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Some Applications of Degree Theory to Bifurcation Problems

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Es werden einige Ergebnisse über das Verschwinden eines verallgemeinerten Abbildungsgrades für mengenwertige Abbildungen bewiesen und auf die Untersuchungen von Bifurkationslösungen der Gleichung $F(\lambda, u) = 0$ in einem Banach-Raum angewandt. Als Spezialfall wird die Abbildung $F(\lambda, u) = u - M(u) - N(\lambda; u) - H(\lambda, u)$ mit dem Parameter λ aus einem normierten Raum betrachtet. Als konkrete Anwendung werden Bifurkationslösungen einer nichtlinearen elliptischen Differentialgleichung zweiter Ordnung untersucht.

Некоторые результаты об исчезновении обобщённой степени отображения для многозначных отображений доказываются и применяются для исследования бифуркационных решений уравнения $F(\lambda, u) = 0$ в банаховом пространстве. Как специальный случай рассматривается отображение $F(\lambda, u) = u - M(u) - N(\lambda, u) - H(\lambda, u)$ с параметром λ из нормированного пространства. В качестве приложения исследуются бифуркационные решения нелинейного эллиптического дифференциального уравнения второго порядка.

Some results on the vanishing of a generalized degree of multivalued mappings are proved and applied to study bifurcation solutions of the equation $F(\lambda, u) = 0$ in a Banach space; as a special case, $F(\lambda, u) = u - M(u) - N(\lambda, u) - H(\lambda, u)$, where the parameter λ is from a normed space. For concrete application there are considered bifurcation solutions of a nonlinear elliptic differential equation of second order.

1. Introduction

The degree theory for nonlinear multivalued mappings, which plays an important role in functional analysis, in theories of ordinary and partial differential equations and in some other fields of applied mathematics, has received very much attention from mathematicians. Granas [8] and Ma [17] extended the Leray-Schauder degree for compact single-valued fields to the degree for compact multivalued fields in locally convex Hausdorff spaces. Browder and Petryshyn [1, 2] defined the degree for a class of multivalued mappings between Banach spaces which are approximation proper (A-proper) with respect to some approximation scheme. This degree can be used to establish a degree theory for some other class of multivalued mappings. Krauss [14] introduced the degree for multivalued mappings satisfying an extremely weak continuity hypothesis, i.e. for triples (A, Ω, p) which are admissible in the sense of Definition 2 in [14]. Let X be a Banach space, $K \subset X$ a closed and convex cone, $T: D_K = D \cap K \to K$ a mapping such that $id - T$ is A-proper with the projectionally complete scheme $\Gamma_0 = \{X_n, P_n\}$ with $P_n(K) \subseteq K$. Fitzpatrick and Petryshyn [6] defined the fixed point index of T on D with respect to K, denoted by $I_K(T, D)$. Further, some other definitions of degree for more general types of multivalued mappings were constructed by different authors.

In what follows, by $\mathbf{Z}$ we denote the space of all integers and by $\mathbf{Z}'$ the set $\mathbf{Z} \cup \{-\infty, +\infty\}$. Let X and Y be real locally convex Hausdorff spaces, K and D

non-empty subsets of X with $D_K = D \cap K \neq \emptyset$. Let $\mathscr{C}_K(D)$ be a family of non-empty subsets of D_K. Suppose that $\Omega \in \mathscr{C}_K(D)$. We write $\partial_K \Omega$ for the boundary of Ω relative to K. Further, by $\mathscr{B}_K(\Omega, Y)$ we denote the class of multivalued mappings F from $\bar{\Omega}$ into Y such that if $0 \notin F(\partial_K \Omega)$ then one can define the topological degree of F on Ω at the zero, $\deg_K(F, \Omega, 0) \subset \mathbf{Z}'$, satisfying the following axioms:

Axiom I: If $\deg_K(F, \Omega, 0) \neq \{0\}$ then there exists a point $\bar{u} \in \bar{\Omega}$ with $0 \in F(\bar{u})$.

Axiom II: Let $H: [0, 1] \times \bar{\Omega} \to 2^Y$ be a multivalued mapping such that for any fixed $t \in [0, 1]$ $H(t, \cdot) \in \mathscr{B}_K(\Omega, Y)$ and $H(\cdot, x)$ is upper semicontinuous uniformly to x from any bounded subset of $\partial_K \Omega$. If $0 \notin H(t, u)$ for all $t \in [0, 1]$ and $u \in \partial_K \Omega$, then

$$\deg_K\big(H(1, \cdot), \Omega, 0\big) = \deg_K\big(H(0, \cdot), \Omega, 0\big).$$

The main purpose of this paper is to show that under some necessary additional assumptions on the multivalued mapping $F \in \mathscr{B}_K(\Omega, Y)$ we have

$$\deg_K(F, \Omega, 0) = \{0\} \quad \text{provided} \quad 0 \notin F(\partial_K \Omega).$$

Further, we shall apply the obtained result on the vanishing of the degree of the mapping F as above to consider the existence of bifurcation solutions of the equation

$$F(\lambda, u) = 0, \qquad (\lambda, u) \in \Lambda \times D_K,$$

where Λ is an open subset of some metric space, D, K are subsets of a Banach space X with $0 \in \operatorname{int} D$ and $D_K = D \cap K \neq \emptyset$ and F is a mapping from $\Lambda \times D_K$ into another Banach space Y.

As a special case we consider the mapping F of the form

$$F(\lambda, u) = u - M(u) - N(\lambda, u) - H(\lambda, u), \qquad (\lambda, u) \in \Lambda \times D_K,$$

where Λ is an open subset of some normed space on which a partial ordering $\prec$ is defined. D is a subset of a Banach space X with $0 \in \operatorname{int} D$, and K is a closed and convex cone in X; for any fixed $\lambda \in \Lambda$, M, $N(\lambda, \cdot)$ are linear continuous mappings from D_K into X, and $H: \Lambda \times D_K \to X$ is a mapping with $H(\lambda, 0) = 0$ for all $\lambda \in \Lambda$ and $\|H(\lambda, u)\| = o(\|u\|)$ as $\|u\| \to 0$. We shall show that if $\lambda_0 \in \Lambda$ is the smallest eigenvalue of the pair (M, N) with respect to K which is isolated from the right side (see the defition below), then $(\lambda_0, 0)$ is a bifurcation solution of the equation

$$F(\lambda, u) = 0, \qquad (\lambda, u) \in \Lambda \times D_K.$$

Lastly, we apply the obtained result to investigate bifurcation solutions of the elliptic partial differential equation of second order in the form

$$L(u) = d(x)\, u + \lambda(x)\, g(x)\, u + h\big(\lambda(x), x, u, Du\big),$$

$$(\lambda, u) \in L_2(G) \times \big(W_2^2(G) \cap \mathring{W}_2^1(G)\big),$$

where L is defined by

$$L(u) = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u.$$

Let λ_0 be the smallest function from $L_2(G)$ satisfying

$$L(u_0) = d(x)\, u_0 + \lambda_0(x)\, g(x)\, u_0, \quad \text{for some} \quad u_0 \in \mathring{W}_2^1(G),$$

$$u_0 \neq 0, \qquad u_0(x) \geqq 0, \qquad x \in G;$$

and λ_0 is isolated from the right side (see the definition below). Under some additional conditions on the functions d, g, and h, we shall show that the above equation has at least two branches from the solution $(\lambda_0, 0)$. This implies that the above equation has $(\lambda_0, 0)$ as a bifurcation solution.

2. Notations and preliminaries

Throughout this paper, by X and Y we denote real locally convex Hausdorff spaces. Given subsets $D, K \subseteqq X$ with $D_K = D \cap K \neq \emptyset$, we denote by $\bar{D}$, $\partial_K D$ and 2^D the closure of D in X, the frontier of D relative to K and the family of all subsets of D, respectively. Further $\emptyset$ stands for the empty set, $\mathbf{N}$ for the set of natural numbers, and $\mathbf{R}$ for the set of real numbers. In the case that X is a normed space, $U(0, r)$ will indicate the open ball with the center at the zero in X and with the radius $r > 0$. For any multivalued mapping F from D into Y we write $F: D \to 2^Y$. We recall that F is called *upper semicontinuous at a point* $u_0 \in D$ if for every open set $Q \subseteqq Y$, $F(u_0) \subseteqq Q$, there exists a neighborhood U of u_0 such that $F(u) \subseteqq Q$ holds for all $u \in U \cap D$. The identity mapping is always denoted by *id*.

Let $\mathscr{C}_K(D)$ be some family of non-empty subsets of $D \cap K$. For $\Omega \in \mathscr{C}_K(D)$ by $\mathscr{B}_K(\Omega, Y)$ we denote the class of multivalued mappings F from Ω into Y for which one can define $\deg_K (F, \Omega, 0)$ (read: the degree of F on Ω at the zero), provided that $0 \notin F(\partial_K \Omega)$, which satisfies Axiom I and Axiom II just mentioned above in the Introduction.

Example 1: Let X be a locally convex Hausdorff space, $D \subseteqq X$ an open subset, $K = X$, let $\mathscr{C}_K(D)$ be a family of non-empty open and bounded subsets of D. For $\Omega \in \mathscr{C}_K(D)$ let $\mathscr{B}_K(\Omega, X)$ be the class of all multivalued mappings F from $\bar{\Omega}$ into X of the form $F = id - G$, where G is an upper semicontinuous compact multivalued mapping from $\bar{\Omega}$ into X with $G(x)$, $x \in \bar{\Omega}$, non-empty, convex and compact. Then $\deg (F, \Omega, 0)$ is well-defined by GRANAS [8] and MA [17], provided that $0 \notin F(\partial\Omega)$. It satisfies, of course, Axioms I and II mentioned above.

Example 2: Let X and Y be Banach spaces. The scheme $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is called a projectionally complete scheme for (X, Y) provided that $X_n \subseteqq X$ and $Y_n \subseteqq Y$ are sequences of monotonically increasing finite-dimensional subspaces with $\dim X_n = \dim Y_n$ for each n, and $P_n: X \to X_n$ and $Q_n: Y \to Y_n$ are linear projections such that $P_n x \to x$ and $Q_n y \to y$ for all $x \in X$ and $y \in Y$. We have

Definition 1 (see [18]): $T: \bar{D} \subseteqq X \to Y$ is said to be *A-proper with respect to the projectionally complete scheme* Γ if

(i) $T_n: D_n \subseteqq X_n \to Y_n$, where $\hat{D}_n = D \cap X_n$, $T_n = Q_n T / D_n$, is continuous for each n and

(ii) if $\{x_{n_j} \mid x_{n_j} \in D_{n_j}\}$ is any bounded sequence such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, then there exist a subsequence $\{x_{n_{j(k)}}\}$ and $x \in D$ such that $x_{n_{j(k)}} \to x$ and $T(x) = g$.

