(1)

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 (3)

On an Abstract Nonlinear Cauchy-Kowalewski Theorem a Variant of L. Nirenberg's and T. Nishida's Proof¹)

W. Tutschke

Die Arbeit beschäftigt sich mit einer Variante des Beweises eines abstrakten nichtlinearen Cauchy-Kowalewski-Satzes. Sowohl der Originalbeweis wie auch dessen hier gegebene Modifikation basieren auf der Methode der sukzessiven Approximationen, wobei in beiden Fällen das Konvergenzintervall durch ein unendliches Produkt charakterisiert wird. Während im Originalbeweis ein festes unendliches Produkt verwendet wird, sind in der vorliegenden Arbeit dessen Faktoren willkürlich wählbar. Dadurch erreicht man, daß in manchen Fällen die Konvergenz in einem größeren Intervall nachgewiesen werden kann.

В статье рассматривается абстрактная форма нелинейной теоремы Коши-Ковалевской. И оригинальное доказательство и рассматриваемая здесь модификация используют метод последовательных приближений, причем в обоих случаях интервал сходимости характеризуется через бесконечное произведение. В настоящей статье факторы этого произведения - произвольные, тогда как в оригинальном доказательстве некоторое фиксированное произведение применяется. Таким образом в некоторых случаях интервал сходимости оказывается больше.

The paper deals with a variant of the proof of an abstract nonlinear Cauchy-Kowalewski theorem. The original proof as well as its modification regarded in this paper make use of the method of successive approximations where in both cases the interval of convergence is characterized by an infinite product. In the present paper the infinite product of the original proof is replaced by one with arbitrary factors. In this way in some cases the convergence is proved in a larger interval.

1. The initial value problem

$$
\frac{du}{dt}=F(t, u), \qquad u(0)=0,
$$

where the right-hand side $F(t, u)$ maps a scale of Banach spaces into itself for every t, is equivalent to the integral equation

$$
u(t) = \int_{\tau=0}^t F(\tau, u(\tau)) d\tau
$$

that may be solved by the method of successive approximations. The present paper contains a modification of L. NIRENBERG's and T. NISHIDA's considerations, cf. $[2-4]$. They are based on the method of successive approximations. Starting with $u_0 = 0$ the $(k + 1)$ th iteration is defined by

$$
\int u_{k+1}(t) = \int_{\tau=0}^{\tau} F(\tau, u_k(\tau)) d\tau.
$$

¹) The paper has also been discussed in the seminar of the research group "Partial complex differential equations" of the Halle university. The author thanks the members of this research group, especially A. Crodel and M. Reissig, for useful discussions.

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The proof of L. Nirenberg's nonlinear Cauchy-Kowalewski theorem makes use of J. MOSER'S method'[1], whereas T. NISIUDA's paper [4] contains a modification of L. Nirenberg's considerations. See also L. V. OvsYANNIKOV [5].

Similarly as in T. NISHIDA'S paper [4] assume the following conditions on the right-hand side $F(t, u)$ of (1), where \mathscr{B}_s , $0 \le s < s_0 < +\infty$, is a given scale of Banach spaces and $||\cdot||_s$ is the norm in \mathscr{B}_s .

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The proof of L. Nirenberg's nonlinear Cauchy-Kowalewski theor

J. MosER's method [1], whereas T. NISHIDA's paper [4] contains

L. Nirenberg's considerations. See also L. V. OVSYANNIKOV [5].

Similarly as (i) There exist positive numbers *R* and *T* such that for any pair s' , s with $0 < s'$ (i) There exist positive numbers R and T such that for any pair s', s with $0 < s'$
 $s < s_0$ the right-hand side $F(t, u)$ is a continuous mapping of $\{t: 0 \le t < T\}$ $\times \{u \in \mathscr{B}_{s} : ||u||_{s} \leq R\}$ into \mathscr{B}_{s} . Fight-hand side $F(t, u)$ of (1), where \mathcal{B}_s , $0 \le s < s_0 < +\infty$, is a given scale of Banach spaces and $||\cdot||_s$ is the norm in \mathcal{B}_s :

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Banach spaces and $||\cdot||_s$ is therefore in the set of the right-hand
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 $\langle u \in \mathcal{B}_s : ||u||_s \leq R$ into $\$

constant K , the estimate^{$\overline{}$} (ii) The continuous function defined by $F(t, 0)$ satisfies, with a fixed positive

K, the estimate
\n
$$
||F(t, 0)||_s \le \frac{K}{s_0 - s}, \qquad 0 \le s < s_0.
$$

\nhere exists a positive constant C such
\n $||F(t, u) - F(t, v)||_s \le \frac{C||u - v||_s}{s - s'}$

