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On an Abstract Nonlinear Cauchy-Kowalewski Theorem — a Variant of L. Nirenberg's and T. Nishida's Proof[1])

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Die Arbeit beschäftigt sich mit einer Variante des Beweises eines abstrakten nichtlinearen Cauchy-Kowalewski-Satzes. Sowohl der Originalbeweis wie auch dessen hier gegebene Modifikation basieren auf der Methode der sukzessiven Approximationen, wobei in beiden Fällen das Konvergenzintervall durch ein unendliches Produkt charakterisiert wird. Während im Originalbeweis ein festes unendliches Produkt verwendet wird, sind in der vorliegenden Arbeit dessen Faktoren willkürlich wählbar. Dadurch erreicht man, daß in manchen Fällen die Konvergenz in einem größeren Intervall nachgewiesen werden kann.

В статье рассматривается абстрактная форма нелинейной теоремы Коши-Ковалевской. И оригинальное доказательство и рассматриваемая здесь модификация используют метод последовательных приближений, причем в обоих случаях интервал сходимости характеризуется через бесконечное произведение. В настоящей статье факторы этого произведения — произвольные, тогда как в оригинальном доказательстве некоторое фиксированное произведение применяется. Таким образом в некоторых случаях интервал сходимости оказывается больше.

The paper deals with a variant of the proof of an abstract nonlinear Cauchy-Kowalewski theorem. The original proof as well as its modification regarded in this paper make use of the method of successive approximations where in both cases the interval of convergence is characterized by an infinite product. In the present paper the infinite product of the original proof is replaced by one with arbitrary factors. In this way in some cases the convergence is proved in a larger interval.

1. The initial value problem

$$\frac{du}{dt} = F(t, u), \qquad u(0) = 0, \tag{1}$$

where the right-hand side $F(t, u)$ maps a scale of Banach spaces into itself for every t, is equivalent to the integral equation

$$u(t) = \int_{\tau=0}^{t} F(\tau, u(\tau))\, d\tau \tag{2}$$

that may be solved by the method of successive approximations. The present paper contains a modification of L. NIRENBERG's and T. NISHIDA's considerations, cf. [2–4]. They are based on the method of successive approximations. Starting with $u_0 = 0$ the $(k+1)$th iteration is defined by

$$u_{k+1}(t) = \int_{\tau=0}^{t} F(\tau, u_k(\tau))\, d\tau. \tag{3}$$

1) The paper has also been discussed in the seminar of the research group "Partial complex differential equations" of the Halle university. The author thanks the members of this research group, especially A. Crodel and M. Reissig, for useful discussions.

The proof of L. Nirenberg's nonlinear Cauchy-Kowalewski theorem makes use of J. MOSER's method [1], whereas T. NISHIDA's paper [4] contains a modification of L. Nirenberg's considerations. See also L. V. OVSYANNIKOV [5].

Similarly as in T. NISHIDA's paper [4] assume the following conditions on the right-hand side $F(t, u)$ of (1), where $\mathcal{B}_s$, $0 \leqq s < s_0 < +\infty$, is a given scale of Banach spaces and $\|\cdot\|_s$ is the norm in $\mathcal{B}_s$:

(i) There exist positive numbers R and T such that for any pair s', s with $0 < s' < s < s_0$ the right-hand side $F(t, u)$ is a continuous mapping of $\{t: 0 \leqq t < T\} \times \{u \in \mathcal{B}_s: \|u\|_s \leqq R\}$ into $\mathcal{B}_{s'}$.

(ii) The continuous function defined by $F(t, 0)$ satisfies, with a fixed positive constant K, the estimate

$$\|F(t, 0)\|_s \leqq \frac{K}{s_0 - s}, \qquad 0 \leqq s < s_0.$$

(iii) There exists a positive constant C such that

$$\|F(t, u) - F(t, v)\|_{s'} \leqq \frac{C\|u - v\|_s}{s - s'}$$

for any pair s', s with $0 \leqq s' < s < s_0$, where C is independent of t, u, v, s and s'.

