# **Maximal Monotone Operators and Saddle Functions I')**

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Wir untersuchen den monotonen Operator  $T_K \subseteq E \times E^*$ ,  $f \in T_K x := [-f, f] \in \partial K(x, x)$ , der uber das Subdifferential einer konkav-konvexen Sattelfunktion *K* definiert ist. Unsere Uber. legungen werdon durch die Tatsache motiviert, daB jeder maximal monotone Operator *A* in der Form  $A = T_K$  darstellbar ist. Wir zeigen, daß  $T_K$  genau dann maximal monoton ist, wenn *K* in einer abgeschwächten Form schiefsymmetrisch ist. Dies erlaubt eine Verallgemeinerung früher erzielter Ergebnisse.

Исследуется монотонный оператор  $T_K \subseteq E \times E^*$ ,  $j \in T_K x := [-f, f] \in \partial K(x, x)$ , определённый с помощью субдифференциала вогнуто-выпуклой седловой функции *K*. Paccyaeiitia o6octionaiiai TeM, *'ITO* ma?+b1fi Mam-c1tMaJ1blmo MOiIOTOiIHbifi olieparop *<sup>A</sup>*  $M$ ожет быть представлен в виде  $A = T_K$ . Показывается, что  $T_K$  максимально моно-T01111bift TOrga it *T0I1,K0* Tora, mora *K 911.1ACTCH B* ocjia6jieuiioft (J)opMe hOCOCIIM: метрической. Это обобщение результатов полученных раньше. Рассуждения обоснованы тем, что каждый максимально монотонный оператор A может быть представлен в виде  $A = T_K$ . Показывается, что  $T_K$  максимально моно-<br>гонный тогда и только тогда, когда K является в сослаблённой форме к

We investigate the monotone operator  $T_K \subseteq E \times E^*$ ,  $f \in T_K x := [-f, f] \in \partial K(x, x)$ , which is defined via the subdifferential of a concave-convex saddle function *K*. Our considerations are motivated by the fact that each maximal monotone operator A possesses a representation of the form  $A = T_K$ . We show that  $T_K$  is maximal monotone if and only if K is in a relaxed form skew-symmetric. This allows a generalization of results obtained previously.

# **1.1 Introduction**<br> **In this paper we**  $T_K \subseteq M$

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T_K \subseteq E \times E^*; \quad f \in T_K x := [-f, f] \in \partial K(x, x),
$$
\ndefined via the subdifferential of a concave

which is defined via the subdifferential of a concave-convex saddle function  $K: E \times E \to \overline{\mathbf{R}}$ . In [3] we imposed a condition on *K* (namely, a relaxed form of skewof the form  $A = T_K$ . We show that  $T_K$  is maximal monotone if and only if K is in a relaxed<br>form skew-symmetric. This allows a generalization of results obtained previously.<br>
I.1 Introduction<br>
In this paper we investigate symmetry) guaranteeing the maximal monotonicity of the operator  $T_K$ . Now we show that this condition is even necessary for the maximality of  $T_K$ . This fact allows to improve several results previously obtained in  $[3-4]$ . In the forthcoming Part II of this paper we show that each maximal monotone operator  $A \subseteq E \times E^*$ has the form  $A = T_K$ , where K is a closed and skew-symmetric saddle function. In general, this saddle function cannot be found constructively. It turns out, however, that several functions K with  $A = T_K$  can be constructed if one relaxes the assumption on the skew-symmetry of *K*. In Section I.5 we are concerned with the question how, from a given saddle function, we can construct a "more regular" one such that both saddle functions generate the same monotone operator  $T$ . The principles stated here allow a short presentation of many results about saddle functions. Here we use them to generalize a principle from  $[4]$  concerning the existence of

<sup>&#</sup>x27;) Part H of the paper will be published in one of the following issues of this journal.

certain skew-symmetric saddle functions connected with the theory of maximal monotone operators (compare Th. 1.6). Besides, we give some estimates for the sizes of the domains of a saddle function *K* and of the monotone operator corresponding to it (compare the Th. I.7, I.8 and the Cor. I.1, I.2). These results are partially strenghtened in the forthcoming Part II. Among other things, we show there that the *topological* interior of the domain of a maximal monotone operator is non-empty, provided that the convex hull of this domain has an *algebraic* interior. Moreover, we shall give the following approach to the solution of the equation  $Ax \ni 0$ , with a maximal monotone operator  $A \subseteq E \times E^*$ : Find an arbitrary saddle function *K* with  $T_K = A$  (since *K* is not supposed to be skew-symmetric this can be done in a constructive manner). Then, for any saddle point  $[x_0, y_0]$  of  $K$ , the element  $(x_0 + y_0)/2$  solves the equation  $Ax \ni 0$ . On the other hand, if  $x_0 \in E$  solves  $Ax \ni 0$ , then  $[x_0, x_0]$  is a saddle point of *K*.

## **1.2 Closed saddle hinetions**

For the readers' convenience, hero we recall some definitions of R. T. Rockafeilar's theory of closed saddle functions. The basic material can be found in  $[1, 2, 7, 8]$  and for the finitedimensional case also in *[5, 6].* 

By *E* and *F* we denote, if not otherwise stated, locally convex Hausdorff spaces over the reals. For the dual pairing between  $E$  and the topologically dual space  $E^*$ both the notations  $f(x)$  and  $(f, x)$  are used. We are dealing with functions with values in the extended real line  $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm \infty\}$ . Let  $p: E \to \overline{\mathbf{R}}$  be *convex* (i.e.) By *E* and *F* we denote, if not otherwise stated, locally convex Hausdorff spaces<br>over the reals. For the dual pairing between *E* and the topologically dual space  $E^*$ <br>both the notations  $f(x)$  and  $(f, x)$  are used. We a over the reals. For the dual pairing between E and that the notations  $f(x)$  and  $(f, x)$  are used. We arvalues in the extended real line  $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm \infty\}$ .<br>
epi  $p := \{[x, t] \in E \times \mathbf{R} : p(x) \leq t\}$  is a convex set). epi  $p := \{ [x, t] \in E \times \mathbb{R} : p(x) \le t \}$  is a convex set). We call p proper if  $p(x) > -\infty$  for all  $x \in E$  and dom  $p := \{ x \in E : p(x) < +\infty \} \neq \emptyset$ . The *closure* cl p of p is the pointwise supremum of all continuous affine minorants both the notations  $f(x)$  and  $(f, x)$  are used. We are dealing with functions with values in the extended real line  $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm \infty\}$ . Let  $p: E \to \overline{\mathbf{R}}$  be convex (i.e.  $f$ ) epi  $p := \{[x, t] \in E \times \mathbf{R} : p(x) \le t\}$ there is no such minorant): ar sase since in [v, 0, *F*,<br>
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for all  $x \in E$  and dom  $p := \{x \in E : p$  $\{ [x, t] \in E \times \mathbf{R} : p(x) \leq t \}$  is a convex set). We<br>  $\in E$  and dom  $p := \{ x \in E : p(x) < +\infty \} \neq \emptyset$ .<br>  $\in$  supremum of all continuous affine minorano<br>  $\infty$  supremum of all continuous affine minorano<br>  $\infty$  such minorant):<br>  $\$ 

$$
\operatorname{cl} p(x) = \sup_{f \in E^*} \inf_{v \in E} \left\{ f(x - v) + p(v) \right\}. \tag{1.1}
$$

In the case  $p = c/p$  we say that p is *closed.* The *subdifferential*  $\partial p \subseteq E \times E^*$  of p is defined by

$$
[x, f] \in \partial p := p(x) + f(v - x) \leq p(v) \quad \text{for all} \ \ v \in E. \tag{I.2}
$$

The same notions also make sense for *concave* functions  $q: F \to \overline{R}$ . One has only to change the roles of  $+\infty$  and  $-\infty$ ,  $\leq$  and  $\geq$ , sup and infinitions above.<br>For example, the closure of a concave function is given by *f*  $p(x) = \sup_{f \in \mathbb{F}^*} \inf_{v \in E} \{f(x - v) + p(v)\}.$  (1.1)<br>  $\lim_{f \in \mathbb{F}^*} \inf_{v \in E} \{f(x - v) + p(v)\}.$  (1.1)<br>  $[\big\{x, f\big\} \in \partial p := p(x) + f(v - x) \leq p(v) \text{ for all } v \in E.$  (1.2)<br>  $[\big\{x, f\big\} \in \partial p := p(x) + f(v - x) \leq p(v) \text{ for all } v \in E.$  (1.2)<br>  $[\big\{x, f\big\} \in \partial p$ 

$$
\operatorname{cl} q(x) := \inf_{f \in F^*} \sup_{v \in F} \{ f(x - v) + q(v) \}, \tag{I.3}
$$

and for the subdifferential  $\partial q$  of  $q$  holds

$$
[x, f] \in \partial q := q(x) + f(v - x) \geq q(v) \quad \text{for all} \ \ v \in F. \tag{I.4}
$$

By a *saddle function*  $K: E \times F \to \overline{\mathbf{R}}$  we shall always understand a function which is concave in the first argument and convex in the second one. The *convex closure*  cl<sub>2</sub> K of K is obtained by closing  $K(x, \cdot)$  as a convex function (for arbitrary  $x \in E$ ),<br>and similarly the *concave closure* cl<sub>1</sub> K. Two saddle functions K and L are said to<br>be *equivalent* if we have cl<sub>i</sub> K = cl<sub>i</sub>L f and similarly the *concave closure* cl<sub>1</sub> K. Two saddle functions K and L are said to be *equivalent* if we have cl<sub>i</sub> K = cl<sub>i</sub> L for  $i = 1, 2$ . If both cl<sub>1</sub> K and cl<sub>2</sub> K are equivalent to *K* we refer to *K* as a *closed* saddle function. In particular, the *lower closure*   $cl_2 \, cl_1 \, K$  and the *upper closure*  $cl_1 \, cl_2 \, K$  of  $K$  are closed saddle functions. For these saddle functions we can even say more. Namely, the lower closure is lower closed and the upper closure is upper closed. Here *lower closed* saddle functions *L* are characterized by the identity cl<sub>2</sub> cl<sub>1</sub>  $L = L$ , and *upper closed* ones by cl<sub>1</sub> cl<sub>2</sub>  $L = L$ . We call *K proper* if dom  $K := \text{dom}_1 K \times \text{dom}_2 K = 0$ , with

$$
\operatorname{dom}_1 K := \{x \in E : \operatorname{cl}_2 K(x, y) > -\infty \text{ for all } y \in F\}
$$

and

$$
\operatorname{dom}_2 K := \{ y \in F : \operatorname{cl}_1 K(x, y) < +\infty \text{ for all } x \in E \}.
$$

Note that a closed saddle function is proper if and only if it is finite in at least one point. If  $\partial_1 K(\cdot, y)$  (resp.  $\partial_2 K(x, \cdot)$ ) is the subdifferential of the concave function  $K(\cdot, y)$  (or of the convex function  $K(x, \cdot)$ , respectively), the mapping

$$
\partial K: E \times F \to 2^{E^* \times F^*}; \ \partial K(x, y) := \partial_1 K(x, y) \times \partial_2 K(x, y),
$$

is referred to as the *subdif/erential* of *K.* An important feature of equivalent saddle functions is that they have the same subdifferentials.

We shall occasionally make use of the conventions inf  $\theta = +\infty$  and sup  $\theta = -\infty$ . Moreover, sometimes we shall identify a multivalued mapping with its graph. call of K. An important feature of equivalent saddle<br>he same subdifferentials.<br>the same subdifferentials.<br>I identify a multivalued mapping with its graph.<br>I identify a multivalued mapping with its graph.<br>anctions<br>differen

## **1.3** Skew-symmetric saddle functions

In this section we investigate different versions of the notion of a skew-symmetric saddle function. All these notions will frequently be used in the sequel. We start with  $P = \sum_{i=1}^{n} P_i$ characteries we shall identify a multivalued mapping with its graph.<br>
symmetric saddle functions<br>
ction we investigate different versions of the notion of a skew-symmetric saddle<br>
All these notions will frequently be used of the notion of a skew-symmetric saddle<br>in the sequel. We start with<br>addle function. Then we say that<br>for all  $x, y \in E$ ,<br> $x \in E$ ,<br> $x, y \in E$ .<br> $x \in E$ ,<br> $\therefore x \in E$ .