In the case $X = Y$, $X_n = Y_n$ and $P_n = Q_n$ we denote the projectionally complete scheme $\{X_n, P_n; X_n, P_n\}$ for (X, X) by $\Gamma_0 = \{X_n, P_n\}$. Furthermore, if $\|P_n\| = 1$ we say that X is a π_1*-space*. In [18] we can see that:

(i) If X is a π_1-space and $F: \bar{D} \subseteqq X \to X$ is ball-condensing (see the definition in [18]), then $T = id - F$ is A-proper.

(ii) Let X be reflexive with a projectionally complete scheme $\Gamma_1 = \{X_n, P_n; Y_n, Q_n\}$ for (X, X^*), where X^* is the dual space of X and $Q_n = P_n^*: X^* \to Y_n = R(Q_n)$, with $R(Q_n)$ being the range of Q_n. Let $T: X \to X^*$ be strongly monotone, i.e. $(Tx - Ty, x - y) \geqq c\,\|x - y\|$ for all $x, y \in X$ and for some $c \geqq 0$, and either continuous, semicontinuous, or weakly continuous, then T is A-proper with respect to Γ_1.

(iii) Let X, X^* be as in (ii) and let $T: X \to X^*$ satisfy *condition* (S), i.e. whenever $x_n \rightharpoonup x$ and $(Tx_n - Tx, x_n - x) \to 0$ imply that $x_n \to x$, where $\rightharpoonup$ and $\to$ denote the weak and the strong convergence, respectively, then T is A-proper with respect to Γ_1.

Further examples of A-proper mappings can be found in [18]. Now, let $D \subseteqq X$ with $\operatorname{int} D \neq \emptyset$, take $K = X$ and $\mathscr{E}_K(D) = \mathscr{E}(D)$, the family of all open and bounded subsets of D, and for $\Omega \in \mathscr{E}(D)$, $\mathscr{B}_K(\Omega, Y) = \mathscr{B}(\Omega, Y)$, the class of all A-proper mappings with respect to $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ from $\bar{\Omega}$ into Y. Thus, for any $\Omega \in \mathscr{E}(D)$, $F \in \mathscr{B}(\Omega, Y)$, $0 \notin F(\partial\Omega)$, $\deg(F, \Omega, 0)$ is defined by BROWDER and PETRYSHYN [1, 2]. It always satisfies Axioms I and II mentioned above.

Example 3: X is supposed to be a real Banach space, $K \subseteqq X$ is a closed convex cone, i.e. $K \subseteqq X$ is a closed subset and $x, y \in K$, $\alpha > 0$ imply $x + y \in K$ and $\alpha x \in K$ and $K \cap (-K) = \{0\}$. Suppose that $D \subseteqq X$ with $D_K = D \cap K \neq \emptyset$, $0 \in \operatorname{int} D$. Let $\mathscr{E}_K(D)$ be a family of non-empty open and bounded subsets of D_K. For $\Omega \in \mathscr{E}_K(D)$ let $\mathscr{B}_K(\Omega, X)$ be the class of all single-valued mappings F from $\bar{\Omega}$ into X of the form $F = id - T$ which are A-proper with respect to Γ_0 (see the definition in Example 2) and $T(\bar{\Omega}) \subseteqq K$. Thus, for $\Omega \in \mathscr{E}_K(D)$, $F \in \mathscr{B}_K(\Omega, X)$, $0 \notin F(\partial_K\Omega)$, the fixed point index of T on Ω with respect to K, denoted by $I_K(T, \Omega)$, is defined by FITZPATRICK and PETRYSHYN [6]. We can put $\deg_K(F, \Omega, 0) = I_K(T, \Omega)$ and we can see that Axioms I and II mentioned above are always satisfied.

Example 4: Let (X, Y) be a pair of dual spaces, $D \subseteqq X$ a subset. Take $K = X$ and $\mathscr{E}_K(D) = \mathscr{E}(D)$, the family of all non-empty open subsets Ω of D which are finitely bounded (i.e. $\Omega \cap X_0$ are bounded subsets for each finite-dimensional subspace X_0 of X). For $\Omega \in \mathscr{E}(D)$ let $\mathscr{B}(\Omega, Y)$ be the class of all multivalued mappings F from Ω into Y such that $(F, \Omega, 0)$ are admissible triples, provided $0 \notin F(\partial\Omega)$, in the sense of Definition 2 in [14]. Then $\deg(F, \Omega, 0)$ is defined, provided $0 \notin F(\partial\Omega)$, for any $F \in \mathscr{B}(\Omega, Y)$ by KRAUSS in [14] and it satisfies Axioms I and II mentioned above.

We recall that a multivalued mapping $F: D \to 2^Y$ is called *of type* S_+ if for any sequence $\{x_n\} \subset D$, $x_n \rightharpoonup x$ and $\{y_n\} \subset Y$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$ and $\lim (x_n, y_n) = (x, y)$, where $(\cdot, \cdot)$ denotes the pairing between elements of X and Y, it implies that $x_n \to x$. It has been shown in [14] that if F is a multivalued mapping of type S_+ and $\Omega \in \mathscr{E}(D)$ is a relative weakly compact subset, then $(F, \Omega, 0)$ is admissible, provided that $0 \notin F(\partial\Omega)$. Thus, $\deg(F, \Omega, 0)$ is defined (see [14: Lemma 2]).

Definition 2: Let $D, K \subseteqq X$ and let Λ be a subset of some space. Suppose that $M: D \to Y$ and $N: \Lambda \times \bar{D} \to Y$ are mappings.

a) A point $\lambda_0 \in \Lambda$ is said to be an *eigenvalue of the pair* (M, N) *with respect to* K if there is a $u_0 \in D_K = D \cap K$, $u_0 \neq 0$ such that

$$u_0 = M(u_0) + N(\lambda_0, u_0). \tag{1}$$

b) Let Λ be an open subset of some normed space on which a partial ordering $\prec$ is defined. The eigenvalue λ_0 of the pair (M, N) with respect to K is called *smallest* if $\lambda \prec \lambda_0$, $\lambda \neq \lambda_0$, implies that λ is not an eigenvalue of the pair (M, N) with respect to K.

c) Let Λ be as in b). The eigenvalue λ_0 of the pair (M, N) with respect to K is called *isolated from the right side* if for any neighborhood V of λ_0 in Λ there exists a $\lambda_1 \in V$ with $\lambda_0 \prec \lambda_1$ and λ_1 is not an eigenvalue of the pair (M, N) with respect to K.

d) The point $u_0 \in D_K$ satisfying (1) is called an *eigenvector of the pair* (M, N) *with respect to* K *associated to* λ_0.

Let us consider the multivalued equation

$$\{0\} \subset F(\lambda, u), \qquad (\lambda, u) \in \Lambda \times \bar{D}_K, \tag{2}$$

where Λ is a metric space, $D \subseteqq X$ is a subset with $0 \in \operatorname{int} D$, $K \subseteqq X$ with $D_K = D \cap K \neq \emptyset$, and $F: \Lambda \times \bar{D}_K \to 2^Y$ is a multivalued mapping with $\{0\} \subset F(\lambda, 0)$ for all $\lambda \in \Lambda$.

Definition 3: a) A point $(\lambda_0, 0) \in \Lambda \times \overline{D}_K$ is said to be a *bifurcation solution of equation* (2) if for any neighborhood V of λ_0 there is a neighborhood U_0 of the zero in X such that for each neighborhood U of the zero, $U \subseteqq U_0$, one can find a solution $(\bar{\lambda}, \bar{u})$ of (2) with $\bar{\lambda} \in V$ and $\bar{u} \in \partial U$.

b) A point $\lambda_0 \in \Lambda$ is said to be an *asymptotic bifurcation point of equation* (2) if for any neighborhood V of λ_0 there is a neighborhood U_0 of the zero in X such that for each neighborhood U of the zero, $U_0 \subseteqq U$, one can find a solution $(\bar{\lambda}, \bar{u})$ of (2) with $\bar{\lambda} \in V$ and $u \in \partial U$.

The bifurcation problem of equation (2) was studied by many authors, especially, in [9] and [10] KLUGE investigated this problem with a parameter from, in general, a metric space.

3. The main results

First of all we prove the following theorem which shows that under some additional assumptions on the mapping F we have $\deg_K (F, \Omega, 0) = \{0\}$ for $\Omega \in \mathscr{C}_K(D)$ and $F \in \mathscr{B}_K(\Omega, Y)$ with $0 \notin F(\partial_K \Omega)$. From this theorem we shall obtain some results on the existence of eigenvalues of some pairs of multivalued mappings and on the existence of bifurcation solutions and of asymptotic points of equations in the form (2). But, we consider in this paper only the existence of bifurcation solution.

Theorem 1: *X, Y, K, D, $\mathscr{C}_K(D)$ and $\mathscr{B}_K(\Omega, Y)$, for $\Omega \in \mathscr{C}_K(D)$, are supposed to be given as in Section 2. Let $F \in \mathscr{B}_K(\Omega, Y)$ be a multivalued mapping. Suppose that there exist an upper semicontinuous multivalued mapping $G: \overline{\Omega} \to 2^Y$ and a closed neighborhood U of the zero in Y and a point $y_0 \in \partial U$ such that the following conditions are satisfied:*

1. $-\overline{G(\partial_K \Omega)} \cap U = \emptyset$,
2. $-\alpha y_0 \notin \overline{G(\partial_K \Omega)}$ *for all* $\alpha \geqq 1$,
3. $F - \alpha G - \beta\{y_0\} \in \mathscr{B}_K(\Omega, Y)$ *for all* $\alpha \geqq 0$, $\beta \geqq 0$,
4. $G(\partial_K \Omega)$ *and* $F(\partial_K \Omega)$ *are bounded subsets in* Y *and for any* $\alpha > 0$ *there exists* $n_\alpha \in \mathbf{N}$ *such that* $n_\alpha y_0 \notin (F - \alpha G)(\overline{\Omega})$,
5. $F(u) \cap \mu G(u) = \emptyset$ *for all* $\mu > 0$ *and* $u \in \partial_K \Omega$. *Then* $\deg_K (F, \Omega, 0) = \{0\}$, *provided that* $0 \notin F(\partial_K \Omega)$.