The proof of the convergence of (3) presented in this paper seems to be a little more immediate and more elementary because it does not make use of the Banach space of functions $u = u(t)$ with values in $\mathcal{B}_s$ for every s, $0 \leqq s < s_0$. Further in some cases we can prove the convergence of the sequence (3) in a larger t-interval.

2. From (3) with $k = 0$ and from (ii) one gets immediately

$$\|u_1(t)\|_s \leqq \frac{Kt}{s_0 - s}, \tag{4}$$

where $u_1(t)$ belongs to every $\mathcal{B}_s$ for $0 \leqq t < T$. Restricting t to the interval $0 < t < a(s_0 - s)$ depending on s (with any $a > 0$), the last inequality may be rewritten as

$$\|u_1(t)\|_s \leqq \frac{Ka}{\dfrac{a(s_0 - s)}{t}} < \frac{Ka}{\dfrac{a(s_0 - s)}{t} - 1},$$

hence

$$\|u_1(t)\|_s \left(\frac{a(s_0 - s)}{t} - 1\right) < Ka. \tag{5}$$

For any $u(t)$ defined for $0 \leqq t < a(s_0 - s)$ and belonging to $\mathcal{B}_s$, $0 \leqq s < s_0$, define the functional

$$M(u) = \sup_{0 \leqq s < s_0} \sup_{0 < t < a(s_0 - s)} \|u(t)\|_s \left(\frac{a(s_0 - s)}{t} - 1\right),$$

which may be equal to $+\infty$. In view of (5) we get the estimate

$$M(u_1) = M(u_1 - u_0) \leqq Ka, \tag{6}$$

therefore $M(u_1)$ is finite, especially[2]). Thus $\|u_1(t)\|_s$ may be estimated by

$$\|u_1(t)\|_s \leqq \frac{M(u_1)}{\frac{a(s_0 - s)}{t} - 1}$$

f $0 < t < a(s_0 - s)$.

Now we want to prove that the sequence (3) converges to a solution of the integral equation (2) and, consequently, to a solution of the initial value problem (1). To this end we must ensure, in addition, that $\|u_k(t)\|_s \leqq R$. This condition is satisfied if

$$\|u_k(t) - u_{k-1}(t)\|_s \leqq \varepsilon_k R \tag{7}$$

for $k = 1, 2, \ldots$, where ε_k are positive numbers satisfying the condition

$$\sum_{k=1}^{\infty} \varepsilon_k \leqq 1. \tag{8}$$

By virtue of (4) the inequality (7) is satisfied for $k = 1$ and $0 < t < a_1(s_0 - s)$ if

$$a_1 s_0 < T \quad \text{and} \quad a_1 \leqq \varepsilon_1 \frac{R}{K}. \tag{9}$$

We shall prove that (7) can be fulfilled for every k if we diminish the t-interval step by step, where we must ensure that the length of the limit interval is positive. The k-th iteration u_k will be defined if

$$0 \leqq t < a_k(s_0 - s), \tag{10}$$

where $a_1 > a_2 > \cdots > a_k > 0$. Corresponding to the t-intervals (10) we must regard a sequence of functionals

$$M_k(u) = \sup_{0 \leqq s < s_0} \sup_{0 < t < a_k(s_0 - s)} \|u(t)\|_s \left(\frac{a_k(s_0 - s)}{t} - 1\right),$$

instead of the only functional $M(u)$ defined above. From this definition one immediately gets the property

$$M_{k+1}(u) \leqq M_k(u) \quad \text{if} \quad a_{k+1} < a_k.$$

Now assume that the $u_j(t)$, $j = 1, 2, \ldots, k$, are defined and belong to $\mathscr{B}_s$ if $0 \leqq t < a_j(s_0 - s)$, $a_1 > a_2 > \cdots > a_k > 0$. Assume by induction, further, that