\n- (., *y*) (or of the convex function 
$$
K(x, \cdot)
$$
, respectively), the mapping  $\partial K : E \times F \to 2^{E^* \times F^*}; \partial K(x, y) := \partial_1 K(x, y) \times \partial_2 K(x, y)$ , referred to as the *subdifferential* of  $K$ . An important feature of equivalent *s* notions is that they have the same subdifferentials.
\n- We shall occasionally make use of the conventions  $0 = +\infty$  and  $\sup \mathcal{O} =$  or  
every, sometimes we shall identify a multivalued mapping with its graph.
\n- **8 Skew-symmetric saddle functions**
\n- **9 Stew-symmetric saddle functions**
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*2. K satisfies the condition (\*\*)* if

$$
\operatorname{cl}_2 K(x, x) \leq 0 \leq \operatorname{cl}_1 K(x, x) \quad \text{for all} \ \ x \in E.
$$

 $cl_2 K(x, y) = -cl_1 K(y, x)$  for all  $x, y \in E$ .

this section we investigate different versions of the notion of a skew-symmetric saddle<br>
content. All these notions will frequently be used in the sequel. We start with<br>
Definition I.1: Let  $K: E \times E \to \overline{R}$  be a saddle fu 3. *K is skew-symmetric* if<br>  $cl_2 K(x, y) = -cl_1 K(y, x)$  for all  $x, y \in E$ .<br>
The relations between these notions are the contents of<br>
Lemma I.1: *For any saddle function*  $K: E \times E \to \overline{\mathbf{R}}$  *the implications* (i)  $\to$  (ii)  $\rightarrow$  (iii) *hold true:* 

*(i) K is a skew-symmetric saddle /unction,* 

*(ii) K satisfies the condition*

cl<sub>2</sub>  $K\left(x, \frac{x+y}{2}\right) \leq c l_1 K\left(\frac{x+y}{2}, y\right)$  for all  $x, y \in E$ ,<br>
2. *K satisfies the condition*  $(**)$  if<br>  $c l_2 K(x, x) \leq 0 \leq c l_1 K(x, x)$  for all  $x \in E$ ,<br>
3. *K is skew-symmetric* if<br>  $c l_2 K(x, y) = -c l_1 K(y, x)$  for all  $x, y \in E$ .<br>
T Proof: The first implication has been proved in [4). Let *K* now satisfy the condition: The first implication has been proved in [4]. Let  $K$  how satisfy the condition (\*\*). Clearly, (\*) holds whenever:  $x \notin \text{dom}_1 K$  or  $y \notin \text{dom}_2 K$ . Assuming  $x \in \text{dom}_1 K$  and  $y \in \text{dom}_2 K$  we get Examples the condition (\*).<br>  $\therefore$  The first implication has been proved in [4]. Let K now<br>  $K$  and  $y \in \text{dom}_2 K$  we get<br>  $\text{cl}_2 K(x, \frac{x+y}{2}) \leq \frac{\text{cl}_2 K(x, x) + \text{cl}_2 K(x, y)}{2} \leq \frac{\text{cl}_2 K(x, y)}{2}$ 

Equations of the equations are a subsubsubproblem, 
$$
f(x)
$$
 satisfies the condition  $(**)$ ,

\n5 satisfies the condition  $(**)$ ,

\n6: The first implication has been proved in [4]. Let  $K$  now is:

\n7: Clearly,  $(*)$  holds whenever  $x \notin \text{dom}_1 K$  or  $y \notin \text{dom}_1 K$  and  $y \in \text{dom}_2 K$  we get

\n8:  $\text{cl}_2 K\left(x, \frac{x+y}{2}\right) \leq \frac{\text{cl}_2 K(x, x) + \text{cl}_2 K(x, y)}{2} \leq \frac{\text{cl}_2 K(x, y)}{2} \leq \frac{\text{cl}_2 K(x, y)}{2} \leq \frac{\text{cl}_1 K(x, y) + \text{cl}_1 K(y, y)}{2} \leq \text{cl}_1 K\left(\frac{x+y}{2}, y\right)$ .

 

Lemma I.2: Any closed saddle function satisfying the condition (\*\*) is proper.

Proof: Suppose that  $K: E \times E \to \overline{\mathbf{R}}$  is, improper. For reasons of symmetry we can assume dom<sub>1</sub>  $K = \emptyset$ . Then cl<sub>2</sub>  $K(s, y) = -\infty$  for all  $x, y \in E$  and hence, cl<sub>1</sub>  $K = cl_1 cl_2 K = -\infty$ . But this is contradictory to the condition  $(**)$ Lemma I.2: Any closed suddle function sutisfying the condition (\*\*)<br>
Proof: Suppose that  $K: E \times E \to \mathbf{R}$  is, improper. For reasons can assume dom<sub>1</sub>  $K = \emptyset$ . Then  $cl_2 K$ ,  $y$ ) =  $-\infty$  for all  $x, y \in E$  a<br>
=  $cl_1 cl_2 K \equiv -\infty$ 1. download the mass of the New York 1. By the new York 1. dog to the new York 1. dong K, 2. K is lower closed (upper closed) if *fying the con*<br>  $\infty$  for all  $x$ <br>  $\infty$  for all  $x$ <br>  $\infty$  for all  $\infty$ <br>  $\infty$ 

Now we give a simple criterion for the closedness of skew-symmetric saddle functions.

Lemma I.3: Let  $K: E \times E \to \mathbf{R}$  be skew-symmetric. Then

1. 
$$
\operatorname{dom}_1 K = \operatorname{dom}_2 K,
$$

2. *K* is lower closed (upper closed) if and only if  $\text{cl}_2 K = K$  (resp.  $\text{cl}_1 K = K$ ) *holds true.* 

*(1.5)* 

For the proof we refer to  $[4]$ . The identity  $(I.5)$  gives rise to the following

**Definition 1.2:** Let  $K: E \times E \rightarrow \overline{\mathbf{R}}$  be skew-symmetric. Then we set  $\text{Dom } K := \text{dom}_1 K = \text{dom}_2 K.$ tions.<br>
Lemma I.3: Let  $K: E \times E \to \mathbf{R}$  be skew-symmetric. Then<br>
1. dom<sub>1</sub>  $K = \text{dom}_2 K$ ,<br>
2. K is lower closed (upper closed) if and only if  $cl_2 K = K$  (re<br>
holds true.<br>
For the proof we refer to [4]. The identity (I.5) give

It'is interesting that a relaxed version of the identity (1.5) remains true if *<sup>K</sup>* merely satisfies the condition (\*\*). We get

Lemma I,4: *For any closed saddle function*  $K: E \times E \to \overline{\mathbf{R}}$  *satisfying the condition*<br>
w) it holds  $\frac{\text{dom}_1 K}{\text{dom}_1 K} = \frac{\text{dom}_2 K}{\text{dom}_2 K}$ . (I.6)

$$
\overline{\text{dom}_1 \, K} = \overline{\text{dom}_2 \, K}.
$$
 (I.6)

a I.3: Let  $K: E \times E \rightarrow \mathbf{R}$  be sket<br>
dom<sub>1</sub>  $K = \text{dom}_2 K$ ,<br>
s lower closed (upper closed) i,<br>
e proof we refer to [4]. The idention 1.2: Let  $K: E \times E = \text{dom}_1 K = \text{dom}_2 K$ ,<br>
tteresting that a relaxed versitisfies the condition (\* In Section 1.5 we shall generalize the statement of Lemma *1.4,* provided that E is a reflexive Banach space (compare Cor. I.2 and also Th. I.8). Moreover, we shall there investigate the question how Lemma I.4 has to be modified if the saddle function  $K$  only satisfies the condition  $(*)$  (compare Cor. I.1). Fig. 15 interesting that a real merely satisfies the condition<br>
Lemma I,4: For any close<br>
(\*\*) it holds<br>
dom<sub>1</sub> K = dom<sub>2</sub> K.<br>
In Section I.5 we shall general flexive Banach space (compare the question how Lemma I.4 h<br>
co dom<sub>1</sub>  $K = \text{dom}_2 K$ .<br>In Section I.5 we shall generalize the statement of Lemma I.4, prov<br>flexive Banach space (compare Cor. I.2 and also Th. I.8). Moreover, we s<br>the question how Lemma I.4 has to be modified if the saddle *c ± h(v — z<sup>0</sup> ) <sup>&</sup>lt;*c1 <sup>2</sup> *K(x<sup>0</sup> , v)* for all *v € K.*  In Section 1.5 we shall generalize the statement of Lemma 1.4, provid<br>flexive Barach space (compare Cor. I.2) and also Th. I.8). Moreover, we sha<br>condition (\*) (compare Cor. I.1).<br> $2 \text{ mod } 1$  and so be modified if the sadd

Proof of Lemma I.4: For reasons of symmetry it suffices to prove dom<sub>1</sub>  $K$  $\overline{\text{dom}_2 K}$ . Thus, let  $x_0 \in \text{dom}_1 K$  be given arbitrarily. Hence, by the definition of V **Proof of Lemma I.4:** For reasons of symmetry it suffices to prove dom<sub>1</sub> *K*  $\subseteq$  dom<sub>2</sub> *K*. Thus, let  $x_0 \in \text{dom}_1 K$  be given arbitrarily. Hence, by the definition of dom<sub>1</sub> *K*, we find an  $h \in E^*$  and a real number c on how Lemma I.4 has to be modified if the saddle<br>
(\*) (compare Cor. I.1).<br>
of Lemma I.4: For reasons of symmetry it<br>  $\overline{K}$ . Thus, let  $x_0 \in \text{dom}_1 K$  be given arbitrarily. I<br>
we find an  $h \in E^*$  and a real number c suc consider the function  $\{x\}$  and  $\{x\}$  *Compare Cor.* I.1).<br>
Proof of Lemma I.4 has to be modified if the saddle function K only satisfies the<br>
Proof of Lemma I.4: For reasons of symmetry it suffices to prove dom<sub>1</sub> K<br> *(g, x<sub>0</sub>* - *v*) + cl<sub>1</sub>*K*(*x<sub>0</sub>*, *v*)  $\geq$  *d e (x<sub>0</sub>*, *v*)  $\geq$  *c*l<sub>2</sub>*K*(*x<sub>0</sub>*, *v*) for all *v*  $\in$  *E*. (1.7)<br> *(x + h*(*v* - *x<sub>0</sub>*)  $\leq$  cl<sub>2</sub>*K*(*x<sub>0</sub>*, *v*) for all *v*  $\in$  *E*. (1.7)<br> *(x + h*(*v* 

$$
c + h(v - x_0) \leq \mathrm{cl}_2 K(x_0, v) \quad \text{for all} \quad v \in E. \tag{I.7}
$$

we find an  $f \in E^*$  and an  $\varepsilon > 0$  with  $\mathfrak{n}_2$  *K*} can be separated, i.e.<br>(1.8)

$$
\varepsilon + f(v - x_0) \leq 0 \quad \text{for all} \quad v \in \text{dom}_2 K. \tag{I.8}
$$

Now consider the functional  $g = \lambda f + h$ , where  $\lambda > 0$  is a fixed number with  $\lambda \varepsilon + c = \lambda > 0$ . According to (1.7) and (1.8) we obtain

$$
g, x_0 - v + c l_2 K(x_0, v) \ge \delta \quad \text{for all} \ \ v \in \text{dom}_2 K \tag{I.9}
$$

and consequently

$$
(g, x_0 - v) + \mathrm{cl}_1 K(x_0, v) \geq \delta \quad \text{for all} \quad v \in E. \tag{I.10}
$$

