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Contribution to the Theory of Generalized Derivatives

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Es werden untere und obere Ableitungen behandelt und für diese eine Reihe von Ausdehnungen grundlegender Aussagen der Differentialrechnung (Produktregel, Quotientenregel, Mittelwertsatz usw.) und als Anwendung eine Verallgemeinerung der Eulerschen Differentialgleichung gebracht.

Рассматриваются нижние и верхние производные и даются для них некоторые расширения основных утверждений дифференциального исчисления (правило произведений, правило частных, теорема о среднем и т. д.) и в качестве применения обобщение дифференциального уравнения Эйлера.

Lower and upper derivatives are considered. For them certain extensions of fundamental assertions of the differential calculus (product rule, quotient rule, mean value theorem etc.) and as application a generalization of the Euler differential equation are given.

To avoid unnecessarily strong differentiability assumptions, in optimization theory several generalizations of the notion of derivative are used. Among them, lower and upper derivatives are of special importance (see the references additionally those of [10]). The underlying paper continues the author's previous investigations [2–6] on this field, using sometimes slightly more restrictive assumptions to make the considerations more transparent.

I. Throughout the paper, let R be a separated topological vector space, $Q \subset R$ be an open subset and $q \in Q$ be a point. Let S denote an ordered topological vector space, that is a topological vector space which is equipped with a partial ordering $\leqq$ given by "$s \leqq s'$ iff $s' - s \in K$" where K is a closed proper cone in S with 0 as vertex, the so-called positive cone. K to be *proper* means that K is convex and $K \cap (-K) = \{0\}$. S is separated. S is said to be *locally order-convex* if there exists an open base at 0 consisting of order-convex sets, that is of sets U such that $(U + K) \cap (U - K) = U$.

Let f be a mapping of Q into S. f is said to be *lower semicontinuous* at q if for every neighbourhood V of the point 0 in S there is a neighbourhood U of the point 0 in R with $q + U \subseteq Q$ such that $f[q + U] \subseteq f(q) + V + K$.

f is said to be *l-differentiable* (*lower differentiable*) at q if there is a mapping $\phi: R \to S$ which has the following properties:

(i) $\phi(\alpha p) - \alpha\phi(p) \in K$ for every $\alpha \in (0, 1)$ and $p \in R$.

(ii) For every point $p \in R$ and every neighbourhood V of the point 0 in S there exists an $\varepsilon > 0$ and a neighbourhood U of the point 0 in R with $q + [0, \varepsilon](p + U) \subseteq Q$ such that

$$\frac{f(q + \varepsilon'(p + p')) - f(q)}{\varepsilon'} \in \phi(p) + V + K$$

whenever $\varepsilon' \in (0, \varepsilon]$ and $p' \in U$.

ϕ is called an *l-derivative* (*lower derivative*) of f at q. From properties (i) and (ii) it follows that $\phi(0) = 0$. The set of all l-derivatives of f at q is denoted by $D_l f(q)$. A mapping $\phi: R \to S$ with property (i) belongs to $D_l f(q)$ if (and only if) there exists a $\phi' \in D_l f(q)$ with $(\phi' - \phi)[R] \subseteqq K$. By [3: Theorem 1] $D_l f(q)$ is convex. From [3: Theorem 2] we know that if $D_l f(q) \neq \emptyset$, then f is lower semicontinuous at q.

If $\alpha \geqq 0$, then $\alpha D_l f(q) \subseteqq D_l(\alpha f)(q)$. For every $g: Q \to S$ with $g(q) = f(q)$ for which there exists a neighbourhood U of q such that $(f - g)[U] \subseteqq K$, $D_l g(q) \subseteqq D_l f(q)$. For every $g: Q \to S$, $D_l f(q) + D_l g(q) \subseteqq D_l(f + g)(q)$. If $\underline{K} \neq \emptyset$ and if for every non-empty finite subset of S there exists the infimum, then $f_i: Q \to S$ and $\phi_i \in D_l f_i(q)$, $i \in \{1, \ldots, n\}$, imply that $\phi \in D_l f(q)$ where $f: Q \to S$ and $\phi: R \to S$ are defined by $f(p) = \inf\limits_i f_i(p)$, $p \in Q$, and $\phi(p) = \inf\limits_i \phi_i(p)$, $p \in R$.

A mapping $f: Q \to S$ is said to be *u-differentiable* (*upper differentiable*) at q if $-f$ is l-differentiable at q. Every $\phi \in D_u f(q) = -D_l(-f)(q)$ is called a *u-derivative* (*upper derivative*) of f at q.