Proof: Let p be the Minkowski function of U (see in [20], for example), i.e. $p(x) = \inf \{\alpha > 0 : x \in \alpha U\}$. Thus, we have $p(y_0) = 1$, because of $y_0 \in \partial U$. Now, we claim that there exists an $\alpha > 0$ such that

$$F(u) \cap \left(\alpha G(u) + nty_0\right) = \emptyset \tag{3}$$

holds for all $n \in \mathbf{N}$, $t \in [0, 1]$ and $u \in \partial_K \Omega$. Indeed, if the assertion were invalid then for each $m \in \mathbf{N}$ we could find $t_m \in [0, 1]$, $u_m \in \partial_K \Omega$ and n_m such that $F(u_m) \cap \left(mG(u_m) + n_m t_m y_0\right) \neq \emptyset$. Therefore we can choose $y_m \in F(u_m) \cap \left(mG(u_m) + n_m t_m y_0\right)$ such that y_m can be written in the form $y_m = m z_m + n_m t_m y_0$, with $z_m \in G(u_m)$. Thus follows

$$p\left(z_m + \frac{n_m}{m} t_m y_0\right) = p\left(\frac{1}{m} y_m\right) = \frac{1}{m} p(y_m).$$

Since $F(\partial_K \Omega)$ is a bounded subset, the left side converges to zero for $m \to \infty$. Further, the boundedness of $G(\partial_K \Omega)$ implies that $p(z_m) < +\infty$ for all $m \in \mathbf{N}$. Hence

$$0 < \frac{n_m}{m} t_m = \frac{n_m}{m} t_m p(y_0) = p\left(\frac{n_m}{m} t_m y_0\right)$$

$$\leqq p(z_m) + p\left(z_m + \frac{n_m}{m} t_m y_0\right) \leqq p(z_m) + \frac{1}{m} p(y_m) < +\infty$$

for all $m \in \mathbf{N}$ greater than some $m_0 \in \mathbf{N}$. Therefore, without loss of generality, we may assume that $\frac{n_m}{m} t_m \to \bar{\alpha}$ as $m \to \infty$, and then

$$0 \leqq p(z_m + \bar{\alpha} y_0) \leqq p\left(z_m + \frac{n_m}{m} t_m y_0\right) + \left|\frac{n_m}{m} t_m - \bar{\alpha}\right| p(y_0) \to 0$$

as $m \to \infty$. This implies that $z_m \to -\bar{\alpha}\bar{y}_0$ as $m \to \infty$. Since $z_m \in G(\partial_K \Omega)$ and $-\overline{G(\partial_K \Omega)} \cap U = \emptyset$, we then deduce $\bar{\alpha} \geqq 1$ and $-\bar{\alpha} y_0 \in \overline{G(\partial_K \Omega)}$, which contradicts the condition 2 that says $-\alpha y_0 \notin \overline{G(\partial_K \Omega)}$ for all $\alpha \geqq 1$. Thus, (3) is proved.

Now, for any $n \in \mathbf{N}$ we define

$$H_n(t, u) = F(u) - \alpha G(u) - nty_0, \qquad (t, u) \in [0, 1] \times \bar{\Omega},$$

where α is taken from (3). Then (3) implies that $0 \notin H_n(t, u)$ for all $t \in [0, 1]$ and $u \in \partial_K \Omega$. Using Axiom II yields

$$\deg_K (F - \alpha G, \Omega, 0) = \deg_K (F - \alpha G - ny_0, \Omega, 0) \tag{4}$$

for all $n \in \mathbf{N}$. Now, we claim that $\deg_K (F - \alpha G, \Omega, 0) = \{0\}$. Indeed, assume in contrary that $\deg_K (F - \alpha G, \Omega, 0) \neq \{0\}$. This implies $\deg_K (F - \alpha G - ny_0, \Omega, 0) \neq \{0\}$ for all $n \in \mathbf{N}$. We use Axiom I to conclude that for any $n \in \mathbf{N}$ there exists $u_n \in \bar{\Omega}$ such that $0 \in \left(F(u_n) - \alpha G(u_n) - ny_0\right)$. Hence, $ny_0 \in (F - \alpha G)(\bar{\Omega})$ for all $n \in \mathbf{N}$. This contradicts the condition 4. Further, we define

$$K(t, u) = F(u) - t\alpha G(u), \qquad (t, u) \in [0, 1] \times \bar{\Omega}.$$

Using condition 5 gives $0 \notin K(t, u)$ for all $t \in [0, 1]$ and $u \in \partial_K \Omega$. We use Axiom II to conclude that $\deg_K (F, \Omega, 0) = \deg_K (F - \alpha G, \Omega, 0)$. As was proved, $\deg_K (F - \alpha G, \Omega, 0) = \{0\}$. Consequently, we obtian $\deg_K (F, \Omega, 0) = \{0\}$ ∎

For the proofs of some corollaries of this theorem we need the following lemma.

Lemma 1: *Let $G: D \to 2^Y$ be a multivalued mapping and U be a neighborhood of the zero in Y such that*

$$\partial U \cap \left(Y \setminus \operatorname{co}\left(\overline{G(\partial_K D)} \cup \{0\}\right)\right) \neq \emptyset.$$

Then there exists a point $y_0 \in \partial U$ such that $\alpha y_0 \notin \overline{G(\partial D)}$ for all $\alpha \geqq 1$.

Proof: Indeed, if the assertion were untrue then for each $y \in \partial U$ we could seek $\alpha(y) \geqq 1$ such that $\alpha(y)\, y \in \overline{G(\partial D)}$. For arbitrary $u \in \partial U$ we have

$$u = \alpha(u) \frac{u}{\alpha(u)} + \left(1 - \frac{1}{\alpha(u)}\right) 0 \in \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right).$$

This implies that $\partial U \subset \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right)$ and hence $\partial U \cap \left(Y \setminus \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right)\right) = \emptyset$, and we have a contradiction ∎

Remark 1: (i) Suppose that Z is a proper closed subspace of Y and $G: D \to 2^Y$ is a multivalued mapping such that $G(\partial D) \subseteqq Z$, then for any neighborhood U of the zero in Y we have

$$\partial U \cap \left(Y \setminus \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right)\right) \neq \emptyset.$$

Indeed, since $G(\partial D) \subseteqq Z$ it follows that $\operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right) \subseteqq Z$. If $\partial U \cap \left(Y \setminus \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right)\right) = \emptyset$ then $\partial U \subseteqq \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right) \subseteqq Z$ and hence $U \subseteqq Z$. This implies $Z = Y$, and we have a contradiction. Consequently, for any neighborhood U of the zero in Y there is a $y_0 \in \partial U$ such that $\alpha y_0 \notin \overline{G(\partial D)}$ for all $\alpha \geqq 1$.

(ii) If K is a closed and convex cone in Y and $G: D \to 2^Y$ is a multivalued mapping such that $G(\partial D) \subseteqq K$, then for any neighborhood U of the zero in Y we have

$$\partial U \cap \left(Y \setminus \operatorname{co}\left(\overline{G(\partial D)} \cup \{0\}\right)\right) \neq \emptyset .$$

The proof is similar to the previous one. Hence, for any neighborhood U of the zero in Y there is a $y_0 \in \partial U$ such that $\alpha y_0 \notin G(\partial D)$ for all $\alpha \geqq 1$. Moreover, if $G(D) \subseteqq K$ then there is a $z_0 \neq 0$ such that $\alpha z_0 \notin G(D)$ for all $\alpha > 0$.

Lemma 2: *Let X be a quasi-complete barrel space with* $\dim X = +\infty$ *and $M \subset X$ be a precompact subset. Denote by Z the smallest closed subspace of X containing the set* $\operatorname{co}(M - M)$, *then Z is a proper closed subspace of X.*

Proof: Suppose that $Z = X$. This implies that $\overline{\operatorname{co}}(M - M)$ is an absolutely convex closed and symmetric subset which is absorbing in X. But, on the other hand, since M is a compact subset, hence $\overline{\operatorname{co}}(M - M)$ is also a compact subset. This contradicts the hypothesis that $\dim X = +\infty$ ∎

Remark 2: It follows from Lemma 2 and Remark 1 that if $F: \bar{D} \to 2^Y$, where D is a bounded subset of X and Y is a quasi-complete barrel space, $\dim Y = +\infty$, is a compact multivalued mapping (i.e. F maps any bounded subset into a precompact subset), then $\overline{F(D)}$ is contained in one proper closed subspace of Y. From this one can conclude that for any neighborhood U of the zero in Y there exists a $y_0 \in \partial U$ such that $\alpha y_0 \notin \overline{F(D)}$ for all $\alpha \geqq 1$.

Corollary 1: *Let X, Y, K, D, $\mathscr{C}_K(D)$ and $\mathscr{B}_K(\Omega, Y)$ for $\Omega \in \mathscr{C}_K(D)$ be the same as in Theorem* 1 *and $F \in \mathscr{B}_K(\Omega, Y)$ with $F(\Omega)$ bounded. In addition, assume that there exists a $z_0 \in Y$, $z_0 \neq 0$ such that $F - \alpha z_0 \in \mathscr{B}_K(\Omega, Y)$ for all $\alpha \geqq 0$ and $\alpha z_0 \notin F(\partial_K \Omega)$ for all $\alpha > 0$. Then $\deg_K(F, \Omega, 0) = \{0\}$, provided that $0 \notin F(\partial_K \Omega)$.*

Proof: Since $z_0 \neq 0$, there exists a closed symmetric neighborhood U of the zero in Y such that $z_0 \notin U$. Let p be the Minkowski function of U. Choose $\alpha_0 > 0$ such that $p(\alpha_0 z_0) = 1$ and put $y_0 = \alpha_0 z_0$. We can easily verify that $y_0 \in \partial U$. Further, set $G(u) = (\alpha_0 + 1) z_0$, $u \in \bar{\Omega}$. It is a simple matter to show that G, U and y_0 satisfy all assumptions of Theorem 1. Hence, to complete the proof it remains to apply Theorem 1 ∎

Corollary 2: *Let X be a quasi-complete barrel infinite-dimensional space, $D \subset X$ a subset, $F: \bar{D} \to 2^X$ a compact upper semicontinuous multivalued mapping on D. Suppose that for an open bounded Ω of D there exists a closed symmetric neighborhood U of the zero in X such that $\Omega \subseteqq U$ and $\overline{F(\partial\Omega)} \cap U = \emptyset$. Then*

$$\deg(id - F, \Omega, 0) = \deg(id + F, \Omega, 0) = \{0\}$$

where $\deg(id \pm F, \Omega, 0)$ *is the Leray-Schauder degree of $id \pm F$ with respect to Ω at 0 defined by* Granas [8] *or by* Ma [17].

Proof: Let $\mathscr{C}(D)$ be the family of all open bounded subsets of D and $\mathscr{B}(\Omega, X)$, for $\Omega \in \mathscr{C}(D)$, be the family of all multivalued mappings T from $\bar{\Omega}$ into X in the form $T = id - H$, where H is a completely continuous multivalued mapping on D. Thus, for any $\Omega \in \mathscr{C}(D)$ and $T \in \mathscr{B}(\Omega, X)$, $\deg(T, \Omega, 0)$ is defined by Ma [17], provided $0 \notin T(\partial\Omega)$. Now, we apply Theorem 1 to show that $\deg(id - F, \Omega, 0) = \{0\}$ (the proof for $\deg(id + F, \Omega, 0)$ is analogical). Indeed, according to Remark 2 there is a $y_0 \in \partial U$ such that $\alpha y_0 \notin -\overline{F(\partial\Omega)}$ for all $\alpha \geqq 1$. Putting $G(u) = F(u)$, for $u \in \bar{\Omega}$, we can easily verify that F, G, U and y_0 satisfy conditions 1—4 of Theorem 1.