By taking into account the closedness of  $K$ ,  $(1.10)$  yields

$$
(g, x_0 - v) + c l_2 K(x_0, v) \ge \delta \quad \text{for all} \quad v \in \text{dom}_2 K
$$
\n
$$
(g, x_0 - v) + c l_1 K(x_0, v) \ge \delta \quad \text{for all} \quad v \in \text{dom}_2 K
$$
\n
$$
(g, x_0 - v) + c l_1 K(x_0, v) \ge \delta \quad \text{for all} \quad v \in E.
$$
\n
$$
(g, x_0 - v) + c l_1 K(x_0, v) \ge \delta \quad \text{for all} \quad v \in E.
$$
\n
$$
c l_2 K(x_0, x_0) = c l_2 c l_1 K(x_0, x_0) = \sup_{g \in E^*} \inf_{v \in E} \{g(x_0 - v) + c l_1 K(x_0, v)\} \ge \delta > 0,
$$
\na contradiction to our assumption  $(**)$ 

which is a contradiction to our assumption *(\*\*)* <sup>I</sup>

The different notions of skew-symmetry we have introduced lead to a characteristic structure of the set of saddle points of these saddle functions. This question will be more detailed 4. and consequently<br>  $(g, x_0 - v) + c I_1 K(x_0, v) \ge$ <br>
By taking into account the closedne<br>  $c I_2 K(x_0, x_0) = c I_2 c I_1 K(x_0, v)$ <br>
which is a contradiction to our assu<br>
The different notions of skew-symme<br>
ture of the set of saddle poi

#### 1.4 A monotone operator corresponding to saddle functions

In this section we study the properties of a certain operator  $T_K \subseteq E \times E^*$  which can be introduced for each saddle function  $\hat{K}$  on the space  $E \times \hat{E}$ . Especially, we show that  $T_K$  is maximal monotone if and only if  $K$  satisfies the condition  $(*)$ . In the forthcoming Part II we demonstrate that the subfamily of the monotone operators of type  $T_K$  is cofinal in the family of all monotone operators. More specifically, for each monotone operator  $A \subseteq E \times E^*$  there is a saddle function  $K: E \times E \to \overline{\mathbf{R}}$  such that  $T_K$  is a monotone extension of *A*. In particular. all maximal monotone operators are of type  $T_K$ . These facts are the background of all our **Example 12** Maximal Monotone Operators a<br> **I.4 A** monotone operator corresponding to saddle functi<br>
In this section we study the properties of a certain operator  $T$ <br>
duced for each saddle function  $K$  on the space  $E \$ **onotione operator corresponding to saddle functions**<br>
cotion we study the properties of a certain operator  $T_K \subseteq E \times E^*$  which can be intro-<br>
each saddle function  $K$  on the space  $E \times E$ . Expecially, we show that  $T_K$  is *x* tion we study the properties of a certain operator  $T_K \subseteq E$ <br>
each saddle function  $K$  on the space  $E \times E$ . Especially, we<br>
if and only if  $K$  satisfies the condition (\*). In the forthcor<br>
the subfamily of the monotone

Definition I.3: Let  $K: E \times E \to \overline{\mathbb{R}}$  be a saddle function. Then we define the operator  $T_K \subseteq E \times E^*$  by

$$
f \in T_K x := [-f, f] \in \partial K(x, x). \tag{I.11}
$$

By the definition of the subdifferential  $\partial K$  the relation  $f \in T_K x$  is equivalent to

Definition I.3: Let 
$$
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 be a saddle function. Then we define the  
\noperator  $T_K \subseteq E \times E^*$  by  
\n $f \in T_K x := [-f, f] \in \partial K(x, x)$ . (I.11)  
\nBy the definition of the subdifferential  $\partial K$  the relation  $f \in T_K x$  is equivalent to  
\n $K(x, x) - f(v - x) \geq K(v, x), v \in E$ ,  
\n $K(x, x) + f(w - x) \leq K(x, w), w \in E$ .  
\nWe show now that the operator  $T_K$  generalizes the notion of the subdifferential of a  
\nconvex function.  
\nTheorem I.1: Let  $p: E \rightarrow \overline{\mathbf{R}}$  be a proper convex function. Then  $T_K = \partial p$  holds  
\ntrue for any saddle function  $K: E \times E \rightarrow \overline{\mathbf{R}}$  with  
\n $K(x, y) = p(y) - p(x)$  if  $x \in \text{dom } p$  or  $y \in \text{dom } p$ . (I.13)  
\nEach such saddle function is skew-symmetric. If, additionally, p is closed, then K is  
\nalso closed and we have Dom  $K = \text{dom } p$ .  
\nFor the proof we refer to [4]. Saddle functions for which the condition (I.13) is  
\nfulfilled are, for example,  
\n $K_1(x, y) = \begin{cases} p(y) - p(x) & \text{for } y \in \text{dom } p \\ +\infty & \text{otherwise} \end{cases}$ 

We show now that the operator  $T_K$  generalizes the notion of the subdifferential of a convex function.

Theorem I.1: Let  $p: E \to \overline{R}$  be a proper convex *function. Then*  $T_K = \partial p$  holds *true for any saddle function*  $K: E \times E \to \overline{\mathbf{R}}$  *with* 

$$
K(x, y) = p(y) - p(x) \quad \text{if} \quad x \in \text{dom } p \quad \text{or} \quad y \in \text{dom } p. \tag{I.13}
$$

*-* 

*Each such saddle function is skew-symmetric. if, additionally, p is closed, then K is*  Theorem I.1: Let  $p: E \to \overline{\mathbf{R}}$  be a proper convention  $K: E \times E \to \overline{\mathbf{R}}$  with<br>  $K(x, y) = p(y) - p(x)$  if  $x \in \text{dom } p$ <br> *Fach such saddle function is skew-symmetric.*<br> *Also closed and we have* Dom  $K = \text{dom } p$ .<br>
For the proof w

For the proof we refer to [4]. Saddle functions for which the condition (1.13) is

$$
K(x, y) = N(x, y)
$$
  
\n
$$
K(x, x) + f(w - x) \leq K(x, w), \quad w \in E.
$$
\nWe show now that the operator  $T_K$  generalizes the notion of the  
\nonvex function.  
\nTheorem I.1: Let  $p: E \to \overline{R}$  be a proper convex function.  
\n
$$
K(x, y) = p(y) - p(x) \quad \text{if} \quad x \in \text{dom } p \quad \text{or} \quad y \in \text{dom } p
$$
  
\n
$$
K(x, y) = p(y) - p(x) \quad \text{if} \quad x \in \text{dom } p \quad \text{or} \quad y \in \text{dom } p
$$
  
\nEach such saddle function is skew-symmetric. If, additionally,  $1$   
\nalso closed and we have Dom  $K = \text{dom } p$ .  
\nFor the proof we refer to [4]. Saddle functions for which th  
\nuifilled are, for example,  
\n
$$
K_1(x, y) = \begin{cases} p(y) - p(x) & \text{for} \quad y \in \text{dom } p \\ +\infty & \text{otherwise} \end{cases}
$$
  
\nand  
\n
$$
K_2(x, y) = \begin{cases} p(y) - p(x) & \text{for} \quad x \in \text{dom } p \\ -\infty & \text{otherwise} \end{cases}
$$
  
\nOur subsequent considerations will rest on the following

and

*/* 

$$
K_1(x, y) = \begin{cases} +\infty & \text{otherwise} \end{cases}
$$
  

$$
K_2(x, y) = \begin{cases} p(y) - p(x) & \text{for } x \in \text{dom } p \\ -\infty & \text{otherwise} \end{cases}
$$

Our subsequent considerations will rest on the following

*Theorem I.2: Let*  $K: E \times E \to \overline{\mathbf{R}}$  *be proper. Then*  $T_K \subseteq E \times E^*$  *is a monotone operator.* 

The proof of this result has been given in [3]. It is easy to see that  $T_K$  is not monotone for improper saddle functions provided that the domain  $D(T_K)$  of  $T_K$  consists of at least two elements. For the investigation of the operator  $T_K$  it will sometimes be convenient to replace the system  $(1.12)$  by a single inequality. This is done in and<br>  $K_2(x, y) = \begin{cases} p(y) - p(x) & \text{for} \\ -\infty & \text{otherwise} \end{cases}$ <br>
Our subsequent considerations will re<br>
Theorem I.2: Let  $K: E \times E \to \overline{R}$  be<br>
operator.<br>
The proof of this result has been given<br>
tone for improper saddle functions prover

$$
K_2(x, y) = \begin{cases} P(y) & P(x) & \text{for } x \in \text{dom } P \\ -\infty & \text{otherwise} \end{cases}
$$
\nOur subsequent considerations will rest on the following

\nTheorem I.2: Let  $K: E \times E \to \mathbf{R}$  be proper. Then  $T_K \subseteq E \times E^*$  is a monotone operator.

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\n(i)  $f \in T_K x$ ;

\n(ii)  $x \in \text{dom}_1 K \cap \text{dom}_2 K$  and

\n $f\left(\frac{v + w}{2} - x\right) \leq \frac{c!_2 K(x, w) - c!_1 K(v, x)}{2}$ ,  $v, w \in E$ .

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*Moreover, if K is closed, then both these conditions are equivalent to* 

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$$
\text{oreover, if } K \text{ is closed, then both these conditions are equivalent to}
$$
\n
$$
\text{(iii)} \quad f\left(\frac{v+w}{2} - x\right) \le \frac{\text{cl}_1 K(x, w) - \text{cl}_2 K(v, x)}{2}, \qquad [v, w] \in \text{dom } K. \tag{I.15}
$$
\n
$$
\text{When dealing with saddle functions with values in the extended real line, } \overline{P} \text{ are has to be } \overline{P}.
$$

**E.** KRAUSS<br> *pver, if* K is closed, then both these conditions are equivalent to<br>  $\left(\frac{v+w}{2} - x\right) \leq \frac{c_1 K(x, w) - c_2 K(v, x)}{2}$ ,  $[v, w] \in \text{dom } K.$  (1.15)<br>  $\left(\frac{v+w}{2} - x\right) \leq \frac{c_1 K(x, w) - c_2 K(v, x)}{2}$ ,  $[v, w] \in \text{dom } K.$  (1.15)<br> When dealing with saddle functions with values in the extended real line  $\overline{R}$ , one has to take care that there does not occur any indefinite expression of the kind  $+\infty - \infty$ . In (I.14) this is achieved by the assumption  $x \in \text{dom}_1 K \cap \text{dom}_2 K$  and in (I.15) by the requirement  $[v, w] \in \text{dom } K.$ . When dealing with saddle functions with values in the extended real line  $\overline{\mathbf{R}}$ , one has to<br>ce care that there does not occur any indefinite expression of the kind  $+\infty - \infty$ . In (I.14)<br>s is achieved by the assumption 338 E. KRAUSS<br>
Moreover, if K is closed, then<br>
(iii)  $f\left(\frac{v+w}{2} - x\right) \leq \frac{c}{v}$ <br>
When dealing with saddle for<br>
take care that there does not oo<br>
this is achieved by the assump<br>  $[v, w] \in \text{dom } K$ .<br>
Proof of Proposition<br>
Con

cake care that there does not occur any indefinite expression of the kind  $+\infty - \infty$ . In (I.14)<br>this is achieved by the assumption  $x \in \text{dom}_1 K \cap \text{dom}_2 K$  and in (I.15) by the requirement<br> $[v, w] \in \text{dom } K$ .<br>Proof of Proposition **•** Proof of Proposition I.1: The equivalence (i)  $\Leftrightarrow$  (iii) has been proved in Concerning the 'implication (i)  $\Rightarrow$  (ii) we refer to [3: (8)]. To prove (ii)  $\Rightarrow$  (i), fr (I.14) we deduce  $K(v, x) + f(v + w - 2x) \leq c l_1 K(v, x) + f(v$ 

$$
K(v, x) + f(v + w - 2x) \leq \mathrm{cl}_1 K(v, x) + f(v + w - 2x) \leq \mathrm{cl}_2 K(x, w) \leq K(x, w)
$$

for all  $v, w \in E$ . Setting here  $v = x$ , or  $w = x$ , respectively, we obtain the system  $(1.12)$ , i.e.  $f \in T_K x$ 

From Prop. 1.1 one can easily derive the following characterization of the domain  $D(T_K)$  of the operator  $T_K$ .