By [3: Theorem 4] we have $\big(D_l f(q) - D_u f(q)\big)[R] \subseteqq -K$ from which it follows that $Df(q) = D_l f(q) \cap D_u f(q)$ contains at most one element. If $Df(q) \neq \emptyset$, then f is said to be *differentiable* at q and the unique mapping $\phi \in Df(q)$ is called a *derivative* of f at q. A derivative ϕ is positive homogeneous, but neither homogeneous, nor additive, nor continuous, in general. Our notion of differentiability is a generalization of the well-known notion of Michal-Bastiani differentiability.

In applications often S is the set $\mathbf{R}$ of all reals which always will be assumed to be equipped with the natural ordering (hence $K = [0, \infty)$) and with the natural topology.

II. Now let S be an ordered inner product space and $(\cdot, \cdot)$ be its inner product. (The fact that the symbol $(\cdot, \cdot)$ is also used to denote the open interval cannot lead to any confusion.) Let us write

$$f(q) \begin{Bmatrix} >_K \\ =_K \\ <_K \end{Bmatrix} 0 \quad \text{if} \quad \big(f(q), K\big) \begin{cases} \subseteqq [0, \infty), & \text{but} \neq \{0\} \\ = \{0\} \\ \subseteqq (-\infty, 0], & \text{but} \neq \{0\} \end{cases}$$

where $\big(f(q), K\big) = \big\{(f(q), k)/k \in K\big\}$. Let $f(q) =_K 0$ if there exists a neighbourhood U of the point 0 in R with $q + U \subseteqq Q$ and $(f[q + U], K) = \{0\}$ where $(f[q + U], K) = \{(s, k)/s \in f[q + K], k \in K\}$. Moreover let $f(q) >_K 0$ ($<_K 0$) if not $f(q) =_K 0$, but if there exists a neighbourhood U of the point 0 in R with $q + U \subseteqq Q$ and $(f[q + U], K) \subseteqq [0, +\infty)$ $\big((-\infty, 0]\big)$. For arbitrary $g: Q \to S$ let

$$\mathscr{D}g_f(q) = \begin{cases} D_l g(q) & \text{if } f(q) >_K 0 \\ D_l g(q) \cup D_u g(q) & \text{if } f(q) =_K 0 \\ D_u g(q) & \text{if } f(q) <_K 0 \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathscr{D}f_g(q) = \begin{cases} D_l f(q) & \text{if } g(q) >_K 0 \\ D_l f(q) \cup D_u f(q) & \text{if } g(q) =_K 0 \\ D_u f(q) & \text{if } g(q) <_K 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorem 1: *If g is continous at q, then*

$$\big(f(q), \mathscr{D}g_f(q)\big) + \big(\mathscr{D}f_g(q), g(q)\big) \subseteqq D_l(f, g)(q).$$

Proof: Assume $\mathcal{D}g_f(q)$ and $\mathcal{D}f_g(q)$ are not empty. Let $f(q) >_K 0$ and $g(q) >_K 0$ and for arbitrary $\gamma \in D_1 g(q)$ and $\phi \in D_1 f(q)$ let $\psi = (f(q), \gamma) + (\phi, g(q))$. Then

$$\psi(\alpha p) - \alpha\psi(p) = (f(q), \gamma(\alpha p) - \alpha\gamma(p)) + (\phi(\alpha p) - \alpha\phi(p), g(q))$$
$$\in (f(q), K) + (K, g(q)) \subseteqq [0, \infty)$$

for every $\alpha \in (0, 1)$ and $p \in K$. Hence, ψ has the property (i) of an l-derivative.

To show that (with respect to (f, g)) ψ has the property (ii) let a point $p \in R$ and an $\eta > 0$ be given and let V be a neighbourhood of the point 0 in S such that

$$(f(q), V) + (\phi(p), V) + (V, g(q)) + (V, V) \subseteqq (-\eta, \eta).$$

Let ε be a positive real number and U be a neighbourhood of the point 0 in R with $q + [0, \varepsilon]\,(p + U) \subseteqq Q$ such that

$$g[q + [0, \varepsilon]\,(p + U)] \subseteqq g(q) + V$$

and

$$(g[q + [0, \varepsilon]\,(p + U)], K) \subseteqq [0, \infty)$$

as well as

$$\frac{f(q + \varepsilon'(p + p')) - f(q)}{\varepsilon'} \in \phi(p) + V + K$$

and

$$\frac{g(q + \varepsilon'(p + p')) - g(q)}{\varepsilon'} \in \gamma(p) + V + K$$

