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Maximal Monotone Operators and Saddle Functions I [1])

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Wir untersuchen den monotonen Operator $T_K \subseteq E \times E^*$, $f \in T_K x := [-f, f] \in \partial K(x, x)$, der über das Subdifferential einer konkav-konvexen Sattelfunktion K definiert ist. Unsere Überlegungen werden durch die Tatsache motiviert, daß jeder maximal monotone Operator A in der Form $A = T_K$ darstellbar ist. Wir zeigen, daß T_K genau dann maximal monoton ist, wenn K in einer abgeschwächten Form schiefsymmetrisch ist. Dies erlaubt eine Verallgemeinerung früher erzielter Ergebnisse.

Исследуется монотонный оператор $T_K \subseteq E \times E^*$, $f \in T_K x := [-f, f] \in \partial K(x, x)$, определённый с помощью субдифференциала вогнуто-выпуклой седловой функции K. Рассуждения обоснованы тем, что каждый максимально монотонный оператор A может быть представлен в виде $A = T_K$. Показывается, что T_K максимально монотонный тогда и только тогда, когда K является в ослаблённой форме кососимметрической. Это обобщение результатов полученных раньше.

We investigate the monotone operator $T_K \subseteq E \times E^*$, $f \in T_K x := [-f, f] \in \partial K(x, x)$, which is defined via the subdifferential of a concave-convex saddle function K. Our considerations are motivated by the fact that each maximal monotone operator A possesses a representation of the form $A = T_K$. We show that T_K is maximal monotone if and only if K is in a relaxed form skew-symmetric. This allows a generalization of results obtained previously.

I.1 Introduction

In this paper we investigate the operator

$$T_K \subseteq E \times E^*; \qquad f \in T_K x := [-f, f] \in \partial K(x, x),$$

which is defined via the subdifferential of a concave-convex saddle function $K : E \times E \to \overline{\mathbf{R}}$. In [3] we imposed a condition on K (namely, a relaxed form of skew-symmetry) guaranteeing the maximal monotonicity of the operator T_K. Now we show that this condition is even necessary for the maximality of T_K. This fact allows to improve several results previously obtained in [3–4]. In the forthcoming Part II of this paper we show that each maximal monotone operator $A \subseteq E \times E^*$ has the form $A = T_K$, where K is a closed and skew-symmetric saddle function. In general, this saddle function cannot be found constructively. It turns out, however, that several functions K with $A = T_K$ can be constructed if one relaxes the assumption on the skew-symmetry of K. In Section I.5 we are concerned with the question how, from a given saddle function, we can construct a "more regular" one such that both saddle functions generate the same monotone operator T. The principles stated here allow a short presentation of many results about saddle functions. Here we use them to generalize a principle from [4] concerning the existence of

[1]) Part II of the paper will be published in one of the following issues of this journal.

certain skew-symmetric saddle functions connected with the theory of maximal monotone operators (compare Th. I.6). Besides, we give some estimates for the sizes of the domains of a saddle function K and of the monotone operator corresponding to it (compare the Th. I.7, I.8 and the Cor. I.1, I.2). These results are partially strenghtened in the forthcoming Part II. Among other things, we show there that the *topological* interior of the domain of a maximal monotone operator is non-empty, provided that the convex hull of this domain has an *algebraic* interior. Moreover, we shall give the following approach to the solution of the equation $Ax \ni 0$, with a maximal monotone operator $A \subseteqq E \times E^*$: Find an arbitrary saddle function K with $T_K = A$ (since K is not supposed to be skew-symmetric this can be done in a constructive manner). Then, for any saddle point $[x_0, y_0]$ of K, the element $(x_0 + y_0)/2$ solves the equation $Ax \ni 0$. On the other hand, if $x_0 \in E$ solves $Ax \ni 0$, then $[x_0, x_0]$ is a saddle point of K.

I.2 Closed saddle functions

For the readers' convenience, here we recall some definitions of R. T. Rockafellar's theory of closed saddle functions. The basic material can be found in [1, 2, 7, 8] and for the finite-dimensional case also in [5, 6].

By E and F we denote, if not otherwise stated, locally convex Hausdorff spaces over the reals. For the dual pairing between E and the topologically dual space E^* both the notations $f(x)$ and (f, x) are used. We are dealing with functions with values in the extended real line $\overline{\mathbf{R}} := \mathbf{R} \cup \{\pm\infty\}$. Let $p: E \to \overline{\mathbf{R}}$ be *convex* (i.e. epi $p := \{[x, t] \in E \times \mathbf{R}: p(x) \leqq t\}$ is a convex set). We call p *proper* if $p(x) > -\infty$ for all $x \in E$ and $\operatorname{dom} p := \{x \in E: p(x) < +\infty\} \neq \emptyset$. The *closure* $\operatorname{cl} p$ of p is the pointwise supremum of all continuous affine minorants of p (with $\operatorname{cl} p \equiv -\infty$ if there is no such minorant):

$$\operatorname{cl} p(x) = \sup_{f \in E^*} \inf_{v \in E} \{f(x - v) + p(v)\}. \tag{I.1}$$

In the case $p = \operatorname{cl} p$ we say that p is *closed*. The *subdifferential* $\partial p \subseteqq E \times E^*$ of p is defined by

$$[x, f] \in \partial p :\Leftrightarrow p(x) + f(v - x) \leqq p(v) \quad \text{for all } v \in E. \tag{I.2}$$

The same notions also make sense for *concave* functions $q: F \to \overline{\mathbf{R}}$. One has only to change the roles of $+\infty$ and $-\infty$, $\leqq$ and $\geqq$, sup and inf in the definitions above. For example, the closure of a concave function is given by

$$\operatorname{cl} q(x) := \inf_{f \in F^*} \sup_{v \in F} \{f(x - v) + q(v)\}, \tag{I.3}$$

and for the subdifferential ∂q of q holds

$$[x, f] \in \partial q :\Leftrightarrow q(x) + f(v - x) \geqq q(v) \quad \text{for all } v \in F. \tag{I.4}$$