**Proposition I.2:** Let  $K: E \times E \rightarrow \overline{\mathbf{R}}$  be proper. Then the following inclusions *hold true:*

$$
e \cdot \text{deduce}
$$
\n
$$
+ f(v + w - 2x) \leq c l_1 K(v, x) + f(v + w - 2x) \leq c l_2 K(x, w) \leq K(x, u)
$$
\n
$$
, w \in E. \text{ Setting here } v = x, \text{ or } w = x, \text{ respectively, we obtain the system of the equation  $v = x$ , and  $v = x$ , respectively, we obtain the system of the equation  $T_K$ .  
\n
$$
f \in T_K x
$$
\n
$$
= \text{Prop. 1.1 one can easily derive the following characterization of the condition  $T_K$ .  
\n
$$
f \in E \times E \implies \overline{R} = \text{Prop. Then the following inclusion  $I$ :\n
$$
D(T_K) \leq \left\{ x \in E : c l_1 K(v, x) \leq c l_2 K(x, w) \text{ for } v; w, w \text{ with } \frac{v + w}{2} = x \right\}
$$
\n
$$
\subseteq \{ x \in \text{dom}_1 K \cap \text{dom}_2 K : c l_1 K(x, x) = c l_2 K(x, x) \}
$$
\n
$$
= \text{post section and in the forthcoming Part II we shall generalize the statement provided that } T_K \text{ is a maximal monotone operator.}
$$
$$
$$
$$

In the next section and in the forthcoming Part II we shall generalize the statement of Prop. I.2 provided that  $T_K$  is a maximal monotone operator.

Proof of Proposition I.2: The first inclusion results from  $(1.14)$ . Setting here In the next section and in the forthcoming Part II we shall generalize the statement of<br>Prop. I.2 provided that  $T_K$  is a maximal monotone operator.<br>Proof of Proposition I.2: The first inclusion results from (1.14). Setti In the next section and in the forthcoming Part II we shall generalize the statement of<br>Prop. I.2 provided that  $T_K$  is a maximal monotone operator.<br>Proof of Proposition I.2: The first inclusion results from (1.14). Setti  $D(T_K) \subseteq \begin{cases} x \in \\ x \in \\ y \in \\ y \in \end{cases}$ <br>
Frop. I.2 provided that T<br>
Prop. I.2 provided that T<br>
Proof of Proposit<br>  $v = w = x$  we find  $c_1$ <br>
more, if in the .inequ<br>
choose  $v \in \text{dom}_1 K \neq$ <br>  $x \in \text{dom}_2 K \blacksquare$ <br>
Now we are going to

Now we are going to ask for the maximal monotonicity of the operator  $T_K$ .

Theorem I.3: Let  $K: E \times E \to \overline{R}$  be a saddle function such that  $T_K \subseteq E \times E^*$  is *maximal monotone. Then. K necessarily satisfies the condition (\*).* 

The proof of this statement is given in Section 1.6. The forthcoming Part If contains some \*results w1ich make the condition (\*) more transparent. For a preliminary interpretation of the condition ( $\ast$ ) one should recall the estimate for the domain of  $T_K$  given in Prop. I.2.

We show now that under quite natural assumptions on *K* and the space *E,* the condition (\*) is also sufficient for the maximality of  $T_K$ .

Theorem I.4: Let  $K: E \times E \rightarrow \overline{\mathbf{R}}$  be a closed proper saddle function on a reflexive. *Banach space. Then*  $T_K$  *is maximal monotone if and only if K satisfies the condition* maximal monotone. Then K necessarily satisfies the condition (\*).<br>
The proof of this statement is given in Section 1.6. The fortheoming Part II contains which make the condition (\*) more transparent. For a preliminary int

Proof: One part of this statement is already contained in Th. 1.3. The remaining one has been proved in [3] (compare the remark following Th. 2 of that paper)  $\blacksquare$ 

It is easy to see how Th.1.4 reads if the saddle function *K* is not-supposed to be

Theorem I.5: Let  $K: E \times E \rightarrow \overline{\mathbf{R}}$  be proper and let E be a reflexive Banach space. Then  $T_K$  is maximal monotone if and only if  $T_K = T_K$  for each  $\tilde{K}$ ,  $\text{cl}_2 K \leq \tilde{K} \leq \text{cl}_1 K$ , and all these saddle functions  $\tilde{K}$  satisfy the condition  $(*)$ .

 $\text{Proof:}$  Let  $T_K$  be maximal monotone. A little thought shows that  $T_K$  is a monotone extension of  $T_K$  for each  $\tilde{K}$ ,  $\text{cl}_2$   $K \leq \tilde{K} \leq \text{cl}_1$   $K$ . This implies  $T_K = T_{\tilde{K}}$ . Hence, on account of Th. I.3,  $T_{\tilde{K}}$  satisfies the condition (\*). On the other hand, the interval cand all these saddle functions K satisfy the condition (\*).<br>
Proof: Let  $T_K$  be maximal monotone. A little thought shows that  $T_{\tilde{K}}$  is a mono-<br>
tone extension of  $T_K$  for each  $\tilde{K}$ ,  $c l_2 K \leq \tilde{K} \leq c l_1 K$ . Thi  $\text{cl}_2 K \leq \tilde{K} \leq \text{cl}_1 K$  contains at least one closed saddle function, namely  $\tilde{K} = \text{cl}_2 \text{cl}_1 K$ .<br>One easily verifies dom  $K \subseteq \text{dom }\tilde{K}$ , so that  $\tilde{K}$  is proper. The maximality of  $T_K = T_{\tilde{K}}$  is now a conse Maximal Monotone Operators and Saddle<br>
Theorem I.5: Let  $K: E \times E \to \overline{R}$  be proper and let E be a refl.<br>
Then  $T_K$  is maximal monotone if and only if  $T_K = T_K$  for each  $\overline{K}$ ,<br>
and all these saddle functions  $\overline{K}$  sat

It is obvious that the operators  $T_K$  and  $T_L$  coincide for equivalent saddle functions *K* and *L.* More important for our purposes is the following simple result, which is an immediate consequ<sub>ence</sub> of the definition of the operator  $T_K$ .

Lemina 1.5: Let  $K: E \times E \to \overline{\mathbf{R}}$  be a saddle function and define  $L: E \times E \to \overline{\mathbf{R}}$ *by*  $L(x, y) := -K(y, x)$ . Then we have  $T_K = T_L$ .

# **1.5 On the construction of** skew-symmetric saddle functions: Results

In this section we show that the characterization of the operator  $T_K$ , as given in (1.12) and (1.14), can be considerably simplified if we have some more information about the skewsymmetry of *K*. Later on we ask for saddle functions  $L$  for which the operator  $T_L$  is a monotone extension of a given operator  $T_K$ . An important role will here be played by skew-symmetric saddle functions *L* and those which satisfy-the inequality

$$
\text{cl}_2 L(x, y) \leq -\text{cl}_2 L(y, x) \quad \text{for all} \quad x, y \in E.
$$

The latter ones are just the saddle functions which can be majorized by a closed skew-symmetric saddle function. We apply these results to get some estimates for the domains of a saddle function and its corresponding monotone operator. Other applications are contained in the forthcoming Part IT. Some of the results are rather technical. The reader who is not interested in too many details is advised only to read Prop. I.3 and the Th.  $1.6-1.8$ , together with the Cor. I.1 and I.2. For the sake of a better reading all proofs are given in the next section. *x* are just the saddle inntitions wind can be majorized by a closed skew-symmetry and its corresponding monotone operator. Other applications are contained ming Part II. Some of the results are rather technical. The read

As already announced, *we* are now looking for a simple characterization of the operator  $T_K$ . A first result is

Lemma I.6: *For any saddle function*  $K: E \times E \rightarrow \overline{\mathbf{R}}$  the *implication* (i)  $\Rightarrow$  (ii) *holds true. If*  $T_K$  *is maximal monotone, then (i) and (ii) are even equivalent:* Frator  $T_K$ . A first result is<br>
Lemma I.6: For any saddle for<br>
ds true. If  $T_K$  is maximal monotorphiant is that if  $\lim_{n \to \infty} \frac{F_K}{n}$ .<br>
ii) Each pair  $[x, f] \in T_K$  satisfies solving for a simple characterization of<br>  $n \ K: E \times E \to \overline{\mathbf{R}}$  the implication (i)  $\Rightarrow$ <br>
en (i) and (ii) are even equivalent:<br>
nequality<br>  $E$ .<br>
at each maximal monotone operator  $A \subseteq E$ <br>  $\downarrow$ ,  $f$   $\in A$ }  $\leq 0$  for each

- 
- (ii) *Each pair*  $[x, f] \in T_K$  *satisfies the inequality*

$$
f(v-x) \leq K(x, v)
$$
 for all  $v \in E$ .

(i)  $cl_1 K(u, u) \geq 0$  for all  $u \in E$ .<br>
(ii) Each pair  $[x, f] \in T_K$  satisfies the inequality<br>  $f(v - x) \leq K(x, v)$  for all  $v \in E$ .<br>
The implication (ii)  $\rightarrow$  (i) reflects the fact that each maximal monotone operator  $A \subseteq E \times E^*$ satisfies the inequality inf  $\{(f - h_0, v - x_0)\}\$ ;  $[v, f] \in A\} \leq 0$  for each  $x_0 \in \overline{E}$ ,  $h_0 \in E^*$ . contributed in the sum of the sum of the section of the section of  $K(u, u) \ge 0$  for all  $u \in E$ .<br>
ch pair  $[x, f] \in T_K$  satisfies the inequality<br>  $f(v-x) \le K(x, v)$  for all  $v \in E$ .<br>
(1.16)<br>
ilication (ii)  $\rightarrow$  (i) reflects the fact

The following result can be viewed as a converse of Lemma I.6.

Lemma 1.7: Let  $K: E \times E \to \overline{\mathbf{R}}$  *be a saddle function with* 

$$
\text{cl}_2 K(x,v) \leq -\text{cl}_1 K(v,x) \quad \text{for all} \ \ x,v \in E.
$$

*Then a pair*  $[x, f] \in E \times E^*$  *belongs to*  $T_K$  *if it obeys the inequality (I.16).* 

The statements of Lemma **1.6 and** Lemma 1.7 are put together in

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Proposition I.3: Let  $K: E \times E \to \overline{\mathbb{R}}$  be a saddle function with<br>  $\text{cl}_2 K(u, v) \leq -\text{cl}_1 K(v, u)$  and  $\text{cl}_1 K(u, u) \geq 0$ ,  $u, v \in E$ . E. KRAUSS<br>
c1<sub>2</sub> *K*(*u, v*)  $\leq$  *C1*<sub>1</sub> *K*(*v, u) and c1*<sub>1</sub> *K*(*u, u)*  $\geq$  0, *u, v*  $\in$  *E*. (I.18)<br> *following conditions are equivalent:* E. KRAUSS<br>
(i)  $[0, \ln x]$ <br>  $\therefore$   $\text{cl}_2 K(u, v) \leq -\text{cl}_1$ <br>  $\therefore$   $\text{cl}_2 K(u, v) \leq -\text{cl}_1$ <br>  $\therefore$   $\text{cl}_2 K(u, v) \leq K(x, v)$ <br>  $\text{cl}_1 (v - x) \leq K(x, v)$  for<br>  $\text{cl}_2 K(x, x)$  and  $\text{cl}_1$   $\text{cl}_2$ <br>  $\text{cl}_2 K(x, x)$  and  $\text{cl}_1$   $\text{cl}_2 K(x, v)$ • (ii) *f(v - x) :5-: K*.