(iii) There exists a positive constant C such that

$$
||F(t, u) - F(t, v)||_{s'} \leq \frac{C||u - v||_{s}}{s - s'}
$$

for any pair *s'*, *s* with $0 \le s' < s < s_0$, where *C* is independent of *t*, *u*, *v*, *s* and *s'*. The proof of the convergence of (3) presented in this paper seems to be a little more immediate and more elementary because it does not make use of the Banach space of functions $u = u(t)$ with values in \mathcal{B}_s for every $s, 0 \leq s < s_0$. Further in some cases we can prove the convergence of the sequence (3) in a larger *t*-interval. *Mush* convergence of (3) presented in this paper seems to be a little mediate and more elementary because it does not make use of the Banach functions $u = u(t)$ with values in \mathscr{B}_s for every $s, 0 \le s < s_0$. Further in

2. From (3) with $k = 0$ and from (ii) one gets immediately

$$
||u_1(t)||_s \le \frac{Kt}{s_0 - s},\tag{4}
$$

where $u_1(t)$ belongs to every \mathcal{B}_s for $0 \le t < T$. Restricting t to the interval $0 < t$ $a(s_0 - s)$ depending on s (with any $a > 0$), the last inequality may be rewritten as gs to every \mathscr{B}_s for $0 \le t < T$. Restricting to minimal on s (with any $a > 0$), the last inequal measurement of $\frac{Ka}{a(s_0 - s)} < \frac{Ka}{a(s_0 - s)} - 1$ 2. From (3) with $k = 0$ and from (ii)
 $||u_1(t)||_s \le \frac{Kt}{s_0 - s}$,

where $u_1(t)$ belongs to every \mathcal{B}_s for $\langle a(s_0 - s)$ depending on s (with ang as
 $||u_1(t)||_s \le \frac{Ka}{a(s_0 - s)} < \frac{I}{a(s_0 - s)}$

hence
 $||u_1(t)||_s \left(\frac{a(s_0 - s)}{t$ some cases we ca $2.$ From (3) with $||u_1(t)||_s \leq$
where $u_1(t)$ belon $< a(s_0 - s)$ depeas $||u_1(t)||_s$: where $u_1(t)$ belongs to every \mathscr{B}_s for $0 \le t < T$. Restricting t to the interva
 $< a(s_0 - s)$ depending on s (with any $a > 0$), the last inequality may be re

as
 $||u_1(t)||_s \le \frac{Ka}{a(s_0 - s)} < \frac{Ka}{a(s_0 - s)} - 1$,

hence
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mediate and more elementary because functions
$$
u = u(t)
$$
 with values in \mathcal{B} as we can prove the convergence of the following:

\n(3) with $k = 0$ and from (ii) one gets

\n
$$
||u_1(t)||_s \leq \frac{Kt}{s_0 - s},
$$

\n(i) belongs to every \mathcal{B}_s for $0 \leq t < s$ depending on s (with any $a > 0$).

\n
$$
||u_1(t)||_s \leq \frac{Ka}{a(s_0 - s)} < \frac{Ka}{a(s_0 - s)} - 1
$$

\n
$$
||u_1(t)||_s \left(\frac{a(s_0 - s)}{t} - 1\right) < Ka.
$$

where
$$
u_1(t)
$$
 belongs to every \mathcal{S}_s for $0 \le t < T$. Restricting t to the interval $0 < t < \alpha(s_0 - s)$ depending on s (with any $a > 0$), the last inequality may be rewritten as\n
$$
||u_1(t)||_s \le \frac{Ka}{a(s_0 - s)} < \frac{Ka}{a(s_0 - s)} - 1
$$
\nhence\n
$$
||u_1(t)||_s \left(\frac{a(s_0 - s)}{t} - 1\right) < Ka.
$$
\nFor any $u(t)$ defined for $0 \le t < a(s_0 - s)$ and belonging to \mathcal{B}_s , $0 \le s < s_0$, define the functional\n
$$
M(u) = \sup_{0 \le s < s_0} \sup_{0 \le t < a(s_0 - s)} ||u(t)||_s \left(\frac{a(s_0 - s)}{t} - 1\right),
$$
\nwhich may be equal to $+\infty$. In view of (5) we get the estimate\n
$$
M(u_1) = M(u_1 - u_0) \le Ka,
$$
\n(6)

$$
\|u_1(t)\|_{s} \left(\frac{t}{t} - 1 \right) < R a.
$$
\n
$$
u(t) \text{ defined for } 0 \leq t < a(s_0 - s) \text{ and belonging}
$$
\n
$$
M(u) = \sup_{0 \leq s < s_0} \sup_{0 < t < a(s_0 - s)} \|u(t)\|_{s} \left(\frac{a(s_0 - s)}{t} - 1 \right),
$$

$$
M(u_1)=M(u_1-u_0)\leq Ka,
$$

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therefore $M(u_1)$ is finite, especially²). Thus $||u_1(t)||_s$ may be estimated by

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$$
M(u_1)
$$
 is finite, especially
\n
$$
||u_1(t)||_s \leqq \frac{M(u_1)}{a(s_0-s)} - 1
$$

f 0 < $t < a(s_0 - s)$.

