \|\psi_1\|^2 \\
&\leqq \iint\limits_{Q_1} \left[\left(\frac{a_1 - \tau_h a_1}{h} \psi_{0z} + \cdots\right) (\psi_{0z})^* \right. \\
&\quad \left. + \left(\frac{a_5 - \tau_h a_5}{h} \psi_{0z^*} + \cdots\right) (\psi_{0z^*})^* \right] dx\, dy\, dt + c_1 \\
&\leqq \max_{i=1,2,\ldots,6} \int\limits_{I_1} \frac{1}{h} \|a_i(\cdot, t) - \tau_h a_i(\cdot, t)\|_{L_\infty(G)}\, dt \cdot 4\, \|\psi_0\|_1^2 + c_1 \leqq c_2.
\end{aligned}$$

This is true since the coefficients a_i satisfy $a_i \equiv 0$ for $t < 0$ and (5). Applying (iv) thus we can continue in (12)

$$\begin{aligned}
&a\, \|w_p\|_1^2 + \|q_p\|^2 \leqq c_2 + \sum_{j=2}^{p} [\mathcal{A}_j(w_{j-1}, w_{j-1}) - \mathcal{A}_{j-1}(w_{j-1}, w_{j-1})] \\
&+ \sum_{j=2}^{p} \left[2K \int\limits_{I_j} \|F\|\, dt \cdot \max_{j=2,3,\ldots,p} \|q_j\| + Kh(\|w_{j-1}\|^2 + \|q_j\|^2) \right. \\
&\left. + \int\limits_{I_j} \|B\|_{L_\infty(G)}\, dt (4\, \|w_j\|_1^2 + 5\, \|q_j\|^2) \right].
\end{aligned}$$

As proved in Lemma 2 the term $\int\limits_{I_j} \|B\|_{L_\infty(G)}\, dt$ can be made arbitrarily small for all $h \leqq h_0$ if h_0 is sufficient small. Now we choose $h_0 = \min \{K/4, h_0(a/8), h_0(1/20)\}$ and obtain, by subtracting the terms with $\|w_p\|_1^2$, $\|q_p\|^2$,

$$\begin{aligned}
&\frac{a}{2}\, \|w_p\|_1^2 + \frac{1}{4}\, \|q_p\|^2 \leqq c_2 + \sum_{j=2}^{p} [\mathcal{A}_j(w_{j-1}, w_{j-1}) - \mathcal{A}_{j-1}(w_{j-1}, w_{j-1})] \\
&+ \max_{j=1,2,\ldots,n} \|q_j\| \cdot 2K\, \|F\|_{L_1(I, L_2(G))} + \sum_{j=1}^{p-1} Kh\, \|w_j\|^2 + \sum_{j=2}^{p-1} Kh\, \|q_j\|^2 \\
&+ \sum_{j=2}^{p-1} \int\limits_{I_j} \|B\|_{L_\infty(G)}\, dt \cdot (4\, \|w_j\|_1^2 + 5\, \|q_j\|^2).
\end{aligned}$$

The Friedrichs inequality for $w_j \in \mathring{W}_2^1(G)$ yields $\|w_j\|^2 \leqq C \cdot \|w_j\|_1^2$, and by further estimation we get

$$2a\, \|w_p\|_1^2 + \|q_p\|^2 \leqq c_3 + c_4 \cdot \max_{j=1,2,\ldots,n} \|q_j\|$$

$$+ 4 \sum_{j=1}^{p-1} [\mathcal{A}_{j+1}(w_j, w_j) - \mathcal{A}_j(w_j, w_j)] + \sum_{j=1}^{p-1} 4Kh(C \|w_j\|_1^2 + \|q_j\|^2)$$

$$+ \sum_{j=1}^{p-1} \int_{I_j} \|B(\cdot, t)\|_{L_\infty(G)} \, dt (16 \|w_j\|_1^2 + 20 \|q_j\|^2).$$