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*Then the following conditions are equivalent:* 

(ii)  $f(v-x) \leq K(x, v)$  for all  $v \in E$ ,

- (iii)  $f \in \partial_2 K(x, x)$  and  $\text{cl}_1 K(x, x) = K(x, x)$ , <br>(iv) there exists a  $g \in E^*$  with  $[-g, f] \in \partial K(x, x)$ ,
- 
- (v) *there exists a*  $y \in E$  with  $[-f, f] \in \partial K(x, y)$ .

*(iv) there. exists a g. € E\* with [—g,* /] *€ bK(x, x),*  The assumption  $(I.\overline{I}8)$  implies that *K* satisfies the condition  $(\rightarrow \rightarrow)$ . It is fulfilled, in particular, for skew-symmetric, saddle functions. Other saddle functionsfor which (1.18) holds true will be considered in the forthcoming Part II.  $\begin{aligned}\n\langle \cdot, \cdot \cdot \rangle &= x \rangle &\leq K(x, v) \text{ for all } v \in E, \\
\langle \cdot \cdot \cdot \cdot \rangle &\leq K(x, x) \text{ and } c_1 K(x, x) = K(x, x), \\
\text{where exists } u, g \in E^* \text{ with } [-g, f] \in \partial K(x, x), \\
\text{where exists } u, y \in E \text{ with } [-f, f] \in \partial K(x, y). \\
\text{function (I.18) implies that } K \text{ satisfies the condition } (\bullet \bullet). \text{ It is fulfilled, in particular,}\\
\text{function (I.18) implies that } K \text{ satisfies the condition } (\bullet \bullet). \text$ 

The background of our subsequent considerations is the following

Proposition I.4: Let  $K: E \times E \rightarrow \overline{\mathbf{R}}$  *be a saddle function such that* 

\n Theorem 1.11:\n \n- the for the image
\n- the form 
$$
\text{Cov}(x, y) = \text{Cov}(x, y) = \text{Cov}(x, y)
$$
\n- the form  $\text{Cov}(x, y) \leq -\text{Cov}(x, y)$  for all  $x, y \in E$ .
\n
\n

\n\n The series of  $\text{Cov}(x, y) \leq -\text{Cov}(x, y)$  for all  $x, y \in E$ .\n

\n\n The series of  $\text{Cov}(x, y) \leq -\text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x, y) \leq -\text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x, y) = \text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x, y) = \text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x, y) = \text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x, y)$  is a linearly independent. The series  $\text{Cov}(x,$ 

*Then there exists a closed skew-symmetric saddle function*  $L: E \times E \rightarrow \overline{R}$  *with* 

$$
\operatorname{cl}_2 K(x, y) \leq L(x, y) \leq -\operatorname{cl}_2 K(y, x) \quad \text{for all} \ \ x, y \in E. \tag{I.19}
$$

The proof which was given in *[4]* is non-constructive. In the forthcoming Part II we shall calculate such a saddle function *L* under certain regularity assumptions on *K.* 

Now it seems natural to ask how the operators  $T_K$  and  $T_L$  relate to each other if the saddle functions  $K$  and  $L$  satisfy (I.19). ckground of our subsequent considerations is the following<br>
sition I.4: Let  $K: E \times E \to \overline{\mathbf{R}}$  be a saddle function such that<br>  $\text{cl}_2 K(x, y) \leq -\text{cl}_2 K(y, x)$  for all  $x, y \in E$ .<br>  $\therefore$ <br>  $\text{e exists a closed skew-symmetric saddle function } L: E \times E \to \overline{\mathbf{R}}$  with<br> **Proposition I.4:** Let  $K: E \times E \rightarrow \mathbf{R}$  be a saddle function  $cl_2 K(x, y) \leq -cl_2 K(y, x)$  for all  $x, y \in E$ .<br> *Then there exists a closed skew-symmetric saddle function I*<br>  $cl_2 K(x, y) \leq L(x, y) \leq -cl_2 K(y, x)$  for all  $x$ ,<br> *The proof* (i) downtown  $L$  under certain regularity assumptions on  $K$ .<br>
(i) downtown  $K$  and  $L$  satisfy (1.19).<br>
(i) downtown  $K, U: E \times E \rightarrow \overline{R}$  with<br>  $\begin{aligned}\nc_1 & K(x, y) \leq L(x, y) \leq -c_1, K(y, x) \quad \text{for all } x, y \in E \\
\text{for each pair of saddle functions } K, L: E \times E \rightarrow \overline{R$ 

Lemma I.8: For each pair of saddle functions  $K, L: E \times E \rightarrow \overline{\mathbf{R}}$  with

$$
\text{cl}_2\ K(x,\ y) \leq L(x,\ y) \leq -\text{cl}_2\ K(y,\ x) \quad \text{for all} \ \ x,\ y \in E \tag{1.20}
$$

(i)  $\text{dom}_1 K \subseteq \text{dom}_1 L \cap \text{dom}_2 L$ ,  $\text{dom}_1 L \cup \text{dom}_2 L \subseteq \text{dom}_2 \text{cl}_2 K$ .<br>
(ii) *Any pair*  $[x, f] \in E \times E^*$  *with*  $f(v - x) \leq K(x, v)$ ,  $v \in E$ , *belongs to*  $T_L$ .<br>
(iii) *If additionally*  $\text{cl}_1 K(u, u) \geq 0$ ,  $u \in E$ , *holds then we al* 

From the proof wind was given in [up] is increased that sumptions on  $K$ .<br>
Now it seems natural to ask how the operators  $T_K$  and  $T_L$  relate to each othe<br>
c saddle functions  $K$  and  $L$  satisfy (I.19).<br>
Lemma I.8: For ea The preceding lemma provides us with a' tool to attack a more general question. Let an arbitrary saddle function K be given. We ask whether we can find a certain interval of saddle functions such that  $T_L \supseteq T_K$  holds for each *L* belonging to this interval. Results of this type are of special. interest if the interval in' consideration contains a skev-synmnetric saddle function. For this purpose we need the following Lemma I.8: For each pair of saddle functions *K*,  $L: E \times$ <br>  $cl_2 K(x, y) \leq L(x, y) \leq -cl_2 K(y, x)$  for all  $x, y$ <br>
the following statements are true:<br>
(i) dom<sub>1</sub>  $K \subseteq$  dom<sub>1</sub>  $L \cap$  dom<sub>2</sub>  $L$ , dom<sub>1</sub>  $L \cup$  dom<sub>2</sub>  $L$  (ii) *Any* pai 2. dom<sub>1</sub> L u dom<sub>2</sub> L  $\subseteq$  do<br>  $f(v - x) \le K(x, v), v \in E$ ,<br>  $v, u \in E$ , holds then we also<br>
with a tool to attack a more given. We ask whether<br>
at  $T_L \supseteq T_K$  holds for each<br>
special interest if the inter-<br>
saddle function  $K: E \times$ <br> Unity statute three form. Are given. We ask whether we can find a certain<br>of saddle functions such that  $T_L \supseteq T_K$  holds for each L belonging to this<br>Results of this type are of special interest if the interval in conside

Definition.I.4: To an arbitrary saddle function  $K: E \times E \rightarrow \bar{R}$  we associate

a skew-symmetric saddle function. For this purpose we need  
ition I.4: To an arbitrary saddle function 
$$
K: E \times E \to \mathbb{R}
$$
  
saddle function  $L_K: E \times E \to \mathbb{R}$  by  

$$
L_K(x, v) := \inf \left\{ \frac{c l_2 K(x, v_1) - c l_1 K(v_2, x)}{2}; \frac{v_1 + v_2}{2} = v \right\},
$$

for each  $v \in E$  and  $x \in \text{dom}_1 K \cap \text{dom}_2 K$ . Otherwise we set  $L_K(x, v) = -\infty$ .

Concerning an interpretation of this saddle function we mention that the condifor each  $v \in E$  and  $x \in \text{dom}_1 K \cap \text{dom}_2 K$ . Otherwise we set  $L_K(x, v) = -\infty$ .<br>Concerning an interpretation of this saddle function we mention that the cortion (1.14) of Prop. 1.1 can be reformulated as  $f(u - x) \leq L_K(x, u)$  for a It can easily be checked that the saddle function  $L_K$  obeys the inequality Concerning an interpretation of this saddle function (I.14) of Prop. I.1 can be reformulated as  $j$ <br>It can easily be checked that the saddle function<br> $L_K(x, y) \le -L_K(y, x)$  for all  $x, y \in E$ .<br>Hence, the following statement make

$$
L_K(x, y) \leq -L_K(y, x) \quad \text{for all} \ \ x, y \in E.
$$

Maximal Monotone Operators and Saddle Functions I 341<br>
Proposition 1.5: Let  $K: E \times E \to \overline{R}$  be proper. Then we have  $T_K \subseteq T_L$  for each *saddle function*  $L: E \times E \rightarrow \overline{\mathbf{R}}$  *with* Maximal Monotone Operators and Saddle Functions I<br>
sition 1.5: Let  $K: E \times E \to \overline{\mathbf{R}}$  be proper. Then we have  $T_K \subseteq T_L$  for each<br>
notion  $L: E \times E \to \overline{\mathbf{R}}$  with<br>  $c!_2 L_K(x, y) \leq L(x, y) \leq -c!_2 L_K(y, x),$   $x, y \in E.$  (1.23)<br>
maxim

$$
\operatorname{cl}_2 L_K(x, y) \leq L(x, y) \leq -\operatorname{cl}_2 L_K(y, x), \qquad x, y \in E. \tag{1.23}
$$

*1/ T<sub>K</sub> is maximal monotone, then each L satisfies the condition (\*\*) and*  $T<sub>L</sub>$  *coincides with*  $T_K$ .

We are now going to derive some consequences of Prop. I.5. The importance of the following theorem will become clear in the forthcoming Part II, when we are concerned with the representation of monotone operators by saddle functions. It allows to pass here from arbitrary saddle functions to skew-s concerned with the representation of monotone operators by saddle functions. It allows to pass here from arbitrary saddle functions to skew-symmetric ones. Th. 1.6 generalizes a result previously obtained in [4]. be now going to derive some consequences of Prop. I.5. The<br>
ving theorem will become clear in the forthcoming Part I<br>
d with the representation of monotone operators by sadd<br>
pass here from arbitrary saddle functions to s

Theorem I.6: Let  $K: E \times E \to \overline{\mathbf{R}}$  be a saddle function,  $dom_1 K \cap dom_2 K \neq \emptyset$ .<br>Then there exists a closed skew-symmetric saddle function  $L: E \times E \to \overline{\mathbf{R}}$  with

*such that*  $T_L$  *is a monotone extension of*  $T_K$ . If E is a reflexive Banach space, then  $T_L$  is  $even$  maximal monotone.

Another consequence of Prop. 1.5 is the following estimate for the domains of K and  $T_K$ .

Theorem 1.7: Let  $K: E \times E \to \overline{\mathbf{R}}$  be a proper saddle function such that  $T_K$  is *maximal monotone. Then the identity*

dom1 *K +* don't, *K -* dom<sup>1</sup> *<sup>K</sup>*n dom *K*  2 = **Co** *D(I)*  (1.24)

*holds.* 

In the forthcoming Part II we shall sharpen this result considerably, provided that  $E_i$  is a reflexive Banach space. The next result shows how the assertion of Lemma I.4 is modified if one there replaces the condition  $(**)$  by the condition  $(*)$ .