To apply Theorem 1 condition 5 remains to be shown condition 5. We assume on the contrary that there exist $\bar{\mu} > 0$ and $\bar{u} \in \partial\Omega$ such that $(\bar{u} - F(\bar{u})) \cap \bar{\mu}F(\bar{u}) \neq 0$. This implies that $\bar{u} \in (1 + \bar{\mu}) F(\bar{u})$. Therefore, we can write $\bar{u} = (1 + \bar{\mu}) z$, for some $z \in F(\bar{u})$. Let p be the Minkowski function of U. Since $\bar{u} \in \partial\Omega \subset U$, $F(\partial\Omega) \cap U = \emptyset$ and $z \in F(\bar{u})$, we get $p(\bar{u}) \leqq 1$ and $p(z) > 1$. We have $1 \geqq p(\bar{u}) = p((1 + \bar{\mu}) z) = (1 + \bar{\mu}) \times p(z) > 1$, which is impossible. Consequently, condition 5 is also fulfilled ∎

Remark 3: Corollary 2 is an extension of Theorem 2.1 in [21], which proves for the case X is an infinite-dimensional linear normed space and F is a single-valued completely continuous mapping on D.

Now we apply Theorem 1 to investigate the existence of bifurcation solutions of the equation (2). For simplicity in what follows we consider only the case that the mapping $F: \Lambda \times \bar{D}_K \to Y$ is single-valued, $F(\lambda, u) = 0$ for all $\lambda \in \Lambda$ and $F(\cdot, u)$ is continuous in λ uniformly for u from any bounded subset of D. More precisely, we consider the existence of a bifurcation solution of the equation

$$F(\lambda, u) = 0, \qquad (\lambda, u) \in \Lambda \times \bar{D}_K, \tag{5}$$

where Λ is an open subset of some normed space, $K, D \subseteqq X$ with $D_K = D \cap K \neq \emptyset$, $0 \in \operatorname{int} D$ and X and Y are supposed to be Banach spaces. We have

Theorem 2: *Let X, Y, D, K, Λ and F be as just mentioned above and let λ_0 be a point such that for any neighborhood V of λ_0 there is a neighborhood U_0 of the zero in X such that for any neighborhood U of the zero, $U \subset U_0$, $U_K = U \cap K \in \mathscr{C}_K(D)$ and one can find $\lambda_1, \lambda_2 \in V$ and a single-valued mapping $A: \bar{U}_K \to K$ with $A \in \mathscr{B}_K(U_K, Y)$ and $\deg_K (A, U, 0) \neq \{0\}$, provided $0 \notin A(\partial_K U_K)$, satisfying the following conditions:*

1. *$tF(\lambda_1, \cdot) + (1 - t) A \in \mathscr{B}_K(U_K, Y)$ for any $t \in [0, 1]$.*
2. *$\|F(\lambda_1, u) - A(u)\| \leqq \|A(u)\|$ for any $u \in \partial_K U_K$.*
3. *There exists a $z_0 \in Y$, $z_0 \neq 0$ such that either $\alpha z_0 \notin F(\lambda_2, \partial_K U_K)$ for all $\alpha > 0$ and $F(\lambda_2, \cdot) - \beta\{z_0\} \in \mathscr{B}_K(U_K, Y)$ for all $\beta \geqq 0$ or $\alpha z_0 \notin A(\partial_K U_K)$ for all $\alpha > 0$ and $A - \beta\{z_0\} \in \mathscr{B}_K(U_K, Y)$ for all $\beta \geqq 0$.*

Then $(\lambda_0, 0)$ is a bifurcation solution of equation (5).

Proof: Let V be an arbitrary neighborhood of λ_0. Without loss of generality we may assume that V is convex. Let U_0 exist corresponding to V by the hypotheses and $U \subset U_0$, $U_K = U \cap K \in \mathscr{C}_K(D)$. We have to show that there exists a point $(\bar{\lambda}, \bar{u}) \in V \times \partial_K U$ with $F(\bar{\lambda}, \bar{u}) = 0$.

Indeed, for U one can find $\lambda_1, \lambda_2 \in V$ and a mapping $A: \bar{U}_K \to Y$ satisfying conditions 1—3. If $F(\lambda_i, u) = 0$ for $i = 1$ or $i = 2$ and for some $u \in \partial_K U$ we have the proof. Therefore in the sequel we assume that $F(\lambda_i, u) \neq 0$ for $i = 1, 2$ and for all $u \in \partial_K U$.

Further, if $A(\bar{u}) = 0$ for some $\bar{u} \in \partial_K U$, then we imply from the inequality of condition 2 that $F(\lambda_1, \bar{u}) = 0$, which contradicts the above assumption. Thus, $\deg_K (A, U_K, 0)$ is defined and by the hypotheses, $\deg_K (A, U_K, 0) \neq \{0\}$. Together with condition 3 this yields $\alpha z_0 \notin F(\lambda_2, \partial_K U)$ for all $\alpha > 0$ and $F(\lambda_2, \cdot) - \beta z_0 \in \mathscr{B}_K(U_K, Y)$; otherwise, we use Corollary 1 to obtain $\deg_K (A, U_K, 0) = \{0\}$, which is impossible. Now, we use Corollary 1 again to conclude

$$\deg_K (F(\lambda_2, \cdot), U_K, 0) = \{0\}. \tag{6}$$

Furher, let us define

$$H(t, u) = (1 - t) A(u) + tF(\lambda_1, u), \qquad (t, u) \in [0, 1] \times \bar{U}. \tag{7}$$

Suppose that $H(\bar{t}, \bar{u}) = 0$ for some $\bar{t} \in [0, 1]$ and $\bar{u} \in \partial_K U$. Since $F(\lambda_i, u) \neq 0$, $i = 1, 2$ and $u \in \partial_K U$ we deduce $\bar{t} \in (0, 1)$. (7) implies that $A(\bar{u}) = \bar{t}\big(A(\bar{u}) - F(\lambda_1, \bar{u})\big)$. Together with condition 2 we obtain

$$\bar{t}\, \|A(\bar{u}) - F(\lambda_1, \bar{u})\| = \|A(\bar{u})\| \geqq \|F(\lambda_1, \bar{u}) - A(\bar{u})\|. \tag{8}$$

If $\|A(\bar{u}) - F(\lambda_1, \bar{u})\| = 0$ then (8) implies $A(\bar{u}) = 0$ and hence we get from condition 2 $F(\lambda_1, \bar{u}) = 0$, which contradicts the assumption that $F(\lambda_i, u) \neq 0$ for $i = 1, 2$ and for all $u \in \partial_K U$. Therefore we have $\|A(\bar{u}) - F(\lambda_1, \bar{u})\| \neq 0$. Consequently, (8) implies that $\bar{t} \geqq 1$, and then we have a contradiction. Hence, $H(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U$. Further, we use Axiom II to obtain

$$\deg_K \big(F(\lambda_1, \cdot), U_K, 0\big) = \deg_K (A, U_K, 0) \neq \{0\}. \tag{9}$$

Finally, we define the mapping $M: [0, 1] \times \bar{U} \to Y$ by

$$M(t, u) = F\big(t\lambda_1 + (1 - t)\,\lambda_2, u\big), \qquad (t, u) \in [0, 1] \times \bar{U}.$$

Suppose that $M(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U$. Using Axiom II gives $\deg_K \big(F(\lambda_1, \cdot), U_K, 0\big) = \deg_K \big(F(\lambda_2, \cdot), U_K, 0\big)$. Together with (6) and (9) we obtain a contradiction. This implies that $M(\bar{t}, \bar{u}) = 0$ for some $\bar{t} \in [0, 1]$ and $\bar{u} \in \partial_K U$. Setting $\bar{\lambda} = \bar{t}\lambda_1 + (1 - \bar{t})\,\lambda_2$, we infer $F(\bar{\lambda}, \bar{u}) = 0$ with $\bar{\lambda} \in V$ and $\bar{u} \in \partial_K U$ ∎

As a special case of (5) we consider the equation

$$u = M(u) + N(\lambda, u) + H(\lambda, u), \qquad (\lambda, u) \in \Lambda \times \bar{D}_K, \tag{10}$$

where Λ is an open subset of some normed space on which a partial ordering $\prec$ is defined that satisfies: if $0 \prec \lambda$ then $t\lambda \prec \lambda$ for all $t \in [0, 1]$, X is a Banach space with a closed and convex cone K, $D \subseteqq X$ is a subset with $0 \in \operatorname{int} D$ and $D_K = D \cap K \neq \emptyset$. In the following we need the following assumptions:

1. $\lambda_0 \in \Lambda$, $0 \prec \lambda_0$, is the smallest eigenvalue of the pair (M, N) with respect to K which is isolated from the right side.

2. M is a linear continuous mapping from $\bar{D}_K$ into X and $u \neq tM(u)$ for all $u \in K$, $u \neq 0$ and $t \in [0, 1]$.

3. For any fixed $\lambda \in \Lambda$, $N(\lambda, \cdot)$ is a linear continuous mapping from $\bar{D}_K$ into X and $N(0, u) = 0$ for all $u \in \bar{D}_K$ and if $\lambda_0 \prec \lambda$ then $\big(N(\lambda, \cdot) - N(\lambda_0, \cdot)\big)(u_0) \in K$, where u_0 is an eigenvector corresponding to λ_0.

4. $H: \Lambda \times \bar{D}_K \to X$, $H(\lambda, 0) = 0$ for all $\lambda \in \Lambda$ and $\|H(\lambda, u)\| = o(\|u\|)$ as $\|u\| \to 0$ uniformly for $\lambda \in \Lambda$ and, $H(\cdot, u)$ is continuous in $\lambda \in \Lambda$ uniformly for u from any bounded subset of $\bar{D}_K$.

5. There exist a neighborhood $\tilde{V}$ of λ_0 and a neighborhood $\tilde{U}$ of the zero in X such that $M(\tilde{U}_K) \subset K$, $N(\lambda, \tilde{U}_K) \subset K$ for all $\lambda \in \tilde{V}$, $0 \prec \lambda$ and $H(\tilde{V}, \tilde{U}_K) \subset K$ and either

a) M, $N(\lambda, \cdot)$, $H(\lambda, \cdot)$ are compact continuous mappings on $\tilde{U}_K$ for any fixed $\lambda \in \tilde{V}$ or

b) the mapping $F: \tilde{U}_K \to X$ defined by

$$F(u) = u - \big(t_1 M(u) + N(\lambda, u) + t_2 H(\lambda, u)\big), \qquad u \in \tilde{U}_K$$

for any fixed $t_1, t_2 \in [0, 1]$ and $\lambda \in \tilde{V}$ is A-proper with respect to Γ_0, where $\Gamma_0 = \{X_n, P_n\}$ is the projectionally complete scheme for (X, X) with $P_n(K) \subset K$ for all $n \in \mathbb{N}$ (see the definition in Example 2).