By a *saddle function* $K: E \times F \to \overline{\mathbf{R}}$ we shall always understand a function which is concave in the first argument and convex in the second one. The *convex closure* $\operatorname{cl}_2 K$ of K is obtained by closing $K(x, \cdot)$ as a convex function (for arbitrary $x \in E$), and similarly the *concave closure* $\operatorname{cl}_1 K$. Two saddle functions K and L are said to be *equivalent* if we have $\operatorname{cl}_i K = \operatorname{cl}_i L$ for $i = 1, 2$. If both $\operatorname{cl}_1 K$ and $\operatorname{cl}_2 K$ are equivalent to K we refer to K as a *closed* saddle function. In particular, the *lower closure* $\operatorname{cl}_2 \operatorname{cl}_1 K$ and the *upper closure* $\operatorname{cl}_1 \operatorname{cl}_2 K$ of K are closed saddle functions. For these saddle functions we can even say more. Namely, the lower closure is lower closed and the upper closure is upper closed. Here *lower closed* saddle functions L are characterized by the identity $\text{cl}_2\, \text{cl}_1\, L = L$, and *upper closed* ones by $\text{cl}_1\, \text{cl}_2\, L = L$. We call K *proper* if $\text{dom}\, K := \text{dom}_1\, K \times \text{dom}_2\, K \neq \emptyset$, with

$$\text{dom}_1\, K := \{x \in E : \text{cl}_2\, K(x, y) > -\infty \text{ for all } y \in F\}$$

and

$$\text{dom}_2\, K := \{y \in F : \text{cl}_1\, K(x, y) < +\infty \text{ for all } x \in E\}.$$

Note that a closed saddle function is proper if and only if it is finite in at least one point. If $\partial_1 K(\cdot, y)$ (resp. $\partial_2 K(x, \cdot)$) is the subdifferential of the concave function $K(\cdot, y)$ (or of the convex function $K(x, \cdot)$, respectively), the mapping

$$\partial K : E \times F \to 2^{E^* \times F^*};\ \partial K(x, y) := \partial_1 K(x, y) \times \partial_2 K(x, y),$$

is referred to as the *subdifferential* of K. An important feature of equivalent saddle functions is that they have the same subdifferentials.

We shall occasionally make use of the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. Moreover, sometimes we shall identify a multivalued mapping with its graph.

I.3 Skew-symmetric saddle functions

In this section we investigate different versions of the notion of a skew-symmetric saddle function. All these notions will frequently be used in the sequel. We start with

Definition I.1: Let $K : E \times E \to \overline{\mathbb{R}}$ be a saddle function. Then we say that

1. K *satisfies the condition* (*) if

$$\text{cl}_2\, K\left(x, \frac{x+y}{2}\right) \leqq \text{cl}_1\, K\left(\frac{x+y}{2}, y\right) \quad \text{for all } x, y \in E, \tag{*}$$

2. K *satisfies the condition* (**) if

$$\text{cl}_2\, K(x, x) \leqq 0 \leqq \text{cl}_1\, K(x, x) \quad \text{for all } x \in E, \tag{**}$$

3. K *is skew-symmetric* if

$$\text{cl}_2\, K(x, y) = -\text{cl}_1\, K(y, x) \quad \text{for all } x, y \in E.$$

The relations between these notions are the contents of

Lemma I.1: *For any saddle function* $K : E \times E \to \overline{\mathbb{R}}$ *the implications* (i) $\to$ (ii) $\to$ (iii) *hold true:*

(i) *K is a skew-symmetric saddle function,*
(ii) *K satisfies the condition* (**),
(iii) *K satisfies the condition* (*).

Proof: The first implication has been proved in [4]. Let K now satisfy the condition (**). Clearly, (*) holds whenever $x \notin \text{dom}_1\, K$ or $y \notin \text{dom}_2\, K$. Assuming $x \in \text{dom}_1\, K$ and $y \in \text{dom}_2\, K$ we get

$$\begin{aligned}\text{cl}_2\, K\left(x, \frac{x+y}{2}\right) &\leqq \frac{\text{cl}_2\, K(x, x) + \text{cl}_2\, K(x, y)}{2} \leqq \frac{\text{cl}_2\, K(x, y)}{2} \\ &\leqq \frac{\text{cl}_1\, K(x, y)}{2} \leqq \frac{\text{cl}_1\, K(x, y) + \text{cl}_1\, K(y, y)}{2} \\ &\leqq \text{cl}_1\, K\left(\frac{x+y}{2}, y\right) \quad \blacksquare\end{aligned}$$

Lemma I.2: *Any closed saddle function satisfying the condition* (**) *is proper.*

Proof: Suppose that $K: E \times E \to \overline{\mathbf{R}}$ is improper. For reasons of symmetry we can assume $\mathrm{dom}_1 K = \emptyset$. Then $\mathrm{cl}_2 K(, y) = -\infty$ for all $x, y \in E$ and hence, $\mathrm{cl}_1 K = \mathrm{cl}_1 \mathrm{cl}_2 K \equiv -\infty$. But this is contradictory to the condition (**) ∎

Now we give a simple criterion for the closedness of skew-symmetric saddle functions.

Lemma I.3: *Let* $K: E \times E \to \overline{\mathbf{R}}$ *be skew-symmetric. Then*

1. $$\mathrm{dom}_1 K = \mathrm{dom}_2 K, \tag{I.5}$$

2. *K is lower closed (upper closed) if and only if* $\mathrm{cl}_2 K = K$ (resp. $\mathrm{cl}_1 K = K$) *holds true.*

For the proof we refer to [4]. The identity (I.5) gives rise to the following

Definition I.2: Let $K: E \times E \to \overline{\mathbf{R}}$ be skew-symmetric. Then we set $\mathrm{Dom}\, K := \mathrm{dom}_1 K = \mathrm{dom}_2 K$.