whenever $\varepsilon' \in (0, \varepsilon]$ and $p' \in U$. For every such ε' and p', then

$$\frac{(f(q + \varepsilon'(p + p')), g(q + \varepsilon'(p + p'))) - (f(q), g(q))}{\varepsilon'}$$

$$= \left(f(q), \frac{g(q + \varepsilon'(p + p')) - g(q)}{\varepsilon'}\right) + \left(\frac{f(q + \varepsilon'(p + p')) - f(q)}{\varepsilon'}, g(q + \varepsilon'(p + p')\right)$$

$$\in (f(q), \gamma(p) + V + K) + (\phi(p) + V + K, g(q + \varepsilon'(p + p')))$$

$$\subseteqq (f(q), \gamma(p)) + (\phi(p), g(q)) + (f(q), V) + (\phi(p), V) + (V, g(q)) + (V, V)$$
$$+ (f(q), K) + (g[q + [0, \varepsilon]\,(p + U)], K)$$

$$\subseteqq \psi(p) + (-\eta, \infty).$$

Hence we have $\psi \in D_1(f, g)\,(q)$ which proves the assertion of the theorem in the case $f(q) >_K 0$ and $g(q) >_K 0$. The other cases can be proved analogously ∎

Corollary 1: *Let S be locally order-convex, f be differentiable at q and g be continuous at q. Then*

$$(f(q), \mathcal{D}g_f(q)) + (\phi, g(q)) \subseteqq D_1(f, g)\,(q),$$

where ϕ denotes the derivative of f at q.

Proof: Analogously to the proof of Theorem 1 ∎

Corollary 2: *Let S be locally order-convex and f and g be differentiable at q. Then (f, g) is differentiable at q with $(f(q), \gamma) + (\phi, g(q))$ being the derivative, where ϕ and γ denote the derivatives of f and g at q, respectively.*

Proof: $D_l g(q) \neq \emptyset$ and $D_l(-g)(q) = -D_u g(q) \neq \emptyset$ imply the lower semicontinuity of g and $-g$ at q from which because of the local order-convexity of S it follows that g is continuous at q. With that, Corollary 2 can be proved analogously as Theorem 1 ∎

III. Let $S = \mathbf{R}$. For arbitrary mappings f and g of Q into S let

$$\mathcal{D}f_g(q) = \begin{cases} D_l f(q) & \text{if } g(q) > 0 \\ \text{any set of mappings of } R \text{ into } S & \text{if } g(q) = 0 \\ D_u f(q) & \text{if } g(q) < 0. \end{cases}$$

Theorem 2: *If g is continuous at q and if $g(q) \neq 0$, then*

$$\frac{g(q)\,\mathcal{D}f_g(q) + f(q)\,\mathcal{D}(-g)_f(q)}{g(q)^2} \subseteq D_l\left(\frac{f}{g}\right)(q).$$

Proof: Assume $\mathcal{D}f_g(q)$ and $\mathcal{D}(-g)_f(q)$ are not empty. Let $g(q) > 0$ and $f(q) > 0$ and for arbitrary $\phi \in D_l f(q)$ and $\gamma \in -D_l(-g)(q) = D_u g(q)$ let $\psi = \big(g(q)\,\phi - f(q)\,\gamma\big)/g(q)^2$.

It is obvious that ψ has the property (i) of an l-derivative. To show that (with respect to f/g) ψ has the property (ii) let be given a point $p \in R$ and an $\eta > 0$. Let $\varkappa$ be a positive real number such that $\varkappa < g(q)/2$ and

$$\frac{g(q)\big(\phi(p) + (-\varkappa, \infty)\big) - f(q)\big(\gamma(p) + (-\infty, \varkappa)\big)}{g(q)^2 + g(q)\,(-\varkappa, \varkappa)}$$

$$\subseteq \frac{g(q)\,\phi(p) - f(q)\,\gamma(p)}{g(q)^2} + (-\eta, \infty).$$

Let ε be a positive real number and U be a neighbourhood of the point 0 in R with $q + [0, \varepsilon]\,(p + U) \subseteq Q$ such that

$$g\big[q + [0, \varepsilon]\,(p + U)\big] \subseteq g(q) + (-\varkappa, \varkappa)$$