Now we want to prove that the sequence (3) converges to a solution of the integral equation (2) and, consequently, to a solution of the initial value problem (1). To this end we must ensure, in addition, that $||u_k(t)||_s \leq R$. This condition is satisfied On an Abstract Nonlinear Cauchy-Kowalewski Theorem

therefore $M(u_1)$ is finite, especially²). Thus $||u_1(t)||_s$ may be estimated by
 $||u_1(t)||_s \le \frac{M(u_1)}{t}$
 $10 < t < a(s_0 - s)$

Now we want to prove that the sequence (3) con 1

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ie sequence (3) converges to a solution of the integral

1, to a solution of the initial value problem (1). To

dition, that $||u_k(t)||_s \leq R$. This condition is satisfied

(7)

sitive numbers satisfying the conditio $\begin{align*}\n\frac{d}{dt}(t) \rVert_s &\leq \frac{d}{dt} \frac{d(s_0-s)}{t}, \\
\frac{d}{dt}(s_0-s) \rVert_s, \\
\text{with } t \geq 0, \text{ and, } t \geq 0. \text{ The case, the function is true, and the$ *a* $\left\{ \begin{array}{ll} 2 & \frac{d(s_0 - s)}{s_0 - s_1}, \\ 2 & \frac{d(s_0 - s)}{s_0 - s_1}, \end{array} \right\}$ **a** $\left\{ \begin{array}{ll} 2 & \frac{d(s_0 - s)}{s_0 - s_1}, \\ 2 & \frac{d(s_0 - s)}{s_0 - s_1}, \end{array} \right\}$ and, consequently, to a solution of the initial value problem (1). To we must

$$
||u_k(t) - u_{k-1}(t)||_s \leq \varepsilon_k R \tag{7}
$$

for $k = 1, 2, \ldots$, where ε_k are positive numbers satisfying the condition

$$
\sum_{k=1}^{\infty} \varepsilon_k \le 1.
$$
\n(8)

By virtue of (4) the inequality (7) is satisfied for $k = 1$ and $0 < t < a_1(s_0 - s)$ if

$$
a_1 s_0 < T \quad \text{and} \quad a_1 \leq \varepsilon_1 \frac{R}{K} \,. \tag{9}
$$

We shall prove that (7) can be fulfilled for every k if we diminish the t-interval step by step, where we must ensure that the length of the limit interval is positive. The k-th iteration u_k will be defined if $\sum_{i=1}^{\infty} \varepsilon_k \le 1.$ (8)
 $\sum_{i=1}^{\infty} \varepsilon_k \le 1.$ (8)
 $a_1s_0 < T$ and $a_1 \le \varepsilon_1 \frac{R}{K}$. (9)
 prove that (7) can be fulfilled for every *k* if we diminish the *t*-interval

term, where we must ensure that the le $a_1 s_0 < T$ and $a_1 \leq \varepsilon_1 \frac{R}{K}$. (9)

We shall prove that (7) can be fulfilled for every k if we diminish the t-interval

step by step, where we must ensure that the length of the limit interval is positive.

The k-t For the set of the set of the set of the set of u_k will be defined if
 $a_k(s_0 - s)$,
 $\cdots > a_k > 0$. Correspond

ctionals
 $\sup_{0 \le s < s_0} \sup_{0 \le t < a_k(s_0 - s)} ||u(t)||_s$
 $\bigg\{$
 $\bigg|$ functional $M(u)$ defined

$$
0\leq t
$$

a sequence of functionals

$$
0 \leq t < a_k(s_0 - s),
$$
\n
$$
a_2 > \cdots > a_k > 0. \text{ Corresponding to the } t \text{-inter}
$$
\nce of functionals

\n
$$
M_k(u) = \sup_{0 \leq s < s_0} \sup_{0 < t < a_k(s_0 - s)} \|u(t)\|_s \left(\frac{a_k(s_0 - s)}{t} - 1 \right)
$$

instead of the only functional $M(u)$ defined above. From this definition one immediately gets the property *M_k*(*u*) = $\sum_{i=1}^{n} I_i \leq a_k(s_0 - s)$,
 $> a_2 > \cdots > a_k > 0$. Corresponding to the *t*-intervals (10)
 $M_k(u) = \sup_{0 \leq s < s_0} \sup_{0 \leq t < a_k(s_0 - s)} ||u(t)||_s \left(\frac{a_k(s_0 - s)}{t} - 1 \right)$
 A f the only functional *M*(*u*) defined above. Fr

$$
M_{k+1}(u) \leq M_k(u) \quad \text{if} \quad a_{k+1} < a_k.
$$

Instead of the only functional $M(u)$ defined above. From this definition one inediately gets the property
 $M_{k+1}(u) \leq M_k(u)$ if $a_{k+1} < a_k$.