It is interesting that a relaxed version of the identity (I.5) remains true if K merely satisfies the condition (**). We get

Lemma I.4: *For any closed saddle function* $K: E \times E \to \overline{\mathbf{R}}$ *satisfying the condition* (**) it holds

$$\overline{\mathrm{dom}_1 K} = \overline{\mathrm{dom}_2 K}. \tag{I.6}$$

In Section I.5 we shall generalize the statement of Lemma I.4, provided that E is a reflexive Banach space (compare Cor. I.2 and also Th. I.8). Moreover, we shall there investigate the question how Lemma I.4 has to be modified if the saddle function K only satisfies the condition (*) (compare Cor. I.1).

Proof of Lemma I.4: For reasons of symmetry it suffices to prove $\mathrm{dom}_1 K \subseteq \overline{\mathrm{dom}_2 K}$. Thus, let $x_0 \in \mathrm{dom}_1 K$ be given arbitrarily. Hence, by the definition of $\mathrm{dom}_1 K$, we find an $h \in E^*$ and a real number c such that

$$c + h(v - x_0) \leqq \mathrm{cl}_2 K(x_0, v) \quad \text{for all } v \in E. \tag{I.7}$$

Let us suppose $x_0 \notin \overline{\mathrm{dom}_2 K}$. Then the sets $\{x_0\}$ and $\{\mathrm{dom}_2 K\}$ can be separated, i.e. we find an $f \in E^*$ and an $\varepsilon > 0$ with

$$\varepsilon + f(v - x_0) \leqq 0 \quad \text{for all } v \in \mathrm{dom}_2 K. \tag{I.8}$$

Now consider the functional $g = \lambda f + h$, where $\lambda > 0$ is a fixed number with $\lambda\varepsilon + c =: \delta > 0$. According to (I.7) and (I.8) we obtain

$$(g, x_0 - v) + \mathrm{cl}_2 K(x_0, v) \geqq \delta \quad \text{for all } v \in \mathrm{dom}_2 K \tag{I.9}$$

and consequently

$$(g, x_0 - v) + \mathrm{cl}_1 K(x_0, v) \geqq \delta \quad \text{for all } v \in E. \tag{I.10}$$

By taking into account the closedness of K, (I.10) yields

$$\mathrm{cl}_2 K(x_0, x_0) = \mathrm{cl}_2 \mathrm{cl}_1 K(x_0, x_0) = \sup_{g \in E^*} \inf_{v \in E} \{g(x_0 - v) + \mathrm{cl}_1 K(x_0, v)\} \geqq \delta > 0,$$

which is a contradiction to our assumption (**) ∎

The different notions of skew-symmetry we have introduced lead to a characteristic structure of the set of saddle points of these saddle functions. This question will be more detailed investigated in the forthcoming Part II.

I.4 A monotone operator corresponding to saddle functions

In this section we study the properties of a certain operator $T_K \subseteqq E \times E^*$ which can be introduced for each saddle function K on the space $E \times E$. Especially, we show that T_K is maximal monotone if and only if K satisfies the condition (*). In the forthcoming Part II we demonstrate that the subfamily of the monotone operators of type T_K is cofinal in the family of all monotone operators. More specifically, for each monotone operator $A \subseteqq E \times E^*$ there is a saddle function $K: E \times E \to \overline{\mathbf{R}}$ such that T_K is a monotone extension of A. In particular, all maximal monotone operators are of type T_K. These facts are the background of all our considerations. We begin with

Definition I.3: Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function. Then we define the operator $T_K \subseteqq E \times E^*$ by

$$f \in T_K x := [-f, f] \in \partial K(x, x). \tag{I.11}$$

By the definition of the subdifferential ∂K the relation $f \in T_K x$ is equivalent to

$$\left.\begin{aligned} K(x, x) - f(v - x) &\geqq K(v, x), & v \in E, \\ K(x, x) + f(w - x) &\leqq K(x, w), & w \in E. \end{aligned}\right\} \tag{I.12}$$

We show now that the operator T_K generalizes the notion of the subdifferential of a convex function.

Theorem I.1: *Let $p: E \to \overline{\mathbf{R}}$ be a proper convex function. Then $T_K = \partial p$ holds true for any saddle function $K: E \times E \to \overline{\mathbf{R}}$ with*

$$K(x, y) = p(y) - p(x) \quad \text{if} \quad x \in \operatorname{dom} p \quad \text{or} \quad y \in \operatorname{dom} p. \tag{I.13}$$

Each such saddle function is skew-symmetric. If, additionally, p is closed, then K is also closed and we have $\operatorname{Dom} K = \operatorname{dom} p$.

For the proof we refer to [4]. Saddle functions for which the condition (I.13) is fulfilled are, for example,

$$K_1(x, y) = \begin{cases} p(y) - p(x) & \text{for } y \in \operatorname{dom} p \\ +\infty & \text{otherwise} \end{cases}$$

and

$$K_2(x, y) = \begin{cases} p(y) - p(x) & \text{for } x \in \operatorname{dom} p \\ -\infty & \text{otherwise} \end{cases}.$$

Our subsequent considerations will rest on the following

Theorem I.2: *Let $K: E \times E \to \overline{\mathbf{R}}$ be proper. Then $T_K \subseteqq E \times E^*$ is a monotone operator.*

The proof of this result has been given in [3]. It is easy to see that T_K is not monotone for improper saddle functions provided that the domain $D(T_K)$ of T_K consists of at least two elements. For the investigation of the operator T_K it will sometimes be convenient to replace the system (I.12) by a single inequality. This is done in

Proposition I.1: *Let $K: E \times E \to \overline{\mathbf{R}}$ be proper. Then the following conditions are equivalent:*

(i) $f \in T_K x$;

(ii) $x \in \operatorname{dom}_1 K \cap \operatorname{dom}_2 K$ *and* (I.14)

$$f\left(\frac{v + w}{2} - x\right) \leqq \frac{\operatorname{cl}_2 K(x, w) - \operatorname{cl}_1 K(v, x)}{2}, \qquad v, w \in E.$$

Moreover, if K is closed, then both these conditions are equivalent to

$$\text{(iii)}\quad f\left(\frac{v+w}{2}-x\right) \leqq \frac{\mathrm{cl}_1 K(x,w) - \mathrm{cl}_2 K(v,x)}{2}, \qquad [v,w] \in \operatorname{dom} K. \qquad \text{(I.15)}$$

When dealing with saddle functions with values in the extended real line $\overline{\mathbf{R}}$, one has to take care that there does not occur any indefinite expression of the kind $+\infty - \infty$. In (I.14) this is achieved by the assumption $x \in \operatorname{dom}_1 K \cap \operatorname{dom}_2 K$ and in (I.15) by the requirement $[v, w] \in \operatorname{dom} K$.

