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# Non-Negative Trigonometric Polynomials with Constraints

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Es werden Extremalprobleme für nicht-negative trigonometrische Polynome mit vorgegebenen Nullstellen behandelt. In Anwendung des allgemeinen Satzes wird als Verschärfung eines Ergebnisses von Fejér das Wachstum eines solchen Polynoms in der Nähe einer Nullstelle diskutiert. Eine weitere Anwendung betrifft das Koeffizientenproblem für typisch reelle (algebraische) Polynome.

Рассматриваются экстремальные задачи для неотрицательных тригонометрических многочленов, обладающих заданными нулями. Применением общей теоремы работы получается усиление результата Фейера о росте такого многочлена в окрестности нуля. Другое применение касается оценки коэффициентов в случае типично вещественных (алгебраических) многочленов.

We discuss extremal problems for non-negative trigonometric polynomials with prescribed zeros. The general result is used to refine a former theorem of Fejér concerning upper bounds of those polynomials near to a zero. Another application deals with the coefficient problem for typically real (algebraic) polynomials.

## 1. Introduction

A real trigonometric polynomial $t$ of degree $n$ is *non-negative* if and only if it has a representation

$$t(\theta) = \operatorname{Re} p(e^{i\theta})$$

where

$$p(z) = \sum_{k=0}^{n} p_k z^k \text{ with } p_0 \in \mathbb{R}, \qquad \operatorname{Re} p(z) \geqq 0, \qquad |z| \leqq 1.$$

Let $\mathcal{R}_n$ denote the class of such polynomials $p$ and let $M_n = \mathbb{R} \times \mathbb{C}^n$. With every $p \in \mathcal{R}_n$ we assign the coefficient vector $\mathbf{p} = (p_0, \dots, p_n) \in M_n$. Let $\mathbf{c} \in M_n$. In the present note we are interested in estimates for linear functionals like $\operatorname{Re} \mathbf{c} \cdot \mathbf{p}$ for $p$ in $\mathcal{R}_n$ or suitable subsets of $\mathcal{R}_n$. It is known since long that such problems for the whole of $\mathcal{R}_n$ are closely related to the eigenvalues of the Toeplitz matrix

$$\mathbf{C} := \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ \overline{c_1} & c_0 & \cdots & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{c_n} & \overline{c_{n-1}} & \cdots & c_0 \end{pmatrix}.$$

The following elegant theorem is due to Szász [7] and has found numerous applications.

Theorem A: *Let $\lambda_{\min}$, $\lambda_{\max}$ denote the smallest and the greatest eigenvalue of* $\mathbf{C}$. *Then for $p \in \mathcal{R}_n$ we have*

$$p(0)\,\lambda_{\min} \leqq \operatorname{Re} \mathbf{c}\cdot\mathbf{p} \leqq p(0)\,\lambda_{\max} \tag{1}$$

*and these bounds are best possible for every $\mathbf{c} \in M_n$.*

Thus the estimation of those linear functionals is reduced to the solution of algebraic equations. We shall deduce similar results for $p \in \mathcal{R}_n$ which have zeros of $\operatorname{Re} p$ at given points on $|z| = 1$. In terms of the associated non-negative trigonometric polynomials this means that we prescribe certain zeros. A first application of our general result is a refinement of an estimate of L. Fejér [1] dealing with the maximum of a non-negative trigonometric polynomial with constant term 1. This in turn can be used to improve a root-finding algorithm for complex polynomials which was recently established [5]. Our theorem applies also to the estimation of linear functionals of typically real polynomials. As an example we solve the coefficient problem for the third coefficient of such polynomials. Partial results for this problem have recently been obtained by Suffridge [6]. We also give a table of the numerical values of the bounds for the coefficients of all typically real polynomials with degree $\leqq 10$.