Now assume that the $u_j(t)$, $j = 1, 2, ..., k$, are defined and belong to \mathcal{B}_s i $M_{k+1}(u) \leq M_k(u)$ if α
ime that the $u_j(t)$, $j = s$, $a_1 > a_2 > \cdots > a_l$
 $|u_j(t) - u_{j-1}(t)||_s \leq \varepsilon_j R$,

$$
||u_i(t) - u_{i-1}(t)||_s \leq \varepsilon_i R,
$$

while $M_i(u_i - u_{i-1})$ is supposed to be finite. In order to prove the existence of u_{k+1} we choose any positive $a_{k+1} < a_k$. Later we shall get a positive lower bound for $a_k - a_{k+1}$. For an arbitrary $s, 0 \le s < s_0$, we define \bar{s} by $M_k(u) = \sup_{0 \le s < s_0} \sup_{0 \le t < a_k(s_s - s)} |u(t)|_s$ $\left(\frac{u}{t}\right)^{-1}$

of the only functional $M(u)$ defined above. From this defined
 $\lim_{s \to s_1} \lim_{s \to s_1} \lim_{s \to s_1} \lim_{s \to s_1} s \lim_{s \to s_1} \lim_{s \to s_1} s \lim_{s \to s_1} \lim_{s \to s_1} s \lim_{s \to s_1} \lim_{s \$

$$
a_{k+1}(s_0 - s) = a_k(s_0 - \tilde{s})
$$

²) From the definition of $M(u)$ one obtains immediately the estimate

$$
||u(t)||_s \leq \frac{tM(u)}{a(s_0-s)-t},
$$

thus $u(0)$ is proved to be equal to 0 provided $M(u)$ is finite.

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such that $s < \tilde{s} < s_0$. The k-th iteration \imath SUCHER 188 W. TUTSCHKE

such that $s < \tilde{s} < s_0$. The k-th iteration $u_k(\tau)$ is defined and belongs to $\mathcal{R}_{\tilde{s}}$ if $0 \leq \tau < a_k(s_0 - \tilde{s})$. In view of (8) we have

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\nsuch that
$$
s < \tilde{s} < s_0
$$
. The k-th iteration $u_k(\tau)$ is defined and be
\n $< a_k(s_0 - \tilde{s})$. In view of (8) we have
\n
$$
||u_k(\tau)||_s \leqq \sum_{j=1}^k ||u_j(\tau) - u_{j-1}(\tau)||_{\tilde{s}} \leqq \sum_{j=1}^k \varepsilon_j R < R,
$$
\nthus $F(\cdot, u_k(\cdot))$ and, consequently, $u_{k+1}(t)$ are defined and be
\n $< a_{k+1}(s_0 - s)$, $0 \leqq \tau \leqq t$. From (3) we get, moreover, that
\n
$$
||u_{k+1}(t) - u_k(t)||_s \leqq \int ||F(\tau, u_k(\tau)) - F(\tau, u_{k-1}(\tau)||_s) d\tau.
$$

thus $F(\cdot, u_k(\cdot))$ and, consequently, $u_{k+1}(t)$ are defined and belong to \mathcal{B}_s if $0 \le t < u_{k+1}(s_0 - s)$, $0 \le \tau \le t$. From (3) we get, moreover, that

$$
\|u_{k+1}(t)-u_k(t)\|_{s} \leq \int_{\tau=0}^{t} \|F(\tau,u_k(\tau))-F(\tau,u_{k-1}(\tau))\|_{s} d\tau.
$$

In order to estimate the norm in the integrand by assumption (iii) we must take the norm of $u_k(\tau) - u_{k-1}(\tau)$ in a space with larger index. This index will be chosen in dependence on τ and will be denoted by $s(\tau)$. The numbers τ and $s(\tau)$ must be in dependence on τ and will be denoted by $s(\tau)$. The numbers τ and $s(\tau)$ must be connected by $\tau < a_k(s_0 - s(\tau))$. This inequality may be rewritten as In order to estimate the norm in the integrand by assumption (iii) we multile norm of $u_k(\tau) - u_{k-1}(\tau)$ in a space with larger index. This index will be in dependence on τ and will be denoted by $s(\tau)$. The numbers τ $||u_{k+1}(t) - u_k(t)||_s \leq \int ||F(\tau, u_k(\tau)) - F(\tau, u_{k-1}(\tau))||_s d\tau.$

to estimate the norm in the integrand by assumption (iii) we must take

of $u_k(\tau) - u_{k-1}(\tau)$ in a space with larger index. This index will be chosen

dence on τ a

$$
s(\tau) < s_0 - \frac{\tau}{a_k}.\tag{11}
$$

Now remark that for any two real numbers *a* and *b* with $a < b$ we have $a + \frac{a - b}{2}$ $< b$. Thus (11) and $s < s(\tau)$ are satisfied if we set

$$
s(\tau) = s + \frac{1}{2} \left(s_0 - \frac{\tau}{a_k} - s \right).
$$
\n
$$
s(\tau) = s + \frac{1}{2} \left(s_0 - \frac{\tau}{a_k} - s \right).
$$
\n
$$
s(\text{iii}) \text{ we get, finally, the estimate}
$$
\n
$$
||u_{k+1}(t) - u_k(t)||_s \leq C \int_{\tau=0}^t \frac{||u_k(\tau) - u_{k-1}(\tau)||_{s(\tau)}}{s(\tau) - s} d\tau
$$
\n(12)