Proof of Proposition I.1: The equivalence (i) $\Leftrightarrow$ (iii) has been proved in [3]. Concerning the implication (i) $\Rightarrow$ (ii) we refer to [3: (8)]. To prove (ii) $\Rightarrow$ (i), from (I.14) we deduce

$$K(v,x) + f(v+w-2x) \leqq \mathrm{cl}_1 K(v,x) + f(v+w-2x) \leqq \mathrm{cl}_2 K(x,w) \leqq K(x,w)$$

for all $v, w \in E$. Setting here $v = x$, or $w = x$, respectively, we obtain the system (I.12), i.e. $f \in T_K x$ ∎

From Prop. I.1 one can easily derive the following characterization of the domain $D(T_K)$ of the operator T_K.

Proposition I.2: *Let $K: E \times E \to \overline{\mathbf{R}}$ be proper. Then the following inclusions hold true:*

$$D(T_K) \subseteqq \left\{x \in E : \mathrm{cl}_1 K(v,x) \leqq \mathrm{cl}_2 K(x,w) \text{ for } v, w \text{ with } \frac{v+w}{2} = x\right\}$$

$$\subseteqq \{x \in \operatorname{dom}_1 K \cap \operatorname{dom}_2 K : \mathrm{cl}_1 K(x,x) = \mathrm{cl}_2 K(x,x)\}.$$

In the next section and in the forthcoming Part II we shall generalize the statement of Prop. I.2 provided that T_K is a maximal monotone operator.

Proof of Proposition I.2: The first inclusion results from (I.14). Setting here $v = w = x$ we find $\mathrm{cl}_1 K(x,x) \leqq \mathrm{cl}_2 K(x,x)$, hence $\mathrm{cl}_2 K(x,x) = \mathrm{cl}_1 K(x,x)$. Furthermore, if in the inequality $\mathrm{cl}_1 K(v,x) \leqq \mathrm{cl}_2 K(x,w)$ $\big(v, w \in E,\ (v+w)/2 = x\big)$ we choose $v \in \operatorname{dom}_1 K \neq \emptyset$ then we get $x \in \operatorname{dom}_1 K$. A similar argument shows $x \in \operatorname{dom}_2 K$ ∎

Now we are going to ask for the maximal monotonicity of the operator T_K.

Theorem I.3: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function such that $T_K \subseteqq E \times E^*$ is maximal monotone. Then K necessarily satisfies the condition $(*)$.*

The proof of this statement is given in Section I.6. The forthcoming Part II contains some results which make the condition $(*)$ more transparent. For a preliminary interpretation of the condition $(*)$ one should recall the estimate for the domain of T_K given in Prop. I.2.

We show now that under quite natural assumptions on K and the space E, the condition $(*)$ is also sufficient for the maximality of T_K.

Theorem I.4: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a closed proper saddle function on a reflexive Banach space. Then T_K is maximal monotone if and only if K satisfies the condition $(*)$.*

Proof: One part of this statement is already contained in Th. I.3. The remaining one has been proved in [3] (compare the remark following Th. 2 of that paper) ∎

It is easy to see how Th. I.4 reads if the saddle function K is not supposed to be closed. In this case we obtain

Theorem I.5: *Let $K: E \times E \to \overline{\mathbf{R}}$ be proper and let E be a reflexive Banach space. Then T_K is maximal monotone if and only if $T_K = T_{\tilde{K}}$ for each $\tilde{K}$, $\mathrm{cl}_2\, K \leqq \tilde{K} \leqq \mathrm{cl}_1 K$, and all these saddle functions $\tilde{K}$ satisfy the condition* (*).

Proof: Let T_K be maximal monotone. A little thought shows that $T_{\tilde{K}}$ is a monotone extension of T_K for each $\tilde{K}$, $\mathrm{cl}_2\, K \leqq \tilde{K} \leqq \mathrm{cl}_1\, K$. This implies $T_K = T_{\tilde{K}}$. Hence, on account of Th. I.3, $T_{\tilde{K}}$ satisfies the condition (*). On the other hand, the interval $\mathrm{cl}_2\, K \leqq \tilde{K} \leqq \mathrm{cl}_1\, K$ contains at least one closed saddle function, namely $\tilde{K} = \mathrm{cl}_2\, \mathrm{cl}_1\, K$. One easily verifies $\mathrm{dom}\, K \subseteq \mathrm{dom}\, \tilde{K}$, so that $\tilde{K}$ is proper. The maximality of $T_K = T_{\tilde{K}}$ is now a consequence of Th. I.4 ∎

It is obvious that the operators T_K and T_L coincide for equivalent saddle functions K and L. More important for our purposes is the following simple result, which is an immediate consequ nce of the definition of the operator T_K.

Lemma I.5: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function and define $L: E \times E \to \overline{\mathbf{R}}$ by $L(x, y) := -K(y, x)$. Then we have $T_K = T_L$.*

I.5 On the construction of skew-symmetric saddle functions. Results

In this section we show that the characterization of the operator T_K, as given in (I.12) and (I.14), can be considerably simplified if we have some more information about the skew-symmetry of K. Later on we ask for saddle functions L for which the operator T_L is a monotone extension of a given operator T_K. An important role will here be played by skew-symmetric saddle functions L and those which satisfy the inequality

$$\mathrm{cl}_2\, L(x, y) \leqq -\mathrm{cl}_2\, L(y, x) \quad \text{for all} \quad x, y \in E.$$

The latter ones are just the saddle functions which can be majorized by a closed skew-symmetric saddle function. We apply these results to get some estimates for the domains of a saddle function and its corresponding monotone operator. Other applications are contained in the forthcoming Part II. Some of the results are rather technical. The reader who is not interested in too many details is advised only to read Prop. I.3 and the Th. I.6—I.8, together with the Cor. I.1 and I.2. For the sake of a better reading all proofs are given in the next section.