## 2. The main result

Let $n \in \mathbb{N}$ be fixed. Let $\Theta = \{z_1, \ldots, z_s\}$, $s \leqq n$, where $z_j \in \mathbb{C}$ are disjoint with $|z_j| = 1$. By $\mathcal{R}_n(\Theta)$ we denote the set of polynomials $p \in \mathcal{R}_n$ with $\operatorname{Re} p(z_j) = 0$, $z_j \in \Theta$. A vector $\mathbf{d} \in M_n$ is called a positive multiplier for $\mathcal{R}_n(\Theta)$ if $\operatorname{Re} \mathbf{d}\cdot\mathbf{p} > 0$ holds in $\mathcal{R}_n(\Theta)$ except for $p \equiv 0$. With $\Theta$ we assign the matrix

$$\mathbf{D}_\Theta := \begin{pmatrix} z_1^0 & \ldots & z_s^0 \\ z_1^1 & \ldots & z_s^1 \\ \cdot & \cdot\;\cdot & \cdot \\ z_1^n & \ldots & z_s^n \end{pmatrix}$$

$\mathbf{0}$ is the null matrix. With $\mathbf{c} \in M_n$, $\Theta$ as above we assign the hermitian matrix

$$\mathbf{T}(\mathbf{c}, \Theta) := \begin{pmatrix} \mathbf{C} & \mathbf{D}_\Theta \\ \overline{\mathbf{D}_\Theta^t} & \mathbf{0} \end{pmatrix}.$$

Theorem 1: *Let $\mathbf{c}, \mathbf{d} \in M_n$, $\mathbf{d}$ a positive multiplier for $\mathcal{R}_n(\Theta)$. Let $\lambda_{\min}$, $\lambda_{\max}$ be the smallest and the greatest solution of the equation*

$$\det\big(\mathbf{T}(\mathbf{c} - \lambda\mathbf{d}, \Theta)\big) = 0. \tag{2}$$

*Then for $p \in \mathcal{R}_n(\Theta)$, $p \not\equiv 0$, we have*

$$\lambda_{\min} \leqq \frac{\operatorname{Re} \mathbf{c}\cdot\mathbf{p}}{\operatorname{Re} \mathbf{d}\cdot\mathbf{p}} \leqq \lambda_{\max}. \tag{3}$$

*These bounds are best possible for any admissible choice of* $\mathbf{c}, \mathbf{d}, \Theta$.

Remarks: 1. The extremal polynomials for (3) can be obtained via the solution of a linear equation system involving $\lambda_{\min}$, $\lambda_{\max}$, respectively. 2. The case $\Theta = \varnothing$, $\mathbf{d} = \mathbf{e} := (1, 0, \ldots, 0)^t$ of Theorem 1 is Theorem A. 3. It is not difficult to see that all solutions of (2) are real (this was observed by Dr. R. Freund).

Theorem 1 itself supplies a necessary and sufficient criterion for $\mathbf{d} \in M_n$ to be a positive multiplier for $\mathcal{R}_n(\Theta)$.

Corollary 1: *$\mathbf{d} \in M_n$ is a positive multiplier for $\mathcal{R}_n(\Theta)$ if and only if all solutions of* $\det\big(\mathbf{T}(\mathbf{d} - \lambda \mathbf{e}, \Theta)\big) = 0$ *are positive.*

Proof of Theorem 1: It follows from Fejér's theorem [1] that $p \in \mathcal{R}_n$ if and only if there exists a polynomial $q(z) = \sum_{k=0}^{n} q_k z^k$ such that $|q(e^{i\theta})|^2 = \operatorname{Re} p(e^{i\theta})$, $\theta \in \mathbb{R}$. Furthermore, $p \in \mathcal{R}_n(\Theta)$ if and only if the corresponding $q$ has zeros at $z_j \in \Theta$. Now let $\mathbf{x} = (q_0, \ldots, q_n, \mu_1, \ldots, \mu_s)^t$ be arbitrary in $\mathbb{C}^{n+s+1}$ and choose $\lambda > \lambda_{\max}$. With the subvector $(q_0, \ldots, q_n)^t$ we construct a polynomial $q$ and then the corresponding $p \in \mathcal{R}_n$. The following relation is easily verified:

$$F(\mathbf{x}) = \mathbf{x}^t \cdot \mathbf{T}(\mathbf{c} - \lambda \mathbf{d}, \Theta) \cdot \bar{\mathbf{x}} = \operatorname{Re}\left[(\mathbf{c} - \lambda \mathbf{d}) \cdot \mathbf{p} + 2 \sum_{j=1}^{s} \mu_j q(z_j)\right].$$