Applying (iii) we get, finally, the estimate

Now remark that for any two real numbers
$$
a
$$
 and b with $a < b$ we have $a + \frac{a - b}{2}$
\n $< b$. Thus (11) and $s < s(\tau)$ are satisfied if we set
\n
$$
s(\tau) = s + \frac{1}{2} \left(s_0 - \frac{\tau}{a_k} - s \right).
$$
\n(12)
\nApplying (iii) we get, finally, the estimate
\n
$$
||u_{k+1}(t) - u_k(t)||_s \leq C \int_0^t \frac{||u_k(\tau) - u_{k-1}(\tau)||_{s(\tau)}}{s(\tau) - s} d\tau
$$
\nif $0 \leq t < a_{k+1}(s_0 - s)$. By assumption, $M_k(u_k - u_{k-1})$ is finite. From the definition of M_k we get $(\tau > 0)$

of M_k we get $(\tau > 0)$ $(k - u_{k-1})$ is finite. From
 $\frac{(-1)}{2}$.

$$
||u_k(\tau)| - u_{k-1}(\tau)||_{s(\tau)} \leqq \frac{M_k(u_k - u_{k-1})}{\frac{a_k(s_0 - s(\tau))}{\tau} - 1}.
$$

Substituting (12), the following estimate follows:

$$
\langle a_{k+1}(s_0-s), \text{ By assumption, } M_k(u_k-u_{k-1}) \text{ is finite. From the}
$$
\n
$$
\text{get } (\tau > 0)
$$
\n
$$
||u_k(\tau) - u_{k-1}(\tau)||_{s(\tau)} \le \frac{M_k(u_k - u_{k-1})}{\tau}
$$
\n
$$
\text{ting (12), the following estimate follows:}
$$
\n
$$
||u_{k+1}(t) - u_k(t)||_s \le 4Ca_k M_k(u_k - u_{k-1}) t \int_{\tau=0}^t \frac{1}{(a_k(s_0 - s) - \tau)^2} d\tau.
$$

Calculating the integral, we get for $0 < t < a_{k+1}(s_0 - s)$ the inequality

$$
||u_{k+1}(t) - u_k(t)||_s \le 4Ca_k M_k(u_k - u_{k-1}) t \int \frac{1}{(a_k(s_0 - s) - \tau)^2} d\tau.
$$

ing the integral, we get for $0 < t < a_{k+1}(s_0 - s)$ the inequality

$$
||u_{k+1}(t) - u_k(t)||_s \le 4a_{k+1}CM_k(u_k - u_{k-1}) \frac{1}{\frac{a_k(s_0 - s)}{t} - 1},
$$

ich we obtain first

$$
M_{k+1}(u_{k+1} - u_k) \le 4a_{k+1}CM_k(u_k - u_{k-1})
$$
 (13)

from which we obtain first

$$
M_{k+1}(u_{k+1}-u_k)\leq 4a_{k+1}CM_k(u_k-u_{k-1})
$$

and second

On an Abstract Nonlinear Cauchy-Kowalewski Theorem 189
and second

$$
||u_{k+1}(t) - u_k(t)||_s \le 4 \frac{a_{k+1}^2}{a_k - a_{k+1}} CM_k(u_k - u_{k-1}),
$$
(14)
since

$$
\frac{a_k(s_0 - s)}{t} - 1 > \frac{a_k(s_0 - s)}{a_{k+1}(s_0 - s)} - 1 = \frac{a_k - a_{k+1}}{a_{k+1}}.
$$

On an Abstract Nonlinear Cauchy-Kowa
\nand second
\n
$$
||u_{k+1}(t) - u_k(t)||_s \leq 4 \frac{a_{k+1}^2}{a_k - a_{k+1}} CM_k(u_k - u_{k-1}),
$$
\nsince\n
$$
\frac{a_k(s_0 - s)}{t} - 1 > \frac{a_k(s_0 - s)}{a_{k+1}(s_0 - s)} - 1 = \frac{a_k - a_{k+1}}{a_{k+1}}.
$$
\nTaking into consideration the estimate (6) with $a = a_1$, the
\n
$$
M_k(u_k - u_{k-1}) \leq (4C)^{k-1} a_k \dots a_2 a_1 K,
$$
\nhence (14) gives
\n
$$
||u_{k+1}(t) - u_k(t)||_s \leq (4C)^k K \frac{a_{k+1}^2 a_k \dots a_1}{a_k - a_{k+1}}.
$$
\nFrom (13) we derive further that $M_{k+1}(u_{k+1} - u_k)$ is finite