Corollary I.1: Let E be a reflexive. Banach space and let  $K: E \times E \to \overline{\mathbf{R}}$  be a *closed proper saddle function satisfying the condition*  $(*)$ . Then the identity  $(1.24)$ *.holds trite.*

Now we show that the assertions of Th. I.7 and Cor. I.1 can be strenghtened, if we additionally suppose the saddle function *K* to satisfy the condition (\*\*). For a reflexive Banach space *E*, we obtain a generalization of Lemma I.4.  $\frac{1}{\text{dom}_1 K \cap \text{dom}_2 K} = \frac{\overline{\text{dom}_1 K + \text{dom}_2 K}}{2} = \overline{\text{co}} D(T_K)$  (I.2<br>
holds.<br>
In the forthcoming Part II we shall sharpen this result considerably, provid<br>
that  $E_j$  is a reflexive Banach space. The next result shows how t Corollary I.1: Let *E* be a reflexive Banach space and let  $K : E$  closed proper saddle function satisfying the condition (\*). Then the holds true.<br>
Now we show that the assertions of Th. I.7 and Cor. I.1 can be st<br>
we addi de Banach space. The next result considerably, provided<br>
e Banach space. The next result shows how the assertion of<br>
ed if one there replaces the condition  $(**)$  by the condition  $(*)$ .<br>
et *K* be a reflexive Banach space an

Theorem 1.8: Let  $K: E \times E \to \overline{\mathbf{R}}$  *be a proper saddle function such that*  $T_K$  *is* eximal monotone. If K satisfies the condition (\*\*) then we have  $\overline{\text{dom}_1 K} = \overline{\text{dom}_2 K} = \text{dom}_1 K \cap \text{dom}_2 K = \overline{\text{co}} D(T_K)$ . (1.25)

$$
\overline{\text{dom}_1 K} = \overline{\text{dom}_2 K} = \overline{\text{dom}_1 K \cap \text{dom}_2 K} = \overline{\text{co}} D(T_K). \tag{I.25}
$$

Corollary 1.2: *Let F be a reflexive Banach space and letK be a closed proper saddle function satisfying the condition (\*\*). 'J'/een the identity* (1.25) *holds true.* 

#### **1.6** On the construction of skew-symmetric saddle functions. Proofs

Now we<sup>-</sup>furnish the proofs for the results of the preceding section.

Proof of Lemma 1.6: (i)  $\Rightarrow$  (ii): For an arbitrary pair  $[x, f] \in T_K$  one has  $cl_1 K(x, x)$  $K(x, x)$  and  $K(x, x) + f(w - x) \leq K(x, w)$  for any  $w \in E$  (compare Prop. 1.2). Due to our assumption  $cl_1 K(x, x) \geq 0$  we get  $f(w - x) \leq K(x, w)$ ,  $w \in E$ , as desired.

 

(ii)  $\Rightarrow$  (i): We supposed  $T_K$  to be maximal monotone. Let us show that the assumption

$$
cl_1 K(v_0, v_0) < 0,\tag{I.26}
$$

E. KRAUSS<br>
i): We supposed  $T_K$  to be maximal monotone. Let us show that the<br>
on<br>
cl<sub>1</sub>  $K(v_0, v_0) < 0$ , (1.26)<br>  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure<br>
(1.3)), (1.26) implies the exis for some  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure (compare (I.3)), (I.26) implies the existence of a  $g_0 \in E^*$  and an  $\varepsilon > 0$  with

$$
g_0(w - v_0) - K(w, v_0) \geq \varepsilon > 0 \quad \text{for any} \quad w \in E. \tag{I.27}
$$

*goue in the supposed*  $T_K$  *to be maximal monotonc. Let us show that the*<br>  $e_1 K(v_0, v_0) < 0$ , (1.26)<br>  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure<br>  $v(1.3)$ ), (1.26) implies the existence of According to our assumption (ii), (1.27) yields  $(f + g_0, w - v_0) \geq \varepsilon$  for each pair  $[w, f] \in T_K$ . Since  $T_K$  was maximal monotone, we get  $-g_0 \in T_K v_0$ . Hence,  $\varepsilon \leq 0$ , which contradicts the choice of  $\varepsilon$ (ii)  $\Rightarrow$  (i): We supposed  $T_K$  to be maximal monotone. Let us show that the assumption<br>  $\text{cl}_1 K(v_0, v_0) < 0$ , (I.26)<br>
for some  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closur<br>
(compare (1.3)), cn  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure  $(f.3)$ ), (I.26) implies the existence of a  $g_0 \in E^*$  and an  $\vare$  $v_0 \in E$ , leads to a contradiction. By the definition of the concave closure (1.3)), (1.26) implies the existence of a  $g_0 \in E^*$  and an  $\varepsilon > 0$  with  $g_0(w - v_0) - K(w, v_0) \ge \varepsilon > 0$  for any  $w \in E$ . (1.27) g to our assumpti yields  $(f + g_0, \hat{u} - v_0) \geq \varepsilon$  for each pair<br>
iotone, we get  $-g_0 \in T_K v_0$ . Hence,  $\varepsilon \leq 0$ ,<br>
g of the inequality (I.20) with respect to<br>
cl<sub>2</sub> K(y, x), x, y E. (I.28)<br>
ment shows<br>
cl<sub>2</sub> K(y, x), x, y E. (I.29)<br>
first

**Proof of Lemma 1.8:** (i): The closing of the inequality  $(1.20)$  with respect to *of* Lemma 1.8: (i): The closing of the inequality (I.20) with respect to<br>  $\text{cx}$  argument yields<br>  $\text{cI}_2 K(x, y) \leq \text{cI}_2 L(x, y) \leq -\text{cI}_1 \text{cI}_2 K(y, x),$   $x, y \in E$ . (I.28)<br> *Procedure for the concave argument shows*<br>  $\text{cI$ 

$$
\operatorname{cl}_2 K(x, y) \leq \operatorname{cl}_2 L(x, y) \leq -\operatorname{cl}_1 \operatorname{cl}_2 K(y, x), \qquad x, y \in E. \tag{I.28}
$$

A similar procedure for the concave argument shows

$$
\operatorname{cl}_1 \operatorname{cl}_2 K(x, y) \leq \operatorname{cl}_1 L(x, y) \leq -\operatorname{cl}_2 K(y, x), \qquad x, y \in E. \tag{I.29}
$$

These inequalities immediately imply the first assertion.

 $(iii)$ : We have to show that the inequality

$$
f(w - x) \le K(x, w) \quad \text{for all} \quad w \in E
$$
 (I.30)

implies  $[x, f] \in T_L$ . From (1.30) follows

$$
f(w-x) \leq \mathrm{cl}_2\,K(x,w), \qquad w\in E, \qquad \qquad \square \qquad (1.31)
$$

which together with the inequalities  $(1.28)$  and  $(1.29)$  leads to

$$
cl_2 K(x, y) \leq cl_2 L(x, y) \leq -cl_1 cl_2 K(y, x), \qquad x, y \in E.
$$
 (I.28)  
lar procedure for the concave argument shows  

$$
cl_1 cl_2 K(x, y) \leq cl_1 L(x, y) \leq -cl_2 K(y, x), \qquad x, y \in E.
$$
 (I.29)  
inequalities immediately imply the first assertion.  
We have to show that the inequality  

$$
f(w - x) \leq K(x, w) \text{ for all } w \in E
$$
 (I.30)  

$$
s[x, f] \in T_L. \text{ From (I.30) follows}
$$

$$
f(w - x) \leq cl_2 K(x, w), \qquad w \in E,
$$
 (I.31)  
together with the inequalities (I.28) and (I.29) leads to  

$$
f(v - x) \leq cl_2 L(x, v), \qquad v \in E,
$$
 (I.32)  

$$
f(w - x) \leq -cl_1 L(w, x), \qquad w \in E.
$$
 (I.33)  
ticular, we get  $x \in dom_1 L \cap dom_2 L$ . Moreover, the adding of (I.32) and

and

• 

$$
f(w-x) \leq -c l_1 L(w,x), \qquad w \in E. \tag{I.33}
$$

procedure for<sup>st</sup>he concave argument shows<br>  $cl_1 cl_2 K(x, y) \leq cl_1 L(x, y) \leq -cl_2 K(y, x), \quad x, y \in E.$  (I.29)<br> *qualities immediately imply the first assertion.*<br>  $\ell(w - x) \leq K(x, w)$  for all  $w \in E$  (I.30)<br>  $x, f] \in T_L$ . From (I.30) follows In particular, we get  $x \in \text{dom}_1 L \cap \text{dom}_2 L$ . Moreover, the adding of (I.32) and These inequalities immed<br>
(ii): We have to show<br>  $f(w-x) \leq K$ <br>
implies  $[x, f] \in T_L$ . From<br>  $f(w-x) \leq c l_2$ <br>
which together with the<br>  $f(v-x) \leq c l_2$ <br>
and<br>  $f(w-x) \leq -1$ <br>
In particular, we get<br>
(1.33) yields<br>  $f\left(\frac{v+w}{2} - x\right)$ 

$$
f(w - x) \le K(x, w) \text{ for all } w \in E
$$
\n
$$
[x, f] \in T_L. \text{ From (I.30) follows}
$$
\n
$$
f(w - x) \le c l_2 K(x, w), \quad w \in E,
$$
\n
$$
f(w - x) \le c l_2 K(x, w), \quad w \in E,
$$
\n
$$
f(v - x) \le c l_2 L(x, v), \quad v \in E,
$$
\n
$$
f(w - x) \le -c l_1 L(w, x), \quad w \in E.
$$
\n
$$
f(w - x) \le -c l_1 L(w, x), \quad w \in E.
$$
\n
$$
f(w - x) \le -c l_1 L(w, x), \quad w \in E.
$$
\n
$$
f(\frac{v + w}{2} - x) \le \frac{c l_2 L(x, v) - c l_1 L(w, x)}{2}, \quad v, w \in E.
$$
\n
$$
f(\frac{v + w}{2} - x) \le \frac{c l_2 L(x, v) - c l_1 L(w, x)}{2}, \quad v, w \in E.
$$
\n
$$
f(\frac{v + w}{2} - x) \le \frac{c l_2 L(x, v) - c l_1 L(w, x)}{2}, \quad v, w \in E.
$$
\n
$$
f(\frac{v + w}{2} - x) \le \frac{c l_2 L(x, v) - c l_1 L(w, x)}{2}, \quad v, w \in E.
$$

Recalling Prop. I.1 we get  $[x, f] \in T_L$  as desired.