as well as

$$\frac{f\big(q + \varepsilon'(p + p')\big) - f(q)}{\varepsilon'} \in \phi(p) + (-\varkappa, \infty)$$

and

$$\frac{g\big(q + \varepsilon'(p + p')\big) - g(q)}{\varepsilon'} \in \gamma(p) + (-\infty, \varkappa)$$

whenever $\varepsilon' \in (0, \varepsilon]$ and $p' \in U$. For every such ε' and p', then

$$\frac{1}{\varepsilon'}\left(\frac{f\big(q + \varepsilon'(p + p')\big)}{g\big(q + \varepsilon'(p + p')\big)} - \frac{f(q)}{g(q)}\right)$$

$$= \frac{g(q)\,\dfrac{f\big(q + \varepsilon'(p + p')\big) - f(q)}{\varepsilon'} - f(q)\,\dfrac{g\big(q + \varepsilon'(p + p')\big) - g(q)}{\varepsilon'}}{g(q)\,g\big(q + \varepsilon'(p + p')\big)}$$

$$\in \frac{g(q)\big(\phi(p) + (-\varkappa, \infty)\big) - f(q)\big(\gamma(p) + (-\infty, \varkappa)\big)}{g(q)^2 + g(q)\,(-\varkappa, \varkappa)}$$

$$\subseteq \psi(p) + (-\eta, \infty).$$

Hence the assertion of the theorem is true in the case where $f(q) > 0$, $g(q) > 0$. In the other cases the proof can be given analogously ∎

Corollary: *Let f and g be differentiable at q and let $g(q) \neq 0$. Then f/g is differentiable at q with $\big(g(q)\,\phi - f(q)\,\gamma\big)/g(q)^2$ being the derivative, where ϕ and γ denote the derivatives of f and g at q, respectively.*

Proof: Since $S = \mathbf{R}$ is locally order-convex, from the proof of Corollary 2 of Theorem 1 we know that g is continuous at q. With that, the corollary becomes a consequence of Theorem 2 ∎

IV. Let S be a locally convex ordered topological vector space (with the positive cone K). In [3: Theorem 3] there is given a mean value theorem which in a slightly weaker form looks as follows.

Theorem 3: *Let p be a point of R such that $q + [0, 1]\,p \subseteq Q$. Assume that the mapping $f: Q \to S$ is continuous and for every $\varepsilon \in (0, 1)$ there exists an l-derivative $\phi(q + \varepsilon p)$ of f at $q + \varepsilon p$. Then*

$$f(q + p) - f(q) \in \overline{\operatorname{co}\{\phi(q + \varepsilon p)\,(p) \mid \varepsilon \in (0, 1)\} + K},$$

where co $\{\cdot\}$ *denotes the convex hull of* $\{\cdot\}$.

Corollary: *Let f be a continuous mapping of $[0, 1]$ into S. Assume that for every $t \in (0, 1)$ there exists an l-derivative $\phi(t)$ of f at t such that $\phi(t)\,(1) \in K$. Then $f(1) - f(0) \in K$.*

Proof: Application of Theorem 3 ∎

Theorem 4: *Let S be an ordered Banach space and f be a continuous mapping of $[0, 1]$ into S. Assume that for every $t \in (0, 1)$ there exist an l-derivative $\phi(t)$ of f at t and a $\varkappa \geqq 1$ such that $\phi(\cdot)\,(\varkappa)$ is continuous and bounded. Then*

$$f(1) - f(0) - \int_0^1 \frac{1}{\varkappa}\,\phi(t)\,(\varkappa)\,dt \in K.$$

Proof: For every $t \in (0, 1)$, let $\gamma(t)$ denote the derivative at t of the mapping $g: [0, 1] \to S$ given by $g(t) = \int_0^t \varkappa^{-1}\phi(s)\,(\varkappa)\,ds$. Obviously, this derivative exists. $\varkappa^{-1}\phi(t)\,(\varkappa\,\cdot) - \gamma(t)\,(\cdot)$ is an l-derivative of $f - g$ at t. Since $\varkappa^{-1}\phi(t)\,(\varkappa) - \gamma(t)\,(1) = 0$, by means of the Corollary of Theorem 3 we get $f(1) - g(1) - \big(f(0) - g(0)\big) \in K$ and hence, taking into account that $g(0) = 0$,

$$f(1) - f(0) - \int_0^1 \frac{1}{\varkappa}\,\phi(t)\,(\varkappa)\,dt = f(1) - g(1) - \big(f(0) - g(0)\big) \in K \ ∎$$

V. Let X and Y be locally order-convex ordered topological vector spaces and f and g be mappings of Q into X or Y, respectively. Let S be a locally convex ordered topological vector space (with positive cone K) and let L be a mapping of an open subset $W \subset X \times Y$ with $(f[Q], g[Q]) \subseteq W$ into S.