Lemma 3: *Under the assumptions 1–5 for any $\lambda \in \tilde{V}$, $\lambda_0 \prec \lambda$, which is not an eigenvalue of the pair (M, N) with respect to K there exists a $r_1 > 0$ such that $U(0, r_1) \subset \tilde{U}$ and*

$$u - \big(M(u) + N(\lambda, u) + H(\lambda, u)\big) \neq \mu u_0$$

for all $u \in U_K(0, r_1) = U(0, r_1) \cap K$ *and* $\mu > 0$, *where* $u_0 \in K$, $u_0 \neq 0$, *is an eigenvector of the pair* (M, N) *with respect to* K *corresponding to* λ_0.

Proof: Suppose that $\lambda \in \tilde{V}$, $\lambda_0 \prec \lambda$, is not an eigenvalue of the pair (M, N) with respect to K. If the claim of the lemma were false then for any $n \in \mathbb{N}$ with $U(0, 1/n) \subset \tilde{U}$ one could find $u_n \in U_K(0, 1/n)$ and $\mu_n > 0$ such that

$$u_n - \big(M(u_n) + N(\lambda, u_n) + H(\lambda, u_n)\big) = \mu_n u_0. \tag{11}$$

Since $H(\lambda, 0) = 0$, $\mu_n > 0$ and $u_0 \neq 0$ we deduce from (11) that $u_n \neq 0$ for all n. Hence, by dividing both sides of (11) with $\|u_n\|$ we obtain

$$\frac{u_n}{\|u_n\|} - \left(M\left(\frac{u_n}{\|u_n\|}\right) + N\left(\lambda, \frac{u_n}{\|u_n\|}\right) + \frac{H(\lambda, u_n)}{\|u_n\|}\right) = \frac{\mu_n}{\|u_n\|} u_0. \tag{12}$$

Because of $\|u_n\| \to 0$ as $n \to \infty$ and $\|H(\lambda, u_n)\| = o(\|u_n\|)$ as $\|u_n\| \to 0$ we infer

$$\|H(\lambda, u_n)\|/\|u_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{13}$$

Further, we obtain from (12) that

$$0 < \frac{\mu_n}{\|u_n\|} \leqq \|u_0\|^{-1} \left(\left\|\frac{u_n}{\|u_n\|} - M\left(\frac{u_n}{\|u_n\|}\right) - N\left(\lambda, \frac{u_n}{\|u_n\|}\right)\right\| + \frac{\|H(\lambda, u_n)\|}{\|u_n\|}\right). \tag{14}$$

Now, by the continuity of M, $N(\lambda, \cdot)$ and the boundedness of the sequence $\{u_n/\|u_n\|\}$ one can find a constant $c > 0$ such that

$$\left\|\frac{u_n}{\|u_n\|} - M\left(\frac{u_n}{\|u_n\|}\right) - N\left(\lambda, \frac{u_n}{\|u_n\|}\right)\right\| \leqq c. \tag{15}$$

Because of (13) there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqq n_0$ one has

$$\|H(\lambda, u_n)\|/\|u_n\| \leqq c. \tag{16}$$

Finally, combining (14)—(16) yields $0 < \mu_n/\|u_n\| \leqq 2\,\|u_0\|^{-1} c$ for all $n \geqq n_0$. Therefore, by extracting a subsequence if necessary, we may assume that $\mu_n/\|u_n\| \to \alpha \geqq 0$. Now we put

$$Q = \overline{\left\{\frac{\mu_n}{\|u_n\|} \cdot u_0 - \frac{H(\lambda, u_n)}{\|u_n\|}\right\}_{n=1}^{+\infty}},$$

then we imply that Q is a compact subset in X. Evidently, it follows from (12) that

$$z_n = u_n/\|u_n\| \in \big(id - M - N(\lambda, \cdot)\big)^{-1} (Q) \cap K.$$

In the case that M, $N(\lambda, \cdot)$ are compact mappings, without loss of generality, we may conclude that $M(z_n) + N(\lambda, z_n)$ converges to some point $y_0 \in X$. Hence, (12) implies $z_n \to z_0 = y_0 + \alpha u_0 \in K$. Since $\|z_n\| = 1$, we deduce $\|z_0\| = 1$.

Now, in the case that the mapping $id - M - N(\lambda, \cdot)$ is A-proper with respect to Γ_0 we apply Proposition 1.1.C in [18] to conclude that the set $\big(id - M - N(\lambda, \cdot)\big)^{-1} \times (Q)$ is also compact. Hence, by extracting a subsequence if necessary, we may assume that $z_n \to z_0$. So, $z_0 \in K$ and $\|z_0\| = 1$.

Further, from the continuity of M, $N(\lambda, \cdot)$ and from (12) and (13) we obtain

$$z_0 - \big(M(z_0) + N(\lambda, z_0)\big) = \alpha u_0. \tag{17}$$

Since λ is not an eigenvalue of the pair (M, N) with respect to K we then conclude $\alpha > 0$. Because of $M, N(\lambda, \cdot)$ being linear mappings and $M(\tilde{U}_K), N(\lambda, \tilde{U}_K) \subset K$ we can easily verify that

$$\left(M + N(\lambda, \cdot)\right)^m (u_0) \in K \tag{18}$$

and

$$\left(M + N(\lambda, \cdot)\right)^m \left(N(\lambda, u_0) - N(\lambda_0, u_0)\right) \in K \tag{19}$$

for an arbitrary integer $m \in \mathbf{N}$.

From now on, for the simplicity of the notations, we define the mapping $E: \tilde{U}_K \to X$ by

$$E(u) = \left(M + N(\lambda, \cdot)\right)(u), \qquad u \in \tilde{U}_K.$$

Next, we claim that for arbitrary m

$$E^m(z_0) = E^{m+1}(z_0) + \sum_{j=0}^{m-1} \alpha E^j\left(N(\lambda, u_0) - N(\lambda_0, u_0)\right) + \alpha u_0. \tag{20}$$

We prove this assertion by induction. Indeed, it follows from (17) that $z_0 = \left(M + N(\lambda, \cdot)\right)(z_0) + \alpha u_0$. Hence

$$\begin{aligned} E(z_0) &= E^2(z_0) + \alpha E(u_0) = E^2(z_0) + \alpha\left(M + N(\lambda, \cdot)\right)(u_0) \\ &= E^2(z_0) + \alpha\left(M + N(\lambda_0, \cdot)\right)(u_0) + \alpha\left(N(\lambda, \cdot) - N(\lambda_0, \cdot)\right)(u_0) \\ &= E^2(z_0) + \alpha\left(N(\lambda, \cdot) - N(\lambda_0, \cdot)\right)(u_0) + \alpha u_0. \end{aligned}$$

This shows that (20) is valid for $m = 1$. Suppose now that (20) is true for some $k \in \mathbf{N}$:

$$E^k(z_0) = E^{k+1}(z_0) + \sum_{j=1}^{k-1} \alpha E^j(x_0) + \alpha u_0,$$

where $x_0 = \left(N(\lambda, \cdot) - N(\lambda_0, \cdot)\right)(u_0)$. Hence

$$\begin{aligned} E^{k+1}(z_0) &= E\left(E^k(z_0)\right) = E\left(E^{k+1}(z_0) + \sum_{j=1}^{k-1} \alpha E^j(x_0) + \alpha u_0\right) \\ &= E^{k+2}(z_0) + \sum_{j=1}^{k-1} \alpha E^{j+1}(x_0) + \alpha E(u_0). \end{aligned} \tag{21}$$

But $E(u_0) = \left(M + N(\lambda, \cdot)\right)(u_0) = \left(M + N(\lambda_0, \cdot)\right)(u_0) + x_0 = u_0 + x_0$. Together with (21) we obtain

$$\begin{aligned} E^{k+1}(z_0) &= E^{k+2}(z_0) + \sum_{j=0}^{k-1} \alpha E^{j+1}(x_0) + \alpha x_0 + \alpha u_0 \\ &= E^{k+2}(z_0) + \sum_{j=0}^{k} \alpha E^j(x_0) + \alpha u_0, \end{aligned}$$

which shows that (20) is also valid for $k + 1$. This proves (20).

Now we set

$$y_m = \sum_{j=0}^{m} \alpha E^j(x_0) \quad \text{for} \quad m = 0, 1, \ldots.$$

It follows from (17) and (20) that

$$z_0 = E(z_0) + \alpha u_0,$$
$$E(z_0) = E^2(z_0) + y_0 + \alpha u_0,$$
$$E^m(z_0) = E^{m+1}(z_0) + y_{m-1} + \alpha u_0.$$

Taking the sum of both sides of this equalities we obtain

$$z_0 = E^{m+1}(z_0) + \sum_{j=0}^{m-1} y_j + \alpha(m+1)\, u_0, \qquad y_{-1} := 0. \tag{22}$$

It follows from (18) and (19) that $E^{m+1}(z_0)$, $(y_0 + y_1 + \cdots + y_{m-1}) \in K$, $m \in \mathbf{N}$. Therefore, (22) yields $z_0 \in \alpha(m+1)\, u_0 + K$, $m \in \mathbf{N}$. Hence, $z_0/\alpha(m+1) - u_0 \in K$, $m \in \mathbf{N}$. Letting $m \to \infty$ we obtain $-u_0 \in K$. Thus, $u_0 \in K \cap (-K) = \{0\}$ and hence $u_0 = 0$, which contradicts $u_0 \in K$, $u_0 \neq 0$ ∎

Lemma 4: *Under the assumptions 1—5 for any $\lambda \in V$ which is not an eigenvalue of the pair (M, N) with respect to K, there exists a $r_2 > 0$ such that $U(0, r_2) \subset \tilde{U}$ and*

$$\|H(\lambda, u)\| \leqq \|u - M(u) - N(\lambda, u)\|$$

for all $u \in U_K(0, r_2) = U(0, r_2) \cap K$.