Hence $F$ is the Lagrange multiplier function for the extremization of

$$\operatorname{Re}(\mathbf{c} - \lambda \mathbf{d}) \cdot \mathbf{p} \tag{4}$$

in $\mathcal{R}_n(\Theta)$. Since the $z_j$ are disjoint the manifold described by the constraints has maximal rank. Hence for an extremum we must have

$$\nabla F = \mathbf{T}(\mathbf{c} - \lambda \mathbf{d}, \Theta) \cdot \bar{\mathbf{x}} = 0. \tag{5}$$

But $\lambda > \lambda_{\max}$ implies that (4) has only the trivial solution and in this case (5) is zero for the extremum. Therefore (4) has constant sign on $\mathcal{R}_n(\Theta)$. But for $\lambda$ large and non-trivial $p \in \mathcal{R}_n(\Theta)$ this sign is obviously $-1$ which, by continuity, implies

$$\frac{\operatorname{Re} \mathbf{c} \cdot \mathbf{p}}{\operatorname{Re} \mathbf{d} \cdot \mathbf{p}} \leqq \lambda, \qquad p \in \mathcal{R}_n(\Theta),\ p \not\equiv 0, \qquad \lambda > \lambda_{\max}.$$

If $\lambda = \lambda_{\max}$ (5) has a non-trivial solution which is also non-trivial in the first $n + 1$ components $q_0, \ldots, q_n$, which produce a non-trivial $p \in \mathcal{R}_n(\Theta)$. Clearly (4) vanishes for that $p$ which proves the sharpness of the upper bound $\lambda_{\max}$. The other estimate follows similarly ∎

## 3. The range of trigonometric polynomials

Let $t$ be a non-negative trigonometric polynomial of degree $n$ with constant term 1. Fejér [1] proved

$$t(\theta) \leqq n + 1, \qquad \theta \in \mathbb{R}, \tag{6}$$

with equality (at $\theta = 0$) only for

$$t_0(\theta) = \frac{1}{n+1} \left( \frac{\sin \frac{n+1}{2}\,\theta}{\sin \frac{1}{2}\,\theta} \right)^2.$$

The following theorem is a refinement of (6).

Theorem 2: *Let $t$ be a non-negative trigonometric polynomial of degree $n$ with constant term 1 and $t(0) = 0$. Then*

$$t(\theta) + t_0(\theta) \leqq n + 1, \qquad \theta \in \mathbb{R}. \tag{7}$$

*For each $\theta_0 \in \mathbb{R}$ there exists an admissible $t$ such that equality holds in (7) for $\theta = \theta_0$.*

Proof: Let $\theta$ be fixed, $z = e^{i\theta}$. Our problem to maximize $t(\theta)$ is obviously equivalent to the following extremal problem:

$$\max \frac{\operatorname{Re} p(1)}{\operatorname{Re} p(0)}, \qquad p \in \mathcal{R}_n(\{z\}), \qquad p \not\equiv 0.$$

In view of Theorem 1 the solution to the latter problem is $\lambda_{\max}$, the greatest solution of $\det\big(\mathbf{T}(\mathbf{d} - \lambda\mathbf{e}, \{z\})\big) = 0$, where $\mathbf{d} = (1, 1, \ldots, 1)^t \in M_n$. Denote this determinant by $D_n(\lambda)$ (it has $n + 2$ rows). We perform the following operations to evaluate $D_n(\lambda)$ (assuming $\lambda > 0$):

1. Subtract the first row from the other rows except for the last one;
2. Add the first column multiplied by $1/\lambda$ to the last one;
3. Add all columns except for the first and the last one to the first column;
4. Expand with respect to the first column (only the first and the last element is non-zero).