The set of all l-derivatives of L with respect to the first argument at a point $(x, y) \in W$ will be denoted by $D_{1,l}L(x, y)$. L is said to be lower semicontinuously l-differentiable with respect to the second argument at (x, y) if there exists a neighbourhood U_x of the point x in X and a neighbourhood U_y of the point y in Y such that $U_x \times U_y \subseteqq W$ and for every $(x', y') \in U_x \times U_y$ there exists an l-derivative $\Lambda_2(x', y')(\cdot)$ of L with respect to the second argument at (x', y') where with respect to all three arguments Λ_2 is lower semicontinuous at (x, y, y^*) for every $y^* \in Y$. The set of all mappings $\Lambda_2(x, y)$ of this kind will be denoted by $\mathbf{D}_{2,1}L(x, y)$. Let $D_1L(f, g)(q)$ be the set of all l-derivatives of $L(f, g)$ at q.

Theorem 5: *If L is continuous with respect to the second argument on a neighbourhood of $(f(q), g(q))$, then*

$$D_{1,1}L(f(q), g(q)) \circ Df(q) + \mathbf{D}_{2,1}L(f(q), g(q)) \circ Dg(q) \subseteqq D_1L(f, g)(q).$$

Proof: Let $x = f(q)$, $y = g(q)$ and $\phi \in Df(q)$, $\gamma \in Dg(q)$, $\Lambda_1 \in D_{1,1}L(x, y)$. Assume, U_x, U_y and Λ_2 are given as above. Let $\mathbf{\Lambda}_2 = \Lambda_2(x, y)$.

Obviously, $\Lambda = \Lambda_1 \circ \phi + \mathbf{\Lambda}_2 \circ \gamma$ has the property (i) of an l-derivative. To show that (with respect to $L(f, g)$) Λ has the property (ii) let a point $p \in R$ and a convex neighbourhood V of the point 0 in S be given. Let $U_{x,0} = U_x - x$ and $U_{y,0} = U_y - y$. We may assume that U_x and U_y are choosen so that L is continuous with respect to the second argument on $U_x \times U_y$ and

$$\cup\, \Lambda_2(x^+, y^+)(y^*) \subseteqq \mathbf{\Lambda}_2 \circ \gamma(p) + V/3 + K,$$

where the union is taken with respect to all $x^+ \in U_x$, $y^+ \in U_y$ and $y^* \in \gamma(p) + U_{y,0}$. We also may assume that there is an $\varepsilon \in (0, 1)$ such that $[0, \varepsilon]\,(\phi(p) + U_{x,0}) \subseteqq U_{x,0}$, $[0, \varepsilon]\,(\gamma(p) + U_{y,0}) \subseteqq U_{y,0}$ and

$$\frac{L(x + \varepsilon'(\phi(p) + x^*), y) - L(x, y)}{\varepsilon'} \in \Lambda_1 \circ \phi(p) + \frac{V}{2} + K$$

whenever $\varepsilon' \in (0, \varepsilon]$ and $x^* \in U_{x,0}$. Moreover we can arrange it so that for a suitable neighbourhood U of the point 0 in R, $q + [0, \varepsilon]\,(p + U) \subseteqq Q$ and that for arbitrary $\varepsilon' \in (0, \varepsilon]$ and $p' \in U$, taking $x' = f(q + \varepsilon'(p + p'))$ and $y' = g(q + \varepsilon'(p + p'))$, we have

$$\frac{x' - x}{\varepsilon'} \in \phi(p) + U_{x,0}, \quad \frac{y' - y}{\varepsilon'} \in \gamma(p) + U_{y,0}.$$