Proof: Let $\lambda \in V$ be no eigenvalue of the pair (M, N) with respect to K. As before, for the simplicity of the notations we put

$$E(u) = \big(M + N(\lambda, \cdot)\big)(u), \qquad u \in \tilde{U}.$$

If the assertion of the lemma were invalid, then for any $n \in \mathbf{N}$ one could find $u_n \in U_K(0, 1/n)$ with

$$\|u_n - E(u_n)\| < \|H(\lambda, u_n)\|. \tag{23}$$

Since $H(\lambda, 0) = 0$ we then deduce from (23) that $u_n \neq 0$ for all $n \in \mathbf{N}$. Hence, by dividing both sides of (23) with $\|u_n\|$ we obtain

$$\left\|\frac{u_n}{\|u_n\|} - E\left(\frac{u_n}{\|u_n\|}\right)\right\| < \frac{\|H(\lambda, u_n)\|}{\|u_n\|}.$$

By $\|H(\lambda, u_n)\| = o(\|u_n\|)$ as $n \to \infty$ we get $\|H(\lambda, u_n)\|/\|u_n\| \to 0$ as $n \to \infty$. Therefore, the last inequality gives

$$\frac{u_n}{\|u_n\|} - E\left(\frac{u_n}{\|u_n\|}\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{24}$$

In the case when M, $N(\lambda, \cdot)$ are compact mappings, E is also a compact mapping. Hence, by extracting a subsequence if necessary, we may assume that $E(u_n/\|u_n\|) \to z_0 \in K$ (because of $u_n/\|u_n\| \in K$ and $E(K) \subset K$). Consequently, it follows from (24) that $z_n = u_n/\|u_n\| \to z_0$, and then $z_0 \in K$ and $\|z_0\| = 1$.

In the case that $id - M - N(\lambda, \cdot)$ is an A-proper mapping with respect to Γ_0, we put

$$Q = \overline{\left\{\frac{u_n}{\|u_n\|} - E\left(\frac{u_n}{\|u_n\|}\right)\right\}_{n=1}^{+\infty}}.$$

(24) implies that Q is a compact subset in X. We apply Proposition 1.1.C in [18] again to deduce that the set $(id - E)^{-1}(Q)$ is also compact in X. We have

$z_n = u_n/\|u_n\| \in (id - E)^{-1}(Q) \cap K$. Hence, by extracting a subsequence if necessary, we infer also that $z_n \to z_0 \in K$ and $\|z_0\| = 1$.

Now, by the continuity of M, $N(\lambda, \cdot)$ and by (24) we obtain $z_0 = E(z_0) = M(z_0) + N(\lambda, z_0)$, $\|z_0\| = 1$. This shows that λ is an eigenvalue of the pair (M, N) with respect to K, which is a contradiction ∎

Now we can prove the following theorem on the existence of bifurcation solutions of equation (10).

Theorem 3: *Under the assumptions* 1—5, $(\lambda_0, 0)$ *is a bifurcation solution of equation* (10). *More precisely, for any given* $\varepsilon > 0$ *there is a* $r_0 > 0$ *such that for each* $r \in (0, r_0)$ *one can find a solution* $(\bar{\lambda}, \bar{u})$ *of* (10) *with* $\|\bar{\lambda} - \lambda_0\|_\Lambda < \varepsilon$ *and* $\bar{u} \in \bar{D}_K$, $\|\bar{u}\| = r$, *where* $\|\cdot\|_\Lambda$ *denotes the norm of the normed space containing* Λ *and* $\|\cdot\|$ *denotes the norm of* X.

Proof: Let $\varepsilon > 0$ be given. By choosing $\varepsilon' \in (0, \varepsilon)$ if necessary we may assume that $V(\lambda_0) = \{\lambda \in \Lambda : \|\lambda - \lambda_0\|_\Lambda < \varepsilon\} \subset \tilde{V}$ (we recall that $\tilde{V}$ comes from assumption 5 mentioned above). Choose $\bar{t} \in [0, 1]$ such that $(1 - \bar{t}) \|\lambda_0\|_\Lambda < \varepsilon$ and put $\lambda_1 = \bar{t}\lambda_0$. We have $\lambda_1 \in V(\lambda_0)$ and $\lambda_1 \prec \lambda_0$. Because of the assumption that λ_0 is the smallest eigenvalue of the pair (M, N) with respect to K we conclude that λ_1 has not this property. We apply Lemma 4 to deduce that there exists a $r_1 > 0$ such that $U(0, r_1) \subset \tilde{U}$ (we recall also that $\tilde{U}$ comes from assumption 5) and

$$\|H(\lambda_1, u)\| \leqq \|u - M(u) - N(\lambda_1, u)\| \tag{25}$$

for all $u \in U_K(0, r_1)$.

Since λ_0 is isolated from the right side there exists $\lambda_2 \in V(\lambda_0)$ with $\lambda_0 \prec \lambda_2$ and λ_2 is not an eigenvalue of the pair (M, N) with respect to K. We apply Lemma 3 to conclude that there exists a $r_2 > 0$ such that $U(0, r_2) \subset \tilde{U}$ and

$$u - \big(M(u) + N(\lambda_2, u) + H(\lambda_2, u)\big) \neq \mu u_0 \tag{26}$$

for all $u \in U_K(0, r_2)$ and $\mu > 0$.

Now we put $r_0 = \min(r_1, r_2)$, then we prove that for each $r \in (0, r_0)$ there exists a solution $(\bar{\lambda}, \bar{u})$ of equation (10) with $\|\bar{\lambda} - \lambda_0\|_\Lambda < \varepsilon$ and $\|\bar{u}\| = r$. Indeed, let $\mathscr{E}_K(U_0)$ be the family of all $U_K(0, r) = U(0, r) \cap K$ with $U(0, r) \subset U_0 = U(0, r_0)$ and $r \in (0, r_0)$. Let $r \in (0, r_0)$, so we have $U_K = U(0, r) \cap K \in \mathscr{E}_K(U_0)$. We consider the two following cases:

a) In the case that M, $N(\lambda_i, \cdot)$, $H(\lambda_i, \cdot)$ are compact mappings we take $\mathscr{B}_K{}^a(U_K, X)$ to be the family of all single-valued mappings F from U_K into X of the form $F = id - G$, where G is a compact continuous mapping from $\bar{U}_K$ into K. Since $\bar{U}_K$ is a closed convex bounded subset in a Banach space X, it follows that there exists a continuous retraction $\alpha : X \to \bar{U}_K$ such that $\alpha(x) = x$ for all $x \in \bar{U}_K$. Now, for $F \in \mathscr{B}_K{}^a(U_K, X)$, $F = id - G$, we set $\tilde{G}(x) = G(\alpha(x))$ and $\tilde{F} = id - \tilde{G}$, for $x \in \bar{U}(0, r)$. It is clear that $\tilde{G}$ is a compact continuous mapping from $\bar{U}(0, r)$ into K. Thus, the Leray-Schauder degree of $\tilde{F}$ on $U(0, r)$ at the zero is defined, provided $0 \notin \tilde{F}(\partial U(0, r))$. We denote it by $\deg(\tilde{F}, U, 0)$.

Now suppose that $x \in \partial U(0, r)$, $\tilde{F}(x) = 0$, we then have $x = \tilde{G}(x) \in K$. From this follows $x \in \partial U(0, r) \cap K = \partial_K U(0, r)$, and hence $\tilde{F}(x) = x - \tilde{G}(x) = x - G(x) = F(x)$. Consequently, $x \in \partial_K U$ and $F(x) = 0$. In other words, if $0 \notin F(\partial_K U)$, then $0 \notin \tilde{F} \times \big(\partial U(0, r)\big)$. Therefore, we can define $\deg_K{}^a(F, U_K, 0) = \deg(\tilde{F}, U, 0)$, provided $0 \notin F(\partial_K U)$. This implies that for any $F \in \mathscr{B}_K{}^a(U_K, X)$, $\deg_K{}^a(F, U_K, 0)$ is defined, provided $0 \notin F(\partial_K U)$. It is easily to verify that Axioms I and II are always satisfied.

b) In the case that the mapping $id - (t_1 M + N(\lambda_i, \cdot) + t_2 H(\lambda_i, \cdot))$, for $i = 1, 2$ and for any fixed $t_1, t_2 \in [0, 1]$ is A-proper with respect to Γ_0 we take $\mathcal{B}_K{}^{\mathfrak{b}}(U_K, X)$ to be the family of all A-proper mappings F with respect to Γ_0 from $\overline{U}_K$ into X of the form $F = id - G$, where G is a mapping from $\overline{U}_K$ into K. So, the fixed point index, denoted by $I_K(G, U)$ of G on U is defined by FITZPATRICK and PETRYSHYN [6]. Hence, we can define $\deg_K{}^{\mathfrak{b}}(F, U_K, 0) = I_K(G, U)$, provided $0 \notin F(\partial_K U)$. Thus, for any $F \in \mathcal{B}_K{}^{\mathfrak{b}}(U_K, X)$, $\deg_K{}^{\mathfrak{b}}(F, U_K, 0)$ is defined, provided $0 \notin F(\partial_K U)$, which satisfies of course Axioms I and II.

In what follows by $\mathcal{B}_K{}^{i}(U_K, X)$ and by $\deg_K{}^{i}(F, U_K, 0)$, $i = \mathfrak{a}$ or $i = \mathfrak{b}$, we denote $\mathcal{B}_K{}^{\mathfrak{a}}(U_K, X)$ and $\deg_K{}^{\mathfrak{a}}(F, U_K, 0)$ or $\mathcal{B}_K{}^{\mathfrak{b}}(U_K, X)$ and $\deg_K{}^{\mathfrak{b}}(F, U_K, 0)$, respectively, according to case a) or case b). Now we claim that

$$\deg_K{}^{i}\big(id - M - N(\lambda_1, \cdot), U_K, 0\big) = \{1\} \tag{27}$$

for $i = \mathfrak{a}, \mathfrak{b}$. Indeed, we define the mapping $R: [0, 1] \times \overline{U}_K \to X$ by

$$R(t, u) = u - M(u) - N(t\lambda_1, u), \quad (t, u) \in [0, 1] \times \overline{U}_K.$$

Since λ_0 is the smallest eigenvalue of the pair (M, N) with respect to K and $t\lambda_1 = t\bar{t}\lambda_0 \prec \lambda_0$, we then conclude $R(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U$. Hence, applying Axiom II yields

$$\deg_K{}^{i}\big(id - M - N(\lambda_1, \cdot), U_K, 0\big) = \deg_K{}^{i}(id - M, U_K, 0). \tag{28}$$

Further, we define the mapping $T: [0, 1] \times \overline{U}_K \to X$ by

$$T(t, u) = u - tM(u), \quad (t, u) \in [0, 1] \times \overline{U}_K.$$

Assumption 2 gives $T(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U$. We apply Axiom II again to get

$$\deg_K{}^{i}(id - M, U_K, 0) = \deg_K{}^{i}(id, U_K, 0) = \{1\}. \tag{29}$$

(In case $i = \mathfrak{a}$, $\deg_K{}^{\mathfrak{a}}(id, U_K, 0) = \deg(id, U, 0) = 1$, it follows from the properties of the Leray-Schauder degree, and in case $i = \mathfrak{b}$, $\deg_K{}^{\mathfrak{b}}(id, U_K, 0) = I_K(0, U) = \{1\}$, it follows from the properties of the fixed point index defined by FITZPATRICK and PETRYSHYN in [6]). Finally, combining (28) and (29) yields (27).