The remaining two determinants (with $n + 1$ rows each) are easily evaluated and we finally obtain

$$D_n(\lambda) = (-\lambda)^{n-1} \left[(n + 1 - \lambda)(n + 1) - \left|\sum_{k=0}^{n} z^k\right|^2\right].$$

Solving $D_n(\lambda) = 0$, $\lambda \neq 0$, yields $\lambda = n + 1 - t_0(\theta)$ which gives (7). Our claim about equality is a consequence of the sharpness of Theorem 1 ∎

As a consequence of Theorem 2 we get

Corollary 2: *Let $p(z) = 1 + \sum_{k=1}^{n} b_k z^k$ be a polynomial with $|p(z) - 1| \leqq 1$ in $|z| \leqq 1$. Then for $0 \leqq \lambda \leqq 1$ there exists an* arc *$\Gamma$ on $|z| = 1$ of length*

$$L(\Gamma) \geqq \frac{1}{n+1} \sqrt{\frac{24(1-\lambda)}{n+1}} \tag{8}$$

*such that*

$$|p(z)|^2 \leqq 1 - \lambda \sum_{k=1}^{n} |b_k|^2, \qquad z \in \Gamma.$$

In [5] we proved Corollary 2 with the bound

$$L(\Gamma) \geqq \frac{1}{n} \sqrt{\frac{8(1-\lambda)}{n+1}}$$

instead of (8). Hence (8) is better by a factor of about $\sqrt{3}$. This can be used to reduce the number of search points in the global descent method for solving polynomial equations described in [5] by about 40%.

Proof of Corollary 2: Let $s = \min |p(e^{i\theta})|$. By the minimum principle we have $s \leqq 1$ and we may assume $s < 1$. Furthermore, we can assume $|p(1)| = s$. The trigonometric polynomial

$$t(\theta) = \left(|p(e^{i\theta})|^2 - s^2\right)/(v^2 - s^2), \quad \text{where } v^2 = 1 + \sum_{k=1}^{n} |b_k|^2,$$

satisfies the assumptions of Theorem 2 and hence

$$|p(e^{i\theta})|^2 \leqq s^2 + (v^2 - s^2)\left(n + 1 - t_0(\theta)\right).$$

It is known [5: Th. 3] that $v^2 \leqq 2 - s^2$ and a simple calculation shows that $|p(e^{i\theta})|^2 \leqq 1 - \lambda(v^2 - 1)$ holds for all $\theta$ with $t_0(\theta) \geqq n + \frac{1+\lambda}{2}$. Now, for $|\theta| \leqq \frac{1}{n+1} \times \sqrt{\frac{6(1-\lambda)}{n+1}} =: \theta_0$ we have

$$t_0(\theta) \geqq \frac{1}{n+1}\left(\frac{\sin \frac{n+1}{2}\theta_0}{\sin \frac{1}{2}\theta_0}\right)^2 \geqq \frac{1}{n+1}\left(\frac{\frac{n+1}{2}\theta_0 - \frac{1}{6}\left(\frac{n+1}{2}\theta_0\right)^3}{\frac{1}{2}\theta_0}\right)^2$$

$$= (n+1)\left(1 - \frac{1-\lambda}{4(n+1)}\right)^2 \geqq n + 1 - \frac{1-\lambda}{2}.$$

Hence the arc $\Gamma = \{e^{i\theta}: |\theta| \leqq \theta_0\}$ has the desired property ∎

## 4. Application to typically real polynomials

If the vectors $\mathbf{c}$, $\mathbf{d}$ have real components and if $\theta \in \Theta$ implies $2\pi - \theta \in \Theta$ one easily deduces that the bounds in (3) are attained for polynomials $p \in \mathcal{R}_n(\Theta)$ with real coefficients. Let $\mathcal{R}_n^{r}(\Theta)$ denote the subset of polynomials $\mathcal{R}_n(\Theta)$ with real coefficents. We have the following corollary to Theorem 1.