Using Theorem 3, for every such ε' and p' we get

$$\frac{L(f(q + \varepsilon'(p + p')), g(q + \varepsilon'(p + p'))) - L(f(q), g(q))}{\varepsilon'}$$

$$= \frac{L\left(x + \varepsilon'\dfrac{x' - x}{\varepsilon'}, y\right) - L(x, y)}{\varepsilon'} + \frac{L(x', y') - L(x', y)}{\varepsilon'}$$

$$\in \bigcup_{x^* \in U_{x,0}} \frac{L(x + \varepsilon'(\phi(p) + x^*), y) - L(x, y)}{\varepsilon'}$$

$$+ \overline{\operatorname{co}\left\{\Lambda_2(x', y + \varepsilon^*(y' - y))\left(\frac{y' - y}{\varepsilon'}\right) \,\middle|\, \varepsilon^* \in (0, 1)\right\} + K}$$

$$\subseteqq \Lambda_1 \circ \phi(p) + V/2 + K$$

$$+ \overline{\operatorname{co}\,\{\Lambda_2(x^+, y^+)(y^*) \mid x^+ \in U_x, y^+ \in U_y, y^* \in \gamma(p) + U_{y,0}\} + K}$$

$$\subseteqq \Lambda_1 \circ \phi(p) + V/2 + K + \mathbf{\Lambda}_2 \circ \gamma(p) + V/2 + K = \Lambda(p) + V + K \quad \blacksquare$$

Corollary [3: Theorem 5]: *Let X and f be as above. Let S be an ordered topological vector space and f' be a mapping of an open subset Q' of X with $f[Q] \subseteqq Q'$ into S. Then* $D_1 f'(f(q)) \circ Df(q) \subseteqq D_1(f' \circ f)(q)$.

In the case of S being locally convex, the corollary is a special case of Theorem 5 since for L which is independent of the second argument, $\mathbf{D}_{2,1}L(x, y)$ contains the zero mapping.

VI. Let S be an ordered Banach space (with positive cone K) and G be a mapping of $Q \times [0, 1]$ into S.

Theorem 6: *Let G be continuous. Assume, there exists an l-derivative $\gamma(p, t)$ of G with respect to the first argument at every point $(p, t) \in Q \times [0, 1]$ such that the following holds: For arbitrary $p \in R$, $\gamma(q, \cdot)(p)$ is continuous and $\gamma(\cdot, t)(\cdot)$ is lower semicontinuous (with respect to the pair consisting of the first and third argument) at (q, p) uniformly for all $t \in [0, 1]$*[1]*). Let $F: Q \to S$ and $\phi: R \to S$ be given by*

$$F(p) = \int_0^1 G(p, t)\,dt, \quad p \in Q, \quad \text{and} \quad \phi(p) = \int_0^1 \gamma(q, t)(p)\,dt, \quad p \in R.$$

Then $\phi \in D_1 F(q)$.

Proof: Obviously ϕ has the property (i) of an l-derivative. To show that (with respect to F) ϕ has the property (ii) let a point $p \in R$ and a convex neighbourhood V of the point 0 in S be given. There exists a neighbourhood U of the point 0 in R with $q + U \subseteqq Q$ such that $\gamma(q^+, t)(p^+) \subseteqq \gamma(q, t)(p) + V/2 + K$ for every $q^+ \in q + U$, $t \in [0, 1]$ and $p^+ \in p + U$. Let $\varepsilon \in (0, 1)$ be such that $[0, \varepsilon](p + U) \subseteqq U$. Using Theorem 3, for every $\varepsilon' \in (0, \varepsilon]$ and $p' \in U$ we get

$$\begin{aligned} &\frac{G(q + \varepsilon'(p + p'), t) - G(q, t)}{\varepsilon'} \\ &\in \varepsilon^{-1} \overline{\mathrm{co}\,\{\gamma(q + \varepsilon^*\varepsilon'(p + p'), t)(\varepsilon'(p + p')) \mid \varepsilon^* \in (0, 1)\} + K} \\ &\subseteqq \overline{\mathrm{co}\,\{\gamma(q^+, t)(p^+) \mid q^+ \in q + U, p^+ \in p + U\} + K} \\ &\subseteqq \overline{\gamma(q, t)(p) + V/2 + K} \subseteqq \gamma(q, t)(p) + 2V/3 + K \end{aligned}$$

hence

$$\begin{aligned} \frac{F(q + \varepsilon'(p + p')) - F(q)}{\varepsilon'} &= \int_0^1 \frac{G(q + \varepsilon'(p + p'), t) - G(q, t)}{\varepsilon'}\,dt \\ &\subseteqq \int_1^0 \gamma(q, t)(p)\,dt + V + K = \phi(p) + V + K \quad \blacksquare \end{aligned}$$

VII. Lower and upper derivatives can be used to get very general optimality conditions in vector optimization (see the references). The following considerations deal with a certain generalization of the Euler differential equation.