Taking into consideration the estimate (6) with $a = a_1$, the inequality (13) leads to

$$
M_k(u_k - u_{k-1}) \leq (4C)^{k-1} a_k \dots a_2 a_1 K,
$$

$$
||u_{k+1}(t) - u_k(t)||_s \le 4 \frac{1}{a_k - a_{k+1}} CM_k(u_k - u_{k-1}),
$$
\n
$$
\frac{a_k(s_0 - s)}{t} - 1 > \frac{a_k(s_0 - s)}{a_{k+1}(s_0 - s)} - 1 = \frac{a_k - a_{k+1}}{a_{k+1}}.
$$
\ninto consideration the estimate (6) with $a' = a_1$, the inequality (13) leads to

\n
$$
M_k(u_k - u_{k-1}) \le (4C)^{k-1} a_k \dots a_2 a_1 K,
$$
\n4) gives

\n
$$
||u_{k+1}(t) - u_k(t)||_s \le (4C)^k K \frac{a_{k+1}^2 a_k \dots a_1}{a_k - a_{k+1}}.
$$
\n3) we derive, further, that $M_{k+1}(u_{k+1} - u_k)$ is finite, too. From (15) one

From (13) we derive, further, that $M_{k+1}(u_{k+1} - u_k)$ is finite, too. From (15) one gets the desired lower bound for $a_k - a_{k+1}$ guaranteeing the estimate $\mathcal{U}_{k+1}(t) = u_k(t) / |s| \leq (10)$
 I is example to the *IU_{k+1}* (*t*) $-u_k(t)$, $\leq \varepsilon_{k+1}R$.
 I is the bound is Taking into consideration the estimate (6) with $a = a_1$, the inequality

Taking into consideration the estimate (6) with $a' = a_1$, the inequality
 $M_k(u_k - u_{k-1}) \leq (4C)^{k-1} a_k \dots a_2 a_1 K$,

hence (14) gives
 $||u_{k+1}(t) - u_k(t)||_s$

$$
||u_{k+1}(t) - u_k(t)||_s \leq \varepsilon_{k+1} R.
$$

$$
(4C)^k \frac{K}{R} a_{k+1}^2 a_k \dots a_1 \frac{1}{\varepsilon_{k+1}} \le a_k - a_{k+1}.
$$
 (16)

 $\frac{a_{k+1}^2 a_k \dots a_1}{a_k - a_{k+1}}$. (15)
 $M_{k+1}(u_{k+1} - u_k)$ is finite, too. From (15) one
 $-a_{k+1}$ guaranteeing the estimate
 $a_k - a_{k+1}$. (16)

e defined for $0 \le t < a(s_0 - s)$ and will belong

3) we must ensure that $a > 0$. T msideration the estimate (6) with $a' = a_1$, the inequality (13) le
 $-u_{k-1} \leq (4C)^{k-1} a_k \dots a_2 a_1 K$,

S
 $\|h\|_s \leq (4C)^k K \frac{a_{k+1}^2 a_k \dots a_1}{a_k - a_{k+1}}$.

derive, further, that $M_{k+1}(u_{k+1} - u_k)$ is finite, too. From (1

I The limit-function $u = \lim u_k$ will be defined for $0 \le t < a(s_0 - s)$ and will belong to \mathscr{B}_s , where $a = \lim a_k$. Besides (16) we must ensure that $a > 0$. To this end we define a_{k+1} as product hence (14) gives
 $||u_{k+1}(t) - u_k(t)||_s \leq (4C)^k K^{\frac{N}{2}}$

From (13) we derive, further, that

gets the desired lower bound for a_k –
 $||u_{k+1}(t) - u_k(t)||_s \leq \varepsilon_{k+1} R$.

This lower bound is
 $(4C)^k \frac{K}{R} a_{k+1}^2 a_{k} \dots a_1 \frac{1$ $||u_{k+1}(t) - u_k(t)||_s \leq (4C)^k K \frac{a_{k+1}^k a_{k} \dots a_1}{a_k - a_{k+1}}.$

3) we derive, further, that $M_{k+1}(u_{k+1} - u_k)$ is fidesired lower bound for $a_k - a_{k+1}$ guaranteeing th
 $||u_{k+1}(t) - u_k(t)||_s \leq \varepsilon_{k+1} R.$

ar bound is
 $(4C)^k \frac{$ desired lower bound for $a_k - a_{k+1}$ guaranteeing the estimate
 $||u_{k+1}(t) - u_k(t)||_s \leq \varepsilon_{k+1}R$.

ar bound is
 $(4C)^k \frac{K}{R} a_{k+1}^2 a_k \dots a_1 \frac{1}{\varepsilon_{k+1}} \leq a_k - a_{k+1}$.