Corollary 3: *Let* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{n+1}$, $\mathbf{d}$ *a positive multiplier for* $\mathcal{R}_n^{r}(\Theta)$. *Then for* $\lambda_{\min}, \lambda_{\max}$ *as in Theorem* 1 *we have*

$$\lambda_{\min} \leqq \frac{\mathbf{c}\cdot\mathbf{p}}{\mathbf{d}\cdot\mathbf{p}} \leqq \lambda_{\max}, \qquad p \in \mathcal{R}_n^{r}(\Theta), \qquad p \not\equiv 0.$$

*These bounds are best possible.*

A polynomial $s(z) = \sum_{k=1}^{n} s_k z^k$ is said to be *typically real* if $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$, and $\operatorname{Im} s(z) \cdot \operatorname{Im} z \geqq 0$ in $|z| \leqq 1$. Let $S_n$ denote the set of those polynomials. It is well-known that $s \in S_n$ if and only if

$$p(z) = (1 - z^2)\,\frac{s(z)}{z} \in \mathcal{R}^{r}_{n+1}(\{-1, 1\}). \tag{9}$$

Let $\mathbf{c} = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$, $\mathbf{c}' = (c_0', \ldots, c_{n-1}', 0, 0)^t \in \mathbb{R}^{n+2}$ where

$$c_j' = \sum_{k=0}^{\left[\frac{n-j-1}{2}\right]} c_{j+1+2k}, \qquad j = 0, 1, \ldots, n-1.$$

If $s$, $p$ are related by (9) we find $\mathbf{c}\cdot\mathbf{s} = \mathbf{c}'\cdot\mathbf{p}$. Hence from Corollary 3 we get

Theorem 3: *Let* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$, $\mathbf{d}$ *a positive multiplier for* $S_n$. *Let* $\lambda_{\min}$, $\lambda_{\max}$ *be the smallest and the greatest solution of*

$$\det\left(\mathbf{T}(\mathbf{c}' - \lambda \mathbf{d}', \{-1, 1\})\right) = 0. \tag{10}$$

*Then we have*

$$\lambda_{\min} \leqq \frac{\mathbf{c} \cdot \mathbf{s}}{\mathbf{d} \cdot \mathbf{s}} \leqq \lambda_{\max}, \qquad s \in S_n, \qquad s \not\equiv 0.$$

*These bounds are sharp.*

We note that in case of "odd" multipliers $\mathbf{c}$, $\mathbf{d}$, i.e. if the components with even index are zero, we can replace (10) by the simpler equation

$$\det\left(\mathbf{T}(\mathbf{c}'' - \lambda \mathbf{d}'', \{1\})\right) = 0 \tag{11}$$

where $\mathbf{c}'' = (c_1', c_3', \ldots; 0)^t \in \mathbb{R}^m$, $\mathbf{d}'' = (d_1', d_3', \ldots; 0)^t \in \mathbb{R}^m$ with $m = [(n+3)/2]$. Note that the determinant in (11) has only $m + 1$ rows instead of $n + 4$ in (10). This simplification is due to the fact that in this case we only need to consider odd polynomials $s \in S_n$ which are in one-to-one correspondence with the polynomials $p \in \mathcal{R}_{m-1}^{\mathrm{r}}(\{1\})$ and we have $\mathbf{c} \cdot \mathbf{s} = \mathbf{c}'' \cdot \mathbf{p}$. We omit the details.

## 5. The coefficient problem for typically real polynomials

Let $S_n^{\mathrm{N}}$ denote the set of normalized typically real polynomials

$$s(z) = z + \sum_{k=2}^{n} s_k z^k.$$

The coefficient problem for $S_n^{\mathrm{N}}$ is the determinant of the best constants $A_k(n)$, $B_k(n)$ such that for $k \in \{2, \ldots, n\}$

$$-B_k(n) \leqq s_k \leqq A_k(n), \qquad s \in S_n^{\mathrm{N}}.$$

This problem has been studied several times (see, for instance, ROYSTER and SUFFRIDGE [4], SUFFRIDGE [6]). For arbitrary $n$ it is solved in the cases $k = 2$, $n - 1$, $n$. Also $A_3(n)$ is known and $B_3(n)$ in the cases $4 \mid n - 1$, $4 \mid n - 2$. Specializing Theorem 3 we obtain