In other notation (see Def. 3) this is equivalent to

$$2a \|\overline{w}^n(\cdot, \hat{t})\|_1^2 + \|\overline{q}^n(\cdot, \hat{t})\|^2 \leqq c_3 + c_4 \sup_{t \in I} \|\overline{q}^n\|$$

$$+ \frac{4}{h} \int_0^{t_{p-1}} \left\{ \iint_G \left[\left((\tau_h{}^+ a_1 - a_1) \, \overline{w}_z{}^n + \cdots \right) (\overline{w}_z{}^n)^* \right. \right.$$

$$\left. \left. + \left((\tau_h{}^+ a_5 - a_5) \, \overline{w}^n_{\bar{z}*} + \cdots \right) (\overline{w}^n_{\bar{z}*})^* \right] dx \, dy \right\} dt$$

$$+ \int_0^{t_{p-1}} [4K(C \|\overline{w}^n\|_1^2 + \|\overline{q}^n\|^2) + \|B(\cdot, t)\|_{L_\infty(G)} (16 \|\overline{w}^n\|_1^2 + 20 \|\overline{q}^n\|^2)] \, dt$$

for $\hat{t} \in I_p$. Hence, we obtain

$$2a \|\overline{w}^n(\cdot, \hat{t})\|_1^2 + \|\overline{q}^n(\cdot, \hat{t})\|^2 \leqq c_3 + c_4 \cdot \sup_{t \in I} \|\overline{q}^n(\cdot, t)\|$$

$$+ \int_0^{\hat{t}} \left[c_5 \cdot \max_{i=1,2,\ldots,6} \frac{1}{h} \|\tau_h{}^+ a_i - a_i\|_{L_\infty(G)} + c_6 + c_7 \|B(\cdot, t)\|_{L_\infty(G)} \right]$$

$$\times (2a \|\overline{w}^n(\cdot, t)\|_1^2 + \|\overline{q}^n(\cdot, t)\|^2 \, dt$$

for all $\hat{t} \in I$ since p, $2 \leqq p \leqq n$, is an arbitrary number (it is obvious that such an inequality holds for $\hat{t} \in [0, h]$, too). On this inequality we can apply Lemma 3, which implies, due to (5) and (6),

$$2a \|\overline{w}^n(\cdot, t)\|_1^2 + \|\overline{q}^n(\cdot, t)\|^2 \leqq \left(c_3 + c_4 \cdot \sup_{t \in I} \|\overline{q}^n(\cdot, t)\| \right) e^{c_8}$$

for $t \in I$, $h \leqq h_0$, that means

$$2a \cdot \sup_{t \in I} \|\overline{w}^n\|_1^2 + \sup_{t \in I} \|\overline{q}^n\|^2 \leqq c_9 + c_{10} \sup_{t \in I} \|\overline{q}^n\|.$$

The Young inequality $2ab \leqq a^2/\varepsilon + \varepsilon b^2$ completes the proof with

$$2a \cdot \sup_{t \in I} \|\overline{w}^n\|_1^2 + \frac{1}{2} \sup_{t \in I} \|\overline{q}^n\|^2 \leqq c_9 + \frac{1}{2} c_{10}^2 \quad \text{for} \quad h \leqq h_0 \; ∎$$

The existence theorem in the next section is based essentially upon this lemma.

5. Existence of a solution

The assertion of Lemma 5 means

$$\sup_{t \in I} \|\tilde{w}^n(\cdot, t)\|_1 \leqq M_1, \qquad \sup_{t \in I} \|\overline{w}^n(\cdot, t)\|_1 \leqq M_1, \tag{13}$$

$$\sup_{t \in I} \left\| \frac{\partial^- \tilde{w}^n}{\partial t} (\cdot, t) \right\| \leqq M_2. \tag{14}$$

Moreover, the estimate $\|\Delta w_j\| \leqq M_2 h$ yields a Lipschitz condition

$$\|\tilde{w}^n(\cdot, t) - \tilde{w}^n(\cdot, t')\| \leqq M_2 \, |t - t'| \quad \text{for} \quad t, t' \in I \tag{15}$$

and the property

$$\sup_{t \in I} \|\tilde{w}^n - \overline{w}^n\| \leqq M_2 h, \qquad \sup_{t \in I} \|\tilde{w}^n - \tau_h \overline{w}^n\| \leqq M_2 h. \tag{16}$$

We can now prove the existence theorem.