(iii): This statement follows from (ii) by an application of Lemma  $I.6$ 

**Proof of Lemma 1.7: Set in Lemma 1.8**  $L = K \blacksquare$ 

calling Prop. I.1 we get  $[x, f] \in T_L$  as desired.<br>
(iii): This statement follows from (ii) by an application of Lemma I.6  $\blacksquare$ <br>
Proof of Lemma I.7: Set in Lemma I.8  $L = K \blacksquare$ <br>
Proof of Proposition I.3: The implications (i (iii): This statement follows from (ii) by an application of Lemma I.6 <br>Proof of Lemma I.7: Set in Lemma I.8  $L = K$ <br>
<br>
'Proof of Proposition I.3: The implications (i)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (v) are<br>
obvious, whi Proof of Proposition I.3: The implications (i)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (v) are obvious, while (ii)  $\Rightarrow$  (i) follows from Lemma I.7. (iii)  $\Rightarrow$  (ii): We consider an element  $f \in \partial_2 K(x, x)$ . The assumption (I.18) i **Proof of Lemma I.7:** Set in Lemma I.8  $L = K$ <br> **Proof of Proposition I.3:** The implications (i)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (v) are<br>
obvious, while (ii)  $\Rightarrow$  (i) follows from Lemma I.7. (iii)  $\Rightarrow$  (ii): We consider a For any *v* i.e. (ii)  $\Rightarrow$  (i) follows from Lemma I.7. (iii)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (v) are obvious, while (ii)  $\Rightarrow$  (i) follows from Lemma I.7. (iii)  $\Rightarrow$  (ii): We consider an element  $f \in \partial_2 K(x, x)$ . The assumption (I.1 We find  $f\left(\frac{v+w}{2} - x\right) \leq \frac{c_1 L(x, v) - c_1 L(w, x)}{2}, \quad v, w \in E.$ <br>
Prop. I.1 we get  $[x, f] \in T_L$  as desired.<br>
this statement follows from (ii) by an application of Lemma I.6 ■<br>
of Lemma I.7: Set in Lemma I.8  $L = K$  ■<br>
of Proposition 1. Prop. I.1 we get  $[x, f] \in T_L$  as desired.<br>
is statement follows from (ii) by an application of Lemma I.6 **I**<br>
of Lemma I.7: Set in Lemma I.8  $L = K$  **I**<br>
of Proposition I.3: The implications (i) ⇒ (iv) ⇒ (iii) and (i) ⇒ (v)

$$
K(x, y) - f(v - x) \geq cl_1 K(v, y), \qquad v \in E,
$$
\n(1.34)

and

$$
K(x, y) + f(w - y) \leq \mathrm{cl}_2 \left\{ K(x, w), \qquad w \in E. \right\} \tag{I.35}
$$

Setting  $v = y$  in (1.34) and  $w = x$  in (1.35) we obtain

$$
cl_1 K(y, y) \le K(x, y) + f(x - y) \le cl_2 K(x, x).
$$
 (I.36)

*Maximal Monotone Operators and Saddle Functions I* 343<br>
= *y* in (I.34) and  $w = x$  in (I.35) we obtain<br>
cl<sub>1</sub>  $K(y, y) \le K(x, y) + f(x - y) \le c l_2 K(x, x)$ . (I.36)<br>
ant of our assumption (I.18) we have cl<sub>1</sub>  $K(y, y) \ge 0$  and cl<sub>2</sub>  $K(x, x$ Setting  $v = y$  in (1.34) and  $w = x$  in (1.35) we obtain<br>
cl<sub>1</sub>  $K(y, y) \leq K(x, y) + f(x - y) \leq c l_2 K(x, x)$ . (1.36)<br>
On account of our assumption (1.18) we have  $c l_1 K(y, y) \geq 0$  and  $c l_2 K(x, x) \leq 0$ .<br>
Thus (1.36) implies  $K(x, y) = f(y - x)$ cl<sub>1</sub>  $K(y, y) \le K(x, y) + f(x)$ <br>
On account of our assumption (I.<br>
Thus (I.36) implies  $K(x, y) = f(y)$ <br>  $f(w - x) \le K(x, w)$  for  $w \in E$ , i.e.  $- x$ ). Inserting this identity into (1.35) we get  $f(w - x) \leq K(x, w)$  for  $w \in E$ , i.e. (ii) **I** 

tting  $v = y$  in (1.34) and  $w = x$  in (1.35) we obtain<br>
cl<sub>1</sub>  $K(y, y) \leq K(x, y) + f(x - y) \leq c l_2 K(x, x)$ . (1.36)<br>
1 account of our assumption (1.18) we have  $c l_1 K(y, y) \geq 0$  and  $c l_2 K(x, x) \leq 0$ .<br>
1 account of our assumption (1.18) w Proof of Proposition I.5: In order to prove the inclusion  $T_K \subseteq T_L$ , we consider an arbitrary element  $[x, f] \in T_K$ . In view of the definition of the saddle func-<sup>t</sup> tion *L<sub>K</sub>* we obtain  $f(v - x) \leq L_K(x, v)$  for all  $v \in E$  (compare Prop. 1.1). The closing of this inequality with respect to the convex argument yields y in (I.34) and  $w = x$  in (I.35) we obtain<br>  $K(y, y) \leq K(x, y) + f(x - y) \leq c_1 K(x, x)$ . (I.36)<br>
of our assumption (I.18) we have  $c_1 K(y, y) \geq 0$  and  $c_1 K(x, x) \leq 0$ .<br>
implies  $K(x, y) = f(y - x)$ . Inserting this identity into (I.35) we ge

$$
f(v-x) \leq c l_2 L_K(x,v). \quad \text{for} \quad v \in E \quad \text{and} \quad [x, f] \in T_K. \tag{I.37}
$$

The inclusion  $T_K \subseteq T_L$  now results from Lemma I.8 if we there replace the saddle function *K* by  $L_K$ . Let us suppose now that  $T_K$  is maximal monotone. Then, from (1.37) and Lemma 1.8 (i) we can conclude this inequality with respect to the convex argume<br>  $f(v - x) \leq c|_2 L_K(x, v)$  for  $v \in E$  and<br>
the inclusion  $T_K \subseteq T_L$  now results from Lemma I.8<br>
notion K by  $L_K$ . Let us suppose now that  $T_K$  is n<br>
37) and Lemma I.8 (i) we can arbitrary element  $[x, f] \in T_K$ . In view of the definition of the saddle func-<br>we obtain  $f(v - x) \leq L_K(x, v)$  for all  $v \in E$  (compare Prop. I.1). The closing<br>equality with respect to the convex argument yields<br> $f(v - x) \leq c_1 L_K(x, v)$ 

$$
\emptyset \neq D(T_K) \subseteq \text{dom}_1 L_K \subseteq \text{dom}_1 L \cap \text{dom}_2 L,
$$

for each saddle function *L* with

 

$$
\operatorname{cl}_2 L_K(x, y) \le L(x, y) \le -\operatorname{cl}_2 L_K(y, x), \qquad x, y \in E. \tag{1.38}
$$

In particular, all these saddle functions are proper. Hence, as a consequence of Th. I.2,  $T_L$  is a monotone operator. Since  $T_K$  was maximal monotone the inclusion  $T_K \subseteq T_L$  implies  $T_L = T_K$ . The inequality (1.37) can now be read as  $f(v - x)$  $\leq$   $c_1 \overline{C_2 L_K(x, v)}$ ;  $v \in \overline{E}$  and  $[x, f] \in T_{(c_1, L_K)} = T_K$ . We can now apply Lemma 1.6 to the saddle function  $\operatorname{cl}_2 L_K$  and obtain d Lemma I.8 (i) we can conclude<br>  $\emptyset \neq D(T_K) \subseteq \text{dom}_1 L_K \subseteq \text{dom}_1 L \cap \text{dom}_2 L$ ,<br>
saddle function L with<br>  $c!_2 L_K(x, y) \leq L(x, y) \leq -c!_2 L_K(y, x)$ ,  $x, y \in E$ . (I.38)<br>
ular, all these saddle functions are proper. Hence, as a consequenc

$$
\operatorname{cl}_1 \operatorname{cl}_2 L_K(x, x) \ge 0 \quad \text{for all} \quad x \in E. \tag{I.39}
$$

The inequality (I.38) implies

The function 
$$
cl_2 L_K
$$
 and obtain  
\n $cl_1 cl_2 L_K(x, x) \ge 0$  for all  $x \in E$ .  
\nquality (I.38) implies  
\n $cl_1 cl_2 L_K(x, y) \le cl_1 L(x, y)$  and  $cl_2 L(x, y) \le -cl_1 cl_2 L_K(y, x)$ . (I.40)

for all  $x, y \in E$ . From (I.39) and (I.40) we can easily deduce the desired inequality cl<sub>1</sub> cl<sub>2</sub>  $L_K(x, y) \leq c$ l<sub>1</sub>  $L(x, y)$  and<br>for all  $x, y \in E$ . From (I.39) and (I.40) we<br>cl<sub>2</sub>  $L(x, x) \leq 0 \leq c$ l<sub>1</sub>  $L(x, x)$  for all  $x \in E$ 

Now we are in the position to give the

Proof of Theorem I.3: For an improper saddle function  $K$  the condition  $(*)$  is Proof of Theorem I.3: For an improper saddle function *K* the condition (\*) is<br>trivially satisfied, so that we can assume *K* to be proper. We introduce a saddle<br>function  $L: E \times E \to \overline{R}$ ,<br> $L(x, y) := \inf \left\{ \frac{c_1 K(x, y_1) - c_2 K(y$ function  $L: E \times E \rightarrow \overline{R}$ ,

$$
L(x, y) := \inf \left\{ \frac{\operatorname{cl}_1 K(x, y_1) - \operatorname{cl}_2 K(y_2, x)}{2} : [y_1, y_2] \in \operatorname{dom} K, \frac{y_1 + y_2}{2} = y \right\}.
$$
  
slly be checked that L fulfils the conditions  

$$
\operatorname{cl}_1 L = L \quad \text{and} \quad L_K(x, y) \le L(x, y) \le -L_K(y, x) \quad \text{for } x, y \in E,
$$

It can easily be checked that *L* fulfils the conditions

$$
cl_1 L = L \quad \text{and} \quad L_K(x, y) \leq L(x, y) \leq -L_K(y, x) \quad \text{for} \quad x, y \in E,
$$

with  $L_K$  as in Def. I.4. According to Prop. 1.5,  $L_K$  has to satisfy the condition (\*\*), so that we obtain  $L(x, x) \ge 0$  for  $x \in E$ . By the definition of L, this just means. that K satisfies the condition  $(*)$ 



$$
L = \mathrm{cl}_2 L, \quad \mathrm{dom}_1 L = \mathrm{dom}_1 K \cap \mathrm{dom}_2 K \neq \emptyset,
$$
  

$$
\mathrm{dom}_2 L \subseteq \overline{\mathrm{dom}_1 K \cap \mathrm{dom}_2 K}
$$

$$
\operatorname{lon}_{2} L \subseteq \operatorname{dom}_{1} K \cap \operatorname{dom}_{2} K
$$

and the inequalities

$$
L = \text{cl}_2 L, \quad \text{dom}_1 L = \text{dom}_1 A \cap \text{dom}_2 K + C
$$
  
dom<sub>2</sub>  $L \subseteq \overline{\text{dom}_1 K \cap \text{dom}_2 K}$   
inequalities  
 $L_K(x, y) \le L(x, y) \le -L(y, x) \le -L_K(y, x)$ ,

with  $L_K$  as in Def. I.4. Hence, we can apply Prop. I.4 to the saddle function  $\hat{L}$ and obtain a closed skew-symmetric saddle function  $L\colon E\times E\to \overline{\mathbf{R}}$  with *L<sub>K</sub>*(*x, y*)  $\leq L(x, y) \leq -L(y, x) \leq -L_K(y, x)$ ,<br>as in Def. I.4. Hence, we can apply Prop. I.4 to the saddle func<br>in a closed skew-symmetric saddle function  $L: E \times E \to \overline{\mathbb{R}}$  with<br> $L_K(x, y) \leq L(x, y) \leq L(x, y) \leq -L(y, x) \leq -L_K(y, x)$ 

$$
L_K(x, y) \leq L(x, y) \leq L(x, y) \leq -L(y, x) \leq -L_K(y, x), \qquad x, y \in E.
$$