Theorem 4: $A_k(n)$, $-B_k(n)$ *are the greatest and the smallest solution* $\lambda$ *of the equation*

$$\begin{aligned} &\det\left(\mathbf{T}(\mathbf{c} - \lambda \mathbf{e}, \{-1, 1\})\right) = 0, \qquad k \text{ even}, \\ &\det\left(\mathbf{T}(\mathbf{d} - \lambda \mathbf{e}, \{1\})\right) = 0, \qquad k \text{ odd}. \end{aligned} \tag{12}$$

*Here* $\mathbf{c} = (c_0, \ldots, c_{n+1})^t$ *with* $c_j = 1$ *for* $j = 1, 3, \ldots, k-1$ *and* $c_j = 0$ *otherwise;* $\mathbf{d} = (d_0, \ldots, d_m)^t$ *with* $m = [(n+1)/2]$ *and* $d_j = 1$ *for* $j = 0, 1, \ldots, (k-1)/2$, $d_j = 0$ *otherwise.*

This result gives a means to calculate $A_k(n)$, $B_k(n)$ at least numerically. This has been done for $n \leqq 10$ and the results — rounded to 6 decimal places — are given in Table 1. It may be possible, however, to simplify (12) considerably and to obtain a theoretically satisfying solution to the coefficient problem. In the sequel we do so for $k = 3$ thereby completing the solution of the third-coefficient-problem for typically real polynomials.

Table 1

| $n$ | $A_2(n)$ | $A_3(n)$ | $A_4(n)$ | $A_5(n)$ | $A_6(n)$ | $A_7(n)$ | $A_8(n)$ | $A_9(n)$ | $A_{10}(n)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1.768177 | 2.246980 | 2.331017 | 2.246980 | 1.889229 | 1.618034 | 1.291726 | 1.000000 | .833333 | |
| 9 | 1.732051 | 2.246980 | 2.135779 | 2.246980 | 1.618034 | 1.618034 | 1.000000 | 1.000000 | .500000 | 2 |
| 8 | 1.677193 | 2.000000 | 1.878133 | 1.618034 | 1.266451 | 1.000000 | .800000 | .333333 | 1.000000 | 3 |
| 7 | 1.618034 | 2.000000 | 1.618034 | 1.618034 | 1.000000 | 1.000000 | .666667 | .333333 | 1.215250 | 4 |
| 6 | 1.520315 | 1.618034 | 1.240597 | 1.000000 | .750000 | .500000 | 1.000000 | .618034 | 1.414214 | 5 |
| 5 | 1.414214 | 1.618034 | 1.000000 | 1.000000 | .750000 | .500000 | 1.240597 | .618034 | 1.520315 | 6 |
| 4 | 1.215250 | 1.000000 | .666667 | .600000 | 1.000000 | .618034 | 1.618034 | .716515 | 1.618034 | 7 |
| 3 | 1.000000 | 1.000000 | .800000 | .600000 | 1.266451 | .618034 | 1.878133 | .716515 | 1.677193 | 8 |
| 2 | .500000 | .666667 | 1.000000 | .618034 | 1.618034 | .801938 | 2.135779 | .801938 | 1.732051 | 9 |
| | .833333 | .666667 | 1.291726 | .618034 | 1.889229 | .801938 | 2.331017 | .801938 | 1.768177 | 10 |
| | $B_{10}(m)$ | $B_9(m)$ | $B_8(m)$ | $B_7(m)$ | $B_6(m)$ | $B_5(m)$ | $B_4(m)$ | $B_3(m)$ | $B_2(m)$ | $m$ |

Theorem 5: *Let* $n \in \mathbb{N}$, $m = \left[\frac{n+1}{2}\right]$. *Then we have*

$$A_3(n) = 1 + 2\cos\frac{2}{m+2}\pi, \quad B_3(n) = -1 - 2\cos\frac{m+1}{m+2}\pi, \quad m \text{ odd}.$$