Theorem: *Let the assumptions* (i)—(iv) *be fulfilled.*

a) *The problem* (2)—(4) *has a weak solution*

$$w \in C^{0,1}\big(I, L_2(G)\big) \cap L_\infty\big(I, \mathring{W}_2^1(G)\big), \quad \frac{\partial w}{\partial t} \in L_\infty\big(I, L_2(G)\big),$$

satisfying the integral relation

$$\mathcal{A}(w, \varphi)_{Q_T} + \langle \mathcal{B} w, \varphi \rangle_{Q_T} - \langle w_t, \varphi_t \rangle_{Q_T} - \langle \psi_1, \varphi(\cdot, 0) \rangle = \langle f, \varphi \rangle_{Q_T}, \tag{17}$$

for $\varphi \in C_0^\infty\big(G \times [0, T)\big)$;

b) *There exists a subsequence* $\{n_k\}$ *of subdivisions of* $I = [0, T]$ *such that the approximate functions calculated from* $(2)_j$, $(3)_j$, $(4)_0$ *have the convergence properties*

$$\tilde{w}^{n_k}, \overline{w}^{n_k} \to w \quad \text{in the norm of } C\big(I, L_2(G)\big), \tag{18}$$

$$\tilde{w}^{n_k}, \overline{w}^{n_k} \rightharpoonup w \quad \text{in } L_\infty\big(I, \mathring{W}_2^1(G)\big), \tag{19}$$

$$\frac{\partial \tilde{w}^{n_k}}{\partial t} \rightharpoonup \frac{\partial w}{\partial t} \quad \text{in } L_\infty\big(I, L_2(G)\big) \tag{20}$$

for $k \to \infty$.

Proof: b) Due to (13) the functions $\tilde{w}^n(\cdot, t)$ are uniformly bounded in $\mathring{W}_2^1(G)$ for all $t \in I$, $n \in \mathbf{N}$. The set of bounded functions in $\mathring{W}_2^1(G)$ is a compact set in $L_2(G)$. Furthermore we have the uniform Lipschitz condition (15) for $t \in I$, $n \in \mathbf{N}$. Thus, by the diagonal method of Arselá-Ascoli, it is possible to choose a subsequence $\{\tilde{w}^{n_k}\}$ with $\tilde{w}^{n_k} \to w$ in $C\big(I, L_2(G)\big)$ (for details see [4, 7]). Together with (16) this yields (18).

For simplicity, in the following let us denote all further subsequences of $\{n_k\}$ by $\{n_k\}$ again. As a consequence of boundedness (13) of $\tilde{w}^{n_k}, \overline{w}^{n_k}$ in $L_\infty\big(I, \mathring{W}_2^1(G)\big)$ there exists a subsequence with $\tilde{w}^{n_k}, \overline{w}^{n_k} \rightharpoonup w$ in $L_\infty\big(I, \mathring{W}_2^1(G)\big)$. The coincidence of these limits with the limit in (18) results from $C\big(I, L_2(G)\big) \subset L_\infty\big(I, \mathring{W}_2^1(G)\big)$ and the uniqueness of a weak limit. Thus we have proved (19).

Formula (14) implies the existence of a subsequence such that $\partial \tilde{w}^{n_k}/\partial t \rightharpoonup q$ in $L_\infty\big(I, L_2(G)\big)$. By a limit process $k \to \infty$ the integral identity

$$\iint_{Q_T} \left(\frac{\partial \tilde{w}^{n_k}}{\partial t} \cdot \varphi + \tilde{w}^{n_k} \cdot \frac{\partial \varphi}{\partial t} \right) dx\, dy\, dt = 0 \quad \text{for} \quad \varphi \in C_0^\infty(Q_T)$$

yields

$$\iint_{Q_T} \left(q \cdot \varphi + w \cdot \frac{\partial \varphi}{\partial t} \right) dx\, dy\, dt = 0 \quad \text{for} \quad \varphi \in C_0^\infty(Q_T).$$

In this way we get $q = \partial w/\partial t$ and thus (20).

a) The regularity assertions on w result from completeness properties of the spaces $C(I, L_2(G))$, $L_\infty(I, L_2(G))$, and $L_\infty(I, \mathring{W}_2^1(G))$, as well as from Lipschitz condition (15). It remains to show that w satisfies the integral relation (17).