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[1]) This means that for every neighbourhood V of the point 0 in S there is a neighbourhood U of the point 0 in R with $q + U \subseteqq Q$ such that $\gamma(q^+, t)(p^+) \subseteqq \gamma(q, t)(p) + V + K$ forevery $q^+ \in q + U$, $t \in [0, 1]$ and $p^+ \in p + U$.

For this let $p_0, p_1 \in Q$ and an open $Q' \subseteq R$ be fixed. Let us consider R to be (trivially) ordered by using $\{0\}$ as positive cone. Let $\mathfrak{R}$ denote the set of all mappings $x: [0, 1] \to R$ with $x(0) = p_0$, $x(1) = p_1$, $x[0, 1] \subseteq Q$ which are differentiable at every point $t \in [0, 1]$[2]), where the derivatives $dx(t)$ are linear and $\dot{x}: [0, 1] \to R$ given by $\dot{x}(t) = dx(t)\,(1)$ is continuous and $\dot{x}[0, 1] \subseteq Q'$. Every $x \in \mathfrak{R}$ is continuous. Assume $\mathfrak{R} \neq \emptyset$. Let $\mathfrak{R}$ be equipped with the unique topology being such that for every $x \in \mathfrak{R}$, for every natural number $n \geqq 1$ and for arbitrary open sets $U_1, \ldots, U_n$ in R with $x[(i-1)/n, i/n] \subseteq U_i$, $i \in \{1, \ldots, n\}$, each subset

$$\left\{y \in \mathfrak{R} \mid y\left[\frac{i-1}{n}, \frac{i}{n}\right] \subseteq U_i \quad \text{for all } \ i \in \{1, \ldots, n\}\right\}$$

of $\mathfrak{R}$ is an open neighbourhood of x and the system of all such subsets is a base of $\mathfrak{R}$.

Let S be an ordered Banach space (with positive cone K) and L be a continuous mapping of $Q \times Q'$ into S. Let I be the mapping of $\mathfrak{R}$ into S given by

$$I(x) = \int_0^1 L(x(t), \dot{x}(t))\, dt.$$

I is said to attain at $x \in \mathfrak{R}$ a local maximum of the first or second kind — the latter also being called a local Pareto maximum — if there exists a neighbourhood $\mathcal{U}$ of x such that

$$I[\mathcal{U}] - I(x) \subseteq -K \quad \text{or} \quad I[\mathcal{U}] - I(x) \subseteq S \setminus (K \setminus \{0\}),$$

respectively. Every local maximum of I of the first kind is a local maximum of I of the second kind. If $S = \mathbf{R}$, both kinds of local maxima are identical.

Theorem 7: *For every $x_1 \in Q$ and $x_2 \in Q'$ let there exist l-derivatives $d_{1,1}L(x_1, x_2)\,(\cdot)$ and $d_{2,1}L(x_1, x_2)\,(\cdot)$ of L at (x_1, x_2) with respect to the first or second argument, respectively, which are continuous (in all three arguments) where, moreover, $d_{2,1}L(x_1, x_2)$ always is positive homogeneous and subadditive.*[3])

Let $x \in \mathfrak{R}$. For every $t \in [0, 1]$ and $y \in R$ let there exist a u-derivative $d_u(d_{2,1}L(x(t), \dot{x}(t))\,(y))$ of $d_{2,1}L(x(t), \dot{x}(t))\,(y)$ with respect to t which is continuous with respect to t and y.

If I attains at x a local maximum of the first or second kind, then

$$d_{1,1}L(x(t), \dot{x}(t))\,(y) - d_u(d_{2,1}L(x(t), \dot{x}(t))\,(y))\,(1) \in -K \text{ resp. } \in S \setminus \underline{K}$$

for every $y \in R$.

Proof: The procedure is analogous to the usual proof of the Euler differential equation.

1. Let us assume at first that I at x attains a local maximum of the first kind. Let $y: [0, 1] \to R$ be a mapping with $y(0) = y(1) = 0$ which is differentiable at every point $t \in [0, 1]$ where the derivatives $dy(t)$ are linear and $\dot{y}: [0, 1] \to R$ given by $\dot{y}(t) = dy(t)\,(1)$ is continuous. y is continuous. There exists an $\eta > 0$ such that

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[2]) At $t = 0, 1$ the derivative of x naturally is one-sided, this notion being obvious. One-sided derivatives and u-derivatives will also appear in the proof of Theorem 7.