We show that *L* fulfils all requirements of Th. I.6. First, according to Lemma I.8 We show that *L* fulfils all requirements of Th. I.6. First, according to Lemma I.8 (i), we get dom<sub>1</sub>  $\hat{L} \subseteq \text{Dom } L \subseteq \text{dom}_2 \hat{L}$ , which together with (1.41) yields  $\emptyset = \text{dom}_1 K$ (i), we get dom<sub>1</sub>  $\hat{L} \subseteq \text{Dom } L \subseteq \text{dom}_2 \hat{L}$ , which together with (1.41) yields  $\emptyset \neq \text{dom}_1 K$ <br>  $\cap$  dom<sub>2</sub>  $K \subseteq \text{Dom } L \subseteq \text{dom}_1 K \cap \text{dom}_2 K$ . On account of Prop. 1.5,  $T_L$  is a monotone extension of  $T_K$ . For a reflexive Banach space  $E$ , the maximal monotonicity  $\cdot$  of  $T_K$  follows from Th. I.4  $\blacksquare$ L<sub>K</sub>(x, y)  $\leq L(x, y) \leq -L(y, x) \leq -L_K(y, x)$ ,<br>
as in Def. I.4. Hence, we can apply Prop. I.4 to the saddle function L<br>
in a closed skew-symmetric saddle function L:  $E \times E \to \overline{R}$  with<br>  $L_K(x, y) \leq L(x, y) \leq L(x, y) \leq -L(y, x) \leq -L_K(y$ In dom<sub>2</sub>  $K \subseteq$  Dom  $L \subseteq$  dom<sub>1</sub>  $K \cap$  dom<sub>2</sub>  $K$ . On account of Prop. I.5,  $T_L$  is a monotone extension of  $T_K$  follows from Th. I.4 **I**<br>
Proof of Theorem I.8: It suffices to verify the inclusion<br>
dom<sub>2</sub>  $K \subseteq \overline{co} D(T_K)$ . with  $L_K$  as in Def. 1.4. Hence, we can apply Prop. 1.4 to the saddle function  $L : E \times E \to \mathbf{R}$  with<br>and obtain a closed skew-symmetric saddle function  $L : E \times E \to \mathbf{R}$  with<br> $L_K(x, y) \leq \hat{L}(x, y) \leq -L(x, x) \leq -L_K(y, x),$   $x, y \in E$ 

Proof of Theorem 1.8: It suffices to verify'the inclusion

$$
\operatorname{dom}_2 K \subseteq \overline{\operatorname{co}}\, D(T_K). \tag{I.42}
$$

com<sub>2</sub>  $K \subseteq \overline{\text{co}} D(T_K)$ . (I.42)<br>or reasons of symmetry, together with (I.42) we obtain dom<sub>1</sub>  $K \subseteq \overline{\text{co}} D(T_K)$ <br>e,<br>co  $D(T_K) \subseteq \text{dom}_1 K \cap \text{dom}_2 K \subseteq \text{dom}_1 K \subseteq \overline{\text{co}} D(T_K)$ ,  $i = 1, 2$  (1.43)<br>Prop. I.2). This relation immediatel follows from 1n. 1.4 **a**<br>
of Theorem 1.8: It suffices to verify the inclusion<br>
dom<sub>2</sub>  $K \subseteq \overline{\infty} D(T_K)$ . (I.42)<br>
or reasons of symmetry, together with (I.42) we obtain dom<sub>1</sub>  $K \subseteq \overline{\infty} D(T_K)$ <br>
e,<br>
co  $D(T_K) \subseteq$  dom<sub>1</sub>  $K \cap \$ 

$$
\text{co } D(T_K) \subseteq \text{dom}_1 K \cap \text{dom}_2 K \subseteq \text{dom}_i K \subseteq \overline{\text{co }} D(T_K), \qquad i = 1, 2 \quad (1.43)
$$

(compare Prop. I.2). This relation immediately implies the desired identity  $(1.25)$ .

To prove (1.42) let an arbitrary element  $x_0 \in \text{dom}_2 K$  be given. By the definition of dom<sub>2</sub> K we find an  $h \in E^*$  and a  $c \in \mathbb{R}$  with

$$
\operatorname{cl}_1 K(v, x_0) \leq h(v - x_0) + c \quad \text{for all} \quad v \in E. \tag{I.44}
$$

Suppose now  $x_0 \notin \overline{co} D(T_K)$ . Then, by the separation theorem for convex sets, there exist a  $g \in E^*$  and an  $\varepsilon > 0$  with

$$
0 \leq g(v - x_0) - \varepsilon \quad \text{for} \quad v \in D(T_K). \tag{I.45}
$$

or reasons of symmetry, together with  $(1.42)$  we obtain dom<sub>1</sub>  $K \subseteq \overline{\infty} D(T_K)$ <br>
e,<br>  $\cos D(T_K) \subseteq \text{dom}_1 K \cap \text{dom}_2 K \subseteq \text{dom}_i K \subseteq \overline{\infty} D(T_K), \quad i = 1, 2 \quad (\text{I.43})$ <br>
Prop. I.2). This relation immediately implies the desired identity Now we set  $f = -h - \lambda g$ , where  $\lambda > 0$  is a real number with  $c \leq \lambda \varepsilon$ . Then, from (I.44) and (I.45) we can conclude Final details of Freasons of symmetry, together with  $(1.42)$  we obtain dom<sub>1</sub>  $K \subseteq \emptyset$   $D(T)$ <br>
conclude  $D(T_K) \subseteq \emptyset$  concludes  $D(T_K)$ .  $i = 1, 2$  (1.4<br>
(compare Prop. I.2). This relation immediately implies the desired iden c<sub>1</sub>  $K(v, x_0) \leq h(v - x_0) + c$  for all  $v \in E$ . (I.44)<br>
now  $x_0 \notin \overline{co} D(T_K)$ . Then, by the separation theorem for convex sets, there<br>  $\in E^*$  and an  $\varepsilon > 0$  with<br>  $0 \leq g(v - x_0) - \varepsilon$  for  $v \in D(T_K)$ . (I.45)<br>
set  $f = -h - \lambda g$ , whe c<sub>1</sub>  $k \in E^*$  and  $a \ c \in \mathbb{R}$  with<br>  $\leq h(v - x_0) + c$  for all  $v \in E$ .<br>  $D(T_K)$ . Then, by the separation theorem for convex sets, there<br>  $x_0$ )  $- \varepsilon$  for  $v \in D(T_K)$ . (1.45)<br>  $- \lambda g$ , where  $\lambda > 0$  is a real number with  $c \leq$ 

$$
\text{cl}_1 K(v, x_0) \leq f(x_0 - v) \quad \text{for all} \quad v \in D(T_K). \tag{I.46}
$$

Now let an arbitrary element  $[z, j] \in T_K$  be given. Since K was supposed to satisfy the condition (\*\*), by Lemma 1.6 we get  $\text{cl}_1 K(v, x_0)$ <br>an arbitrary<br>*j*(*w* - *z*)  $\leq$ <br>*j*(*w* - *z*)  $\leq$ 

$$
j(w-z) \leq \mathrm{cl}_2 K(z,w) \quad \text{for all} \quad w \in E. \tag{I.47}
$$

(1.48)

Setting  $v = z$  in (I.46) and  $w = x_0$  in (I.47) yields  $(f - j, x_0 - z) \ge 0$  for 'each pair  $[z, j] \in T_K$ . Since  $T_K$  was supposed to be maximal monotone, we can conclude Setting  $v = z$  in (I.46) and  $w = x_0$  in (I.47) yields  $(j - j, x_0 - z)$ <br>
[ $z, j$ ]  $\in T_K$ . Since  $T_K$  was supposed to be maximal monotone,<br>
[ $x_0, f$ ]  $\in T_K$ , which is a contradiction to the assumption  $x_0 \notin \overline{\infty} D(T)$ .<br>
Proof of

Proof of Corollary I.2: Under our assumptions the operator  $T_K$  is maximal monotone (compare Th. I.4). Hence, the assertion follows from-Th. I.8

Proof<sub>1</sub> of Theorem I.7: Let us consider the saddle function  $L: E \times E \to \mathbb{R}$ ,  $L := cl_2 L$ , where  $\hat{L}$  is defined by

$$
[x_0, f] \in T_K
$$
, which is a contradiction to the assumption  $x_0 \notin \overline{\text{co}} D(T_K)$   
\nProof of Corollary I.2: Under our assumptions the operator  $T_K$  is maxim  
\nmonotone (compare Th. I.4). Hence, the assertion follows from Th. I.8  
\nProof<sub>1</sub> of Theorem I.7: Let us consider the saddle function  $L: E \times E \to I$   
\n $L := \text{cl}_2 L$ , where  $L$  is defined by  
\n
$$
L(x, y) := \inf \left\{ \frac{\text{cl}_1 K(x, y_1) - \text{cl}_2 K(y_2, x)}{2}; y = \frac{y_1 + y_2}{2}, [y_1, y_2] \in \text{dom } K \right\},
$$

for each  $y \in E$  and  $x \in \text{dom}_1 K \cap \text{dom}_2 K$ . For  $x \notin \text{dom}_1 K \cap \text{dom}_2 K$  we set

$$
f(x, y) := \inf \left\{ \frac{\operatorname{cl}_1 K(x, y_1) - \operatorname{cl}_2 K(y_2, x)}{2}; y = \frac{y_1 + y_2}{2}, [y_1, y_2] \in \operatorname{dom} K \right\},
$$
  
\n
$$
f(x, y) := \inf \left\{ \frac{\operatorname{cl}_1 K(x, y_1) - \operatorname{cl}_2 K(y_2, x)}{2}; y = \frac{y_1 + y_2}{2}, [y_1, y_2] \in \operatorname{dom} K \right\},
$$
  
\n
$$
f(x, y) = -\infty.
$$
 One easily verifies the inclusions  
\n
$$
\operatorname{dom}_1 L \subseteq \operatorname{dom}_1 K \cap \operatorname{dom}_2 K, \quad \operatorname{dom}_2 L \supseteq \operatorname{dom}_2 L = \frac{\operatorname{dom}_1 K + \operatorname{dom}_2 K}{2}.
$$

Obviously *L* satisfies the inequality

 

$$
\mathrm{cl}_2 L_K(x,y) \leqq L(x,y) \leqq -L_K(y,x) \quad \text{for } x,y \in E,
$$

where  $L_K$  is taken from Def. I.4. Since  $T_K$  was supposed to be maximal monotone, we can conclude from Prop. 1.5 that also  $T_L = T_K$  is maximal monotone and that *L* satisfies the condition (\*\*). Moreover, we have  $L = c_1 L$   $\dot{L} = c_2 L$   $\dot{L} = c_1 L$ , i.e. *L* is a closed saddle function. We can now apply Th. I.8 to L and obtain  $\overline{\text{dom}_1 L} = \overline{\text{dom}_2 L}$ wh<br>we<br> $L$  is<br> $\frac{1}{2}$ <br> $\frac{1}{2}$  $\overline{c} = \overline{co} D(T_L) = \overline{co} D(T_K)$ . By (I.48) this leads to s taken from Def. 1.4. donclude from Prop. I.5 the condition (\*\*). Mo<br>s the condition (\*\*). Mo<br>ddle function. We can not<br> $T_L$ ) =  $\overline{co} D(T_K)$ . By (I.48)<br> $\frac{\text{d} \overline{on}_1 K + \text{d} \overline{on}_2 K}{2} \subseteq \overline{co}$ 

$$
\frac{\mathrm{dom}_1 K + \mathrm{dom}_2 K}{2} \subseteq \overline{\mathrm{co}} D(T_K) \subseteq \overline{\mathrm{dom}_1 K \cap \mathrm{dom}_2 K}.
$$

Since the inclusion  $\overline{\text{dom}_1 K} \cap \overline{\text{dom}_2 K} \subseteq 1/2$  ( $\overline{\text{dom}_1 K + \text{dom}_2 K}$ ) is trivially satisfied, we get the desired identity  $(1.24)$ 

Proof of Corollary I.1: Under our assumptions the operator  $T_K$  is maximal monotone (compare Th. I.4), so that we can apply Th. I.7  $\blacksquare$ Proof of Corollary I.1: Under our assumptions the operator  $T_K$  is m<br>notone (compare Th. I.4), so that we can apply Th. I.7  $\blacksquare$ <br>FERENCES<br>BARBU, V., and Th. PRECUPANU: Convexity and optimization in Banach spaces. Bt<br>Edi

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