As already announced, we are now looking for a simple characterization of the operator T_K. A first result is

Lemma I.6: *For any saddle function $K: E \times E \to \overline{\mathbf{R}}$ the implication* (i) ⇒ (ii) *holds true. If T_K is maximal monotone, then* (i) *and* (ii) *are even equivalent:*

(i) $\mathrm{cl}_1\, K(u, u) \geqq 0$ *for all $u \in E$.*
(ii) *Each pair $[x, f] \in T_K$ satisfies the inequality*

$$f(v - x) \leqq K(x, v) \quad \textit{for all} \quad v \in E. \tag{I.16}$$

The implication (ii) → (i) reflects the fact that each maximal monotone operator $A \subseteq E \times E^*$ satisfies the inequality $\inf \{(f - h_0, v - x_0); [v, f] \in A\} \leqq 0$ for each $x_0 \in E$, $h_0 \in E^*$.

The following result can be viewed as a converse of Lemma I.6.

Lemma I.7: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function with*

$$\mathrm{cl}_2\, K(x, v) \leqq -\mathrm{cl}_1\, K(v, x) \quad \textit{for all} \quad x, v \in E. \tag{I.17}$$

Then a pair $[x, f] \in E \times E^*$ belongs to T_K if it obeys the inequality (I.16).

The statements of Lemma I.6 and Lemma I.7 are put together in

Proposition I.3: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function with*

$$\mathrm{cl}_2 K(u, v) \leqq -\mathrm{cl}_1 K(v, u) \quad \text{and} \quad \mathrm{cl}_1 K(u, u) \geqq 0, \qquad u, v \in E. \tag{I.18}$$

Then the following conditions are equivalent:

(i) $[x, f] \in T_K$,
(ii) $f(v - x) \leqq K(x, v)$ *for all* $v \in E$,
(iii) $f \in \partial_2 K(x, x)$ *and* $\mathrm{cl}_1 K(x, x) = K(x, x)$,
(iv) *there exists a* $g \in E^*$ *with* $[-g, f] \in \partial K(x, x)$,
(v) *there exists a* $y \in E$ *with* $[-f, f] \in \partial K(x, y)$.

The assumption (I.18) implies that K satisfies the condition (**). It is fulfilled, in particular, for skew-symmetric saddle functions. Other saddle functions for which (I.18) holds true will be considered in the forthcoming Part II.

The background of our subsequent considerations is the following

Proposition I.4: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function such that*

$$\mathrm{cl}_2 K(x, y) \leqq -\mathrm{cl}_2 K(y, x) \quad \text{for all} \quad x, y \in E.$$

Then there exists a closed skew-symmetric saddle function $L: E \times E \to \overline{\mathbf{R}}$ with

$$\mathrm{cl}_2 K(x, y) \leqq L(x, y) \leqq -\mathrm{cl}_2 K(y, x) \quad \text{for all} \quad x, y \in E. \tag{I.19}$$

The proof which was given in [4] is non-constructive. In the forthcoming Part II we shall calculate such a saddle function L under certain regularity assumptions on K.

Now it seems natural to ask how the operators T_K and T_L relate to each other if the saddle functions K and L satisfy (I.19).

Lemma I.8: *For each pair of saddle functions $K, L: E \times E \to \overline{\mathbf{R}}$ with*

$$\mathrm{cl}_2 K(x, y) \leqq L(x, y) \leqq -\mathrm{cl}_2 K(y, x) \quad \text{for all} \quad x, y \in E \tag{I.20}$$

the following statements are true:

(i) $\mathrm{dom}_1 K \subseteqq \mathrm{dom}_1 L \cap \mathrm{dom}_2 L, \quad \mathrm{dom}_1 L \cup \mathrm{dom}_2 L \subseteqq \mathrm{dom}_2 \mathrm{cl}_2 K.$ (I.21)
(ii) *Any pair* $[x, f] \in E \times E^*$ *with* $f(v - x) \leqq K(x, v)$, $v \in E$, *belongs to* T_L.
(iii) *If additionally* $\mathrm{cl}_1 K(u, u) \geqq 0$, $u \in E$, *holds then we also have* $T_K \subseteqq T_L$.

The preceding lemma provides us with a tool to attack a more general question. Let an arbitrary saddle function K be given. We ask whether we can find a certain interval of saddle functions such that $T_L \supseteqq T_K$ holds for each L belonging to this interval. Results of this type are of special interest if the interval in consideration contains a skew-symmetric saddle function. For this purpose we need the following

Definition I.4: To an arbitrary saddle function $K: E \times E \to \overline{\mathbf{R}}$ we associate another saddle function $L_K: E \times E \to \overline{\mathbf{R}}$ by

$$L_K(x, v) := \inf \left\{ \frac{\mathrm{cl}_2 K(x, v_1) - \mathrm{cl}_1 K(v_2, x)}{2} ; \frac{v_1 + v_2}{2} = v \right\},$$

for each $v \in E$ and $x \in \mathrm{dom}_1 K \cap \mathrm{dom}_2 K$. Otherwise we set $L_K(x, v) = -\infty$.

Concerning an interpretation of this saddle function we mention that the condition (I.14) of Prop. I.1 can be reformulated as $f(u - x) \leqq L_K(x, u)$ for all $u \in E$. It can easily be checked that the saddle function L_K obeys the inequality

$$L_K(x, y) \leqq -L_K(y, x) \quad \text{for all} \quad x, y \in E. \tag{I.22}$$

Hence, the following statement makes sense.

Proposition I.5: *Let $K: E \times E \to \overline{\mathbf{R}}$ be proper. Then we have $T_K \subseteqq T_L$ for each saddle function $L: E \times E \to \overline{\mathbf{R}}$ with*

$$\mathrm{cl}_2\, L_K(x, y) \leqq L(x, y) \leqq -\mathrm{cl}_2\, L_K(y, x), \qquad x, y \in E. \tag{I.23}$$

If T_K is maximal monotone, then each L satisfies the condition (**) and T_L coincides with T_K.