We choose an arbitrary but fixed test function $\varphi \in C_0^\infty(G \times [0, T))$. For the n-th subdivision of Q_T this function generates a step function $\bar{\varphi}^n(z, t)$ (see Def. 3). Then, by summation of (9) for $j = 2, 3, \ldots, n$ with $\varphi = \varphi_j$, we get in equivalent notation

$$\int_h^T \mathcal{A}(\overline{w}^n, \bar{\varphi}^n)\, dt + \int_h^T \langle \mathcal{B}^n \overline{w}^n, \bar{\varphi}^n \rangle\, dt + \frac{1}{h} \int_h^T \left\langle \Delta_h \frac{\partial^- \tilde{w}^n}{\partial t}, \bar{\varphi}^n \right\rangle dt = \int_h^T \langle f^n, \bar{\varphi}^n \rangle\, dt. \tag{21}$$

Here the coefficients in $\mathcal{B}^n$ are $b_i^n = b_i(z, t, \tau_h \overline{w}^n)$; analogously $f^n = f(z, t, \tau_h \overline{w}^n)$. Defining $\bar{\varphi}^n(z, t) \equiv 0$ for $t \geq T$ we have the identity

$$\int_h^T \frac{1}{h} \left\langle \Delta_h \frac{\partial^- \tilde{w}^n}{\partial t}, \bar{\varphi}^n \right\rangle dt = -\int_h^T \left\langle \frac{\partial^- \tilde{w}^n}{\partial t}, \tau_h^+ \frac{\partial^- \tilde{\varphi}^n}{\partial t} \right\rangle dt - \langle \psi_1, \varphi(\cdot, 2h) \rangle.$$

Furthermore, owing to the estimates of $\overline{w}^n$, $\tilde{w}^n$ and Lemma 2, we change (21) only by a quantity $\varrho(h)$ with $\lim \varrho(h) = 0$ for $h \to 0$ if we integrate over the whole interval I. Thus, we have

$$\begin{aligned} &\mathcal{A}(\overline{w}^n, \bar{\varphi}^n)_{Q_T} + \langle \mathcal{B}^n \overline{w}^n, \bar{\varphi}^n \rangle_{Q_T} - \left\langle \frac{\partial \tilde{w}^n}{\partial t}, \tau_h^+ \frac{\partial \tilde{\varphi}^n}{\partial t} \right\rangle_{Q_T} - \langle \psi_1, \varphi(\cdot, 2h) \rangle \\ &= \langle f^n, \bar{\varphi}^n \rangle_{Q_T} + \varrho(h). \end{aligned} \tag{22}$$

As a consequence of assumption (iii) the functions $b_i(z, t, w)$ and $f(z, t, w)$ define continuous and bounded Nemyzki operators mapping $L_\infty(I, L_2(G))$ into $L_1(I, L_2(G))$ (cf. [7, 8]). Due to (16) and (18) the convergence $\tau_h \overline{w}^{n_k} \to w$ in $L_\infty(I, L_2(G))$ takes place and, consequently,

$$b_i^{n_k}, f^{n_k} \to b_i(z, t, w), f(z, t, w) \quad \text{in} \quad L_1(I, L_2(G)) \quad \text{for } k \to \infty. \tag{23}$$

The approximations of φ converge uniformly on Q_T to φ and the derivatives of φ, respectively. Passing to the limit $n_k \to \infty$ in (22) the already proved convergence properties (18)—(20) together with (23) thus yield the integral relation (17). Since the solution w belongs to $L_\infty(I, \mathring{W}_2^1(G))$ it satisfies the boundary condition (3) in the sense of traces. The initial condition (4a) is satisfied due to $\tilde{w}^{n_k}(\cdot, 0) = w_0 = \psi_0$ for every n_k and (18). The initial condition (4b) is fulfilled in a weak sense defined by the test relation (17) ∎

Remark 4: The space of test functions in (17) may be a Sobolev space, too. One can also show that w belongs to the space $C_{\mathcal{W}}(I, \mathring{W}_2^1(G))$ of weakly continuous functions $I \to \mathring{W}_2^1(G)$ (see [4]). The weak assumptions (i)—(iv) do not ensure uniqueness of the solution w.

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