Aunction $u = \lim u_k$ will be defined for $0 \leq t < a(s_0 - s)$ $t < a(s_0 - s_1)$
that $a > 0$
nn
m
onvergence if Finition $u = \lim_{\alpha} a_k$. Besides (16) we must ensure that $a > 0$. To, this end
here $a = \lim_{k \to \infty} a_k$. Besides (16) we must ensure that $a > 0$. To, this end
 $a_{k+1} = a_k(1 - \delta_k)$, $0 < \delta_k < 1$
as in Nirenberg's and Nishida's pape

$$
a_{k+1} = a_k(1 - \delta_k), \qquad 0 < \delta_k < 1
$$

similarly as in Nirenberg's and Nishida's papers³). Then

$$
a=a_1\prod_{k=1}^{\infty}(1-\delta_k)
$$

is positive if the infinite product converges. For this convergence the condition

where
$$
a = \lim a_k
$$
. Besides (16) we must ensure that $a > 0$. To this end we
\n
$$
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$$
\ny as in Nirenberg's and Nishida's papers³). Then
\n
$$
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$$
\nive if the infinite product converges. For this convergence the condition
\n
$$
\sum_{k=1}^{\infty} \delta_k < +\infty
$$
\n
$$
\sum_{k=1}^{\infty} \delta_k < +\infty
$$
\n(17)\n
$$
(4C)^k \frac{K}{R} a_{k+1}^2 a_{k-1} a_{k-2} \dots a_1 \le \varepsilon_{k+1} \delta_k, \qquad k = 1, 2, \dots
$$
\n
$$
\lim g \varepsilon_{k+1} \text{ and } \delta_k. \text{ This relation is fulfilled if}
$$
\n
$$
(4a_1 C)^k a_1 \frac{K}{R} \le \varepsilon_{k+1} \delta_k
$$
\n
$$
(18)
$$
\n
$$
\text{try } k = 1, 2, \dots
$$
\nThe existence of a solution of the initial value problem (1)

is sufficient and necessary. Condition (16) may be rewritten to become the relation

$$
(4C)^k \frac{K}{R} a_{k+1}^2 a_{k-1} a_{k-2} \dots a_1 \leq \varepsilon_{k+1} \delta_k, \qquad k = 1, 2, \dots
$$

connecting ε_{k+1} and δ_k . This relation is fulfilled if

$$
\sum_{k=1}^{L} b_k < +\infty
$$
\nis sufficient and necessary. Condition (16) may be rewritten to become the relation\n
$$
(4C)^k \frac{K}{R} a_{k+1}^2 a_{k-1} a_{k-2} \dots a_1 \le \varepsilon_{k+1} \delta_k, \qquad k = 1, 2, \dots
$$
\nconnecting ε_{k+1} and δ_k . This relation is fulfilled if\n
$$
(4a_1 C)^k a_1 \frac{K}{R} \le \varepsilon_{k+1} \delta_k
$$
\nfor every $k = 1, 2, \dots$ The existence of a solution of the initial value problem (belonging to \mathcal{B}_s for\n
$$
0 \le t < a(s_0 - s), \qquad a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k) > 0,
$$
\nis consequently reduced to the following auxiliary problem:\n
$$
\delta^3
$$
 The basic idea also taken over from the papers mentioned above is the consideration of the functionals M_k .

-.

for every $k = 1, 2, \ldots$ The existence of a solution of the initial value problem (1) belonging to \mathscr{B}_{s} for

$$
0 \leq t < a(s_0 - s), \quad a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k) > 0
$$

3) 'rhe basic idea also taken over from the papers mentioned above is the consideration of the functionals M_k .

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Find positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ such that the conditions (8) and (17) are satisfied. Find, further, a positive number a_1 satisfying (9) as well as (18) for every k .

There are various possibilities to solve the auxiliary problem. In the next section we shall realize only two of them. In view of $\varepsilon_k \to 0$ and $\delta_k \to 0$ for $k \to \infty$ from (18) one immediately gets that a_1 must in any case satisfy the inequality $4a_1C < 1$.

3. In order to fulfill condition (8) we first choose $\varepsilon_k = 1/2^k$. Then in view of (18) we may choose $\delta_k = 2(8a_1C)^k a_1K/R$. Taking into consideration condition (17) this means that a_1 must satisfy the inequality $8a_1C < 1$ instead of $4a_1C < 1$. In order to ensure that every δ_k is less than 1 we restrict a_1 by $2a_1K \leq R$. This condition is identic with the second one of (9). Summarizing (9) and $8a_1C < 1$ we get a

First choice of a: Let a_1 be a positive number satisfying the inequality

$$
a_1 < \min\left(\frac{T}{s_0}, \frac{R}{2K}, \frac{1}{8C}\right).
$$

Then

$$
a = a_1 \prod_{k=1}^{\infty} \left(1 - 2(8a_1C)^k a_1 \frac{K}{R}\right).
$$

Second we put $\varepsilon_1 = 1/2$ and $\delta_k = \lambda \varepsilon_{k+1}$. Then (18) is satisfied for $\lambda \varepsilon_{k+1}^2 = (4a_1C)^k$ $\times a_1K/R$. Provided that $4a_1C<1$ from (8) we get

$$
\sqrt{\lambda} = 4a_1 \sqrt{\frac{KC}{R}} \frac{1}{1 - 2\sqrt{a_1} \sqrt{C}}
$$

and, finally,

$$
\delta_{\pmb{k}} = \frac{K}{2RC} \frac{(2\sqrt{a_1}\sqrt{C})^{k+3}}{1-2\sqrt{a_1}\sqrt{C}}
$$