We are now going to derive some consequences of Prop. I.5. The importance of the following theorem will become clear in the forthcoming Part II, when we are concerned with the representation of monotone operators by saddle functions. It allows to pass here from arbitrary saddle functions to skew-symmetric ones. Th. I.6 generalizes a result previously obtained in [4].

Theorem I.6: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a saddle function,* $\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K \neq \emptyset$. *Then there exists a closed skew-symmetric saddle function $L: E \times E \to \overline{\mathbf{R}}$ with*

$$\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K \subseteqq \mathrm{Dom}\, L \subseteqq \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K},$$

such that T_L is a monotone extension of T_K. If E is a reflexive Banach space, then T_L is even maximal monotone.

Another consequence of Prop. I.5 is the following estimate for the domains of K and T_K.

Theorem I.7: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a proper saddle function such that T_K is maximal monotone. Then the identity*

$$\overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K} = \overline{\frac{\mathrm{dom}_1\, K + \mathrm{dom}_2\, K}{2}} = \overline{\mathrm{co}}\, D(T_K) \tag{I.24}$$

holds.

In the forthcoming Part II we shall sharpen this result considerably, provided that E is a reflexive Banach space. The next result shows how the assertion of Lemma I.4 is modified if one there replaces the condition (**) by the condition (*).

Corollary I.1: *Let E be a reflexive Banach space and let $K: E \times E \to \overline{\mathbf{R}}$ be a closed proper saddle function satisfying the condition (*). Then the identity* (I.24) *holds true.*

Now we show that the assertions of Th. I.7 and Cor. I.1 can be strenghtened, if we additionally suppose the saddle function K to satisfy the condition (**). For a reflexive Banach space E, we obtain a generalization of Lemma I.4.

Theorem I.8: *Let $K: E \times E \to \overline{\mathbf{R}}$ be a proper saddle function such that T_K is maximal monotone. If K satisfies the condition (**) then we have*

$$\overline{\mathrm{dom}_1\, K} = \overline{\mathrm{dom}_2\, K} = \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K} = \overline{\mathrm{co}}\, D(T_K). \tag{I.25}$$

Corollary I.2: *Let E be a reflexive Banach space and let K be a closed proper saddle function satisfying the condition (**). Then the identity* (I.25) *holds true.*

I.6 On the construction of skew-symmetric saddle functions. Proofs

Now we furnish the proofs for the results of the preceding section.

Proof of Lemma 1.6: (i) $\Rightarrow$ (ii): For an arbitrary pair $[x, f] \in T_K$ one has $\mathrm{cl}_1\, K(x, x) = K(x, x)$ and $K(x, x) + f(w - x) \leq K(x, w)$ for any $w \in E$ (compare Prop. I.2). Due to our assumption $\mathrm{cl}_1\, K(x, x) \geqq 0$ we get $f(w - x) \leq K(x, w)$, $w \in E$, as desired.

(ii) $\Rightarrow$ (i): We supposed T_K to be maximal monotone. Let us show that the assumption

$$\mathrm{cl}_1 K(v_0, v_0) < 0, \tag{I.26}$$

for some $v_0 \in E$, leads to a contradiction. By the definition of the concave closure (compare (I.3)), (I.26) implies the existence of a $g_0 \in E^*$ and an $\varepsilon > 0$ with

$$g_0(w - v_0) - K(w, v_0) \geqq \varepsilon > 0 \quad \text{for any} \quad w \in E. \tag{I.27}$$

According to our assumption (ii), (I.27) yields $(f + g_0, w - v_0) \geqq \varepsilon$ for each pair $[w, f] \in T_K$. Since T_K was maximal monotone, we get $-g_0 \in T_K v_0$. Hence, $\varepsilon \leqq 0$, which contradicts the choice of ε ∎

Proof of Lemma 1.8: (i): The closing of the inequality (I.20) with respect to the convex argument yields

$$\mathrm{cl}_2 K(x, y) \leqq \mathrm{cl}_2 L(x, y) \leqq -\mathrm{cl}_1 \mathrm{cl}_2 K(y, x), \qquad x, y \in E. \tag{I.28}$$

A similar procedure for the concave argument shows

$$\mathrm{cl}_1 \mathrm{cl}_2 K(x, y) \leqq \mathrm{cl}_1 L(x, y) \leqq -\mathrm{cl}_2 K(y, x), \qquad x, y \in E. \tag{I.29}$$

These inequalities immediately imply the first assertion.

(ii): We have to show that the inequality

$$f(w - x) \leqq K(x, w) \quad \text{for all} \quad w \in E \tag{I.30}$$

implies $[x, f] \in T_L$. From (I.30) follows

$$f(w - x) \leqq \mathrm{cl}_2 K(x, w), \qquad w \in E, \tag{I.31}$$

which together with the inequalities (I.28) and (I.29) leads to

$$f(v - x) \leqq \mathrm{cl}_2 L(x, v), \qquad v \in E, \tag{I.32}$$

and

$$f(w - x) \leqq -\mathrm{cl}_1 L(w, x), \qquad w \in E. \tag{I.33}$$

In particular, we get $x \in \mathrm{dom}_1 L \cap \mathrm{dom}_2 L$. Moreover, the adding of (I.32) and (I.33) yields

$$f\left(\frac{v + w}{2} - x\right) \leqq \frac{\mathrm{cl}_2 L(x, v) - \mathrm{cl}_1 L(w, x)}{2}, \qquad v, w \in E.$$

Recalling Prop. I.1 we get $[x, f] \in T_L$ as desired.