Now, to complete the proof of the theorem we apply Theorem 2 with the mapping $F: \Lambda \times U_0 \to X$ defined by

$$F(\lambda, u) = u - M(u) - N(\lambda, u) - H(\lambda, u), \quad (\lambda, u) \in \Lambda \times U_0,$$

and the mapping $A: \overline{U}_K \to X$ defined by

$$A(u) = u - M(u) - N(\lambda_1, u), \quad u \in \overline{U}_K.$$

It follows from (27) that $\deg_K{}^{i}(A, U_K, 0) = \{1\}$. Evidently, for any fixed $t \in [0, 1]$

$$tF(\lambda_1, \cdot) + (1 - t) A = id - \big(M + N(\lambda_1, \cdot) + tH(\lambda_1, \cdot)\big) \in \mathcal{B}_K{}^{i}(U_K, X).$$

This shows that the condition 1 of Theorem 2 is satisfied. Further, we have

$$\|F(\lambda_1, u) - A(u)\| = \|H(\lambda_1, u)\| \leqq \|u - M(u) - N(\lambda_1, u)\| = \|A(u)\|$$

for all $u \in U_K(0, r)$ (this follows from (25)). Therefore, the condition 2 of Theorem 2 is also fulfilled. Now, it follows from inequality (26) that $\mu u_0 \notin F(\lambda_2, \partial_K U_K)$ for all $\mu > 0$. This implies that the condition 3 of Theorem 2 is also true. Consequently, the further proof of this theorem follows immediately from Theorem 2 ∎

4. Application

In this section we shall apply Theorem 2 to consider the existence of bifurcation solutions of elliptic differential equations of second order. Let G be a smooth bounded domain in $\mathbf{R}^n$ so that the Sobolev embedding theorem can be applied. By $L_p(G)$ we denote the space of all integrable functions f from G into $\mathbf{R}$ with the norm

$$\|f\|_{L_p} = \Big(\int_G |f(x)|^p \, dx\Big)^{1/p} < +\infty .$$

Further, let $\beta = (\beta_1, \ldots, \beta_n)$ be a n-tuple of nonnegative integers and

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}, \qquad |\beta| = \sum_{i=1}^n \beta_i,$$

be the standard derivative notation. Also let $C^k(\overline{G})$, $k = 0, 1, \ldots$ denote the collection of all functions from $\overline{G}$ into $\mathbf{R}$ having continuous extensions to ∂G. For $u \in C^k(\overline{G})$ we define

$$\|u\|_k = \sum_{j=1}^k \sup_{\substack{x \in G \\ |\beta| = j}} |D^\beta u(x)| .$$

Further, let

$$W_p{}^l(G) = \{u \in L_p(G) : D^\beta u \in L_p(G), 0 \leqq |\beta| \leqq l\}$$

with the norm

$$\|u\|_{l,p} = \Big(\sum_{0 \leqq |\beta| \leqq l} \|D^\beta u\|_{L_p}^p\Big)^{1/p} .$$

Now, denoting by $\mathcal{D}(G)$ the set of all infinitely many times differentiable functions on G with compact supports in G, we define $\mathring{W}_p{}^l(G)$ as the closure of $\mathcal{D}(G)$ in $W_p{}^l(G)$. Then it is well-known that $W_p{}^l(G)$, $\mathring{W}_p{}^l(G)$ are separable Banach spaces, reflexive for $p > 1$ and Hilbertian for $p = 2$.

Let the operator L be defined by

$$L(u) = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x) \, u ,$$

where a_{ij} have Hölder continuous first partial derivatives and b_i and c are Hölder continuous and

$$\sum_{i,j=1}^n a_{ij}(x) \, \xi_i \xi_j \geqq e \sum_{i=1}^n \xi_i^2, \qquad (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n,$$

for some constant $e > 0$. Furthermore, we assume that $(Lu, u) > 0$ for all $u \neq 0$, $u \in W_2{}^1(G)$, where $(\cdot, \cdot)$ is the inner product in $W_2{}^1(G)$ defined by

$$(u, v) = \int_G u(x) \, v(x) \, dx + \int_G Du(x) \, Dv(x) \, dx .$$

In the sequel we study the existence of bifurcation solutions of the elliptic partial differential equation

$$L(u) = d(x) \, u + \lambda(x) \, g(x) \, u + h\big(\lambda(x), x, u, Du\big), \tag{30}$$

$(\lambda, u) \in L_2(G) \times \big(W_2{}^2(G) \cap \mathring{W}_2{}^1(G)\big)$.

We need the following assumptions:

1. d and g are nonnegative functions of class $C^0(\bar{G})$.
2. The function h is continuos and satisfies:
   i) $h(s, x, 0, q) = 0$ for all $s \in \mathbf{R}$, $x \in \bar{G}$, $q \in \mathbf{R}^n$,
   ii) $|h(s, x, p_1, q_1) - h(s, x, p_2, q_2)| \leqq \alpha(|p_1 - p_2| + |q_1 - q_2|_n)$ for some constant $\alpha > 0$ which does not depend on $s \in \mathbf{R}$, $x \in \bar{G}$, where $|\cdot|_n$ is the Euclidean norm in $\mathbf{R}^n$,
   iii) $h(s, x, p, q) = o(|p| + |q|_n)$ as $|p| + |q|_n \to 0$ uniformly for $s \in \mathbf{R}$, $x \in \bar{G}$,
   iv) $h(\cdot, x, p, q)$ is continuous in $s \in \mathbf{R}$ uniformly for (x, p, q) from any bounded subset of $\bar{G} \times \mathbf{R} \times \mathbf{R}^n$,
   v) $ph(s, x, p, q) \geqq 0$ for all $(s, x, p, q) \in \mathbf{R} \times \bar{G} \times \mathbf{R} \times \mathbf{R}^n$.

3. $L(u) \neq td(x)\, u$ for all $t \in [0, 1]$ and $u \in \mathring{W}_2^1(G)$ with $u \neq 0$ and $u(x) \geqq 0$ for all $x \in G$.

4. Let $\lambda_1, \lambda_2 \in L_2(G)$. We define $\lambda_1 \prec \lambda_2$ iff $\lambda_1(x) \leqq \lambda_2(x)$ for almost all $x \in \bar{G}$. In the following we assume that $\lambda_0 \in L_2(G)$ is the smallest function which satisfies the equation

$$L(u_0) = d(x)\, u_0 + \lambda_0(x)\, g(x)\, u_0$$

for some $u_0 \in K = \{u \in \mathring{W}_2^1(G): u(x) \geqq 0 \text{ for all } x \in G\}$ and $u_0 \neq 0$. Furthermore, we assume that for any neighborhood V of λ_0 in $L_2(G)$ there exists $\bar{\lambda} \in V$ such that $\lambda_0 \prec \bar{\lambda}$ and that the equation

$$L(u) = d(x)\, u + \bar{\lambda}(x)\, g(x)\, u$$

has no solution $u \in K$, $u \neq 0$.

Theorem 4: *Let assumptions* 1—4 *be satisfied. Then* $(\lambda_0, 0)$ *is a bifurcation solution of equation* (30) *in the following sense: for any given* $\varepsilon > 0$ *there is a* $r_0 > 0$ *such that for each* $r \in (0, r_0)$ *one can find at least two solutions* (λ_1, u_1) *and* (λ_2, u_2) *of equation* (30) *with* $\lambda_i \in L_2(G)$, $\|\lambda_i - \lambda_0\|_{L_2} < \varepsilon$ *and* $u_i \in W_2^2(G) \cap \mathring{W}_2^1(G)$ *and* $\|u_i\|_{1,2} = r$, $i = 1, 2$. *Furthermore,* $u_1(x) \geqq 0$ *and* $u_2(x) \leqq 0$ *for all* $x \in \bar{G}$.

Proof: Put $H_0 = W_2^2(G) \cap \mathring{W}_2^1(G)$. By Stampacchia's maximum principle (see [22: Theorem 3.6]) there exists a constant $\gamma_0 > 0$ such that for each $\gamma > \gamma_0$ the operator $P_\gamma = \gamma id + L$ is a linear homeomorphism from H_0 onto $L_2(G)$ and there exists a constant $c > 0$ such that

$$\sum_{|\beta|=2} \|D^\beta u\| \leqq c\, \|P_\gamma(u)\|_{L_2} \quad \text{for all } u \in H_0.$$

Take a fixed number $\bar{\gamma} > \gamma_0$ large enough so that Stampacchia's maximum principle (which states that if, when γ is sufficiently large, $u \in W_2^2(G)$, $u(x) = 0$ on ∂G and $L(u)\,(x) + \gamma u(x) \geqq 0$ for allmost all $x \in G$, then $u(x) \geqq 0$ for all $x \in G$) can be applied to $P_{\bar{\gamma}} = \bar{\gamma} id + L$. We have $P_{\bar{\gamma}}: H_0 \to L_2(G)$ and hence $P_{\bar{\gamma}}^{-1}: L_2(G) \to H_0$, furthermore, this maximum principle implies that $P_{\bar{\gamma}}^{-1}(K) \subset K$ and $P_{\bar{\gamma}}^{-1}(-K) \subset -K$. Indeed, let $u \in K$, we have $P_{\bar{\gamma}}^{-1}(u) \in H_0 = W_2^2 \cap \mathring{W}_2^1$ and thus $\big(P_{\bar{\gamma}}^{-1}(u)\big)\,(x) = 0$ on ∂G and

$$L\big(P_{\bar{\gamma}}^{-1}(u)\big)\,(x) + \bar{\gamma}\big(P_\gamma^{-1}(u)\big)\,(x) = P_{\bar{\gamma}}\big(P_{\bar{\gamma}}^{-1}(u)\big)\,(x) = u(x) \geqq 0$$

for all $x \in G$. Applying the maximum principle just mentioned above yields $P_{\bar{\gamma}}^{-1}(u)\,(x) \geqq 0$ for all $x \in G$. This shows that $P_{\bar{\gamma}}^{-1}(u) \in K$ and hence $P_{\bar{\gamma}}^{-1}(K) \subset K$. Analogically, we have $P_{\bar{\gamma}}^{-1}(-K) \subset -K$.