(where we took the equality in (8)). In order to ensure that $\delta_k < 1$ for every k we require

$$
\frac{K}{2RC}\frac{2\,\sqrt{a_1}\,\sqrt{C}}{1\,-\,2\,\sqrt{a_1}\,\sqrt{C}}\,<1\,.
$$

This leads to the condition

$$
a_1 \ll \frac{1}{4C} \left(\frac{2RC}{2RC + K}\right)^2 \tag{19}
$$

for a_1 , which is more restrictive than $4a_1C < 1$. Summarizing (9) and (19) we get a Second choice of a : Let a_1 be a positive number satisfying the inequality

$$
u_1 < \min\left(\frac{T}{s_0}, \frac{R}{2K}, \frac{1}{4C}\left(\frac{2RC}{2RC + K}\right)^2\right).
$$

Then the number a is given by

$$
a = a_1 \prod_{k=1}^{\infty} \left(1 - \frac{K}{2RC} \frac{\left(2 \sqrt{a_1} \sqrt{C} \right)^{k+3}}{1 - 2 \sqrt{a_1} \sqrt{C}} \right).
$$

4. Now let a number $a_1 > 0$ and two sequences $\{\varepsilon_k\}_{k=1,2,...}$ $\{\delta_k\}_{k=1,2...}$ satisfying the conditions (8) , (9) , (17) and (18) be given. Suppose, moreover, that the conditions (i) – (iii) are satisfied. Then the following theorem holds. **12.13** On an Abstract Notice 10.
 12.13 On an Abstract Noticions (8), (9), (17) and (18) be give

are satisfied. Then the following t

rem: The limit function $u(t) = \text{lin}$

is a solution of the initial value pro
 $0 \le t <$

Theorem: The limit function $u(t) = \lim u_k(t)$ *of the functions* u_k *defined by (3),* $u_0 = 0$, is a solution of the initial value problem (1) belonging to \mathcal{B}_s if \cdot

$$
0 \le t < a(s_0 - s), \qquad a = a_1 \prod_{k=1}^{\infty} (1 - \delta_k).
$$
\nallar considerations (cf. the bibliography) th

\n ∂_s can be proved, too.

\nIn $(1 - \alpha q^k) = -\sum_{j=1}^{\infty} \frac{1}{j} (\alpha q^k)^j$

\n $\leq 1, \alpha > 0, \alpha q < 1$ and since

By similar considerations (cf. the bibliography) the uniqueness of the solution in the scale \mathscr{B}_{s} can be proved, too. By similar considerations (cf. the bit
the scale \mathscr{B}_s can be proved, too.

5. Since
 $\ln (1 - \alpha q^k) = -\sum_{j=1}^{\infty} \frac{1}{j} (\alpha q^k)^j$

if $0 < q < 1$, $\alpha > 0$, $\alpha q < 1$, and since.
 $\sum_{i=1}^{\infty} q^{jk} = \frac{q^j}{1 - q^j} > \frac{q^j}{1 - q^j$

5. Since

$$
\ln (1 - \alpha q^k) = -\sum_{j=1}^{\infty} \frac{1}{j} (\alpha q^k)^j
$$

$$
\sum_{k=1}^{\infty} q^{jk} = \frac{q^j}{1-q^j} > \frac{q^j}{1-q}.
$$

one can easily check that

$$
q < 1, \, \alpha > 0, \, \alpha q < 1, \, \text{and since}
$$
\n
$$
\sum_{k=1}^{\infty} q^{jk} = \frac{q^j}{1 - q^j} > \frac{q^j}{1 - q}
$$
\nn easily check that

\n
$$
\prod_{k=1}^{\infty} (1 - \alpha q^k) > (1 - \alpha q)^{\frac{1}{1 - q}}.
$$

This inequality allows to obtain lower bounds for the infinite products regarded in Section 3. In this way one is able to estimate the t -interval in which the solution $u = u(t)$ of the initial value problem (1) exists. In order to illustrate this method we assume $s_0 = 1$, $T = 1$, $K = R = 1$, $C = 1/8$. Then the first choice of the numwe assume $s_0 = 1$, $T = 1$, $K = R = 1$, $C = 1/8$. Then the first choice of the number *a* formulated in Section 3 leads to the condition $a_1 < 1/2$ and we get for *a* the estimate
 $a = a_1 \prod_{k=1}^{\infty} (1 - 2a_1^{1+k}) > a_1(1 - 2a_1$ estimate

$$
a = a_1 \prod_{k=1}^{\infty} (1 - 2a_1^{1+k}) > a_1(1 - 2a_1^2)^{\frac{1}{1-a_1}}.
$$

Finally one gets with $a_1 = 0.35$ the estimate $a > 0.2389$, thus the limit function $u = u(t)$ exists and belongs to \mathcal{B}_s at least for *I* with $0 \le t < 0.2389(1-s)$. In this case Nishida's estimates prove the convergence of the $u_k = u_k(t)$ only for $a < 1/16$.

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