(iii): This statement follows from (ii) by an application of Lemma I.6 ∎

Proof of Lemma I.7: Set in Lemma I.8 $L = K$ ∎

Proof of Proposition I.3: The implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (v) are obvious, while (ii) $\Rightarrow$ (i) follows from Lemma I.7. (iii) $\Rightarrow$ (ii): We consider an element $f \in \partial_2 K(x, x)$. The assumption (I.18) implies $\mathrm{cl}_1 K(x, x) \geqq 0$. Since in (iii) we supposed $\mathrm{cl}_1 K(x, x) = K(x, x)$, we obtain $f(v - x) \leqq K(x, x) + f(v - x) \leqq K(x, v)$ for any $v \in E$, i.e. (ii). (v) $\Rightarrow$ (ii): For this purpose we assume $[-f, f] \in \partial K(x, y)$. We find

$$K(x, y) - f(v - x) \geqq \mathrm{cl}_1 K(v, y), \qquad v \in E, \tag{I.34}$$

and

$$K(x, y) + f(w - y) \leqq \mathrm{cl}_2 K(x, w), \qquad w \in E. \tag{I.35}$$

Setting $v = y$ in (I.34) and $w = x$ in (I.35) we obtain

$$\mathrm{cl}_1\, K(y, y) \leqq K(x, y) + f(x - y) \leqq \mathrm{cl}_2\, K(x, x). \tag{I.36}$$

On account of our assumption (I.18) we have $\mathrm{cl}_1\, K(y, y) \geqq 0$ and $\mathrm{cl}_2\, K(x, x) \leqq 0$. Thus (I.36) implies $K(x, y) = f(y - x)$. Inserting this identity into (I.35) we get $f(w - x) \leqq K(x, w)$ for $w \in E$, i.e. (ii) ∎

Proof of Proposition I.5: In order to prove the inclusion $T_K \subseteqq T_L$, we consider an arbitrary element $[x, f] \in T_K$. In view of the definition of the saddle function L_K we obtain $f(v - x) \leqq L_K(x, v)$ for all $v \in E$ (compare Prop. I.1). The closing of this inequality with respect to the convex argument yields

$$f(v - x) \leqq \mathrm{cl}_2\, L_K(x, v) \quad \text{for} \quad v \in E \quad \text{and} \quad [x, f] \in T_K. \tag{I.37}$$

The inclusion $T_K \subseteqq T_L$ now results from Lemma I.8 if we there replace the saddle function K by L_K. Let us suppose now that T_K is maximal monotone. Then, from (I.37) and Lemma I.8 (i) we can conclude

$$\emptyset \neq D(T_K) \subseteqq \mathrm{dom}_1\, L_K \subseteqq \mathrm{dom}_1\, L \cap \mathrm{dom}_2\, L,$$

for each saddle function L with

$$\mathrm{cl}_2\, L_K(x, y) \leqq L(x, y) \leqq -\mathrm{cl}_2\, L_K(y, x), \qquad x, y \in E. \tag{I.38}$$

In particular, all these saddle functions are proper. Hence, as a consequence of Th. I.2, T_L is a monotone operator. Since T_K was maximal monotone the inclusion $T_K \subseteqq T_L$ implies $T_L = T_K$. The inequality (I.37) can now be read as $f(v - x) \leqq \mathrm{cl}_2\, L_K(x, v)$; $v \in E$ and $[x, f] \in T_{(\mathrm{cl}_2 L_K)} = T_K$. We can now apply Lemma I.6 to the saddle function $\mathrm{cl}_2\, L_K$ and obtain

$$\mathrm{cl}_1\, \mathrm{cl}_2\, L_K(x, x) \geqq 0 \quad \text{for all} \quad x \in E. \tag{I.39}$$

The inequality (I.38) implies

$$\mathrm{cl}_1\, \mathrm{cl}_2\, L_K(x, y) \leqq \mathrm{cl}_1\, L(x, y) \quad \text{and} \quad \mathrm{cl}_2\, L(x, y) \leqq -\mathrm{cl}_1\, \mathrm{cl}_2\, L_K(y, x) \tag{I.40}$$

for all $x, y \in E$. From (I.39) and (I.40) we can easily deduce the desired inequality $\mathrm{cl}_2\, L(x, x) \leqq 0 \leqq \mathrm{cl}_1\, L(x, x)$ for all $x \in E$ ∎

Now we are in the position to give the

Proof of Theorem I.3: For an improper saddle function K the condition (*) is trivially satisfied, so that we can assume K to be proper. We introduce a saddle function $L: E \times E \to \overline{\mathbf{R}}$,

$$L(x, y) := \inf \left\{ \frac{\mathrm{cl}_1\, K(x, y_1) - \mathrm{cl}_2\, K(y_2, x)}{2} : [y_1, y_2] \in \mathrm{dom}\, K,\ \frac{y_1 + y_2}{2} = y \right\}.$$

It can easily be checked that L fulfils the conditions

$$\mathrm{cl}_1\, L = L \quad \text{and} \quad L_K(x, y) \leqq L(x, y) \leqq -L_K(y, x) \quad \text{for} \quad x, y \in E,$$

with L_K as in Def. I.4. According to Prop. I.5, L_K has to satisfy the condition (**), so that we obtain $L(x, x) \geqq 0$ for $x \in E$. By the definition of L, this just means that K satisfies the condition (*) ∎

Proof of Theorem I.6: We define a saddle function $\hat{L}\colon E \times E \to \overline{\mathbf{R}}$ by

$$\hat{L}(x, y) := \begin{cases} \dfrac{\mathrm{cl}_2\, K(x, y) - \mathrm{cl}_1\, K(y, x)}{2} & \text{for} \quad x \in \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K, \\ & \quad\quad y \in \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K} \\ +\infty & \text{for} \quad x \in \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K, \\ & \quad\quad y \notin \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K} \\ -\infty & \text{for} \quad x \notin \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K. \end{cases}$$

One easily verifies the relations

$$\hat{L} = \mathrm{cl}_2\, \hat{L}, \quad \mathrm{dom}_1\, \hat{L} = \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K \neq \emptyset, \tag{I.41}$$
$$\mathrm{dom}_2\, \hat{L} \subseteqq \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K}$$

and the inequalities

$$L_K(x, y) \leqq \hat{L}(x, y) \leqq -\hat{L}(y, x) \leqq -L_K(y, x),$$

with L_K as in Def. I.4. Hence, we can apply Prop. I.4 to the saddle function $\hat{L}$ and obtain a closed skew-symmetric saddle function $L\colon E \times E \to \overline{\mathbf{R}}$ with

$$L_K(x, y) \leqq \hat{L}(x, y) \leqq L(x, y) \leqq -\hat{L}(y, x) \leqq -L_K(y, x), \qquad x, y \in E.$$