Now, we can easily verify that the equation (10) is equivalent to the equation

$$\bar{\gamma} u + L(u) = \big(\bar{\gamma} + d(x)\big) u + \lambda(x) g(x) u + h\big(\lambda(x), x, u, Du\big)$$

and then to

$$u = P_{\bar{\gamma}}^{-1}\big((\bar{\gamma} + d(x)) u + \lambda(x) g(x) u + h(\lambda(x), x, u, Du)\big).$$

Further, we define the mappings

$$M(u) = P_{\bar{\gamma}}^{-1}\big((\bar{\gamma} + d(x)) u\big), \qquad N(\lambda, u) = P_{\bar{\gamma}}^{-1}\big(\lambda(x) g(x) u\big),$$
$$H(\lambda, u) = P_{\bar{\gamma}}^{-1}\big(h(\lambda(x), x, u, Du)\big), \qquad (\lambda, u) \in L_2(G) \times W_2{}^1(G).$$

First of all we remark that any bounded subset in $W_2{}^2(G)$ is precompact in $W_2{}^1(G)$ by the Sobolev embedding theorem. Therefore, $P_{\bar{\gamma}}^{-1}$ is a compact continuous linear mapping from $L_2(G)$ into $W_2{}^1(G)$. Since d, g are nonnegative functions and h has the property 2.v) we then conclude that $M(K)$, $N(\lambda, K)$, $H(\lambda, K) \subseteqq K$ and $M(-K)$, $N(\lambda, -K)$, $H(\lambda, -K) \subseteqq -K$ for all $\lambda > 0$. Further, it is easy to see that for any fixed λ, M, $N(\lambda, \cdot)$ are continuous mappings, moreover, the mapping $N(\cdot, u)$ is continuous in $\lambda \in L_2(G)$ uniformly for u from any bounded subset of $W_2{}^1(G)$.

We now prove that for any fixed $\lambda \in L_2(G)$, $H(\lambda, \cdot)$ is continuous and $H(\lambda, u) = o(\|u\|_{1,2})$ as $\|u\|_{1,2} \to 0$ uniformly for $\lambda \in L_2(G)$. Indeed, by the hypotheses in assumption 2 we have

$$\big|h\big(\lambda(x), x, u(x), Du(x)\big)\big| = \big|h\big(\lambda(x), x, u(x), Du(x)\big) - h\big(\lambda(x), x, 0, 0\big)\big|$$
$$\leqq \alpha\big(|u(x)| + |Du(x)|_n\big).$$

Hence,

$$\int_G \big|h\big(\lambda(x), x, u(x), Du(x)\big)\big|^2 \, dG \leqq \alpha^2 \int_G \big(|u(x)| + |Du(x)|_n\big)^2 \, dG$$
$$\leqq \alpha^2 \int_G \big(|u(x)|^2 + |Du(x)|_n^2 + 2\,|u(x)|\,|Du(x)|_n\big)\, dG$$
$$\leqq \alpha^2(\|u\|_{L_2}^2 + \|Du\|_{L_2}^2 + 2\,\|u\|_{L_2}\,\|Du\|_{L_2}).$$

This shows that if $u \in W_2{}^1(G)$, then $h\big(\lambda(x), x, u, Du\big) \in L_2(G)$. Therefore, we can define the mapping $R(\lambda, \cdot)\colon W_2{}^1(G) \to L_2(G)$ for any fixed $\lambda \in L_2(G)$ by

$$R(\lambda, u)(x) = h\big(\lambda(x), u(x), Du(x)\big).$$

We have

$$\|R(\lambda, u_m) - R(\lambda, u)\|_{L_2}^2$$
$$= \int_G \big|h\big(\lambda(x), x, u_m(x), Du_m(x)\big) - h\big(\lambda(x), x, u(x), Du(x)\big)\big|^2 \, dG$$
$$\leqq \alpha^2 \int_G \big(|u_m(x) - u(x)| + |Du_m(x) - Du(x)|_n\big)^2 \, dG$$
$$\leqq \alpha^2(\|u_m - u\|_{L_2}^2 + \|Du_m - Du\|_{L_2} + 2\,\|u_m - u\|_{L_2}\,\|Du_m - Du\|_{L_2},$$

and hence,

$$\|R(\lambda, u_m) - R(\lambda, u)\|_{L_2}$$
$$\leqq \alpha(\|u_m - u\|_{L_2}^2 + \|Du_m - Du\|_{L_2} + 2\,\|u_m - u\|_{L_2}\,\|Du_m - Du\|_{L_2})^{1/2}$$

for any sequence $\{u_m\}$ in $W_2^1(G)$, $u_n \to u$. This shows that $R(\lambda, \cdot)$ is a continuous mapping. It is also easy to verify that $R(\lambda, \cdot)$ is bounded and $\|R(\lambda, u)\| = o(\|u\|_{1,2})$ as $\|u\|_{1,2} \to 0$ uniformly for $\lambda \in L_2(G)$. We have $H(\lambda, u) = P_{\bar{\gamma}}^{-1}(R(\lambda, u))$, and thus for any fixed $\lambda \in L_2(G)$, $H(\lambda, \cdot)$ is a compact continuous mapping and $\|H(\lambda, u)\| = o(\|u\|_{1,2})$ as $\|u\|_{1,2} \to 0$ uniformly for $\lambda \in L_2(G)$. Furthermore, we can easily prove from assumption 2 that $H(\cdot, u)$ is continuous in λ uniformly for u from any bounded subset of $W_2^1(G)$.

Further, assumption 4 implies that λ_0 is the smallest eigenvalue of the pair (M, N) with respect to K and also with respect to $-K$ which is isolated from the right side. Therefore, for given $\varepsilon > 0$ we can choose $\lambda_i \in L_2(G)$, $\|\lambda_i - \lambda_0\|_{L_2} < \varepsilon$, $i = 1, 2$, $\lambda_1 \prec \lambda_0 \prec \lambda_2$ and λ_i is not an eigenvalue of the pair (M, N) with respect to K (or $-K$, respectively), $i = 1, 2$. Let u_0 be an eigenvector of the pair (M, N) with respect to K corresponding to λ_0. It is clear that $-u_0$ is an eigenvector of the pair (M, N) with respect to $-K$. Therefore, we can apply Lemma 3 and Lemma 4 to conclude that there exists a $r_0 > 0$ such that

$$\|H(\lambda_1, u)\|_{1,2} \leqq \|u - M(u) - N(\lambda_1, u)\|_{1,2}$$

and

$$u - M(u) - N(\lambda_2, u) - H(\lambda_2, u) \neq \mu u_0 \quad (\neq -\mu u_0)$$

hold for all $u \in U_0^+ = U(0, r_0) \cap K$, $\mu > 0$ (or $u \in U_0^- = U(0, r_0) \cap (-K)$, $\mu > 0$, respectively).

Let $\mathscr{C}_K(U_0^+)$ $(\mathscr{C}_K(U_0^-))$ be the family of all subsets of the form $U^+ = U(0, r) \cap K$ (or $U^- = U(0, r) \cap (-K)$, respectively) with $0 < r < r_0$. By $\mathscr{B}_K(U^+, X)$ $(\mathscr{B}_K(U^-, X))$ with $X = W_2^1(G)$ we denote the class of all functions F from $U(0, r)$ to X of the form $F = id - T$, where T is a compact continuous mapping and $T(U^+) \subset K$ (or $T(U^-) \subset -K$, respectively). Thus, for any $F \in \mathscr{B}_K(U^+, X)$ $(F \in \mathscr{B}_K(U^-, X))$ $\deg_K(F, U^+, 0)$ (or $\deg_K(F, U^-, 0)$, respectively), is defined as in part a) of the proof of Theorem 3.

We now claim that

$$\deg_K(A, U^\pm, 0) = \{1\}, \tag{31}$$

where the mapping A is defined by

$$A(u) = u - M(u) - N(\lambda_1, u), \qquad u \in U(0, r).$$

Indeed, we define the mapping $B\colon [0, 1] \times U_0 \to X$ by

$$B(t, u) = u - M(u) - N(t\lambda_1, u).$$

Since λ_0 is the smallest eigenvalue of the pair (M, N) with respect to K $(-K$, respectively), we then conclude that $B(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U^\pm$. Using Axiom II yields

$$\deg_K\big(id - M - N(\lambda_1, \cdot), U^\pm, 0\big) = \deg_K(id - M, U^\pm, 0). \tag{32}$$

Further, we define the mapping $C\colon [0, 1] \times U_0 \to X$ by

$$C(t, u) = u - P_{\bar{\gamma}}^{-1}\big((\bar{\gamma} + td(x))\, u\big).$$

Suppose that $C(\bar{t}, \bar{u}) = 0$ for some $\bar{t} \in [0, 1]$ and $\bar{u} \in \partial_K U^\pm$, we then imply $\overline{\gamma u} + L(\bar{u}) = \overline{\gamma u} + \bar{t}d(x)\, \bar{u}$ and hence $L(\bar{u}) = \bar{t}d(x)\, \bar{u}$. This contradicts assumption 3. Therefore, we conclude $C(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U^\pm$. Using Axiom II again we have

$$\deg_K(id - M, U^\pm, 0) = \deg_K\big(id - P_{\bar{\gamma}}^{-1}(\bar{\gamma}\, \cdot), U^\pm, 0\big). \tag{33}$$

Finally, we define the mapping $D\colon [0, 1] \times U_0 \to X$ by

$$D(t, u) = u - tP_{\bar{\gamma}}^{-1}(\bar{\gamma}u).$$

Suppose that $D(\bar{t}, \bar{u}) = 0$ for some $\bar{t} \in [0, 1]$ and $u \in \partial_K U^{\pm}$. We infer $\bar{\gamma}\bar{u} + L(\bar{u}) = \bar{t}\bar{\gamma}\bar{u}$ or $L(\bar{u}) = (\bar{t} - 1)\,\bar{\gamma}\bar{u}$, and hence $0 < \big(L(\bar{u}), \bar{u}\big) = (\bar{t} - 1)\,\bar{\gamma}(\bar{u}, \bar{u}) \leqq 0$, which is impossible. This implies $D(t, u) \neq 0$ for all $t \in [0, 1]$ and $u \in \partial_K U^{\pm}$. We use Axiom II again to get

$$\deg_K \big(id - P_{\bar{\gamma}}^{-1}(\bar{\gamma}\,\cdot), U^{\pm}, 0\big) = \deg_K (id, U^{\pm}, 0) = \{1\}. \tag{34}$$

Combining (32)–(34) gives (31).

Lastly we put $X = Y = W_2^1(G)$, $D = U_0$ and

$$\left.\begin{aligned} F(\lambda, u) &= u - M(u) - N(\lambda, u) - H(\lambda, u) \\ A(u) &= u - M(u) - N(\lambda_1, u) \end{aligned}\right\} \quad \big(\lambda \in L_2(G),\ u \in \bar{U}_0\big).$$

Then, to complete the proof it remains to apply two times Theorem 2: with X, Y, D, F, A, $\mathscr{E}_K(\bar{U}_0)$, $\mathscr{B}_K(U^+, X)$ and $K = \{u \in W_2^1(G)\colon u(x) \geqq 0 \text{ for all } x \in G\}$ and with X, Y, D, F, A, $\mathscr{E}_K(\bar{U}_0)$ as above, $\mathscr{B}_K(U^-, X)$ and $-K = \{u \in W_2^1(G)\colon u(x) \leqq 0 \text{ for all } x \in G\}$ ∎

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