[3]) A mapping $\varphi: R \to S$ is said to be *subadditive* if for every $p, p' \in R$, $\Phi(p) + \Phi(p') - \Phi(p + p') \in K$.

$x + (-\eta, \eta)\, y \subseteqq \mathcal{R}$. For arbitrary $\varepsilon \in (-\eta, \eta)$ let

$$J(\varepsilon) = \int_0^1 L\big(x(t) + \varepsilon y(t), \dot{x}(t) + \varepsilon \dot{y}(t)\big)\, dt.$$

It is easily seen [3: Theorem 12] that for every l-derivative $d_l J(0)$ of $J: (-\eta, \eta) \to S$ at 0,

$$d_l J(0)\,(1) \in -K. \tag{1}$$

Theorem 5 ensures that

$$\begin{aligned}\psi(\tau) = {} & d_{1,l} L\big(x(t) + \varepsilon y(t), \dot{x}(t) + \varepsilon \dot{y}(t)\big)\big(\tau y(t)\big) \\ & + d_{2,l} L\big(x(t) + \varepsilon y(t), \dot{x}(t) + \varepsilon \dot{y}(t)\big)\big(\tau \dot{y}(t)\big),\end{aligned}$$

$\tau \in \mathbf{R}$, defines an l-derivative ψ of $L\big(x(t) + \varepsilon y(t), \dot{x}(t) + \varepsilon \dot{y}(t)\big)$ with respect to ε. By means of Theorem 6 and (1),

$$\int_0^1 \big(d_{1,l} L(x(t), \dot{x}(t))\,(y(t)) + d_{2,l} L(x(t), \dot{x}(t))\,(\dot{y}(t))\big)\, dt \in -K. \tag{2}$$

The assumptions on $d_{2,l} L(x_1, x_2)$ yield that for fixed $x_1 \in Q$ and $x_2 \in Q'$, $d_{2,l} L(x_1, x_2)$ is a u-derivative of the mapping $d_{2,l} L(x_1, x_2): R \to S$. Theorem 5 ensures that a u-derivative $d_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)$ of $d_{2,l} L\big(x(t), \dot{x}(t)\big)\big(y(t)\big)$ with respect to t is given by

$$\begin{aligned} & d_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)\,(\tau) \\ & = \partial_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)\,(\tau) + d_{2,l} L\big(x(t), \dot{x}(t)\big)\big(\tau \dot{y}(t)\big),\end{aligned}$$

$\tau \in \mathbf{R}$, where $\partial_u\big(d_{2,l} L(x(t), x(t))\,(y(t))\big)\,(\tau)$ is the u-derivative of $d_{2,l} L\big(x(t), x(t)\big)\big(y(t)\big)$ with respect to t occuring in x and $\dot{x}$, but not in y. By means of Theorem 4,

$$\begin{aligned} & d_{2,l} L\big(x(1), \dot{x}(1)\big)\big(y(1)\big) - d_{2,l}\big(Lx(0), \dot{x}(0)\big)\big(y(0)\big) \\ & - \int_0^1 d_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)\,(1)\, dt \in -K.\end{aligned}$$

Because of $y(0) = y(1) = 0$, we have

$$d_{2,l} L\big(x(0), \dot{x}(0)\big)\big(y(0)\big) = d_{2,l} L\big(x(1), \dot{x}(1)\big)\big(y(1)\big) = 0.$$

Therefore

$$\int_0^1 \partial_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)\,(1)\, dt + \int_0^1 d_{2,l} L\big(x(t), \dot{x}(t)\big)\big(\dot{y}(t)\big)\, dt \in K$$

and because of (2) hence

$$\int_0^1 \big(d_{1,l} L(x(t), \dot{x}(t))\,(y(t)) - \partial_u\big(d_{2,l} L(x(t), \dot{x}(t))\,(y(t))\big)\,(1)\big)\, dt \in -K.$$

From this, without difficulty for arbitrary $y \in R$ we get

$$d_{1,1}L(x(t), \dot{x}(t))(y) - d_u(d_{2,1}L(x(t), \dot{x}(t))(y))(1) \in -K$$

which proves the assertion of the theorem in the underlying case.

2. If I at x attains a local maximum of the second kind, the assertion of the theorem can be proved analogously where (1) is to be replaced by $d_1J(0)(1) \in S \setminus \underline{K}$ (vid. [3: Theorem 12]) ∎

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