We show that L fulfils all requirements of Th. I.6. First, according to Lemma I.8 (i), we get $\mathrm{dom}_1\, \hat{L} \subseteqq \mathrm{Dom}\, L \subseteqq \mathrm{dom}_2\, \hat{L}$, which together with (I.41) yields $\emptyset \neq \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K \subseteqq \mathrm{Dom}\, L \subseteqq \overline{\mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K}$. On account of Prop. I.5, T_L is a monotone extension of T_K. For a reflexive Banach space E, the maximal monotonicity of T_K follows from Th. I.4 ∎

Proof of Theorem I.8: It suffices to verify the inclusion

$$\mathrm{dom}_2\, K \subseteqq \overline{\mathrm{co}}\, D(T_K). \tag{I.42}$$

Indeed, for reasons of symmetry, together with (I.42) we obtain $\mathrm{dom}_1\, K \subseteqq \overline{\mathrm{co}}\, D(T_K)$ and hence,

$$\mathrm{co}\, D(T_K) \subseteqq \mathrm{dom}_1\, K \cap \mathrm{dom}_2\, K \subseteqq \mathrm{dom}_i\, K \subseteqq \overline{\mathrm{co}}\, D(T_K), \qquad i = 1, 2 \tag{I.43}$$

(compare Prop. I.2). This relation immediately implies the desired identity (I.25).

To prove (I.42) let an arbitrary element $x_0 \in \mathrm{dom}_2\, K$ be given. By the definition of $\mathrm{dom}_2\, K$ we find an $h \in E^*$ and a $c \in \mathbf{R}$ with

$$\mathrm{cl}_1\, K(v, x_0) \leqq h(v - x_0) + c \quad \text{for all} \ \ v \in E. \tag{I.44}$$

Suppose now $x_0 \notin \overline{\mathrm{co}}\, D(T_K)$. Then, by the separation theorem for convex sets, there exist a $g \in E^*$ and an $\varepsilon > 0$ with

$$0 \leqq g(v - x_0) - \varepsilon \quad \text{for} \ \ v \in D(T_K). \tag{I.45}$$

Now we set $f = -h - \lambda g$, where $\lambda > 0$ is a real number with $c \leqq \lambda\varepsilon$. Then, from (I.44) and (I.45) we can conclude

$$\mathrm{cl}_1\, K(v, x_0) \leqq f(x_0 - v) \quad \text{for all} \ \ v \in D(T_K). \tag{I.46}$$

Now let an arbitrary element $[z, j] \in T_K$ be given. Since K was supposed to satisfy the condition (**), by Lemma I.6 we get

$$j(w - z) \leqq \mathrm{cl}_2\, K(z, w) \quad \text{for all} \ \ w \in E. \tag{I.47}$$

Setting $v = z$ in (I.46) and $w = x_0$ in (I.47) yields $(f - j, x_0 - z) \geqq 0$ for each pair $[z, j] \in T_K$. Since T_K was supposed to be maximal monotone, we can conclude $[x_0, f] \in T_K$, which is a contradiction to the assumption $x_0 \notin \overline{\text{co}}\, D(T_K)$ ▮

Proof of Corollary I.2: Under our assumptions the operator T_K is maximal monotone (compare Th. I.4). Hence, the assertion follows from Th. I.8 ▮

Proof of Theorem I.7: Let us consider the saddle function $L: E \times E \to \overline{\mathbf{R}}$, $L := \text{cl}_2 \hat{L}$, where $\hat{L}$ is defined by

$$\hat{L}(x, y) := \inf \left\{ \frac{\text{cl}_1 K(x, y_1) - \text{cl}_2 K(y_2, x)}{2} ; y = \frac{y_1 + y_2}{2}, [y_1, y_2] \in \text{dom } K \right\},$$

for each $y \in E$ and $x \in \text{dom}_1 K \cap \text{dom}_2 K$. For $x \notin \text{dom}_1 K \cap \text{dom}_2 K$ we set $\hat{L}(x, y) \equiv -\infty$. One easily verifies the inclusions

$$\text{dom}_1 L \subseteqq \text{dom}_1 K \cap \text{dom}_2 K, \quad \text{dom}_2 L \supseteqq \text{dom}_2 \hat{L} = \frac{\text{dom}_1 K + \text{dom}_2 K}{2}. \tag{I.48}$$

Obviously L satisfies the inequality

$$\text{cl}_2 L_K(x, y) \leqq L(x, y) \leqq -L_K(y, x) \quad \text{for } x, y \in E,$$

where L_K is taken from Def. I.4. Since T_K was supposed to be maximal monotone, we can conclude from Prop. I.5 that also $T_L = T_K$ is maximal monotone and that L satisfies the condition (**). Moreover, we have $L = \text{cl}_2 \hat{L} = \text{cl}_2 \text{cl}_1 \hat{L}$, i.e. L is a closed saddle function. We can now apply Th. I.8 to L and obtain $\overline{\text{dom}_1 L} = \overline{\text{dom}_2 L} = \overline{\text{co}}\, D(T_L) = \overline{\text{co}}\, D(T_K)$. By (I.48) this leads to

$$\overline{\frac{\text{dom}_1 K + \text{dom}_2 K}{2}} \subseteqq \overline{\text{co}}\, D(T_K) \subseteqq \overline{\text{dom}_1 K \cap \text{dom}_2 K}.$$

Since the inclusion $\overline{\text{dom}_1 K \cap \text{dom}_2 K} \subseteqq 1/2\, (\overline{\text{dom}_1 K + \text{dom}_2 K})$ is trivially satisfied, we get the desired identity (1.24) ▮

Proof of Corollary I.1: Under our assumptions the operator T_K is maximal monotone (compare Th. I.4), so that we can apply Th. I.7 ▮

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