$$\|u_j(t) - u_{j-1}(t)\|_s \leqq \varepsilon_j R,$$

while $M_j(u_j - u_{j-1})$ is supposed to be finite. In order to prove the existence of u_{k+1} we choose any positive $a_{k+1} < a_k$. Later we shall get a positive lower bound for $a_k - a_{k+1}$. For an arbitrary s, $0 \leqq s < s_0$, we define $\tilde{s}$ by

$$a_{k+1}(s_0 - s) = a_k(s_0 - \tilde{s})$$

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[2]) From the definition of $M(u)$ one obtains immediately the estimate

$$\|u(t)\|_s \leqq \frac{tM(u)}{a(s_0 - s) - t},$$

thus $u(0)$ is proved to be equal to 0 provided $M(u)$ is finite.

such that $s < \tilde{s} < s_0$. The k-th iteration $u_k(\tau)$ is defined and belongs to $\mathcal{B}_{\tilde{s}}$ if $0 \leqq \tau < a_k(s_0 - \tilde{s})$. In view of (8) we have

$$\|u_k(\tau)\|_s \leqq \sum_{j=1}^{k} \|u_j(\tau) - u_{j-1}(\tau)\|_{\tilde{s}} \leqq \sum_{j=1}^{k} \varepsilon_j R < R,$$

thus $F(\cdot, u_k(\cdot))$ and, consequently, $u_{k+1}(t)$ are defined and belong to $\mathcal{B}_s$ if $0 \leqq t < a_{k+1}(s_0 - s)$, $0 \leqq \tau \leqq t$. From (3) we get, moreover, that

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq \int_{\tau=0}^{t} \|F(\tau, u_k(\tau)) - F(\tau, u_{k-1}(\tau))\|_s \, d\tau.$$

In order to estimate the norm in the integrand by assumption (iii) we must take the norm of $u_k(\tau) - u_{k-1}(\tau)$ in a space with larger index. This index will be chosen in dependence on τ and will be denoted by $s(\tau)$. The numbers τ and $s(\tau)$ must be connected by $\tau < a_k(s_0 - s(\tau))$. This inequality may be rewritten as

$$s(\tau) < s_0 - \frac{\tau}{a_k}. \tag{11}$$

Now remark that for any two real numbers a and b with $a < b$ we have $a + \frac{a-b}{2} < b$. Thus (11) and $s < s(\tau)$ are satisfied if we set

$$s(\tau) = s + \frac{1}{2}\left(s_0 - \frac{\tau}{a_k} - s\right). \tag{12}$$

Applying (iii) we get, finally, the estimate

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq C \int_{\tau=0}^{t} \frac{\|u_k(\tau) - u_{k-1}(\tau)\|_{s(\tau)}}{s(\tau) - s} \, d\tau$$

if $0 \leqq t < a_{k+1}(s_0 - s)$. By assumption, $M_k(u_k - u_{k-1})$ is finite. From the definition of M_k we get $(\tau > 0)$

$$\|u_k(\tau) - u_{k-1}(\tau)\|_{s(\tau)} \leqq \frac{M_k(u_k - u_{k-1})}{\dfrac{a_k(s_0 - s(\tau))}{\tau} - 1}.$$

Substituting (12), the following estimate follows:

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq 4Ca_k M_k(u_k - u_{k-1})\, t \int_{\tau=0}^{t} \frac{1}{(a_k(s_0 - s) - \tau)^2} \, d\tau.$$

Calculating the integral, we get for $0 < t < a_{k+1}(s_0 - s)$ the inequality

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq 4a_{k+1} C M_k(u_k - u_{k-1}) \frac{1}{\dfrac{a_k(s_0 - s)}{t} - 1},$$

from which we obtain first

$$M_{k+1}(u_{k+1} - u_k) \leqq 4a_{k+1} C M_k(u_k - u_{k-1}) \tag{13}$$

and second

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq 4 \frac{a_{k+1}^2}{a_k - a_{k+1}} CM_k(u_k - u_{k-1}), \tag{14}$$

since

$$\frac{a_k(s_0 - s)}{t} - 1 > \frac{a_k(s_0 - s)}{a_{k+1}(s_0 - s)} - 1 = \frac{a_k - a_{k+1}}{a_{k+1}}.$$