*If $m$ is even, $B_3(n)$ is the largest root $\lambda \leqq 1$ of the equation* $T''_{m+2}\left(\sqrt{\frac{1}{4}(1-\lambda)}\right) = 0.$

$T_k$, $U_k$ denote the Chebychev polynomials of the first and second kind, respectively. For the proof we need a lemma

Lemma: *Let*

$$E_m(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & 0 & . & 0 & 1 \\ 1 & -\lambda & 1 & 0 & . & 0 & 1 \\ 0 & 1 & -\lambda & 1 & . & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & . & -\lambda & 1 \\ 1 & 1 & 1 & 1 & . & 1 & 0 \end{vmatrix}_{(m+2)}.$$

*Then we have*

$$E_m(\lambda) \cong (-1)^{m+1} \frac{\partial}{\partial\lambda} U^2_{m+1}\left(\sqrt{\frac{1}{4}(\lambda+2)}\right). \tag{13}$$

Proof: We expand $E_m(\lambda)$ with respect to the first column. One of the three resulting determinants is $E_{m-1}(\lambda)$. After expanding the remaining two determinants with respect to the first row we arrive at

$$E_m(\lambda) = -E_{m-1}(\lambda) - E_{m-2}(\lambda) - 2(-1)^m Q_m(\lambda) - R_m(\lambda) \tag{14}$$

where

$$Q_m(\lambda) = \begin{vmatrix} 1 & -\lambda & 1 & 0 & . & 0 & 0 \\ 0 & 1 & -\lambda & 1 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 1 & -\lambda \\ 1 & 1 & 1 & 1 & . & 1 & 1 \end{vmatrix}_{(m)}$$

and

$$R_m(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 & . & 0 & 0 \\ 1 & -\lambda & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & -\lambda \end{vmatrix}_{(m)}.$$

Expansion of $Q_m(\lambda)$ with respect to the first column yields $Q_m(\lambda) = Q_{m-1}(\lambda) + (-1)^{m-1} \times R_{m-1}(\lambda)$ which, by induction, leads to

$$Q_m(\lambda) = \sum_{k=0}^{m-1} (-1)^k R_k(\lambda). \tag{15}$$

It is known [3: p. 528] that $\sum R_k(\lambda)(-x)^k = 1/(1 - \lambda x + x^2)$ and together with (15) we obtain $\sum Q_m(\lambda)\, x^m = x/(1-x)(1 - \lambda x + x^2)$. Multiplying (14) by $x^m$ and summing with respect to $m$ gives

$$\sum_{m=0}^{\infty} E_m(\lambda)\, x^m = \frac{x-1}{(1+x)(1+\lambda x + x^2)^2} = \frac{1}{x} \cdot \frac{\partial}{\partial \lambda} \frac{1-x^2}{(1+x)^2(1+\lambda x + x^2)}.$$

Fejér [2] has shown that

$$\frac{1-x^2}{(1+x)^2(1+\lambda x+x^2)} = \sum_{m=0}^{\infty} (-1)^m U_m{}^2 \left(\sqrt{\frac{1}{4}(\lambda+2)}\right) x^m$$

which completes the proof of the lemma ∎

A simple discussion of the representation (13) shows that the largest root of $E_m(\lambda) = 0$ is $2 \cos \frac{2}{m+2}\pi$. If $m$ is odd, the smallest root is $2\cos\frac{m+1}{m+2}\pi$ while for even $m$ the smallest root coincides with the smallest root $> -2$ of $U'_{m+1}\left(\sqrt{\frac{1}{4}(\lambda+2)}\right)$.

The proof of Theorem 5 follows now from Theorem 4 since

$$E_m(\lambda - 1) + \det\big(\mathbf{T}(\mathbf{d} - \lambda \mathbf{e}, \{1\})\big) \quad \text{with} \quad d = (1, 1, 0, \ldots, 0)^t \in \mathbb{R}^{m+1}$$

and from the relation $(m+2)\, U'_{m+1} = T''_{m+2}$ ∎

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