Taking into consideration the estimate (6) with $a = a_1$, the inequality (13) leads to

$$M_k(u_k - u_{k-1}) \leqq (4C)^{k-1} a_k \dots a_2 a_1 K,$$

hence (14) gives

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq (4C)^k K \frac{a_{k+1}^2 a_k \dots a_1}{a_k - a_{k+1}}. \tag{15}$$

From (13) we derive, further, that $M_{k+1}(u_{k+1} - u_k)$ is finite, too. From (15) one gets the desired lower bound for $a_k - a_{k+1}$ guaranteeing the estimate

$$\|u_{k+1}(t) - u_k(t)\|_s \leqq \varepsilon_{k+1} R.$$

This lower bound is

$$(4C)^k \frac{K}{R} a_{k+1}^2 a_k \dots a_1 \frac{1}{\varepsilon_{k+1}} \leqq a_k - a_{k+1}. \tag{16}$$

The limit function $u = \lim u_k$ will be defined for $0 \leqq t < a(s_0 - s)$ and will belong to $\mathscr{B}_s$, where $a = \lim a_k$. Besides (16) we must ensure that $a > 0$. To this end we define a_{k+1} as product

$$a_{k+1} = a_k(1 - \delta_k), \quad 0 < \delta_k < 1$$

similarly as in Nirenberg's and Nishida's papers[3]). Then

$$a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k)$$

is positive if the infinite product converges. For this convergence the condition

$$\sum_{k=1}^{\infty} \delta_k < +\infty \tag{17}$$

is sufficient and necessary. Condition (16) may be rewritten to become the relation

$$(4C)^k \frac{K}{R} a_{k+1}^2 a_{k-1} a_{k-2} \dots a_1 \leqq \varepsilon_{k+1} \delta_k, \qquad k = 1, 2, \dots$$

connecting ε_{k+1} and δ_k. This relation is fulfilled if

$$(4a_1 C)^k a_1 \frac{K}{R} \leqq \varepsilon_{k+1} \delta_k \tag{18}$$

for every $k = 1, 2, \dots$ The existence of a solution of the initial value problem (1) belonging to $\mathscr{B}_s$ for

$$0 \leqq t < a(s_0 - s), \qquad a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k) > 0,$$

is consequently reduced to the following auxialiary problem:

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[3]) The basic idea also taken over from the papers mentioned above is the consideration of the functionals M_k.

Find positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ such that the conditions (8) and (17) are satisfied. Find, further, a positive number a_1 satisfying (9) as well as (18) for every k.

There are various possibilities to solve the auxiliary problem. In the next section we shall realize only two of them. In view of $\varepsilon_k \to 0$ and $\delta_k \to 0$ for $k \to \infty$ from (18) one immediately gets that a_1 must in any case satisfy the inequality $4a_1C < 1$.

3. In order to fulfill condition (8) we first choose $\varepsilon_k = 1/2^k$. Then in view of (18) we may choose $\delta_k = 2(8a_1C)^k a_1K/R$. Taking into consideration condition (17) this means that a_1 must satisfy the inequality $8a_1C < 1$ instead of $4a_1C < 1$. In order to ensure that every δ_k is less than 1 we restrict a_1 by $2a_1K \leqq R$. This condition is identic with the second one of (9). Summarizing (9) and $8a_1C < 1$ we get a

First choice of a: Let a_1 be a positive number satisfying the inequality

$$a_1 < \min\left(\frac{T}{s_0}, \frac{R}{2K}, \frac{1}{8C}\right).$$

Then

$$a = a_1 \prod_{k=1}^{\infty} \left(1 - 2(8a_1C)^k a_1 \frac{K}{R}\right).$$

Second we put $\varepsilon_1 = 1/2$ and $\delta_k = \lambda\varepsilon_{k+1}$. Then (18) is satisfied for $\lambda\varepsilon_{k+1}^2 = (4a_1C)^k \times a_1K/R$. Provided that $4a_1C < 1$ from (8) we get

$$\sqrt{\lambda} = 4a_1 \sqrt{\frac{KC}{R}} \frac{1}{1 - 2\sqrt{a_1}\sqrt{C}}$$

and, finally,

$$\delta_k = \frac{K}{2RC} \frac{(2\sqrt{a_1}\sqrt{C})^{k+3}}{1 - 2\sqrt{a_1}\sqrt{C}}$$

(where we took the equality in (8)). In order to ensure that $\delta_k < 1$ for every k we require

$$\frac{K}{2RC} \frac{2\sqrt{a_1}\sqrt{C}}{1 - 2\sqrt{a_1}\sqrt{C}} < 1.$$

This leads to the condition

$$a_1 < \frac{1}{4C}\left(\frac{2RC}{2RC + K}\right)^2 \tag{19}$$

for a_1, which is more restrictive than $4a_1C < 1$. Summarizing (9) and (19) we get a

Second choice of a: Let a_1 be a positive number satisfying the inequality

$$a_1 < \min\left(\frac{T}{s_0}, \frac{R}{2K}, \frac{1}{4C}\left(\frac{2RC}{2RC + K}\right)^2\right).$$

Then the number a is given by

$$a = a_1 \prod_{k=1}^{\infty} \left(1 - \frac{K}{2RC} \frac{(2\sqrt{a_1}\sqrt{C})^{k+3}}{1 - 2\sqrt{a_1}\sqrt{C}}\right).$$

4. Now let a number $a_1 > 0$ and two sequences $\{\varepsilon_k\}_{k=1,2,\ldots}$, $\{\delta_k\}_{k=1,2\ldots}$ satisfying the conditions (8), (9), (17) and (18) be given. Suppose, moreover, that the conditions (i)—(iii) are satisfied. Then the following theorem holds.

Theorem: *The limit function $u(t) = \lim u_k(t)$ of the functions u_k defined by* (3), $u_0 = 0$, *is a solution of the initial value problem* (1) *belonging to $\mathscr{B}_s$ if*

$$0 \leqq t < a(s_0 - s), \qquad a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k).$$

By similar considerations (cf. the bibliography) the uniqueness of the solution in the scale $\mathscr{B}_s$ can be proved, too.

5. Since

$$\ln (1 - \alpha q^k) = -\sum_{j=1}^{\infty} \frac{1}{j} (\alpha q^k)^j$$

if $0 < q < 1$, $\alpha > 0$, $\alpha q < 1$, and since

$$\sum_{k=1}^{\infty} q^{jk} = \frac{q^j}{1 - q^j} > \frac{q^j}{1 - q}$$

one can easily check that

$$\prod_{k=1}^{\infty} (1 - \alpha q^k) > (1 - \alpha q)^{\frac{1}{1-q}}.$$

This inequality allows to obtain lower bounds for the infinite products regarded in Section 3. In this way one is able to estimate the t-interval in which the solution $u = u(t)$ of the initial value problem (1) exists. In order to illustrate this method we assume $s_0 = 1$, $T = 1$, $K = R = 1$, $C = 1/8$. Then the first choice of the number a formulated in Section 3 leads to the condition $a_1 < 1/2$ and we get for a the estimate

$$a = a_1 \prod_{k=1}^{\infty} (1 - 2a_1^{1+k}) > a_1(1 - 2a_1^2)^{\frac{1}{1-a_1}}.$$

Finally one gets with $a_1 = 0{,}35$ the estimate $a > 0{,}2389$, thus the limit function $u = u(t)$ exists and belongs to $\mathscr{B}_s$ at least for t with $0 \leqq t < 0{,}2389(1 - s)$. In this case Nishida's estimates prove the convergence of the $u_k = u_k(t)$ only for $a < 1/16$.

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